Similarity and Dimensional Analysis

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1. Introduction & Overview

The importance of similarity in comprehending things and reasoning about them was recognized before the time of Plato. Similarity continues to be important in philosophy, science, and technology to this day. The historical roots of the concepts of similarity, ratio and proportion will be accorded a brief mention in this article order to provide a fuller understanding of the relationship between dimensionless parameters and reasoning from similarity. The main topic of this article, however, is the use of dimensional analysis in similarity-based reasoning in current contexts.[1]

1.1 Ratios and Similarity

The use of ratios in reasoning from similarity has a long history. The Pythagoreans, an intellectual community that predated Plato's Academy, discerned a relationship between observable phenomena and ratios. Certain musical phenomena were correlated with ratios of lengths of unconstrained portions of a lyre string; these ratios were equal to ratios of the first few counting numbers. This discovery seemed to them good reason to expect that all physical phenomena could ultimately be accounted for or described in terms of ratios of whole numbers. The ratios that arose in studying harmony appeared in other important mathematical representations, such as the tetractys, a triangular arrangement of ten marks composed of four rows containing one, two, three and four marks, respectively.

[1] Dimensional analysis is not discussed very much in contemporary philosophy of science. For a discussion of the relationship of dimensional analysis and explanation, however, there is a paper forthcoming in the philosophy journal *Nous* by Marc Lange, entitled "Dimensional Explanations." I also discuss it in many of my own works [Sterrett 2000, Sterrett 2002, Sterrett 2005, and Sterrett 2006 (esp. Chapters 5 and 6)]

Many other numbers had geometrical shapes closely associated with them. This was due to the use of two-dimensional arrays of marks representing whole numbers, which were generated using a certain standard procedure. Thus there was a canonical graphical representation for every number generated using the procedure. The ratios between the sides of the array (i.e., the sides of the square or rectangle formed by the array) were also investigated and associated with particular numbers. Thus the study of geometrical similarity was at first associated with ratios of whole numbers. The simplest example of this was that the arrays for all "square numbers" (the numbers 4, 9, 16, . . ., which had a ratio of sides of 2:2, 3:3, 4:4, . . . respectively) were all square (and so geometrically similar). [McKirahan, 1994, Chapter 9]

That reasoning from geometrical similarity involved ratio and proportion is later reflected in Euclid's *Geometry*, albeit there the ratios are thought of primarily as ratios of line segments per se, rather than as ratios of numbers. Many proofs in Euclid's *Geometry* employ the strategy of establishing that two figures are similar in order to draw inferences about other geometrical entities. Such proofs, or portions of proofs, usually proceed by way of identifying certain ratios of lines and then establishing the similarity of figures by means of proportions. Proportions are statements that two ratios are equal. An example is a statement that the ratio of two lengths x and y in a geometrical figure A is equal to the ratio of the corresponding two lengths x' and y' in another geometrical figure B, which can be put as: x is to y as x' is to y', or, in a familiar notation, as $x : y :: x' : y'$. (Equality of angles is required for geometric similarity, but inasmuch as establishing the equality of angles is based upon establishing the similarity of right triangles, some of which may be constructed for just this purpose, equality of angles is an instance of establishing similarity of geometrical figures by means of proportion as well.) Once the similarity of two geometrical figures (such as similar triangles or similar parallelograms) has been established, other equalities between ratios are then known to hold, from which conclusions about the quantities that occur in those ratios can then be drawn.

There is a more formal conception of geometrical similarity. Geometric similarity in a metric space can be defined in terms of the metric defined on the space and a mapping between corresponding points of the two similar geometrical figures: the distance between any two points m and n in geometrical figure A is equal to some constant times the distance between the corresponding points m' and n' in geometrical figure B. (Put formally, the condition is: g (m, n) = r g (m', n'), where g is a metric[2], r is some numerical constant, and m' and n' are points in figure B that are the images of the points m and n in figure A mapped to figure B.) For the special case of Euclidean geometry where g is distance in its usual sense, r would be the scaling factor between two geometrically similar figures A and B. Every pair of figures whose similarity is established using proportions in Euclidean geometry will come out as geometrically similar on this conception as well.

1.2 Physical Similarity

Can the notion of similarity developed in geometry be generalized from geometry to natural science? The answer is yes. It is common to regard the notion of physical similarity as just such a generalization; in order to generalize the notion of similarity from geometry to natural science, *both* the notion of ratio *and* the notion of shape must be generalized. The correct way to carry out such a generalization of geometrical similarity is to generalize from similarity of *geometrical figures* to similarity of *physical systems*.[3] The notion of similar systems goes back to at least Newton, who used the phrase himself, and some less formal versions of the idea are found in Galileo. The formal statement of the notion of similar systems as it is used today did not appear in English-language literature until the early twentieth century.

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[2] A metric is defined on a set. The set along with the metric constitutes a metric space. A metric is defined as follows. If X is a set, a function g: X xX -> R on X is a *metric* on X if it satisfies the following conditions for all x and y in X: 1. g(x,y) is greater than or equal to 0 (thought of as a distance, function, no "distances" are negative) ; 2. g(x, y) = 0 if and only if x=y (i.e., thought of as a distance function, only the "distance" between an element and itself is zero) and 3. the function g (x, y) is symmetric, i.e., g (x, y) = g (y, x) (the "distance" between x and y is the same as the distance between y and x); 4. the function g (x, y) satisfies the inequality g(x, y) + g(y, z) ≥ g (x, z) (the "triangle inequality").

[3] Peter Kroes [Kroes, 1989] has urged the point that analogies between systems is important. I believe the question has received too little attention among those writing about similarity in philosophy of science.

Instead of ratios restricted so as to involve only lengths (of line segments), similarity of physical systems is established using ratios that may involve other quantities as well, such as time, mass, and force. As will be explained later, the quantities are organized into a system, and associated with a system of dimensions. There are some constraints on such a system: physical quantities of the same kind all have the same dimension, and any quantity can be expressed in terms of the dimensions of the set of quantities designated as the set of base quantities. The ratios used to establish physical similarity are themselves dimensionless, just as the ratios relevant to establishing geometric similarity are. They may be as simple as a ratio of two quantities of the same dimension (such as Mach number, which is a ratio of two velocities) or significantly more complex (such as Reynolds number, which contains density, velocity, length, and kinematic viscosity. They are known as dimensionless parameters, or dimensionless groups, and many have been given proper names due to their usefulness in science and engineering.

The generalization of similarity from geometrical similarity to physical similarity is nontrivial, for the notion of shape is generalized from the shape of a geometrical figure to the configuration of a physical system. The identification of an appropriate physical system may require practical empirical knowledge and the application of fundamental principles, and the physical quantities involved must be identified with an eye to the behavior of interest or phenomenon of interest. There are different species of physical similarity. Often a certain species of similarity is identified with a particular dimensionless parameter. There is a progression of physical similarity from geometric similarity (similarity of the linear relations, i.e., the same ratios between distances) to kinematic similarity (geometric similarity plus similarity of motions, i.e., the same paths and ratios between velocities), to dynamic similarity (kinematic similarity plus the same ratios between forces). All of these arise in hydrodynamics, and are discussed in more detail in SCALE MODELS IN ENGINEERING in this volume. There are other kinds of physical similarity used for experimental model testing, though: writing in 1964, Pankhurst[4] listed elastic similarity, similarity of mass distribution, electrical similarity, magnetic similarity, and thermal similarity among the kinds of similarity between model

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[4] [Pankhurst, 1964, p. 18 and pp. 77 -80]. (Pankhurst was Superintendent of the Aerodynamics Division of the National Physical Laboratory from 1964 until 1970.)

and prototype. More recently, dimensionless parameters for physical similarity with respect to phenomena in chemical engineering and fire modeling have been developed. [Slokarnik, 2006] [Becker, 1976] [Hottel, 1961] [Sterrett, 2006]

1. 3 Physical Similarity and Dimensionless Parameters

The generalization from geometrical similarity to physical similarity was not the work of any single thinker, and it took some time for it to be developed into its present form. Now, over two millennia after Pythagoras, the method of physical similarity is a mature, well-established method used in physics, engineering and some applied sciences. The method began in the Renaissance but was not developed and formalized until the late nineteenth and early twentieth century. Edgar Buckingham of the National Bureau of Standards set out the method in use as of 1914 explicitly, though the development of the method was due to others, and was later made more mathematically rigorous.[5] His 1914 paper "On Physically Similar Systems: Illustrations of the Use of Dimensional Equations" [Buckingham, 1914] has become a landmark reference, and the theorem it contains regarding the reduction of a relation between physical quantities to a relation between dimensionless parameters that can be used to establish physical similarity has consequently become widely known as *Buckingham's Theorem* (in spite of his own remarks in later papers ceding credit for priority to Riabouchinsky). [Sterrett 2005, p. 188] Some of the ideas in Buckingham's paper appeared elsewhere much earlier, though, notably in France by Vaschy [Vaschy 1892] and in the context of a practical problem in aerodynamics by Helmholtz.

