Two Comments on the Common Cause Principle in Algebraic Quantum Field Theory[*]

[Draft. Please do not quote]

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Until the 1990's philosophers took it almost for granted that the common cause principle is at odds with quantum theory. Roughly, they argued that a common cause explanation of correlations between four pairs of events leads inevitably to Bell inequalities, and since Bell inequalities are violated in quantum theory, there cannot be a common cause explanation of quantum correlations. Redei and his collaborators have made a two-fold effort in order to under-cut the implication from the assumption of common causes to Bell Inequalities. First, they claimed that it's not the assumption of a common cause for each pair of correlated events that leads to the inequalities but the distinct assumption that there is a common cause for all four pairs of projection operators that are correlated; this is the *common-common cause hypothesis* to which I shall return below. The other important contribution is the formulation of the principle of common cause in algebraic quantum field theory and the proof of the existence of a common cause that explains quantum correlations which are prescribed by the violation of Bell inequalities for a state of the system. Hence, not only there is nothing odd in the common cause explanation of quantum correlations, but moreover, the violation of Bell inequalities for a pair of spacelike regions and for a state of the system is a sufficient condition for the existence of quantum correlations, that may be explainable in terms of common causes.

In this talk, I shall present two relatively independent sets of remarks on common causes and the violation of Bell inequalities in algebraic quantum field theory. The first set of remarks concerns the possibility of reconciling Reichenbachian ideas on common causes with quantum field theory in the face of an already known difficulty: the event shown to satisfy statistical relations for being the common cause of two correlated events has been associated with the union, rather than the intersection, of the backward light cones of the correlated events. I explore a way of overcoming this difficulty by considering the common cause to be a conjunction of suitably located events. But I show that this line of thought too is beset with interpretational problems. My second set of remarks concerns the type of inequality one may

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derive from the common-common cause hypothesis: I argue, on grounds of interpretation, that the Clauser-Horne type, and not the Bell type, of inequalities emerge more naturally in this context. Let me, however, start by fleshing out in more detail the setting for the first set of remarks.

Common Cause as Multiple Events

In a series of papers, Redei and Summers (1996,2002, and 2005) formulated Reichenbach's common cause principle in the context of algebraic quantum field theory in Minkowski spacetime and proved that it can be satisfied *in a sense*. An algebraic quantum field theory in Minkowski spacetime can be regarded, in the Haag-Araki version, as a collection of models of the form

$$\left\langle \mathbb{R}^4, \eta_{\mu\nu}, \mathcal{H}, O \mapsto \mathcal{R}(O), \{\phi\}, G, g \mapsto U_g \right\rangle, \tag{1}$$

- where, $\left\langle \mathbb{R}^4, \eta_{\mu\nu} \right\rangle$ is Minkowski spacetime; $\mathcal{H}$ is a separable Hilbert space; $O \mapsto \mathcal{R}(O)$ is a net of von Neumann algebras on $\mathcal{H}$, indexed by open bounded regions O of Minkowski spacetime; ϕ is a locally normal and locally faithful state on the quasilocal algebra $\mathcal{A}$ associated with the net $O \mapsto \mathcal{R}(O)$; G is a group that contains the proper orthochronous Poincaré group; and $g \mapsto U_g$ is a unitary representation of G on $\mathcal{H}$. The models satisfy the following postulates: isotony, weak additivity, locality, relativistic covariance, spectrum condition, local primitive causality and the existence of the vacuum as a Poincaré invariant vector state.

Redei and Summers have proved that for every local system of the form

$$< \mathcal{R}(V_1), \mathcal{R}(V_2), \phi > \tag{2}$$

with V_1, V_2 two nonempty convex open subsets of $\mathbb{R}^4$ which have spacelike separated double cones V_1'', V_2'', as causal completions; and for any pair of projections A,B with $A \in \mathcal{R}(V_1),\ B \in \mathcal{R}(V_2)$ which satisfy inequality (3),

$$\phi(A \wedge B) > \phi(A)\phi(B)\,, \tag{3}$$

there exists a projection operator C which commutes with A and B and satisfies relations (4) to (7):

$$\frac{\phi(A \wedge B \wedge C)}{\phi(C)} = \frac{\phi(A \wedge C)}{\phi(C)} \cdot \frac{\phi(B \wedge C)}{\phi(C)}\,, \tag{4}$$

$$\frac{\phi(A \wedge B \wedge C^{\perp})}{\phi(C^{\perp})} = \frac{\phi(A \wedge C^{\perp})}{\phi(C^{\perp})} \cdot \frac{\phi(B \wedge C^{\perp})}{\phi(C^{\perp})}, \tag{5}$$

$$\frac{\phi(A \wedge C)}{\phi(C)} > \frac{\phi(A \wedge C^{\perp})}{\phi(C^{\perp})}, \tag{6}$$

$$\frac{\phi(B \wedge C)}{\phi(C)} > \frac{\phi(B \wedge C^{\perp})}{\phi(C^{\perp})}. \tag{7}$$

Moreover, C is associated with a region O contained in the weak common past of the two regions V_1, V_2:

$$O \subseteq wcpast(V_1, V_2) = \left(BLC(V_1) \setminus V_1\right) \cup \left(BLC(V_2) \setminus V_2\right). \tag{8}$$

.

Note that (4) - (7) are the transcriptions into the algebraic formalism of quantum theory of the mathematical relations which Reichenbach claimed that a common cause must satisfy. Note also, that the weak common past of two regions V_1, V_2 is a subset of the union of the backward light cones of these regions.

