Theory of Dimensions

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Introduction

This chapter concerns dimensions as the term is used in the physical sciences today. Quantities of the same kind have the same dimension. That two quantities have the same dimension, however, does not necessarily mean they are of the same kind.[1] A more precise definition of dimension will be given within, which will reveal that the dimension of a quantity is not determined for a single quantity in isolation, but is determined relative to a system of quantities and the relations that hold between them.

The concept of dimension is considered to be more generally applicable than to just the physical sciences: some have speculated about the conditions under which the concept might (or might not) apply in social sciences such as sociology, anthropology, and economics. [Grudzewski 2013; Folsom 2005; Barnett II 2004; de Jong 1967; Barenblatt 2003; Wormser 1986; McGuire 1986] In physics, the concept of dimension is already institutionalized in that dimensions play a role in the foundation and development of some of the systems of units used in physics, particularly the SI (*Le Système International d'Unités)*, also known as the International System of Units: dimensions are used in organizing the system of quantities that the units are to be associated with. (BIPM 2014, Sec. 1.3) In terms of logical priority, dimensions are logically prior to units. (Although the formulations of the concept of dimension we use do refer to units, (Fourier 1878, Barenblatt 2003, Sterrett 2009) all that the concept of dimension relies upon regarding units is the possibility of the existence of a system of units of a certain kind.)

Probably the most well-known use of dimensional analysis in physics, aside from checking that a formula does not violate dimensional homogeneity is to reveal physical relationships. The principle that equations of physics must be dimensionally homogeneous can be used to deduce

1 An example of two different quantities that have the same dimension: heat capacity and entropy.

relationships between quantities. Dimensional analysis is somewhat like a logical or mathematical technique in that respect. The advantage of using dimensional analysis is that it requires less effort and information than the usual methods of mathematical derivation would require. This article indicates why that is so.. Its aim is philosophical: to explain the role of dimension, so that it will become comprehensible that the basis for knowledge of such physical relationships as the use of dimensional analysis uncovers has already been incorporated in the notation of dimensions, quantities, and units employed in physics. Then it will be seen that methods using dimensional analysis, while elegant and impressively effective, are but a way of more fully developing the consequences of information that has already been built into the notation used in physics. We shall see that dimensional analysis is a deductive method of proving consequences that follow from the laws of physics (collectively) built into the system of units developed for quantitative sciences.[2] It works as it does only when the system of units has the feature of being a "coherent system of units."

Another powerful application of dimensional analysis is to identify physically similar systems: a system's behavior described in terms of dimensioned quantities can usually be redescribed using a function whose arguments are dimensionless parameters. Informally, a system's behavior can be more succinctly and perspicaciously seen to follow from a set of dimensionless ratios, each of which expresses the interrelatedness of some of the quantities. The number of possible dimensionless parameters is unlimited and the set is nonunique; the set of dimensionless parameters is usually chosen such that the dimensionless parameters have some physical significance, such as a ratio of forces.

Dimensional analysis is employed in the method of physically similar system to identify a suitable set of dimensionless parameters. A dimensionless parameter is a product of quantities (some may have negative exponents) such that in the corresponding dimensional formula, the exponent of each dimension is zero (e.g., Mach number, Reynolds number, Richardson number, and indefinitely many others). (Buckingham 1914, Pankhurst 1964, Sterrett 2017a, Sterrett 2017b)

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[2] Philosophers Robert Batterman (2002; 2009) and Marc Lange (2009) address the question of the explanatory role of dimensional analysis in science. Batterman's discussion follows along the general lines of Barenblatt (esp Barenblatt 1996 on asymptotics), and Lange's of Bridgman. Each makes a novel programmatic suggestion about dimensional analysis in relation to explanation in philosophy of science. However, neither appeals to the features emphasized in this chapter, i.e., what is built into the notation and concept of dimensions and the method of dimensional analysis when a coherent system of units is used in physics. This article aims to show the overarching significance of these features.

Mario Bunge's aim in Bunge (1971) is closer to the aim of this paper. However, his approach is much more formal and set-theoretical. Also, his characterization of coherence of a system of units seems very different from that used in this paper, which makes it difficult to determine Bunge's view about the role of coherence to my points here.