Another, less celebrated, point in Buckingham's 1914 paper is that, even after considering all logical and mathematical constraints, there is often still some leeway left in choosing which dimensionless parameters to use to characterize a given physical

[5] . T. Ehrenfest-Afanassjewa [1916] provided more mathematically rigorous concepts (the concepts of homogenous functions and homogeneous equations), which J. Palacios [1964, p. 48 -53] employed to present a more rigorous formalization of the concepts and assumptions involved in deriving a version of Buckingham's result. Langhaar [1951] also contains a mathematically rigorous treatment of dimensional analysis and similarity. However, Buckingham's original statement has retained its prominence and is generally the one cited in the engineering literature.

system and to establish physical similarity of a model prototype to the system it is intended to model. Here additional conceptualization of the situation and empirical knowledge are involved, which raises philosophical questions about the interrelationship of mathematics, logic, natural science, applied science, and engineering science.[6]

In this article we are interested in dimensionless parameters and dimensional analysis insofar as they bear on identifying and reasoning about physically similar systems, but it is noteworthy that dimensional analysis has often been discussed apart from its use in establishing similarity of physical systems. Not only was dimensional analysis discussed as a philosophical topic separately from the topic of similarity during the twentieth century, but similarity was discussed in philosophy and, even, by some in philosophy of science, separately from the use of dimensional equations and dimensionless parameters.

Writers in philosophy and philosophy of science in the twentieth century who wrote about similarity often did so without mentioning the theory of dimensions or dimensionless ratios. Some formal investigations into the topics of dimensional analysis and physical similarity in philosophy of science were carried out [Krantz, 1971, p. 454-544], but the work was not very well assimilated into philosophy. Often what philosophers mean by similarity is simply sharing a property, such as red; or that the shade of red of one object is very close to the shade of red of another. The philosopher of science Giere gives as an example of "exploiting similarities between a model and that aspect of the world it is being used to represent" the example of a mathematical model of a pendulum in terms of similarities: a scientist picks out "some specific features of the model that are then claimed to be similar to features of the designated real system to some (perhaps fairly loosely indicated) degree of fit." [Giere 2004, p. 745] That is not quite what is meant by similarity in this article, unless "feature" is generalized in the manner indicated below, i.e., as the value of one or more of the dimensionless parameters that characterize a system.

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[6] The relationship of science to engineering science as currently practiced, especially as revealed in the development of experimental models, is discussed in the entry on SCALE MODELS IN ENGINEERING . The relationship between pure science and engineering regarding ideas involving similarity and dimensional analysis in the years leading up to and immediately following the Buckingham's 1914 paper is discussed in *Wittgenstein Flies A Kite* by Susan G. Sterrett [2005]

In this article, similarity refers to a generalized notion of geometric similarity. On such a notion of similarity, similarity is a matter of proportion, rather than a matter of having some specific feature in common. The notion of similarity we use here (physical similarity of systems) might thus be of interest to those who do discuss similarity in philosophy. One often hears it said that "anything is similar to anything else in countless respects" [Giere, 2004, p. 747], and many philosophers thus consider establishing similarity as involving little more than picking and choosing features to suit a pragmatic purpose, rather than being a formal notion capable of providing a rich resource for investigations. Goodman enunciated such a position long ago [Goodman, 1972, p. 446], granting that statements of similarity "are still serviceable in the streets", but "cannot be trusted in the philosopher's study." For, he concluded, "As it occurs in philosophy, similarity tends under analysis either to vanish entirely or to require for its explanation just what it purports to explain."

The prevailing attitude concerning similarity "as it occurs in philosophy" is hence somewhat at odds with that in scientific fields, given that (i) the concept of geometrical similarity is a rich resource for proofs in the discipline of geometry, (ii) the concept of physical similarity, though including more of an empirical element than geometrical similarity and hence requiring more empirical knowledge in order to be used effectively, is nevertheless a widely used, indispensable and highly regarded method in many natural and engineering sciences, and (iii) the value of the prospect that the notion of physical similarity could be generalized even further. Contemporary discussions in philosophy and philosophy of science might thus be enriched by taking into account the relationship between physical similarity and dimensional analysis in science. Many treatments of similarity in philosophy of science do not even associate reasoning about similarity with ratio and proportion. It is hoped that, in addition to serving as a reference on the topic, this article might contribute to rectifying that situation.

The discussion in this article is centered on the relationship between similarity and dimensional analysis, and some of the philosophical issues involved in understanding and making use of that relationship. Since the connection between similarity and dimensional analysis is via dimensionless parameters, we begin with basics about quantities, units and dimensions.

2. Quantities, Units, and Dimensions

2.1 Quantities, Units, and Quantity Equations

A science such as physics involves using equations or proportionality relations to describe physical phenomena. Equations and relations in the physical sciences relate physical quantities to each other. This is so whether they are expressed in words or in symbolic formulae.

What exactly is meant by a quantity in this context? Here different approaches have been taken.

One approach to the concept of a quantity emphasizes that the physical quantities appearing in physical equations are required to be measurable, and so that the value of a quantity (e.g., "six feet") consists of two portions: a numerical portion and a unit to which all quantities of that kind can be compared. Maxwell explicitly discussed this conception of a quantity, while recognizing ambiguities in the notation of physical quantities as used in equations in scientific practice. He noted that symbols used as variables in equations of physics lent themselves to two different interpretations: (i) as denoting the lines, masses, times, and so on themselves, and (ii) "as denoting only the numerical value of the corresponding quantity, the concrete unit to which it is referred being tacitly understood." [Maxwell 1890, p. 241]

Each of these interpretations presents a problem, though: The first interpretation doesn't really apply during the process of performing the numerical calculations. (This is because, on the account of quantities Maxwell was using, arithmetic operations apply to numbers, not to the quantities themselves.) The second interpretation doesn't satisfy the requirement that "every term [of an equation of physics] has to be interpreted in a physical sense." Maxwell's way of resolving the ambiguity he identified was to take a sort of hybrid approach. During the process of calculation, he said, we should regard the written symbols as numerical quantities; they are accordingly governed by the rules of arithmetical operations. However, he could not completely dispense with the first interpretation; he wrote that ". . . in the original equations and the final equations, in which every term has to be interpreted in a physical sense, we must convert every numerical quantity into a concrete quantity . . ." by "conversion into a concrete quantity," Maxwell meant multiplying the numerical expression by "the unit for that kind of quantity.' [Maxwell 1890, p. 241]

An alternate approach to the concept of quantity is to regard the operations of multiplication and division as applicable to what Maxwell regarded as concrete quantities themselves. On this view, the equations of physics can be regarded as expressing relations between physical quantities, and there is no need for an additional step of interpreting the terms in the equation to yield physical quantities. This alternate approach is the one favored here. Unlike the approach on which a quantity is considered to consist of a numerical portion and a unit, on this alternate approach a quantity can be thought of independently of reference to a unit. To take an example, the definition of the quantity velocity in terms of length and time is thought of as stating a relation between concrete quantities, without involving units or numerical expressions. The relation $V = l / t$ (velocity is proportional to length and inversely proportional to time) is regarded as expressing the relationship between the concrete quantity velocity, a concrete length, and a concrete time, without any hedging regarding the applicability of the arithmetical operations of multiplication and division to concrete quantities. Lodge[7] argued for the recognition of fundamental equations of mechanics and physics as *quantity equations*, i.e., equations that "express relations among *quantities*." [Lodge 1888, p. 281 - 283]