However, what one would expect from a common cause of two events is to be associated with a region contained in the common past of the correlated events, i.e. in the intersection of their backward light cones,

$$cpast(V_1, V_2) = \left(BLC(V_1) \setminus V_1\right) \cap \left(BLC(V_2) \setminus V_2\right). \tag{9}$$

A common cause of two events affects causally both of them and, according to the traditional interpretation of relativity theory, causal affection propagates between spacetime points connectable by causal (timelike, lightlike) curves; hence, the common cause must be associated with the intersection of backward light cones of the regions of the correlated events. But then, how are we to understand the association of the common cause with the weak common past? Do we have to think of it as an *intermediate step* towards establishing the common cause principle in this context? Or, should we rest content with it and accept a "broader" interpretation of the common cause principle in this setting? And by "*broader*", I mean an interpretation which incorporates the traditional one as a subcase.

I shall explore the latter option. I claim that we can analyze the projection operator C, representing the common cause, into a conjunction of projection operators. These projection operators can be associated with disjoint regions in the weak common past of the correlated events. Hence, we can regard the presumed common cause as representing more than one event that take place in the weak common past. What's more, if we consider a family of events to be the explanans of the correlation, rather than a single event, then there is nothing odd in associating this family with a region in the weak common past. For each event of the family can causally affect at least one of the correlated events and also be associated with a region in the backward light cone of at least one of them. In this manner one can salvage a weak common cause principle: given two correlated events at spacelike separation, there is a family of events, each member of which is associated with a region in the weak common past of the correlated events, which explains the correlation.

But as I will show this suggestion encounters some problems too. The first problem concerns the arbitrariness in selecting the family of events in which the projection operator representing the common cause can be analyzed. The second, and more important problem, has to do with whether the selected family of events is indeed suitable for explaining the distant correlation. Although the family of events follows from the analysis of the common cause, that satisfies (4) - (7), it is not straightforward that it has an explanatory value for the correlation. Hence, I examine whether this family of events can meet the conditions of a Reichenbachian Common Cause System, as these are transcribed in the algebraic formulation of quantum field theory. I show that the answer is negative – a result that casts doubt on the viability of the whole project.

Analyzing the common cause

I shall now prove, first, that the projection operator representing the common cause in Redei and Summers' theorem can be written as a conjunction, an intersection, of projection operators; and, second, that these projection operators can be understood as describing different events in the weak common past, since they can be associated with disjoint spacetime regions.

Redei and Summers considered the common cause as a projection operator associated with some region O contained in the weak common past of the two regions V_1, V_2. The corresponding local algebra, $\mathcal{R}(O)$, is a type III von Neumann algebra. But every projection operator in a type III von Neumann algebra can be written as an intersection of two projection operators. This results from the following proposition:

> Let $\mathcal{R}$ be a W*-algebra with no abelian projections and let P be a projection in $\mathcal{R}$. Then P can be written as follows: $P = Q + R$, with Q, R mutually orthogonal equivalent projections (2.2.13 Sakai 1971:86).

Since the local algebra $\mathcal{R}(O)$ is of type III, it has no abelian projections and, by the above proposition, the orthogonal completion of the common cause projection operator can take the form ,

$$C^{\perp} = Q + R = Q \vee R \tag{10}$$

for two mutually orthogonal equivalent projections Q, R. By De Morgan rule we get the desired result,

$$C = C_1 \wedge C_2 , \tag{11}$$

with $C_1 = Q^{\perp},\ C_2 = R^{\perp}$ and $[C, C_1] = [C, C_2] = [C_1, C_2] = 0$. Moreover, we can analyze further each of the $C_i,\ (i = 1, 2)$ in two projection operators and write C as an intersection of four projection operators, and so on; at the *m*-th step of this procedure we can write C as an intersection of 2^m projection operators. Thus we arrive at the following conclusion:

> For every $m \in \mathbb{N}^*$, there is a family $\left\{C_j^{(m)}\right\}_{j=1...2^m} \subseteq \mathcal{R}(O)$ of projection operators such that $C = \bigwedge_{j=1}^{2^m} C_j^{(m)}$.

Next, I shall show that the events described by such a family of projections can be thought to occur in distinct regions in the weak common past. In order to do this, I will follow Redei and Summers' approach (2002) in setting a family of mutually disjoint open regions $\{O_n\}_{n \in \mathbb{N}}$ which can be associated with different events that serve as common causes.

Since V_1, V_2 are bounded and convex, one has $BLC(V_i) = BLC(V_i''),\ i = \{1, 2\}$. In addition, since V_i'' is a double cone, if $V_i'' = V_-(x^{(i)}) \cap V_+(y^{(i)}),$ then $BLC(V_i'') = V_-(x^{(i)}).$ Let $y^{(i)} = \left(y_0^{(i)}, y_1^{(i)}, y_2^{(i)}, y_3^{(i)}\right),\ i = \{1, 2\}$ and $t = \min\left\{y_0^{(1)}, y_0^{(2)}\right\}$. For $T > 0$ and a natural number $n = 2^m \in \mathbb{N}^*$ - corresponding to the cardinal number of the family of projection operators in which C has been analyzed previously – we define a finite family of time slices $\{S_{\lambda T}\}_{\lambda \in \{1,...,n\}}$ of thickness T,

$$S_{\lambda T} = \left\{\left(x_0, x_1, x_2, x_3\right) \in \mathbb{R}^4 : t - (\lambda + 1)T < x_0 < t - \lambda T\right\},\quad \lambda \in \{1, ..., n\} \tag{12}$$

and a corresponding family of open regions $\{O_\lambda\}_{\lambda \in \{1,...,n\}}$ which are the intersection of $S_{\lambda T}$ with the union of the backward light cones of the regions of the correlated events.

$$O_\lambda = S_{\lambda T} \cap \left(V_-(x^{(1)}) \cap V_-(x^{(2)})\right) = S_{\lambda T} \cap \left(BLC(V_1) \cup BLC(V_2)\right) \qquad (13)$$