Mach number is one of the simplest examples, as it is the ratio of two velocities: the ratio of velocity of something (e.g., fluid flow, a projectile) to the velocity of sound in that fluid, both with reference to the flow at the existent fluid conditions. The corresponding dimensional formula would be $[L]\,[L]^{-1}$, or $[L]^{0}$. For a physical system whose behavior is parameterized by Mach number, all the systems physically similar to it will exhibit that same behavior at the same numerical value of the Mach number. Similarity of systems is always relative to some kind of behavior (e.g., kinetic, dynamic, electrical, thermal, magnetohydrodynamic). Hence claims that two systems are physically similar systems always need to make clear what sort of behavior is claimed to be corresponding in both systems.

One useful application to take advantage of the characterization of a system's behavior in terms of dimensionless parameters is thus to build a system one can observe or experiment upon that bears this relation of "physically similar system" to a system that one wishes to study but that is experimentally inaccessible for some reason --- and then to design an experiment or test such that the value of the dimensionless parameter(s) in the test and the actual system are the same[3] . Thus one can conduct experiments on a system that is more convenient to experiment upon. The method of physically similar systems can be used to establish the bases for the correlations that exist between analogue models and what they model in many areas of engineering and applied physics. It is not uncommon to build several different models of the same thing one is trying to model, in order to get good models of several different kinds of phenomenon associated with it. Being able to show that a certain physical system is equivalent to another with respect to a certain kind of behavior (dynamical, kinematical, buckling, and so on) can be useful in exploring theoretical questions as well. The potential of the method of physically similar systems is not nearly as well recognized for theoretical investigations, however.

Dimensions in Physics

Dimensional analysis does not have the institutional status of a discipline. As a result, when it is discussed and taught, it is usually done so in response to a need or opportunity arising in the

[3] Of course in practice one usually has to settle for approximations, but that does not affect the statement of the criterion of similarity; it just means something less than exact similarity has been achieved --- just as when building a reproduction, exact identity is seldom if ever achieved. The criterion of *similarity of system behavior* is *identity of dimensionless parameters* -- just as everyone ought to recognize the familiar point that similarity of triangles is a matter of identity of ratios. Similar triangles have *the same angles*, even if in practice when we draw triangles, we can only achieve approximating the angles. Stating the criterion in terms of parameters that are equal (invariant, if one figure is thought of as obtained via a transformation of the other) is a matter of stating similarity conditions correctly. Though two exactly similar systems might never exist in the world, it is still the case that *Similarity of systems* is defined in terms of *identity of dimensionless parameters*. Period.

context of a specific field or discipline. Consequently references and publications that discuss dimensional analysis occur in a variety of disciplines (e.g., physics, philosophy, fluid dynamics, thermodynamics, mechanical engineering, metrology), are often specialized treatments, and are authored by a diverse group of professionals and academics with differing backgrounds, goals and training. Unfortunately, not all are correct concerning what they state about the theory of dimensions; some curating is required on the part of the reader who ventures in this literature.

In theoretical physics and philosophy of physics, the most well-known classic work is Percy Bridgman's *Dimensional Analysis* (Bridgman 1922), which consists of five lectures given in 1920. Bridgman in turn acknowledges the work of Edgar Buckingham[4] , whose most well-known paper on the topic, "On Physically Similar Systems: Illustrations of the Use of Dimensional Equations", which appeared in *Physical Review* in 1914, is also a classic of dimensional analysis (Buckingham 1914). Since many readers will be familiar with, and rely upon, Bridgman (1922), it is mentioned here in order to point out how and where the views in this article diverge from it.

Bridgman's stated aim is to provide the fundamentals of dimensional analysis, his motivation being to "remove" some misconceptions about dimensional analysis that abound. In most uses of dimensions in physics, the existence and applicability of a certain set of dimensions is simply taken for granted. Occasionally, the nature and role of dimensions is even conflated with those of units. In contrast, there are some discussions of dimensions and dimensional analysis in physics in which the notion of dimension is explicitly examined and deliberately applied. According to his stated aim, Bridgman means to be doing the latter. Yet, Bridgman proceeds by assuming the existence of a system of measurement, including units. The topic of how dimensions are involved in the process of developing a system of units does not arise.