Understood as quantity equations, the fundamental equations of a science "are independent of the mode of measurement of such quantities; much as one may say that two lengths are equal without inquiring whether they are going to be measured in feet or metres; and, indeed, even though one may be measured in feet and the other in metres." [Lodge 1888, p. 281]

Lodge proposed that the arithmetical operations of multiplication and division be applied to concrete quantities of different kinds. In quantity equations, he said, when quantities are represented by numbers or numerical expressions, "that number is the ratio

[7] Cornelius [1965a] credits Lodge as the first to endorse a calculus of quantities. Lodge did so an 1888 article in Nature entitled "The Multiplication and Division of Concrete Quantities" [Lodge 1888]. It should be noted that a calculus of quantities for the quantities of length, area, and volume in Euclidean geometry existed before Lodge's 1888 paper.

of the quantity to some standard of the same kind [of quantity]." Lodge made some important points about quantity equations that distinguished them from the numerical equations (which he regarded as derived from quantity equations). Physical (quantity) equations "can only be among quantities of the same kind, or . . . if there are quantities of different kinds in the equation, then the equation is really made up of two or more independent equations which must be separately satisfied, each of these being only among quantities of the same kind. " He also referred to the dimensions of a quantity, noting that the dimensions of a quantity may not always determine the kind of quantity it is. The examples he gave were of two different concrete quantities having the same dimension, such as work and moment of a force. The significance of maintaining a distinction between dimensions and kinds of quantities will be considered in more detail below.

The mention of the ratio of a quantity to a standard of the same kind of quantity may seem to bring in the use of units under another guise. Does the alternate approach of a quantity calculus as urged by Lodge really use a notion of quantity that is independent of units after all? The answer is yes, because the need to refer to the particular units used is avoided by an important further step: the requirement that relates units and quantities is a requirement placed on the entire *system* of units employed in a science, rather than on each unit separately. This requirement is that the system of units be coherent.

A system of units is coherent if the relations between the units used for the quantities are the same as the relation between the quantities in the fundamental equations of the science. On this view, of course, the fundamental equations are regarded as quantity equations. Using the example of velocity again, the fundamental equation relating velocity, length and time is that velocity is proportional to length and inversely proportional to time, with a constant of proportionality of unity. The requirement that the system of units used be coherent demands that the relationship between the unit used for velocity, the unit used for length, and the unit used for time is the same as the relationship in the fundamental equation, i.e., the quantity equation $V = l / t$. . As Lodge noted, the constant of proportionality of 1 is a choice we have made in selecting our system of units so that it will be coherent with the fundamental laws of physics. The numerical equation derived from the quantity equation V = l / t will then have the same form as the quantity equation.

Thus, if it is known that the system of units is coherent, it follows that the numerical equation has the same form as the fundamental relation. *The form of the numerical equation can thus be known independently* of actually using units and numerical expressions to express the quantities and then deriving the numerical equation from the quantity equation -- so long as the requirement that the system of units is coherent is met. Lodge noted that there is an element of conventionality in our choice of units of length: "If *a, b* are two lengths, the product *ab* is always used to represent a rectangle whose sides are *a, b* respectively; though we *might* have agreed to use it as a representation of a parallelogram with sides *a, b* containing an angle of (say) 60 [degrees]; " [Lodge 1888, p. 282] The arithmetical operations do not rule out this possibility. The requirement of coherence of a system of units does determine some such choices, however.

The step of thinking in terms of requirements placed on a *system* of units, rather than in terms of requirements placed on the units chosen for each kind of quantity individually, turns out to be crucial to settling one of the most puzzling philosophical disputes on the topic: how to understand the role of quantities, units, and laws when converting from one system of units to another used in electrodynamics. In classical mechanics, conversions from one system of units to another were comparatively straightforward and the questions that arose about the kinds of quantities or form of fundamental equations involved were easily settled by further clarification. This was not so in electrodynamics. The resolution of the issues in electrodynamics is discussed below. The resolution of the issues there contributes to our understanding of the conventional aspects (i.e., the aspects that are a matter of choice) of systems of units and physical quantities in other sciences. We shall see that the requirement that a system of units be coherent turns out to be far more central to understanding the relationship between quantities, dimensions and units than one might have suspected.

2.2 Physical Quantities

There have been various philosophical views on the logical relationship between quantities, units, and systems of measurement. Our interest here is the *logical* priority of physical quantities with respect to units and means of measurement, not historical order of the development of the concepts. Though there is some disagreement among philosophers of science as to what the symbols in a scientific equation represent [Palacios 1964, p. vi], most would at least regard scientific equations as relating quantities only insofar as they are at least in principle measurable. The question then arises as to which is logically prior: scales, units of measurement or the quantities measured? If physical quantities are taken to be logically prior, then the question arises as to what sense it makes to talk of quantities existing independently of the means of measuring them. How, one might ask, can a quantity such as six feet tall be logically prior to a measurement system and specification of a unit? On the other hand, if physical quantities are not logically prior to units and measurement scales, one might ask what basis there could be for claiming that a quantity is anything more than the outcome of a measuring process.

One influential view on the relationship of physical quantities and units that is familiar to philosophers of science is Brian Ellis' view in his early work on the topic of quantities in *Basic Concepts of Measurement*. His view there is that "the existence of a quantity entails and is entailed by a set of linear ordering relationships." [Ellis, 1966, p. 32] [8] Like Percy Bridgman's view in his *Dimensional Analysis* [Bridgman, 1963], Ellis thinks that relative magnitude is invariant (so long as the same dimension is associated with the quantities whose magnitudes are being compared). [Ellis, 1966, p. 141] However, unlike Bridgman, Ellis does not take this point as the starting point of a theory of dimensions. Rather, he uses invariance of relative magnitude of like physical quantities as a criterion for similarity of scales, and then defines dimensions as classes of similar scales. [Ellis, 1966, p. 140] Then, Ellis says, "[w]e may say that two scales X

[8] Ellis distinguishes criteria for the existence of a quantity from criteria for identity of quantities: it is the order, and not the ordering relations, that provides the criteria for identity of quantities. [Ellis, 1966, p. 32]

and X' belong to the same dimension if and only if the ratio of any two measurements on X is the same as the ratios of the same measurements on X'."

As for the logical priority of physical quantities and dimensions, many philosophers, including Ellis, are concerned to distinguish their view from what Ellis refers to as a "naïve realist" view. Ellis describes the naïve realist view that most philosophers want to reject as the view that quantities such as yellow or six feet tall are properties of objects, and that such quantities exist prior to specifying a means of measurement. The problem with such a naïve realist view is obvious, as mentioned earlier: how could quantities such as these exist prior to specifying a means of measurement or defining units for measuring it? As numerous philosophers writing on dimensional analysis have noted, dimensional analysis takes place only in the context of the use of scientific equations, many of which can be expressed in such a way that they hold for any coherent set of units (a concept defined below). Ellis' interest was in identifying the elements of conventionality in means of measurement and units of measurement. He later criticized his own proposal presented in *Basic Concepts of Measurement*, on which all quantitative properties supervene on quantitative relationships, concluding in 1992 that that account, though it works for spatiotemporal locations, "will not do" for properties such as charge, which he came to think was, unike length, an intrinsic property. [Ellis, 1992, p. 177] An historical survey of philosophical works on the topic is beyond the scope of this handbook article. Some of the well-known critical studies are: [Ehrenfest- Afanassjewa, 1916] [Campbell, 1920] [Bridgman, 1931] [Langhaar, 1951] [Duncan, 1953] [Sedov, 1959] [Birkhoff, 1960] [Palacios, 1964] [Pankhurst, 1964] [Krantz et al, 1971] [Becker, 1976]

2.3 Systems of Units

In this article, we forgo further discussion of the philosophical questions about the quantities of physical science discussed in the previous section. We now consider the situation faced by someone today making practical use of the theory of physically similar systems. Much of the work that needs to be done in order to identify the quantities of physical science --- including not only the empirical work involved in formulating the equations of physical science but the analytical work in interpreting the constants in those equations, determining which equations are to be considered fundamental laws or principles and which relations and equations are to be considered definitional -- is done *prior to* the specification of a particular system of coherent units. Thus if one is already committed to the use of a certain coherent system of units, some things will no longer be in question: the decisions and conventions that affect the number of quantities required in physical science and which ones will be considered basic quantities are no longer in question, so long as the system of units is considered satisfactory. We will use the almost universally-accepted SI system (*Le Systéme International d'Unités*) here, not as an authority on these foundational questions, but to illustrate the kinds of foundational issues that can arise in developing a system of units.