Hence, for every $\lambda \in \{1,...,n\}$ the region O_λ is contained in the weak common past of the two correlated events,

$$O_\lambda \subseteq \left(BLC(V_1) \setminus V_1\right) \cup \left(BLC(V_2) \setminus V_2\right) \qquad (14)$$

The causal completions of the O_λ 's contain both regions associated with the correlated events. Further, the family $\{O_\lambda''\}_{\lambda\in\{1,...,n\}}$ of the causal completions of the O_λ 's is a directed family, i.e.,

$$V_1 \cup V_2 \subset O_\lambda'' \qquad (15)$$

$$O_\lambda'' \subset O_{\lambda+1}'' , \quad \lambda \in \{1,...,n\} \qquad (16)$$

Now, by isotony and local primitive causality, we have the following relation between the local von Neumann algebras for every $\lambda \in \{1,...,n\}$

$$\mathcal{R}\left(V_1 \cup V_2\right) \subset \mathcal{R}\left(O_\lambda''\right) = \mathcal{R}\left(O_\lambda\right) \subset \mathcal{R}\left(O_{\lambda+1}''\right) = \mathcal{R}\left(O_{\lambda+1}\right) \qquad (17)$$

Moreover, following the line of argument of Redei and Summers (2002), one concludes that for every $\lambda \in \{1,...,n\}$ the local algebras $\mathcal{R}\left(O_\lambda\right)$ are type III. This implies the existence of a projection operator $C \in \mathcal{R}\left(O_\lambda\right)$, representing a different event for different $\lambda \in \{1,...,n\}$, which satisfies the common cause statistical relations. However, according to our previous analysis, every local algebra $\mathcal{R}\left(O_\lambda\right)$ also contains the whole family of projection operator $\left\{C_j^{(m)}\right\}_{j=1...2^m}$ in terms of which C can be written. Thus, if we assume that each projection operator of this family $C_\lambda^{(m)} \in \left\{C_j^{(m)}\right\}_{j=1...2^m}$, $\lambda \in \{1,...,n\}$, describes an event, we may associate with this event a spacetime region O_λ, contained in the weak common past, in which the event took place.

It's obvious that in this approach there is arbitrariness with regard to the events in which the common cause can be analyzed. First, there is arbitrariness with regard to the number of events. A common cause may be regarded as a conjunction of two, four or any power-of-2-number of events. Second, these events may occur in various disjoint spacetime regions as long as some presuppositions, pertaining to the application of the local primitive causality postulate, are satisfied. The feeling of arbitrariness may be mitigated, if one brings to mind the idea that a causal relation is a many-to-one relation - i.e., that an event may be the effect of multiple causes. Thus, one may consider different combinations of events that may occur in various spacetime regions which can have as a cumulative effect the correlated events.

Explaining Correlations: Reichenbach Common Cause Systems

However, the above approach has a more important problem to deal with. I have said nothing so far regarding the statistical relations that the events in a family must satisfy in order to be considered as an explanans of the correlation. A single event, viewed as a common cause, except from being associated with a region in the common causal past of the correlated events, satisfies also certain statistical conditions which may not be sufficient for a causal relation but are assumed to be necessary. The fact that I have constructed a family of events on the condition that their conjunction satisfies the common cause statistical relations does not imply that these events explain the correlation. What we need is to impose further conditions on a pre-selected family of events.

Hofer-Szabo and Redei (2004, 2006) have generalized Reichenbach's common cause principle in such a way that permits us to consider a family of events as the explanans of a correlation. They defined the notion of a Reichenbachian common cause system in a classical probability measure space:

> Let (S,p) be a probability space and A,B two events in S. The partition $\{C_i\}_{i\in I}$ of S is said to be a *Reichenbachian common cause system* (RCCS, for short) for the pair A,B if the following to conditions are satisfied
>
> $$p(A\wedge B|C_i)=p(A|C_i)\,p(B|C_i)\ \text{ for all } i\in I, \qquad (18)$$
>
> $$\left[p(A|C_i)-p(A|C_j)\right]\left[p(B|C_i)-p(B|C_j)\right]>0,\ (i\neq j). \qquad (19)$$

According to Hofer-Szabo and Redei, "the intuitive idea behind the definition of RCCS is that the correlation between *A* and *B* may not be explainable by displaying a single common cause but may be the cumulative result of a (possibly large) number of different "partial common causes ", none of which can in and by itself yield a complete common – cause – type explanation of the correlation, all of which, taken together, can however account for the entire correlation." (2006:747). Moreover, these philosophers explicate what it means to explain a correlation by a system of partial common causes: "one can partition the statistical ensemble into more than two subensembles in such a manner that (i) the correlation disappears in *each* of the subensembles, (ii) any pair of such subensembles behaves like the two subensembles determined by the common cause and its negation, and (iii) the totality of "partial common causes" explains the correlation in the sense of entailing it." (2006:747).