More recent books that are cited in the philosophy of physics literature include I. G. Barenblatt's *Scaling, self-Similarity, and intermediate asymptotics* (1996), *Dimensional Analysis* (1987), and *Scaling* (2003), which, its author says "follows [Bridgman's *Dimensional Analysis*] in its general ideas." (Barenblatt 1996, p 14; Barenblatt 2003, p. 37) Barenblatt, too, recognizes the power and unrealized potential of dimensional analysis, writing that "using dimensional analysis, researchers have been able to obtain remarkably deep results that have sometimes changed entire branches of science." (Barenblatt 1996, 1) He emphasizes that the power is in the use of dimensions rather

[4] Bridgman writes: "I am under especial obligation to the papers of Dr. Edgar Buckingham on this subject. I am also much indebted to Mr. M. D. Hersey of the Bureau of Standards, who a number of years ago presented Dr. Buckingham's results to the Conference in a series of lectures." Preface, Bridgman 1922 (preface is dated September 1920)

than in advanced mathematics: "The mathematical techniques required to derive these results turn out to be simple and accessible to all." (Barenblatt 1987, 1) [5]

Barenblatt describes dimensional analysis as encapsulated in one simple idea, an idea about features of physical law, rather than about systems of units or about dimensions: "physical laws do not depend on arbitrarily chosen basic units of measurement." (Barenblatt 2003, p. 9) From this, he says, follows the requirement on physical laws that they possess what he terms "generalized homogeneity or symmetry." Buckingham called it dimensional homogeneity, and credited Fourier with the idea; both Buckingham and Fourier pointed out that the requirement yields a system of linear equations (one for each dimension, expressing the constraint of homogeneity) The idea serves the same kind of purposes as grammatical rules about how to use passive and active verbs, or about subject and verb agreement do: the most obvious is to check that one's construction of a statement or equation is correct.[6] But there are other uses, too: just as, if one is deciphering a sentence that is encoded or is missing letters here and there, grammatical constraints help one infer results from partial information, so the principle of dimensional homogeneity is also helpful in deriving relationships expressed in physical equations using the principle of dimensional homogeneity.

Barenblatt's treatment of dimension follows Fourier's However, like Bridgman, Barenblatt does not describe or address what is involved in the *development* of a system of units. Barenblatt is explicit about the existence, or possibility, of different classes of systems of units. He defines what it means for systems of units to be "of the same class." [Barenblatt, 2003] He states criteria for being *sufficient* -- "sufficient for measuring the properties of the class of phenomena under consideration" -- before a set of fundamental units can be called a system of units. But, as in Bridgman, there is very little room in the discussion for the role of dimension in the construction or development of a system of units to arise. Here I mean to do more than report on the current treatment of dimension in these classic works on dimensional analysis used in physics and

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[5] There are other works regarded as classics in different academic communities. Here Birkhoff's *Hydrodynamics: a study in logic, fact and similitude* is worth mentioning in that Birkhoff regards inspectional analysis (proposed by Tatiana Ehrenfest) as more general than dimensional analysis. Palacios's *Dimensional Analysis* (1964) also developed some of Ehrenfest's ideas, and is admired by some philosophers for its logical rigor. Mario Bunge (1971) has also provided a formal treatment. Other classics often cited in physics venues are Langhaar (1951), Pankhurst (1964), and Gibbings (2011). In history of mathematics and metrology, John Roche's *The Mathematics of Measurement*, (1998) has become an important study.

[6] De Clark 2016 remarks on this use of dimensional analysis: "When a result is found by inspection to violate this rule [dimensional homogeneity], this signals that a mistake must have been made in the course of the derivation. This is systematically used as a first check on results, and it seems fair to say that it constitutes the most widespread use of dimensional analysis in physics today." p. 295 I mention, but have not emphasized, this use as it sometimes encourages conflating the application of the principle of dimensional homogeneity with checking units.

philosophy of physics. I mean to go a bit beyond them by showing the fundamental role that the notion of dimension has in the construction of systems of units[7] -- as well as the role it has in applying dimensional analysis (the requirement of dimensional homogeneity) -- to obtain theoretical and practical results in physics.

Dimensions and Systems of Units

Critical evaluation of the nature and role of dimensions only seldom arises in the normal practice of science, but there are some kinds of developments in the history of science around which such discussions tend to cluster. They are: (i) when a new kind of quantity is to be included in the foundations of a science (a key past example here would be temperature (Fourier's *Analytical Theory of Heat* (1878))), and (ii) when the foundations of a system of units is being reformulated (a past example here would be Giorgi's resolution of two conflicting systems of measurement (the electrostatic cgs and the magnetostatic cgs). The Giorgi system is considered the first forerunner of the current SI system. An especially important aspect of Giorgi's suggestion was to show that "the "absolute" system of practical units could be combined with the three mechanical units metre, kilogram and second to constitute a single coherent four-dimensional system of units." (Teichmann 2001)

As this book goes to press, we are at another such point in history: there is to be a major revision of the International System of Units which, though it is not the first, and may not be the last, such revision, is of singular significance. [BIPM 2014] It is referred to as the Proposed Revision of the SI, or, sometimes, as the "New SI." It is of singular significance in that it would achieve the centuries-old aim of a measurement system in which all the units are defined in terms of fundamental constants of nature.