When new scientific developments suggesting that additional quantities might need to be added arise -- as happened with the development of electromagnetism in the late nineteenth century -- then these decisions need to be reevaluated and the system may need to be amended or revised. At one time, the CGS system provided a coherent set of units for Newtonian mechanics, yet it was not clear what to say at that time about a coherent set of units for electromagnetism. There was a CGS system for Newtonian mechanics, a CGS-M system of units for magnetism, and a CGS-E system of units for electrical phenomena. It was shown in 1901[9] by Giorgi that the CGS system could be amended in various ways to provide a coherent set of units for the quantities in the equations describing electromagnetic phenomena, so that a choice had to be made. The decision made by the committees governing the SI system was to include a base unit for the physical quantity of electric current; the SI system now provides a coherent set of units with respect to Maxwell's equations as well as for Newton's. When used for electromagnetism, the SI system (which follows Giorgi) and the older CGS systems are really different systems of units; for electromagnetic phenomena, unlike for mechanics, switching from one of these systems to the other involves more than a simple change of units, for there will be some equations whose form differs depending on which system

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9 See Taylor & Thompson [2008, p. 16]

one is using.[10] Thus some of the units for the older CGS systems are referred to as non-SI units.[11] [Taylor & Thompson, 2008, p. 37]

The calculations needed to switch from cgs units to the Giorgi/SI units were rather complicated. Attempts to explain the source of the complications gave rise to deep-seated disagreements about the nature of quantities, dimensions and units. In 1964, Cornelius summarized the disagreements and presented a resolution that drew heavily on the role of coherence of (a system of) units: "Considering the questions used in electricity to be quantity equations has raised a lengthy dispute. No agreement was reached on the question of whether the dissimilarities between the different systems are caused by a difference in units or in quantities. . . . [T]he real difference lies in a change in the form of the equations and hence in a change in the coherence of units." [Cornelius 1964, p. 1446] He also provides an explanation of the relationship between quantity and dimension: dimension does not characterize a quantity because dimension is not totally independent of the choice of system of units. Quantities such as charge, length, and current, however, are independent of the specification of a system of units. When a change in system of units involves a change in the form of the quantity equations with respect to which that system of units is coherent, the dimension of the quantity may change. To put it another way: The dimension of a quantity is relative to a system of units inasmuch as a system of units involves selecting the quantity equation(s) that relates the basic quantities of a science. The point is general to all sciences.

This resolution of the disagreements in electromagnetism draws on an important point made by Lodge in proposing a quantity calculus: The meanings of products and quotients in a quantity calculus do not have unambiguous meanings. Different choices

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[10] The interdependence of equations of electromagnetism and systems of units is nontrivial. For an entire book devoted to the subject see [Cohen, 2001]. For a more philosophical and comprehensive treatment see the set of four papers by Cornelius on the topic [Cornelius 1964; Cornelius 1965a, 1965b, 1965c].

[11] As for other points that ought to be noted about the range of applicability of the SI system: The current SI System was developed using equations that do not reflect relativistic effects; this is discussed in [Taylor & Thompson, 2008, p. 13] One area where revisions or amendments to the SI System could possibly occur in the future is in the development of units for quantities that measure biological effects; the difficulties that would need to be addressed are discussed in [Taylor & Thompson, 2008, p. 14] Currently such units are recognized, but as non-SI units.

can be made, even for two different coherent systems. (As explained earlier, this is because coherence is with respect to a particular quantity equation; one could make various choices about interpreting the product *ab* of two lengths *a* and *b*.) According to his explanation, the difference between the Giorgi and various cgs systems of units is really a matter of choosing a different *meaning* for the *ratio* of current to length. It is not a matter of a change in quantity. [Cornelius 1964, 1965a, 1965b, 1965c]

Taking physical quantities to be logically prior to units as described earlier allows Ellis' point that nothing rules out the possibility that a given physical quantity might be measurable by two dissimilar scales, for choices between two sets of similar scales associated with the physical quantity might be among the items that need to be settled by convention, if one of the organizing principles of a system of units is that each quantity is to have only one dimension associated with it. Thus, although Ellis says that the same physical quantity could be associated with more than one dimension (since, for him, a dimension is a class of similar scales) and the SI System takes the approach that each quantity has only one dimension associated with it, these views are addressing slightly different questions. The statement that "each of the seven base quantities used in the SI system is regarded as having its own dimension" [Taylor & Thompson, 2008, p. 11] is made about a coherent system of units containing some conventional aspects, whereas Ellis' statement is a statement about the logical possibility of the existence of a physical quantity for which it is possible to construct two dissimilar measurement scales that meet his criteria for a measurement scale.[12]

The approach taken by the SI system is that it is kinds of physical quantities such as *amount of substance* or *length* (rather than quantities or properties such as twenty moles or six feet tall) that are logically prior to units and measuring systems. The physical quantities that the system is committed to providing a means of describing are those that are related by the equations and relations that constitute empirical science, such as masses, volumes, densities, amount of substance, times, distances, velocities, and so on. The SI system is developed within the context of accepted scientific equations, and, as

[12] In the section on "Dimensional Analysis and Numerical Laws" in the now-classic study *Foundations of Measurement* [Krantz et al, 1971, p. 454], it is stated at the outset that all the physical measures being discussed will be treated as if they are ratio scales.

explained below, is a coherent system of units. As remarked in [Krantz, 1971, p. 464], it is true of all well-known physical laws that they "are of such a form that it is unnecessary to specify the units in terms of which the several physical quantities are reported, provided only that a fixed system of coherent units is used."

To explain what is meant here in saying that the kinds of quantities used in physical science are logically prior to the units used in physical science, we begin again with geometry and then consider generalizations to the physical sciences. We can make sense of such claims as the claim that a square has sides of equal length or that the ratio of the circumference of a circle to its diameter is always the same, without specifying a system of units, or making any mention of a unit at all. Thus the notion of length can be said to be logically prior to that of units of length.

Next, we consider whether we can likewise talk about relationships between physical quantities without specifying units or a means of measurement. Here the empirical is involved as well, as was explained above: the physical quantities are those having a place in a system of quantities used in the equations of an established physical science. The relations between the systems of quantities used in physical science are provided by the fundamental equations and laws of that physical science. Given the relations and equations of empirical science used to identify and define the relations between these quantities, we can make sense of claims that two quantities with which we associate the same dimension (two lengths, two masses, two (local) time intervals, two densities) are equal. In some cases it may turn out that determining the truth or falsity of such a claim requires further investigation, which may involve the use of a measurement systems and units. However, this does not conflict with views on which the dimensions (kinds of quantities) involved can be regarded as logically prior to the specification of a particular system of units. On the approach that it is the relations and equations of science that determine what kinds of quantities there are, kinds of quantities can be regarded as independent of the specification of a system of units. Such a realism is thus not the naïve realism Ellis wished to avoid; it concerns the kinds of quantities that are postulated by the current equations and principles of science, based upon the role they play in a mature science that meets the standards of current scientific practice. It is beyond the scope of this article to deal with questions about the particulars of these standards; what is important to our topic here is that such standards involve checks and balances of various kinds, so that capricious invention of quantities is ruled out.