In the algebraic context we may take the probability space to be the triple $\left(\mathcal{R},\mathcal{P}(\mathcal{R}),\phi\right)$, where $\mathcal{P}(\mathcal{R})$ is the lattice of projections of a von Neumann algebra $\mathcal{R}$ and ϕ is a normal state of $\mathcal{R}$. A partition is represented by a family of commuting projection operators $\{C_i\}_{i\in I}$ which satisfies the following relations

$$C_i \wedge C_j = 0 \text{ and } \vee_{i\in I} C_i = I \tag{20}$$

Then, a *Reichenbachian common cause system* for a pair of projections A, B is a partition of $\mathcal{P}(\mathcal{R})$ which satisfies conditions

$$\left[A, C_i\right] = \left[B, C_i\right] = 0 \text{ for all } i \in I \tag{21}$$

$$\phi(A \wedge B\left|C_i\right.) = \phi(A\left|C_i\right.)\phi(B\left|C_i\right.) \text{ for all } i \in I\,, \tag{22}$$

$$\left[\phi(A\left|C_i\right.) - \phi(A\left|C_j\right.)\right]\left[\phi(B\left|C_i\right.) - \phi(B\left|C_j\right.)\right] > 0\,,\ (i \neq j)\,. \tag{23}$$

Although the family of projection operators in which I have analyzed the presumed common cause is not a partition, we can define a partition with the aid of this family. Namely, consider a family of projection operators $\left\{C_1, C_2\right\}$ which are representing events, in terms of which we can analyze the presumed common cause *C* , and satisfying the relations:

$$C = C_1 \wedge C_2 \tag{24}$$

$$C^{\perp} = C_1^{\perp} + C_2^{\perp} \tag{25}$$

$$C_1^{\perp} \wedge C_2^{\perp} = 0 \tag{26}$$

$$\left[C, C_1\right] = \left[C, C_2\right] = \left[C_1, C_2\right] = 0 \tag{27}$$

The subensembles are defined by the following conditions: a) both, C_1, C_2 take place; b) C_2 takes place but not C_1; c) C_1 takes place but not C_2; d) neither C_1 nor C_2 take place. We write these conditions in terms of projection operators as follows:

$$C_1 \wedge C_2\,,\ C_1^{\perp} \wedge C_2\,,\ C_1 \wedge C_2^{\perp}\,, C_1^{\perp} \wedge C_2^{\perp}$$

Using relations (24)-(27) we find that the partition is represented by the following family of projection operators:

$$\left\{C, C_1^{\perp}, C_2^{\perp}\right\} \tag{28}$$

In order to say that (28) describes a Reichenbachian common cause, (21) - (23) must be satisfied. I will assume that (21) is satisfied and then I will show that (22) and (23) cannot be both true.

Since, by assumption, *C* is a common cause, it satisfies (22); also for its orthogonal complement , $C^{\perp}$ holds,

$$\phi(A \wedge B \big| C^{\perp}) = \phi(A \big| C^{\perp})\,\phi(B \big| C^{\perp}) \tag{29}$$

Using (25) to analyze $C^{\perp}$, we have that,

$$a\phi(A \wedge B \big| C_1^{\perp}) + (1-a)\phi(A \wedge B \big| C_2^{\perp}) = \phi(A \wedge B \big| C^{\perp})\,, \tag{30}$$

and

$$\begin{aligned} a^2\phi(A \big| C_1^{\perp})\,\phi(B \big| C_1^{\perp}) + (1-a)^2\,\phi(A \big| C_2^{\perp})\,\phi(B \big| C_2^{\perp}) \\ + a(1-a)\phi(A \big| C_1^{\perp})\,\phi(B \big| C_2^{\perp}) + a(1-a)\phi(A \big| C_2^{\perp})\,\phi(B \big| C_1^{\perp}) \\ = \phi(A \big| C^{\perp})\,\phi(B \big| C^{\perp}) \end{aligned} \tag{31}$$

for $a = \dfrac{\phi(C_1^{\perp})}{\phi(C_1^{\perp}) + \phi(C_2^{\perp})}$.

Next, we substitute (30) and (31) in (29), and we factorize $\phi(A \wedge B \big| C_1^{\perp})$, $\phi(A \wedge B \big| C_2^{\perp})$ according to (22). Thus, by simple calculations, we arrive at the following relation for conditional probabilities of single events:

$$\phi(A \big| C_1^{\perp})\left\{\phi(B \big| C_1^{\perp}) - \phi(B \big| C_2^{\perp})\right\} = \phi(A \big| C_2^{\perp})\left\{\phi(B \big| C_1^{\perp}) - \phi(B \big| C_2^{\perp})\right\}.$$

This, in turn, implies that

$$\phi(A \big| C_1^{\perp}) = \phi(A \big| C_2^{\perp}) \quad \text{or} \quad \phi(B \big| C_1^{\perp}) = \phi(B \big| C_2^{\perp})\,. \tag{32}$$

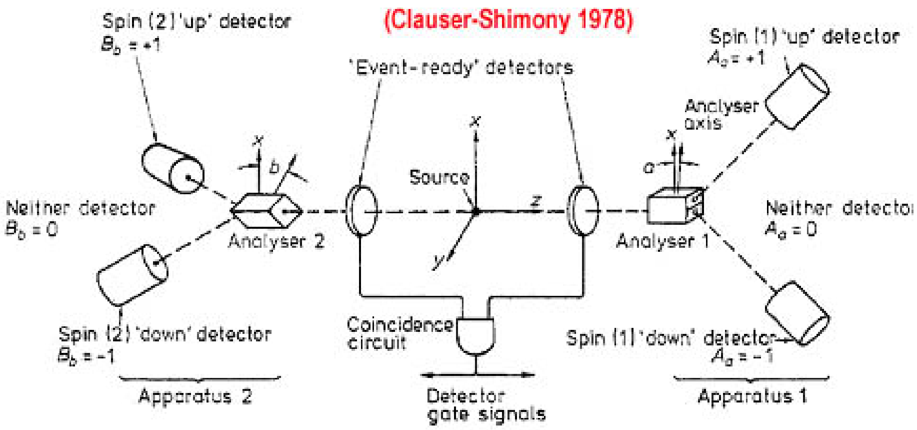
However, if any of these relations is satisfied, then (23) is not satisfied for this partition; hence, the events $\left\{C_1, C_2\right\}$ cannot be regarded as partial causes of a Reichenbachian common cause system.