The reason that the role of dimensions in constructing and developing a system of units will be discussed in this article, in spite of the absence of such discussions in the classic works mentioned above, is that in order to understand why dimensional analysis yields the information that it does, it is important to understand what is built into a system of units. The crucial feature built into a system of units that accounts for much of the power of the method of dimensional analysis is *coherence of a system of units*. [8]

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[7] De Courtenay (2015) is a rare and valuable paper in that it is devoted to a critical examination of the construction of a system of units; the treatment is broadly in accordance with the view in Roche (1998).
[8] I have argued for this point regarding similarity and dimensional analysis in Sterrett 2009,

One way to characterize coherence of a system of units is that "a system of units is coherent if the relations between the units used for the quantities are the same as the relations between the quantities in the fundamental equations of science." [9] Here fundamental equations of science, including definitions, are the source of the relations between quantities. (BIPM 2014, section 1.1) Here is a very simple example: since velocity is proportional to length, and inversely proportional to time, with a constant of proportionality of unity, the requirement that the system be a coherent system of units requires that the units for velocity, length and time must bear the same relations to each other that the quantities do. The same point about relations expressed by definitions holds for relations expressed by fundamental laws of nature, too. So, as de Clark illustrates using Newton's second law, which states that force equals mass times the second derivative of x with respect to time, $F = m\, d^2x/d^2t$ "where F stands for the force on the object, m its mass, x its position and t for time—all understood as the physical quantities themselves. [. . .] The dimensions of a derivative are the same as that of a normal ratio; therefore, the dimensions of force are $[F] = MLT^{-2}$." (de Clark 2016, p. 295) These are constraints we impose on the system of units in the course of developing it. Notice that there is an element of choice involved, i.e., applying the requirement of coherence of a system of units actually does some work in formulating a system of units. For, other than coherence of a system of units, there is no constraint that forces the choice of a constant of proportionality of 1 in this example, for instance.[10]

Once it can be presumed that the system of units in use in physics is a coherent system of units (in the sense that relations between units are the same as relations between quantities), dimensional analysis can be employed to spin out consequences that seem to have been generated out of practically nothing. This is because, as explained in earlier works (Sterrett 2009), "if it is known that the system of units is coherent, it follows that the numerical equation has the same form as the fundamental relations [which are relations between quantities]." Thus, the requirement of dimensional homogeneity is about the logic of the interrelation of quantities used in physics, and it contains content gained empirically, since the content of many of the fundamental laws of physics was obtained empirically.

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[9] A different way to characterize the coherence of a system of units is in terms of base units and derived units. The coherence of a system of units on this characterization is stated as a requirement placed on the selection of base units: a requirement "to select basic units which would produce derived units by multiplication or division without introducing numerical coefficients; . . ." [Burdun, 1960, p. 913-914]. In this article, we will use the former characterization of coherence of a system of units, as it does not presume there must be base units in a system of units.

[10] The element of choice involved is emphasized in Lodge (1888) and discussed in Sterrett 2009.

A system of units constructed in this way thus has built into it many interdependencies between all the various units; equations expressing these relationships will thus have the same form as the interdependencies between quantities encoded in the fundamental laws or relations of physics. In a nutshell: the relations between units in a coherent system of units will reflect the relations between quantities (given by fundamental physical laws) and dimensional analysis (using the principle of dimensional homogeneity) will help spin out these relationships between quantities. Constructing a system of units is done as a whole; a system of units is much more than a collection of units constructed individually for each quantity. The units are interrelated, and they are interrelated in accordance with fundamental physical laws identified in the process of constructing that system of units.

Dimensions and Quantity Equations

We have seen that in a coherent system of units, the relations between units of the system reflect relations drawn from physical laws. This constraint can be applied even before identifying the value of any of the units (as will be explained below). Coherence of a system of units will be relative to the physical laws that one identifies as fundamental laws of physics, so the choice of physical laws on which to base a system of units is logically prior to the system of units. It is at this point in the logical reconstruction of a system of units that dimensions are involved.