There is no limit in principle to the quantities (and the dimensions associated with them) that can be defined in this way, and many of them can be expressed in terms of others (e.g., the dimension of velocity can be expressed in terms of the dimensions of length and time). The question of which of the many kinds of quantities used in science are to be considered fundamental or basic ones and which are to be considered derived is not completely determined by empirical science, but involves some arbitrary choices.

2.4 Dimensions and Coherent Systems of Units

In developing a system of units, the advantage of taking an approach on which the physical quantities and the dimensions associated with them are considered logically prior to units is that it allows for each unit to be related to the others. The units can be chosen such that they are related to each other by the same equations and relations as the kinds of quantities are. Thus it is possible to define and choose to use what is known as a *coherent* set of units. As explained in section 2.1, coherence of a set of units is *relative to* a set of quantity equations of a science that are taken as so basic that they are regarded as fundamental laws of that science, and when a system of units is coherent with a set of equations, the equations between the numerical values of quantities have the same form as the equations between quantities themselves. [13] This is a crucially important advantage; the aim of a coherent set of units motivated choices about how to handle quantities as they were added in the current SI system, and the fact that other competing alternative systems of units were not coherent counted against choosing them.

In the SI system, the basic or fundamental quantities are termed "base quantities." The base quantities of the SI system are length, mass, time, electric current, thermodynamic temperature, amount of substance, and luminous intensity. All other quantities are considered derived quantities. It is a consequence of the coherence of a system of units

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[13] The official description of the SI System explicitly notes that it is a consequence of using a coherent system of units that "equations between the numerical values of quantities take exactly the same form as the equations between quantities themselves." [Taylor & Thompson, 2008, p. 12].

that every derived quantity can be written in terms of these base quantities (since these relations are those given by the equations of physics). Each of the base quantities has a dimension associated with it, and the dimension of every derived quantity can be written in terms of the dimensions of the base quantities. Thus, there is a canonical form in which to express the dimension of any derived physical quantity Q. To illustrate, for the SI System, this is stated:

$$\dim Q = \mathsf{L}^a\, \mathsf{M}^b\, \mathsf{T}^c\, \mathsf{I}^d\, \theta^\varepsilon\, \mathsf{N}^f\, \mathsf{J}^g$$

where L is the symbol for the dimension of the base quantity *length*, M is the symbol for the dimension of the base quantity *mass*, T is the symbol for the dimension of the base quantity *time*, I is the symbol for the dimension of the base quantity *electric current*, θ is the symbol for the dimension of base quantity *temperature*, N is the symbol for the dimension of the base quantity of *amount of substance*, and J is the symbol for the dimension of the base quantity of *luminous intensity*. [Taylor & Thompson, 2008, p. 11] The superscripts a, b, c, d, e, f, and g denote dimensional exponents. (The expression above indicates dimension by the sans serif Roman capital letters shown rather than by the use of brackets.) To put the point in general terms: for any coherent system of units, there is a canonical form in which to express the dimension of any derived physical quantity Q, i.e., using our notation in which brackets indicate the dimension of the quantity:

$$\dim Q = [Q_1]^m\, [Q_2]^n\, [Q_3]^o\, [Q_4]^p\, [Q_5]^r\, [Q_6]^s\, [Q_7]^t$$

where $Q_1, \ldots, Q_7$ are the symbols for the dimension of the base quantities, whatever they may be, and $[Q_1], \ldots, [Q_7]$ are symbols for the dimensions of the base quantities. The superscripts m, n, o, p, r, s, and t denote dimensional exponents.

The emphasis in the discussion above on the symbols used to represent dimensions is intentional; the proofs deriving consequences from principles of dimensional analysis rely almost entirely on facts about the symbolic representation of dimensions and their manipulation. The dimensions can be manipulated according to the rules of algebraic manipulation, though dimensional equations are of a different sort than the equations of physical science from which they were obtained. The discussion above is meant to provide an appreciation of what is built into the symbolic representation of dimension: it reflects some structural aspects of the empirical content of physics as a whole [14], some simple rules of algebra and standard arithmetic, and conventions of various sorts. It also helps us to see where some of the conventions of the SI system are involved, and in some cases, whether or not they are arbitrary. For example, the choice as to which quantities are taken as base quantities is arbitrary in that there are other sets of quantities that could serve as the set of base quantities. However, once the base quantities are chosen, how the base units are related to each other is not arbitrary, in that the relations between them are determined by the equations of physical science.

One of the principles used for the SI System is that the dimension of every quantity, whether base or derived, is unique; that is, there is only one such dimension of canonical form associated with each quantity Q. Since the number of derived quantities is unlimited, and the number of dimensions of canonical form is unlimited, one may ask whether there is a unique quantity associated with each dimension of canonical form. The answer is no: more than one quantity may have a given dimension associated with it, just as more than one quantity may have the same units (*heat capacity* and *entropy* are considered physically distinct quantities, though they are both measured in joule/Kelvin; *electric current* and *magnetomotive force* are both measured in amperes [Taylor & Thompson, 2008, p. 26]).

Thus, one cannot infer from a dimension, the quantity with which that dimension is associated, for the quantity is not uniquely determined. One can make some inferences from dimension, though: if one is using a coherent system of units, one can at least infer what units a quantity has from the dimensions of that quantity. If the dimension is written in canonical form, replacing the dimension by the units used to measure the base quantity with which that dimension is associated will provide the units for that quantity. This is not surprising, for it follows from the fact that every base quantity has a unique

[14] By "structural aspects of the empirical content of physics as a whole", I mean the relations between derived and fundamental quantities implied by whatever scientific equations are used in developing a system of units.

dimension and a unique unit associated with it, and that every derived quantity has a unique canonical form expressed in terms of the base quantities, that each quantity will have associated with it a unique set of units expressed in terms of the seven base units.

3. Dimensionless Quantities

For some quantities Q, all the dimensional exponents in the expression dim Q = $[Q_1]^m [Q_2]^n [Q_3]^o [Q_4]^p [Q_5]^r [Q_6]^s [Q_7]^t$ will equal zero. Then dim Q = 1 , and the dimension of Q is said to be *dimensionless*, or of *dimension one*. The unit associated with it may be considered a derived unit; it is said that such a dimensionless quantity has a coherent derived unit of one. Any ratio of two quantities of the same dimension will be dimensionless, and have dimension one. Thus Mach number and refractive index are dimensionless. The dimension of a plane angle, for which the unit radian is used, is also dimensionless.

The SI System also assigns the dimension of one to "counting quantities", as follows:

> "There are also some quantities that cannot be described in terms of the seven base quantities of the SI at all, but have the nature of a count. Examples are number of molecules, degeneracy in quantum mechanics (the number of independent states of the same energy), and the partition function in statistical thermodynamics (the number of thermally accessible states). Such counting quantities are also usually regarded as dimensionless quantities, or quantities of dimension one, with the unit one, 1." [Taylor & Thompson, 2008, p. 12]

The fact that both counting quantities (which take on integral values) and dimensionless parameters used to establish physical similarity (which take on values of real numbers) are of dimension one is not paradoxical. For, in general, although it is true that there is only one canonical dimension per quantity, it is also true that, given a certain dimension, there may be more than one quantity that has the given dimension associated with it.

As mentioned in section 1.2 above, similarity of physical systems is established by showing that the dimensionless parameters that characterize the system's behavior of interest have the same value. The physical meanings of different dimensionless parameters are of course very different, so, if more than one dimensionless parameter is required to characterize the behavior of the system, this is equivalent to saying that the systems need to be similar in both ways in order to be alike with respect to the behavior of interest. The simplest case is the case where the behavior of interest was characterized by a single dimensionless parameter. Some examples of this are that behavior with respect to the existence of shock waves is characterized by Mach number, and behavior with respect to turbulent flow is characterized by Reynolds number. Many dimensionless parameters were in use before the formal bases for the method of physical similarity was developed. These were often conceived of as ratios of two like things, such as two velocities (Mach Number), two lengths (slenderness ratio), or two forces (Froude number, which can be conceived of as the ratio of inertial forces to gravitational forces). Many of the dimensionless ratios in use can be conceived of as the ratio of two like things by rearrangement of the quantities that appear in them (even if they do not appear to be so at first, and even though some algebraic manipulation is required to regroup them into expressions that reveal two like quantities that are physically meaningful).