My motivation for exploring the possibility that different events in the weak common past of two correlated events, taken in conjunction, may satisfy Reichenbach's relations for the common cause and consequently may explain the correlation, was to render possible, on mathematical grounds, an independent interpretation of the fact that the explanans of the correlation has been "found" only in the weak common past. As I have already mentioned, if the explanans is a single event, the common cause of the correlated events, then classical relativistic intuitions suggest that it has to be associated with the common past. Hence, the association of the common cause with the weak common past can only be considered as an intermediate step in the process of restriction of the presumed common cause in the common past of the correlated events. Instead, if it is assumed that the explanans of the correlation is not, generally, a single event, but a family of events which, taken in conjunction, satisfy Reichenbach's relations, there is no problem in assuming that these events take place in the weak common past. In this case we are not obliged to consider one event as the common cause. We may speculate that some events of the family may influence causally only one of the correlated events, although we cannot assume this to happen for all the events of family; and others may influence causally the other event or both the events of the correlated pair. In this way, each event of the explaining family may be considered as a causal factor for at least one of the correlated events, which means that it should be associated with a spacetime region contained in the backward light cone of either event of the pair, i.e. in the union of the backward light cones. On this view, the work of Redei and Summers is not an unfinished derivation of the common cause principle in algebraic quantum field theory but a full derivation of the weak common cause principle, which has its own meaning if we regard the explanans of the correlated pair as a family of events.

Still, this interpretation has some difficulties which can make us skeptical about the viability of the whole approach. First, the common cause can be analyzed in many different families of events and the number of events contained in each family can be selected at will. Second, the association of events with spacetime regions is also quite arbitrary. But the third, and in my view more important problem is that the two or more events in which a common cause can be analyzed cannot constitute a Reichenbach Common Cause System and, in this sense, explain the correlation.

Experimental Inequalities: Bell vs. Clauser-Horne

Let me now proceed to my second set of remarks on common causes in algebraic quantum field theory. First, I want to pinpoint some differences between Bell and Clauser-Horne inequalities. In order to do this, I will remind you the difference between the experimental setups used by Bell (1971) and Clauser and Horne (1974) in the derivation of the corresponding experimental inequalities. Bell considered a two wing experimental setup with a particle's spin analyzer and two particle detectors placed on each wing. For a given position *α* of the analyzer axis, a particle passing through the analyzer could trigger either of the detectors or none of them. Hence, there are three possible outcomes for each measurement of an observable on each wing: +1 for spin-up, -1 for spin-down and 0 if neither detector responds.



Bell inequalities are formulated in terms of expectation values of these three-valued observables. If parameters *α, α´* denote possible orientations of the axis of the analyzer in apparatus 1 and *b, b´*, possible orientations of the axis of the analyzer in apparatus 2, Bell inequalities take the form

$$-2 \leq E(a,b) - E(a',b') + E(a',b) + E(a,b') \leq 2\,, \tag{33}$$

where $E(a,b)$ etc. are the expectation values of the product of observables that correspond to each wing of the experiment.

On the other hand, Clauser and Horne considered a simpler experimental setup of two analyzers and two detectors, one set in each wing of the setup.

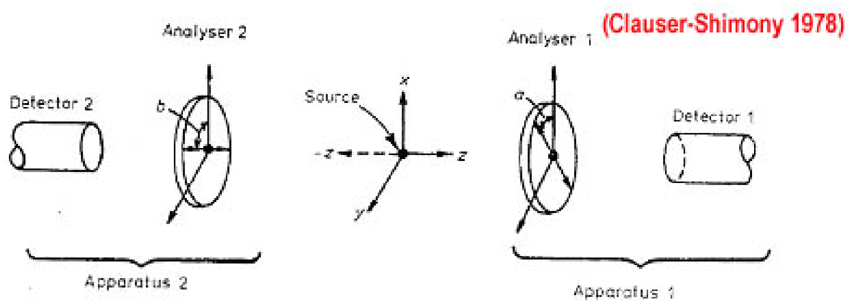


Figure 2. Apparatus configuration used in the proofs by CHSH and by CH. A source emitting particle pairs is viewed by two apparatuses. Each apparatus consists of an analyser and an associated detector. The analysers have parameters *a* and *b* respectively, which are externally adjustable. In the above example *a* and *b* represent the angles between the analyser axes and a fixed reference axis.

The derived inequalities are formulated in terms of probabilities of joint and single events that take place in each one of the detectors for two different orientations of the analyzer axis on each wing,

$$-1 \leq \Pr_{12}(P_a \,\&\, Q_b) - \Pr_{12}(P_{a'} \,\&\, Q_{b'}) + \Pr_{12}(P_a \,\&\, Q_{b'}) + \Pr_{12}(P_{a'} \,\&\, Q_b) - \Pr_1(P_a) - \Pr_2(Q_b) \leq 0 \tag{34}$$

In the above inequalities the numerical indices denote the two apparatuses, one in each wing of the setup, whereas the letter indices, *α, α΄* and *b, b΄* denote the different orientations of the analyzer axis in apparatus 1and 2 respectively.

In appendix B of their paper (1974), Clauser and Horne try to prove the equivalence of their inequalities with those of Bell's. In order to prove that the Bell inequalities are a corollary of theirs, they state that "in an experiment employing two detectors (+ and -) behind each double-channel analyzer, inequalities (4) [Clauser – Horne inequalities] are still applicable providing four sets of inequalities,…". The four sets of inequalities correspond to pairs of detectors or measurement outcomes. From the proper combination of the probabilities for the possible events that can take place on each wing of the setup, Clauser and Horne construct the expectation values of the Bell observables and prove the corresponding inequalities. For the proof of the inverse implication, Clauser and Horne propose that instead of formulating Bell inequalities in terms of the expectation value of products of the Bell observables, we should formulate them in terms of another correlation function. The modification of Bell formulation *"at the beginning*" regards the three valued Bell observables as two-valued, considering only probabilities of events - i.e., of detector measurement outcomes - and not probabilities of "nothing happening", as they say. Using this modification, they prove from the Bell inequalities the Clauser-Horne inequalities.