It might seem puzzling at first to talk about laws or equations prior to having identified a system of units, if one thinks that laws or equations must be expressed numerically. However, there is in fact a way to talk about kinds of quantities such as length, mass, time, charge, and temperature even before the units in which to measure them have been chosen: equations of physics can be regarded as expressing relations between quantities, or *quantity equations*. Alfred Lodge [Lodge 1888, p. 281 - 283] articulated this approach in the late nineteenth century. Lodge pointed out that, understood as quantity equations, the fundamental equations of a science 'are independent of the mode of measurement of such quantities; much as one may say that two lengths are equal without inquiring whether they are going to be measured in feet or metres; and, indeed, even though one may be measured in feet and the other in metres." An article written to serve as a reference on the topic, "Units and Dimensions" (by Lodge's student Guggenheim), argues that "we are entitled to multiply together any two entities, provided our definition of multiplication is self-consistent and obeys the associative and distributive laws."[Guggenheim 1942] (Here it may be helpful to recall that it is possible to add, multiply, divide and take square roots using only an unmarked compass and straightedge (no ruler) in elementary school geometry.) The term "quantity equation" currently appears in the Vocabulary of International Metrology, which gives its meaning as "mathematical relation between quantities in a given system of quantities, independent of measurement units." (VIM 1.22 at http://jcgm.bipm.org/vim/en/1.22.html)"

The question of what counts as a quantity and how relations between quantities are determined is important to understanding what quantity equations are. The notion of dimension is used here. In this paper, we shall use brackets around an uppercase letter to indicate we are talking about dimension, rather than a specific quantity or a unit. (Using brackets is a very common way to indicate dimension, although the SI (International System of Units) uses a different notation to indicate dimension instead (BIPM 2014, Section 1.3 "Dimensions of Quantities"; Sterrett 2009, p. 812)

Fourier uses the notion of dimension to exact some discipline on his analysis, as he attempts to develop fundamental equations for a new science. He writes "the terms of one and the same equation could not be compared, if they had not the same component of dimension. We have introduced this consideration into the theory of heat, in order to make our definitions more exact, and to serve to verify the analysis; it is derived from primary notions on quantities;" (Fourier 1878, section 160, p. 128) The notion of dimension is introduced by considering how magnitudes change when the size of units are changed. For him, dimension is *with respect to a unit* for some kind of quantity; an example here is finding the dimension of the quantity "surface conducibility h" with respect to length. (Fourier 1878, section 161, p. 130) He reasons the exponent of dimension is -2. In our notation, this part of the formula would be written as $[L]^{-2}$ Fourier lays out the method for determining exponents of dimension in a way that is generalizable and thus provides a template, or canonical format.

Note that we might have two different quantities that yield the same exponents of dimension. This definitively illustrates the need for the notion of dimension in addition to the notion of quantity, and underscores the point that dimensions are not quantities. Lodge remarked on the fact that dimension does not determine quantity, using the example of work and moment of force. Work and moment of force are two distinct quantities that have the same dimension. That two different quantities have the same dimension is not a reductio of the notion of dimension, however, pace some commentators, e.g., Emerson (2005). There is nothing objectionable about it; it is not inconsistent with the concept of dimension or of quantity.

Fourier uses the exponents of dimension in a statement expressing the constraint of dimensional homogeneity, i.e., that all the terms in the equation have the same dimension. He was presuming there would be a system of units with three fundamental units, one each for length, temperature, and duration. Each term in a physical law regarded as a quantity equation (which presumes no specific system of units) must have the same exponents of dimension, so there will be three equations (e.g., one for exponents with respect to length, one for exponents of dimension with respect to temperature, and one for exponents of dimension with respect to duration.) We would speak of the dimension for length, the dimension for temperature, and the dimension for duration.

The dimensional equation associated with a quantity equation is often not as informative about the physical situation as the quantity equation is. That is not a shortcoming, however, since its purpose is different. Dimensional equations might be thought of as showing grammatical features of physical equations, analogous to helping us meet grammatical constraints such as subject-verb agreement in an English sentence. The constraint, of course, is dimensional homogeneity, and it is a constraint on equations of physics that express relations between quantities. Consider a simple formula from mechanics: distance traveled equals acceleration times the square of the elapsed time, often written as $s = a\, t^2$. The dimension associated with distance would be simply [L]; the dimension associated with acceleration would be $[L][T]^{-2}$ (in Fourier's way of putting things, the exponent of the dimension of acceleration with respect to the unit for length is 1, and the exponent of the dimension of acceleration with respect to the unit for time is -2). The dimension for the square of elapsed time is, of course, $[T]^2$. The dimensional equation is thus $[L] = [L][T]^{-2}[T]^2$. If we were to write an equation for the exponents of dimension with respect to the unit of length we would get 1 = 1, and for time, we would get 0 = -2 + 2. Both of these are equalities, confirming that the formula is dimensionally homogenous.