4. Dimensionless Parameters and Physical Systems

4.1 Dimensional Equations and Dimensional Homogeneity

Given an equation relating physical quantities, one can write a dimensional equation corresponding to it. This is done by replacing the symbol for the physical quantity with the symbol for dimension associated with it; the standard notation used here is to put square brackets around the symbol for a quantity. If the relation between the quantity and the base quantities of the system is known for each of the physical quantities occurring in the equation, then we have a dimensional equation written solely in terms of the dimensions associated with the basic or fundamental quantities. Thus, for an equation expressing the fact that an elapsed time is equal to the sum of two different distances divided by the two different velocities attained in traversing them that is written:

$$(D_1 / v_1) + (D_2 / v_2) = t_T \qquad \text{(i)}$$

(where D_1 and D_2 stand for designated distances, v_1 and v_2 stand for designated velocities, and t_T stands for the time taken to traverse D_1 and D_2). The dimensional equation corresponding to it, using brackets to indicate dimension of the quantity enclosed in the bracket, does not look that different on first glance:

$$[D_1 / v_1] + [D_2 / v_2] = [t_T] \qquad \text{(ii)}$$

which, when expressed in terms of the dimensions of the basic quantities of the system, becomes:

$$[L][L]^{-1}[T] + [L][L]^{-1}[T] = [T] \qquad \text{(iii)}$$

However, the dimensional equations (ii) and (iii) are not the same type of equation as equation (i) in that they relate dimensions. Here L designates the quantity length and T designates the quantity time. No distinction is made between the two different lengths traversed or the two different velocities attained; what is being related are the dimensions associated with each quantity. Since length and time are taken to be among the basic quantities in a system, we can express the dimensional equation in the basic dimensions [L] and [T], as we have done in (iii) above. The statement that the equation is dimensionally homogeneous is reflected by a syntactic fact: the exponents of the dimensions [L] and [T] are the same for the terms on each side of the equation. The point is general: if a dimensional equation is stated in terms of a set of dimensions of the base quantities, the criterion of dimensional homogeneity is that the exponents of the dimensions for each basic quantity in a dimensional equation must be the same on both sides of the equation.

4.2 Dimensional Homogeneity, Dimensionless Parameters, and Similarity-Based Inference

Much of the practical import of dimensional analysis arises from the powerful and simple logical (or, perhaps, grammatical) principle regarding the requirement of dimensional homogeneity, which is applied to dimensional equations. The principle is known as the principle of dimensional homogeneity. Thus, even if the solution to the associated physical equation is not known, the principle can be employed to obtain information by applying it to dimensional equations. What is somewhat suprising, very striking, and especially significant about the principle of dimensional homogeneity as far as our inquiry into the use of similarity, is that the principle of dimensional homogeneity can produce significant and useful results *even in the absence* of an equation describing the behavior of interest. Instead, we can apply the principle in a way that makes use of the dimensions of the quantities that must occur in such an equation, even though the equation itself remains unknown. Thus the principle of dimensional homogeneity is a method distinct from the method of transforming equations into dimensionless form.

The principle of dimensional homogeneity applied in this way provides useful, significant results by providing *similarity criteria* from the knowledge of which physical quantities are relevant to the systems and phenomenon of interest. These similarity criteria can then be used to obtain information about one system based on the knowledge of another that is similar to it. The fact that such a significant amount of information can be obtained from a list of the quantities on which a phenomenon depends has often been found somewhat mysterious. Of course, there is a lot of background information that is implicitly being drawn upon in using the principle in this manner: if we are using a coherent system of units, the fact that the dimensions themselves are related by the fundamental equations of a physical science is implicitly involved, so that the information drawn upon in coming up with the similarity criteria certainly involves *some* scientific equations (e.g., fundamental laws and principles of the science) implicitly, even when an equation describing the behavior or phenomenon of interest in a particular physical system is not at hand. The behavior of interest might be a particular phenomenon, such as the transition to turbulent flow or the generation of shock waves, or it might refer to one aspect of the system's behavior, such as a temperature distribution, the paths traced out by a particular point or points in the system, a flow velocity profile, or the distribution of forces and stresses within the system.

If all that one knows is *which* physical quantities are the ones upon which a certain phenomenon depends, the principle of dimensional homogeneity can be employed to identify a set of dimensionless ratios upon which the phenomenon depends. The analytical process leading to the identification of the physical quantities upon which the behavior of interest for the system depends when one does not have a governing equation in hand is not entirely a matter of logic and mathematics. A crucial initial step is identifying the physical system whose behavior is to be characterized, and this often involves practical and empirical knowledge required to make the appropriate idealizations and simplifications of the actual system. Then, identifying the physical quantities upon which that system's behavior depends might involve invoking general scientific principles, such as conservation laws, to identify the quantities involved in force balances or flow balances. For instance, one might use the principle of the conservation of mass and write out mass balances in order to identify all the quantities involved in a system in which mass crosses the system boundary, or one might use principles of equilibrium and draw a free body diagram in order to help identify the physical quantities involved in a mechanical system. The set of dimensionless ratios characterizing the behavior of interest of a certain physical system (and hence, characterizing a class of systems that will be physically similar with respect to the behavior or phenomenon of interest) is not unique.

It might seem on first glance that having a set of dimensionless ratios upon which the behavior of interest for a given system depends would not provide much more information, or be of much more use, than merely having a list of the physical quantities upon which the behavior of interest for that system depends. The reason that it is in fact more useful to do so is that, unlike the list of physical quantities, the set of dimensionless ratios that characterizes the behavior of interest for a particular system provides similarity conditions: criteria of similarity of systems with respect to a behavior of interest. The behavior of interest is often indicated by the context and will not always be explicitly identified, but it is important to recognize that although what we are speaking about here is providing formal conditions of similarity, it is always similarity regarding, and hence relative to, some behavior or phenomenon of interest. If all of the dimensionless ratios in such a set have the same value in one particular system as they do in another particular system, then those two systems are physically similar. The set of dimensionless ratios characterizes an unspecified number of physically similar systems. There might be an infinite number of systems whose behavior is characterized by the same set of dimensionless ratios and in which those dimensionless ratios can take on the same values as they take on in the given system. Or, at the other extreme, if the similarity conditions are very restrictive, there might be very few other systems -- possibly even none -- that are physically similar to a given system.

The practical advantage of knowing which ratios (dimensionless parameters) the phenomenon or behavior of interest depends upon, rather than merely knowing which physical quantities the phenomenon or behavior depend upon is that the set of dimensionless ratios that characterizes a certain system's behavior also provides a way to establish that two different systems are physically similar and, hence, informs a researcher of how to *construct* a system that will be similar to the given system with respect to the behavior of interest. This is not something that can be done directly using a list of physical quantities alone.[15] The situation here is just as it is for geometrical similarity: we characterize geometrical shapes by the (dimensionless) ratios relevant to characterizing them, and we establish that two geometrical figures are similar (have the same shape) by showing that all the relevant ratios are the same in one figure as in another. Knowing the dimensionless ratios relevant to geometrically similar figures allows us to construct geometrically similar figures, a strategy used in geometry. Without the knowledge that a certain set of ratios characterizes a certain shape (it need not be a unique set), we could not establish that two geometrical figures are physically similar; whereas, with that knowledge, we are able to do so.