The primary sources for the formulation of Bell Inequalities in algebraic quantum field theory are two 1987 papers – the first by L. Landau and the other by S. Summers and R. Werner. Both of them propose an analogous treatment of Bell Inequalities in terms of observables that lie between – I and I, where I is a properly defined unit element. I will restrict my presentation to Summer and Werner's formulation of Bell inequalities in terms of a correlation duality

$$< p, \mathcal{A}, \mathcal{B} > \tag{35}$$

where $(\mathcal{A},\geq,1_{\mathcal{A}})$ *and* $(\mathcal{B},\geq,1_{\mathcal{B}})$ are two order-unit spaces, and $p:\mathcal{A}\times\mathcal{B}\to\mathbb{R}$ is a bilinear functional such that for every $a\in\mathcal{A}, b\in\mathcal{B}$, and $a,b\geq 0$ one has $p(a,b)\geq 0$ and $p(1_{\mathcal{A}},1_{\mathcal{B}})=1$.

In order to define the notion of an admissible quadruple they consider a standard version of measurements which admits two outcomes, $\{+,-\}$ which are represented by positive pairs of elements of the order-unit space that constitute a partition of the unit element. $\{P_+,P_-\}\subset\mathcal{A}$ with $P_+,P_-\geq 0$ and $P_+ + P_- = 1$. Every pair of measurement outcomes $\{P_+,P_-\}$ is in one-to-one correspondence with an element $A\in\mathcal{A}$ which represents a Bell observable, or "a measurement" in their terminology, and is defined by equation

$$P_\pm = \frac{1}{2}(1\pm A). \tag{36}$$

An admissible quadruple consists of two measurements on each wing of the measuring apparatus. Hence, they define a quadruple (A_1,A_2,B_1,B_2) as admissible if the following conditions are satisfied:

$$A_1,A_2\in\mathcal{A}, B_1,B_2\in\mathcal{B}, \text{ and } -1_{\mathcal{A}}\leq A_j\leq 1_{\mathcal{A}} \text{ and } -1_{\mathcal{B}}\leq B_j\leq 1_{\mathcal{B}}\ , j=1,2 \tag{37}$$

.

Furthermore, by definition an admissible quadruple (A_1,A_2,B_1,B_2) is said to satisfy Bell inequalities if

$$\left|p(A_1,B_1)+p(A_1,B_2)+p(A_2,B_1)-p(A_2,B_2)\right|\leq 2. \tag{38}$$

In the C* or von Neumann- algebraic setting $\mathcal{A},\mathcal{B}$ are commuting subalgebras of a larger algebra $\mathcal{C}$, the functional p is given by a state ϕ on $\mathcal{C}$ with $p(a,b)\equiv\phi(ab)$ and the Bell inequalities take the form

$$\left|\phi(A_1B_1)+\phi(A_1B_2)+\phi(A_2B_1)-\phi(A_2B_2)\right|\leq 2. \tag{39}$$

In order to formulate Clauser – Horne inequalities using the mathematical apparatus of Summer and Werner, first notice that for a pair (P_i,Q_j), $i,j\in\{+,-\}$, of positive elements which represent possible outcomes of measurements in the two different wings of the experiment the quantity $p(P_i,Q_j)$ gives us the probability for obtaining both the result i at the one wing and j at the other. Hence, for admissible quadruples of positive elements

$\left(P_i^1, P_i^2, Q_j^1, Q_j^2\right)$with $0 \le P_i^k, Q_j^m \le 1,$ for $i, j \in \{+,-\}$ and $k, m \in \{1,2\}$, (34) takes the form,

$$-1 \le p(P_i^1, Q_j^1) + p(P_i^1, Q_j^2) + p(P_i^2, Q_j^1) - p(P_i^2, Q_j^2) - p(P_i^1, 1) - p(1, Q_j^1) \le 0 , \tag{40}$$

where the upper indices correspond to the two different orientations of the analyzer axis and the lower indices the respective outcomes. In the C* or von Neumann- algebraic setting, for a state ϕ on $\mathcal{C}$ we have,

$$-1 \le \phi(P_i^1 Q_j^1) + \phi(P_i^1 Q_j^2) + \phi(P_i^2 Q_j^1) - \phi(P_i^2 Q_j^2) - \phi(P_i^1) - \phi(Q_j^1) \le 0 \tag{41}$$

Note that for different values of i, j ,(40) and (41) describe four sets of Clauser Horne inequalities. Further, if we consider a quadruple of projection operators $\left(P_1 , P_2 , Q_1 , Q_2\right)$ with $[P_k, Q_m] = 0$ for $k, m \in \{1, 2\}$, belonging to the lattice of projections of a von Neumann algebra $\mathcal{R}$ and a normal state ϕ of the algebra, then the Clauser – Horne inequalities take the form,

$$-1 \le \phi(P_1 \wedge Q_1) + \phi(P_1 \wedge Q_2) + \phi(P_2 \wedge Q_1) - \phi(P_2 \wedge Q_2) - \phi(P_1) - \phi(Q_1) \le 0 . \tag{42}$$

Projection Operators: Observables or Measurement Outcomes

For the positive elements of a C* algebra that are smaller than the unit element, usually called effects, there are two possible interpretations, which bear different consequences in relation to Bell and Clauser- Horne inequalities. Namely, a positive element that is smaller than the unit element can be interpreted either as representing an observable having more than one possible measurement outcomes in the sense explained above, or as representing an observable that corresponds to a single measurement outcome. In the first case, according to the preceding discussion, it is *natural* to examine whether a quadruple of such elements satisfies Bell-type inequalities, while, in the latter case, to examine whether this quadruple satisfies Clauser-Horne-type inequalities. Hence, when we derive an inequality for a given quadruple of effects from some premises, we have to keep in mind the interpretational commitments that are implicitly or explicitly included in these premises: if in these premises the elements of the quadruple are considered as representing single measurement outcomes then it makes more sense to derive Clauser-Horne inequalities than Bell inequalities. On the contrary, if in these premises the elements of the quadruple are considered as representing observables that correspond to more than one measurement outcome, then it makes more sense to derive Bell inequalities.