Fourier's development of the notion of dimension was occasioned by his development of a theory of heat, which called for adding a new kind of quantity not included in mechanics. But it was not just analytical theory of heat that called out for a new unit not already included in mechanics. The notions of quantity equations, coherence of a system of units, and dimensions were important in reaching a resolution in one of the most important advances towards the current SI system of units: the change from the CGS systems in absolute units, which used three base units, to a four dimensional one, which went beyond the three-unit system used in mechanics and made room for a new unit specifically associated with electrodynamic phenomena.

Absolute Units

The word 'absolute' in the term 'absolute units' is meant to contrast with 'relative' or 'comparative.'

The historical context in which absolute units first arose was Humboldt's work aggregating magnetic measurements taken around the globe, which were relative measurements; he used the value of the instrument reading taken at the equator as a reference for other measurements taken with the same instrument. Gauss and Weber devised a method of providing measurements of the earth's magnetism in dimensions used in mechanics: mass, length, and time. (Gauss 1832/1995; Main 2007) The term absolute units is sometimes used even now to refer to a system of units that makes use of only units in these three dimensions. Such systems are sometimes referred to as 3D systems, and the base and derived units of CGS referred to as 'mechanical units'.

Absolute units were also seen as having the benefit of relating a concept in a newer, comparatively unknown science about which there were still many questions about quantities and measurement, to the known science of mechanics about which there was much confidence: all the units of the new science could be conceived of in terms of units that were already familiar from mechanics. Two additional systems of units, each of which used the three base units used in mechanics, arose: CGS-M for magnetic phenomena and CGS-E for electrical phenomena. All three -- CGS, CGS-M, and CGS-E -- were coherent systems of units, but they differed in the set of physical equations that served as the fundamental laws on which to base the coherence of their system. The dimensions of a quantity (e.g., charge) could differ depending on which of the systems one was working in, which was the source of much discussion. (Cornelius 1964, 1965a, 1965b, 1965c) There has been a resurgence of interest in dimensional analysis among philosophers of science, and some rigorous critical-historical scholarship on the topic has appeared in just the last few years. One recent study notes that work on dimensional formulae by Maxwell is often left out of editions of his collected works, and describes some of the confusions that have been left behind as a result. (Mitchell 2017) In a scholarly critical-historical study of debates related to the dimensions of the magnetic pole in the nineteenth century, de Clark (2016) points out that the discussants sometimes did not even agree as to whether a purported problem concerning dimensions actually constituted an inconsistency or not (de Clark 2016) As for related philosophical works, Nadine deCourtenay's 2015 "The double interpretation of the equations of physics" gives a critical-historical account from the standpoint of the kinds of equations used in physics, explaining the significance of the change from proportionalities to numerical equations. This is a very valuable piece of work for philosophers of science, as it explains how systems of units are constructed, and how the problem raised by the change made in the modern era to the use of numerical equations is now "hidden." These three recent papers provide socio-cultural and intellectual historical context that lend insight into the nature and significance of the theory of dimensions, and provide historical context that richly supplements other works on the topic, including S. G. Sterrett's "Similarity and dimensional analysis" (2009) and the classic works on dimensional analysis mentioned earlier,

Role of Dimension in Current SI

That the development of a coherent system of units involves identifying scientific equations on which the system is to be based, that there is sometimes a choice involved in the form of the equations chosen, and that the choice can make a difference to the features of a system of units is evident from the current official description of the SI (BIPM 2014.) First, as to the need to identify scientific equations:

> "In order to establish a system of units, such as the International System of Units, the SI, it is necessary first to establish a system of quantities, including a set of equations defining the relations between those quantities. This is necessary because the equations between the quantities determine the equations relating the units, . . ." (BIPM 2014.)

Thus, it is concluded that "in a logical development of this subject, the choice of quantities and the equations relating the quantities comes first, and the choice of units comes second." (BIPM 2014, Section 1.1)

The relations between quantities are just the usual equations of science and engineering, the kind of relations found in textbooks and used on a daily basis in labs and engineering office. These have been identified in various standards and an effort made to consolidate them in an international standard (ISO 80000).

In the current SI, the role of quantities, units, and dimensions is presented in official publications from the BIPM, as follows.