Further, the knowledge that certain ratios are equal in the two similar figures or the two similar systems enables the analyst or experimenter to infer the values of a certain

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[15] The topic of physically similar systems and scientific inference is discussed further in Sterrett (2003).

physical quantity if the values of all the other physical quantities are known. This step is straightforward: the values can be computed using the fact that the dimensionless ratios are numerically equal, by simple algebraic manipulation. If this manipulation is carried out algebraically, equations relating the value of a given physical quantity in one figure or system in terms of the values of that physical quantity in the second one, to which it is similar. These relations are sometimes referred to as modeling laws, and the relation between corresponding geometrically similar figures or physically similar systems is linear and can be fully described in terms of a *scale factor*. A common example is the relation expressing the velocities occurring in one system in terms of the velocities in another; these pairs were referred to as *corresponding velocities*. Often it is desired to know the time interval in one system expressed as a multiple of the time interval in the system modeling it; generally time goes faster in a small scale model than it does in the prototype being modeled. These relations will hold only insofar as the similarity conditions are met, of course.

5. Buckingham's Theorem

The ability to correctly infer the values of quantities in one physical system from knowledge of the values of the quantities in a system that is physically similar to it rests on the ability to correctly establish that the two systems are physically similar. In turn, as we have seen, the ability to correctly establish that two systems are physically similar rests on the ability to identify a set of dimensionless parameters that characterizes the system's behavior with respect to the behavior or phenomenon of interest. The set of dimensionless parameters is not unique; any such set is sufficient to establish similarity. We now address the question of identifying such sets of dimensionless parameters for a specific problem. One needs not only to find a sufficient set of dimensionless parameters, but also to *know* that one has found such a sufficient set.

One significant point laid out in Buckingham's paper is that such sets of dimensionless parameters are determined by the principle of dimensional homogeneity, in conjunction with a list of all and only the physical quantities that are relevant to the behavior or phenomenon of interest. Buckingham's paper also included a result about the minimum number of dimensionless parameters that are required to characterize the system's behavior and hence the minimum number of dimensionless parameters that are required to establish that two systems are physically similar. This has a practical importance of great value in that it allows one to minimize the number of laboratory experiments that need to be performed in investigations with experimental models, and the paper is often cited for this result. The reasoning in the proof he gave of that result has been criticized for lack of mathematical rigor in some respects. It is, however, quite useful in that it illustrates how to set up dimensional equations and the relations that the exponents must have to each other. The principle of dimensional homogeneity is then applied, and the solution yields the dimensionless parameters in the set.

Buckingham's analysis in setting up the conditions for the theorem proceeds by considering a system undergoing a transformation rather than by considering two distinct physical systems in which the points of one are mapped to the points of the other, reasoning as follows: "Let S be a physical system, and let a relation subsist among a number of quantities Q which pertain to S. Let us imagine S to be transformed into another system S' so that S' 'corresponds' to S as regards the essential quantities. There is no point of the transformation at which we can suppose the quantities cease to be dependent on one another; hence we must suppose that some relation will subsist among the quantities Q' in S' which correspond to the quantities Q in S. . . . We have to enquire what sort of transformation would lead to [the result that] two systems shall be similar as regards a given physical relation." [Buckingham, 1914, p. 353] This conception of similarity thus involves the preservation of whatever it is that is required to ensure similarity between the system at any point in the process and the system at any other point in the process (i.e., this might be thought of as the "shape" of the system, which is what sameness of the dimensionless parameters that characterize the system throughout the process ensures). Buckingham presented the application to experimental scale models as a special case of a system undergoing a transformation from one size to another. Some later treatments providing a more mathematically-based proof are: [Pankhurst, 1964] [Langhaar , 1951] [Duncan, 1953] and [Palacios, 1964].

Here we put aside controversies about the proof of the theorem itself, as we are mainly concerned with understanding the premises of the theorem, the conclusion of the theorem, and the role of the principle of dimensional homogeneity in it.

The premises of the theorem are that:

(a) There is a dimensionally homogeneous relation between the n quantities $p_1, p_2, p_3, \ldots p_n$ upon which a certain phenomenon or behavior depends. We denote this by the equation $f(p_1, p_2, p_3, \ldots p_n) = 0$. The quantities $p_1, p_2, p_3, \ldots p_n$ related by this equation include all the ones upon which the phenomenon depends, and the phenomenon depends upon all the quantities on the list.

(b) Applying the principle of dimensional homogeneity to the dimensional equation associated with $f(p_1, p_2, p_3, \ldots p_n) = 0$ yields k independent equations relating the exponents in the dimensional equation. In general, there will be one independent equation for each of k base quantities $Q_1, Q_2, Q_3, \ldots, Q_k$, where k is the minimum number of base quantities required to express the n quantities $p_1, p_2, p_3, \ldots p_n$ in terms of base quantities.

The conclusion of the theorem is this: It is a consequence of the principle of dimensional homogeneity that the equation $f(p_1, p_2, p_3, \ldots p_n) = 0$ is reducible to an equation relating (n - k) independent dimensionless parameters $\pi_1, \pi_2, \pi_3, \ldots, \pi_{n-k}$, where each dimensionless parameter is composed only of products that are among the k base quantities. That is, $f(p_1, p_2, p_3, \ldots p_n) = 0$ is reducible to an equation of the form $f(\pi_1, \pi_2, \pi_3, \ldots, \pi_{n-k})$, i.e., an equation relating (n-k) dimensionless parameters. The set $(\pi_1, \pi_2, \pi_3, \ldots, \pi_{n-k})$ will not in general be unique. As explained in section 4.2 above, the advantage of knowing that there exists an equation relating the dimensionless π's is that this knowledge provides similarity criteria for physically similar systems -- even if the equation relating the π's is not known. Knowledge of such a set of dimensionless parameters is sufficient for us to determine *how* to construct a physical system such that it will be physically similar (with respect to a certain behavior or phenomenon of interest) to another given system, if it is possible to do so.

Buckingham illustrated the method of physically similar systems with a concrete but fairly involved example: dynamic similarity of a screw propeller. Writing an equation that describes the dynamic behavior of the propeller is difficult, but identifying the physical quantities upon which the behavior of the propeller depends is less so. He identifies the following seven physical quantities as those on which the propeller's dynamic behavior depends: force, characteristic length and all the length ratios that are required to characterize the shape of the propeller, revolutions per unit time, speed of advance, density and viscosity of the fluid the propeller is immersed in, and gravitational acceleration. Thus there is an equation relating all seven of these quantities; what it is is not specified. There are only three base quantities required (which, in the SI system, would be mass, length, and time) to express the quantities in this list. Thus the minimum number of dimensionless parameters required to characterize the system is 7 - 3 = 4. Here is one of the sets of dimensionless parameters he derives for this problem:

$$\left(\frac{\rho D^2 S^2}{F}, \quad \frac{\rho D^4 n^2}{F}, \quad \frac{\mu^2}{F\rho}, \quad \frac{\rho D^3 g}{F}\right)$$

where ρ is density, D is diameter (chosen as the characteristic length), S is speed of advance, F is force exerted by the propeller, n is the number of revolutions per unit time, μ is viscosity, and g is gravitational acceleration. These are the π's in the equation mentioned earlier to which the equation in terms of the variables f(ρ, D, S, F, n, μ, g) can be reduced: $f(\pi_1, \pi_2, \pi_3, \ldots, \pi_{n-k})$, where f is an undetermined function.

6. Similarity "In Certain Respects", Complete Similarity, and Partial Similarity

We now consider more specifically what similarity amounts to. To establish that two physical systems, each of which is characterized by the same set of dimensionless quantities $(\pi_1, \pi_2, \pi_3, \ldots, \pi_k)$ are physically similar, what we have to establish is that the value of π_1 in System A is equal to the value of π_1 in System B, *and* that the value of π_2 in System A is equal to the value of π_2 in System B, and so on for each of the k dimensionless parameter π's . That is what is required to achieve full physical similarity between two systems, with respect to the phenomena and behavior of interest. However, in practice, often only partial similarity is attempted. There are several different reasons why only partial similarity might be attempted.