Let's take a closer look at the two cases, restricting our treatment to von Neumann algebras $\mathcal{A}, \mathcal{B}$ and to quadruples of projection operators. Let (P_1, P_2, Q_1, Q_2) be a quadruple of projection operators such that $P_1, P_2 \in \mathcal{A},\ Q_1, Q_2 \in \mathcal{B}$. Obviously this is an admissible quadruple according to the definition, since $-I \le P_1, P_2, Q_1, Q_2 \le I$.

One can consider these projection operators as representing observables or measurements and analyze them further in terms of effects, which are not projections, using the following substitutions

$$A_{\pm}^{\ i} = \frac{1}{2}\left(I \pm P_i\right) \text{ and } B_{\pm}^{\ j} = \frac{1}{2}\left(I \pm P_j\right) \text{ for } i, j = 1, 2 \tag{43}$$

.

Each of these elements, $A_{\pm}^{\ i}$, $B_{\pm}^{\ j}$, can now be taken as representing measurement outcomes according to Summer and Werner's analysis. In this case it does make sense to examine whether the quadruple (P_1, P_2, Q_1, Q_2) satisfies Bell inequalities which are given by (39) and whether any of the quadruples $\left(A_i^1, A_i^2, B_j^1, B_j^2\right)$ for $i, j \in \{+, -\}$ satisfies Clauser-Horne inequalities, (41).

If, on the other hand, we consider these projection operators as representing measurement outcomes then in order to formulate Bell inequalities we have to define an admissible quadruple of observables (A_1, A_2, B_1, B_2) by setting

$$A_i = 2P_i - I \text{ and } B_j = 2Q_j - I\ ,\ i, j = 1, 2\ ; \tag{44}$$

in terms of these elements Bell inequalities (39) can be formulated. In addition, we can formulate Clauser-Horne inequalities (41) in terms of the quadruple (P_1, P_2, Q_1, Q_2). Moreover, if Clauser-Horne inequalities hold for a quadruple of projection operators (P_1, P_2, Q_1, Q_2) then, by simple algebraic calculations, we can derive Bell inequalities for the corresponding observables (A_1, A_2, B_1, B_2).

The philosophical significance of this analysis hinges on a mathematical hypothesis which, for the time being, I can neither prove nor disprove: the equivalence of Bell and Clauser-Horne inequalities for a given quadruple of projections. If the hypothesis is true, then the different interpretational attitude toward the projection operators doesn't make much difference after all with respect to the inequalities that are satisfied. Now, it's easy to see that Clauser-Horne inequalities imply Bell inequalities. For, from (41) it follows that,

$$-1+\phi(P_1)+\phi(Q_1)\le\phi(P_1Q_1)+\phi(P_1Q_2)+\phi(P_2Q_1)-\phi(P_2Q_2)\le\phi(P_1)+\phi(Q_1)\,;$$

since, $0\le\phi(P_i),\phi(Q_j)\le 1$,

$$-2\le -1\le\phi(P_1Q_1)+\phi(P_1Q_2)+\phi(P_2Q_1)-\phi(P_2Q_2)\le 2\,.$$

But I do not know whether the inverse implication holds, although I'm skeptical about it.

Common-Common Cause Hypothesis and Clauser-Horne Inequalities

As I said before when we derive an inequality for a given quadruple of effects, or projection operators from some premises, we have to keep in mind the interpretational commitments that are implicitly or explicitly included in these premises.

The common-common cause hypothesis deals with events, with measurement outcomes. In general, the hypothesis states that if we consider two families of events, $\{P_i\}_{i=1,..,m}$ and $\{Q_j\}_{j=1,...,m}$ for $m\in\mathbb{N}^*$, such that a) there is no direct causal relation between the events of the two families (i.e., for every i,j neither P_i causally affects Q_j nor Q_j causally affects P_i) and b) every pair (P_i,Q_j) manifests positive statistical correlation, then there is a an event C, the "common-common cause", which is the common cause of every pair (P_i,Q_j), i.e., satisfies the usual statistical relations for the common cause for every pair of events.

In the context of algebraic quantum theory, events are represented by projection operators of a von Neumann algebra $\mathcal{R}$ and probabilities are prescribed by normal states of the algebra. Hence, assuming that P_i,Q_j commute $\left[P_i,Q_j\right]=0$, for every i,j, and also that for a normal state ϕ these events manifest positive statistical correlation ,

$$\phi(P_i\wedge Q_j)>\phi(P_i)\phi(Q_j)\,, \qquad (45)$$

for every pair of projection operators (P_i,Q_j) ,then, according to the common-common cause hypothesis there is a projection operator C, that commutes with P_i,Q_j, for every i,j, and satisfies relations (11) to (14):

$$\phi(P_i\wedge Q_j\,|\,C)=\phi(P_i\,|\,C)\phi(Q_j\,|\,C)\,, \qquad (46)$$

$$\phi(P_i\wedge Q_j\,|\,C^\perp)=\phi(P_i\,|\,C^\perp)\phi(Q_j\,|\,C^\perp)\,, \qquad (47)$$

$$\phi(P_i | C) > \phi(P_i | C^{\perp}) , \tag{48}$$

$$\phi(Q_j | C) > \phi(Q_j | C^{\perp}) \tag{49}$$

for $i, j = 1,...,m$; where $C^{\perp} = I - C$ is the orthogonal complement of *C* and for every pair of commuting projections *X, Y* in $\mathcal{R}$, $\phi(X|Y) = \frac{\phi(X \wedge Y)}{\phi(Y)}$, given that $\phi(Y) \neq 0$.