> "physical quantities are organized in a system of dimensions. Each of the seven base quantities used in the SI is regarded as having its own dimension" [...] "All other quantities are derived quantities, which may be written in terms of the base quantities by the equations of physics. The dimensions of the derived quantities are written as products of powers of the dimensions of the base quantities using the equations that relate the derived quantities to the base quantities." (BIPM 2014, Section 1.3)

To a philosopher keen on identifying sources of knowledge in the practice of science, the preceding passage indicates even more than that the relations between the units are made to be the same as the relations between the quantities according to the equations of physics, science, and engineering with which we are familiar. It tells us that these relations are encoded in the notation for dimensions of the quantities used in physics. Thus, in using the notation for dimension in the SI and the principle of dimensional analysis to deduce equations, we are actually deducing some consequences of the equations of physics used in establishing the system of units as a coherent system of units.

Anyone who understands how mathematical derivations can likewise yield a great deal of interesting constructions and theorems from just a few postulates and definitions will understand what is meant here. Dimensional analysis is a genuine supplement to the methods of pure mathematics, though, in that it employs the notation of dimensions, and hence can take advantage of what is encoded, or built into, that notation, to obtain results from much less information. More generally, the fact that physical equations (equations of physics), must satisfy the requirement of dimensional homogeneity, provides an explanation of how mathematical equations as used in physics can be informative about the world. The answer does not lie in mathematics alone, but in an understanding of how physical equations, in conjunction with the use of a coherent system of units, will provide information about the world.

Natural Units

Not all science is carried out using the SI system, however. There are a number of alternatives loosely grouped under the rubric "natural unit systems." These systems of units employ non-SI units, and dimensional analysis is not always readily applicable when using equations expressed in terms of them (as explained in van Remortel 2016).

One reason for favoring the use of such natural unit systems in certain subspecialties of physics is that the past and current SI systems of units are inconvenient for very large or very small scales. A more philosophical discomfort with the SI that led to preferring alternatives is that the determination of values for units in the SI relies on "precision measurements of standard prototypes [such as the standard kilogram or, in the past, the standard meter], objects, or systems that define a physical unit." [van Remortal 2016.],whereas, in natural unit systems, units are defined with reference to some fundamental physical constant. In some fields such as particle physics and general relativity, it is much more convenient to express results, theoretical as well as experimental, in reference to a fundamental physical constant of nature, such as the speed of light or mass of some subatomic particle. (BIPM 2014, Section 4.1)

Role of Dimension in New SI

The ideal that natural units aim at remains a desirable goal: a system in which units are defined in terms of fundamental constants of nature. There have already been a number of changes in how specific units have been defined over the years. Here the meter is an iconic example, going from being defined with reference to the earth, then to a prototype kept in a vault, and ultimately to its present definition in terms of the velocity of light. (BIPM 2017) At the other end of the extreme is the kilogram, which is currently (in 2017) still defined in terms of an artefact, which is a prototype kilogram kept in a vault. However, all the work required to enable a major overhaul of the way units are defined has been carried out, and will be implemented in the next revision of the SI, due in 2018. The ideal of defining all units in terms of fundamental constants of nature (per Figure 1) will finally be achieved.

The value of the fundamental constants will be set in terms of the units to be defined, and the units will be collectively defined in terms of the fundamental constants as a result. For example, as shown in Figure 1, the velocity of light is no longer a measured quantity. (Rather, its value is set in units of second and meter, which has the effect of defining the meter in terms of it as a fundamental constant of nature. All seven units that have historically been designated as base units are collectively defined in terms of the seven fundamental constants shown in Figure 1.) Since the value of the fundamental constants is set, rather than measured, the values of the fundamental constants have zero uncertainty.

A draft version of the official wording of the SI system, including the definition of the units, has already been developed and approved. (Figure 1) Here we are especially interested in how dimensions will be treated after the switch to defining all units in terms of fundamental constants. The Draft Ninth SI Brochure reads as follows:

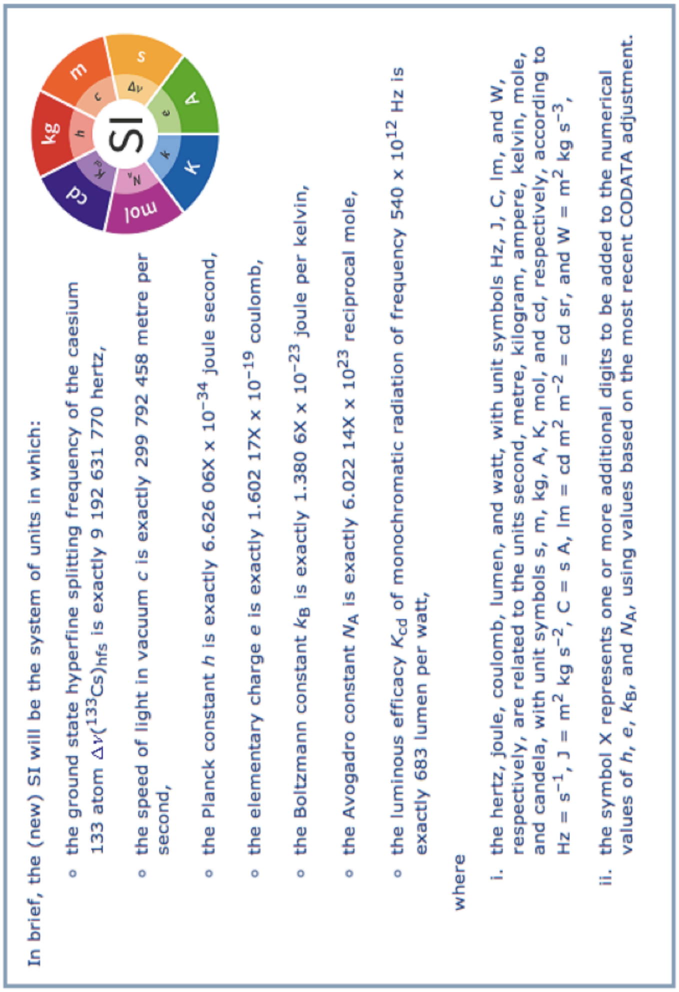
> **"2.2.3 Dimensions of quantities**
>
> Physical quantities can be organised in a system of dimensions, where the system used is decided by convention. Each of the seven base quantities used in the SI is regarded as having its own dimension.

Importance of dimensions in philosophy of science

Dimensions, units, and quantities are distinct notions. We have seen above how they are related in the design of coherent systems of units; the account involves the equations of physics. When the use of a coherent system of units can be presumed, dimensional analysis is a powerful logico-mathematical method for deriving equations and relations in physics, and for parameterizing equations in terms of dimensionless parameters, which allows identifying physically similar systems. The source of the information yielded by dimensional analysis is not yet well understood in philosophy of physics. This article has attempted to reveal the role of dimensions not only in applications of dimensional analysis to obtain information by involving the principle of dimensional homogeneity, but to the role of dimensions in encoding information about physical relationships in the language of dimensions, specifically via the feature of coherence of a system of units.

Philosophers of mathematics and philosophers of science have been concerned to address the question of the effectiveness of mathematics in science. Given the role of dimensions as explained in this chapter, an important part of the answer is that the notation of dimensions is a powerful means of including the content of physical theories into a system of units, and of providing the means of deducing valuable information about the consequences of them, collectively, afterwards. Thus, no philosophical analysis of the question of the applicability of mathematics to science is complete without including dimensions and dimensional analysis in the picture.

Figure 1



In brief, the (new) SI will be the system of units in which:

- the ground state hyperfine splitting frequency of the caesium 133 atom $\Delta\nu(^{133}Cs)_{hfs}$ is exactly 9 192 631 770 hertz,
- the speed of light in vacuum c is exactly 299 792 458 metre per second,
- the Planck constant h is exactly 6.626 06X × 10^{-34} joule second,
- the elementary charge e is exactly 1.602 17X × 10^{-19} coulomb,
- the Boltzmann constant k_B is exactly 1.380 6X × 10^{-23} joule per kelvin,
- the Avogadro constant N_A is exactly 6.022 14X × 10^{23} reciprocal mole,
- the luminous efficacy K_{cd} of monochromatic radiation of frequency 540 × 10^{12} Hz is exactly 683 lumen per watt,

where

i. the hertz, joule, coulomb, lumen, and watt, with unit symbols Hz, J, C, lm, and W, respectively, are related to the units second, metre, kilogram, ampere, kelvin, mole, and candela, with unit symbols s, m, kg, A, K, mol, and cd, respectively, according to Hz = s^{-1}, J = m^2 kg s^{-2}, C = s A, lm = cd m^2 m^{-2} = cd sr, and W = m^2 kg s^{-3},

ii. the symbol X represents one or more additional digits to be added to the numerical values of h, e, k_B, and N_A, using values based on the most recent CODATA adjustment.

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