Sometimes the requirement of geometric similarity is deliberately violated in the construction of a model. There are cases in which the benefits of complete similarity, i.e., of keeping the distance relations the same between two systems, are outweighed by some other benefit. Geometrically distorted models (sometimes referred to simply as *distorted models*) are used extensively in coastal and river models. In order to be able to model large areas with reasonably sized models yet not have the surface tension of the water in the model play an exaggerated role in the model, the vertical dimensions are not scaled down as much as the horizontal dimensions are. Then, there will be more than one scale factor involved for the quantity of length. This is putting things in a somewhat oversimplified manner, as the reasoning about the details of modeling can be quite involved and employ sophisticated mathematics and physics. [Hughes, 1993]

Sometimes full similarity is preferred, but it is not possible to achieve it. There are many modeling problems for which the only possible system that is physically similar to the given system is one of the same size, so that, no matter what the resources at hand, it is not possible to build a scale model of a size other than a full-size one, and still achieve full similarity. This is not a matter of practicality; it is often the case that even in principle it is impossible to simultaneously satisfy all the requirements specified by keeping the value of all the π's on which the phenomena of interest depend the same in the model as in the prototype. So, most models, even geometrically similar ones, will be distorted in the technical sense that not all the dimensionless π's are the same in the model as in the prototype. [Hughes, 1993] The term *distorted model* is generally reserved for geometrically distorted models, though, and the fact that not all the criteria for physical similarity are met is instead indicated by saying that only partial similarity has been achieved between model and prototype. What is often done is to build one model that aims for similarity with respect to one or only some of the π's, and another model that aims for similarity with respect to another one of the π's. Thus, the usual situation is that one has several scale models, none of which has achieved complete physical similarity with the prototype, but each of which is partially similar to it in a prescribed way, and which is used only for inferences with respect to the phenomena or behavior associated with the dimensionless parameters that are the same between model and prototype.

The points about the practice of scale models hold for the use of physical similarity between systems in general. That is, statements of similarity are by their very nature similarity relative to a certain phenomenon or behavior of the system. Even full similarity is only similarity relative to a certain phenomenon or behavior of the system. Yet there are methods of establishing similarity relative to different phenomena and different system behaviors -- as we have seen, we can derive the dimensionless parameters (ratios) that characterize a certain phenomenon or behavior of a system from knowledge about it that does not always mean we can describe the system behavior in terms of an equation, much less in terms of one for which we know how to obtain solutions. So similarity is always relative similarity, just as in mechanics velocity is always relative velocity. Partial similarity is a different notion than relative similarity, but it, too, is a matter of similarity with respect to a behavior of interest. However, none of these points implies that we cannot say what similarity consists in. Far from it. We can say rather clearly what similarity consists in and what it is based upon, even in the cases in which it is not achievable, and we can say a lot about the consequences that we can reliably draw using it.

References for "Similarity and Dimensional Analysis" handbook article

Becker, Henry A. *Dimensionless Parameters: Theory and Methodology*. Wiley, 1976.

Birkhoff, Garrett. *Hydrodynamics: A Study in Logic, Fact and Similitude*. Princeton University Press, 1960.

Bridgman, P. *Dimensional Analysis*. Yale University Press, 1931.

Buckingham, Edgar. "On Physically Similar Systems: Illustrations of the Use of Dimensional Equations" *Physical Review* 4: 345 - 376.

Campbell, Norman Robert. *Physics: The Elements*. Cambridge, 1920.

Cohen, Douglas L. *Demystifying Electromagnetic Equations: A Complete Explanation of EM Unit Systems and Equation Transformations*. SPIE Press, 2001.

Cornelius, P. , W. de Groot and R. Vermeulen. "Quantity Equations and System Variation in Electricity" *Physica* 30: 1446 - 1452. 1964.

Cornelius, P. , W. de Groot and R. Vermeulen. "Quantity equations, rationalization and change of number of fundamental quantities I" *Applied Scientific Research*, Section B, Vol. 12, No. 1. Pp. 1 - 17. 1965a.

Cornelius, P. , W. de Groot and R. Vermeulen. "Quantity equations, rationalization and change of number of fundamental quantities II" *Applied Scientific Research*, Section B, Vol. 12, No. 1. Pp. 235 - 247. 1965b.

Cornelius, P. , W. de Groot and R. Vermeulen. "Quantity equations, rationalization and change of number of fundamental quantities III" *Applied Scientific Research*, Section B, Vol. 12, No. 1, January 1965. Pp. 248-265. 1965c.

Duncan, W. J. *Physical Similarity and Dimensional Analysis*. Edward Arnold & Co., 1953.

Ehrenfest-Afanassjewa, T. "Der Dimensionsbegriff und der analytische Bau physikalischer Gleichungen." *Mathematische Annalen*, 1916.

Ellis, Brian D. *Basic Concepts of Measurement*. Cambridge University Press, 1966.

Ellis, Brian D. "Conventialism in Measurement Theory," in Savage, C. Wade and Ehrlich, Philip (Eds.) *Philosophical and Foundational Issues in Measurement Theory*. Lawrence Erlbaum, 1992, p. 167-180.

Giere, Ronald "How Models Are Used To Represent Reality." *Philosophy of Science*, Vol. 71, December 2004.

Goodman, Nelson. *Problems and Projects*. Bobbs-Merrill, 1972.

Hottel, H. C. "Fire Modeling" in *The Use of Models in Fire Research*, Publication 786, National Academy of Sciences, National Research Council, Washington, D. C., 1961.

Hughes, Stephen A. *Physical Models and Laboratory Techniques in Coastal Engineering*. World Scientific, 1993.

Krantz, David H. et al. *Foundations of Measurement, Vol. I Additive and Polynomial Representations*. Academic Press, 1971. Republished by Dover Publications, 2007.

Kroes, Peter. "Structural Analogies Between Physical Systems." *British Journal for the Philosophy of Science* **40**, 145-154, 1989.

Langhaar, H. L. *Dimensional Analysis and Theory of Models*. John Wiley, 1951.

Lodge, A. "The Multiplication and Division of Concrete Quantities" *Nature* **38**: 281 - 283. 1888.

Maxwell, J. C. "Dimensions" in *The Encyclopaedia Britannica: A Dictionary of Arts, Sciences, and General Literature*, Ninth Edition, Volume VII. Henry G. Allen Company, Publishers, 1890. Pp. 240 -242.

McKirahan, Richard *Philosophy Before Socrates: An Introduction with texts and commentaries*. Hackett Publishing, 1994.

Palacios, J. *Dimensional Analysis.* (Translated from the Spanish by P. Lee and L. Roth) Macmillan, 1964.

Pankhurst, R. C. *Dimensional Analysis and Scale Factors*. Butler and Tanner, 1964.

Sedov, L. I. *Similarity and Dimensional Methods in Mechanics*. Academic Press, 1959.

Slokarnik, Marko *Scale-Up in Chemical Engineering*. Wiley, 2006.

Sterrett, Susan G. "Models of Machines and Models of Phenomena", *International Studies in Philosophy of Science*. Vol. 20 2006, p. 69-80.

Sterrett, Susan G. *Wittgenstein Flies A Kite: A Story of Models of Wings and Models of the World*. Pi Press (Penguin Books), 2005.

Sterrett, Susan G. "Physical Models and Fundamental Laws: Using One Piece of the World to Tell About Another". *Mind and Society*, Vol. 3, March 2002, p. 51-66.

Sterrett, Susan G. "Physical Pictures: Engineering Models circa 1914 and in Wittgenstein's Tractatus." (long version, presented at UNC-Chapel Hill). http://philsci-archive.pitt.edu/archive/00000661/ 2000.

Taylor, Barry N. and Thompson, Ambler (Eds.) *The International System of Units (SI)*. NIST Special Publication 330, 2008 Edition. (United States version of the English text of the eighth edition (2006) of the International Bureau of Weights and Measures *publication Le Systéme International d' Unités (SI)*). National Institute of Standards and Technology. Issued March 2008. http://physics.nist.gov/Pubs/SP330/sp330.pdf downloaded June 23, 2008.

Vaschy, A. *Annales telegraphignes*, 1892. Pp. 25 - 28.