Hofer-Szabo et al. (1999) considered a quadruple of projection operators (P_1, P_2, Q_1, Q_2) which represent four correlated events for a given normal state ϕ of $\mathcal{R}$,

$$\phi(P_i \wedge Q_j) > \phi(P_i)\phi(Q_j) ,\ i, j = 1, 2 . \tag{50}$$

By application of the common-common cause hypothesis, the authors, deduced Bell inequalities for this quadruple:

$$|\phi(P_1 \wedge Q_1) + \phi(P_1 \wedge Q_2) + \phi(P_2 \wedge Q_1) - \phi(P_2 \wedge Q_2)| \leq 2 . \tag{51}$$

and they arrived to the conclusion that the common-common cause hypothesis implies Bell inequalities.

Due to the interpretational commitment of considering the projection operators as representing events, according to the preceding discussion, I claim that it is more natural to derive Clauser-Horne inequalities from such premises, rather than the Bell inequalities Szabo et al (1999) derived. The derivation of Clauser-Horne inequalities can be accomplished using the mathematical procedure of Szabo et al with a suitable modification. Instead of using the arithmetic inequality

$$|a_i b_i + a_i b_j + a_j b_i - a_j b_j| \leq 2 , \tag{52}$$

for numbers $a_i, b_j \in [0,1]$, $(i, j = 1, 2)$, we must employ the following lemma, proved by Clauser and Horne (1974):

<u>Lemma</u>

Given six numbers a_1, a_2, b_1, b_2, *X* and *Y* such that

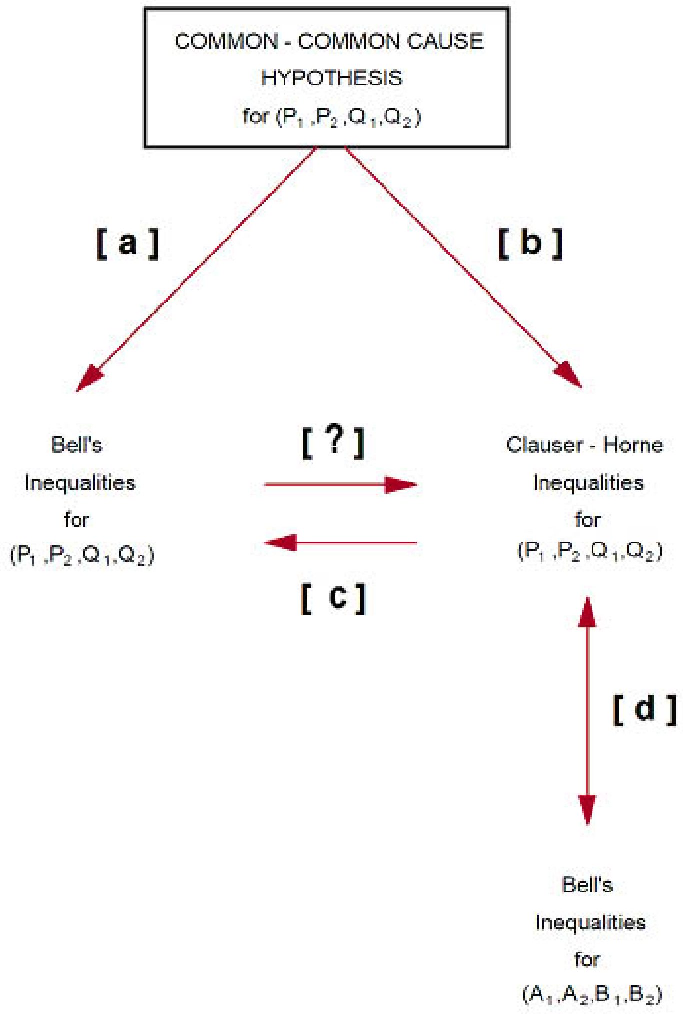
$$\begin{gathered} 0 \leq a_1 \leq X,\ 0 \leq a_2 \leq X, \\ 0 \leq b_1 \leq Y,\ \ 0 \leq b_2 \leq Y, \end{gathered} \tag{53}$$

then the quantity $U = a_1 b_1 - a_1 b_2 + a_2 b_1 + a_2 b_2 - Y a_2 - X b_1$ is constrained by the inequalities

$$-XY \leq U \leq 0. \tag{54}$$

The conclusion of the discussion on the common cause hypothesis and the derivation of inequalities can be presented schematically as follows:

- The use of common-common cause hypothesis as a premise makes more natural the derivation [b] of Clauser-Horne Inequalities than the received derivation [a] of the Bell Inequalities, for a quadruple of projections.
- If a given quadruple of projection operators satisfies Clauser – Horne Inequalities then it satisfies Bell inequalities [c]. If the converse also holds, then my first claim is considerably weakened.
- A given quadruple of projection operators satisfies Clauser – Horne Inequalities if and only if the quadruple of observables (A_1, A_2, B_1, B_2), defined by setting $A_i = 2P_i - I$ and $B_j = 2Q_j - I$, $i, j = 1, 2$, satisfies Bell Inequalities.



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