Jagiellonian University
Institute of Philosophy

Intuitionistic logic versus paraconsistent logic. Categorical approach

Mariusz Stopa

A dissertation submitted for the degree of
Doctor of Philosophy

Supervised by
Prof. Michał Heller

Kraków, 2022

To my family and my Dominican brothers and sisters

Acknowledgements

I would like to thank my supervisor, Prof. Michał Heller, for example of his life and scientific work, for his great cordiality, for everything I have learned from him and for introducing me to the world of categories. I also thank Prof. Jerzy Gołosz for his scientific help and great kindness.

I would also like to express my great gratitude to my family, especially my parents and brothers. I owe everything to them, directly or indirectly.

I would like to thank my Dominican brothers and sisters for their spiritual support, understanding and community of life.

Last but not least, I would like to thank all my friends. First of all, but not only, I would like to thank Grzesiek, Marcin, Piotrek and Radek.

Contents

Introduction

Category theory (CT) is a remarkable, relatively young mathematical theory that plays an important role in mathematics today. Its origin can be traced to the work of Samuel Eilenberg[1] and Saunders Mac Lane from 1945 (Eilenberg and Mac Lane, 1945). The importance of CT, apart from its relevance for mathematical logic and philosophy of mathematics, extends to physics (see e.g. Abramsky and Coecke, 2008; Döring and Isham, 2011; Heller and Król, 2017; Isham and Butterfield, 1999), computer science (see e.g. Pierce et al., 1991), and more general philosophy (see e.g. Landry, 2018). CT in itself as a mathematical theory, in its applications and philosophical aspects, creates a very extensive part of human knowledge.

Intuitionist logic (IL) and, to a lesser extent, paraconsistent logic (PL) are among the best known and analyzed non-classical logics in the literature. Regardless of the context of their origin, they have also appeared in recent times within the framework of CT.[2]

The relationship between CT and mathematical logic is extensive and fundamental. Sufficiently rich categories have their own internal logic. An exceptional

[1]Samuel Eilenberg is related to the Polish thread in the development of CT, as he was a Polish Jew born in Warsaw in 1913. There, he received his doctorate in mathematics at the University of Warsaw. He was an important representative of the Warsaw school of mathematics. Later he was also a member of the Bourbaki group. He was "widely known as Sammy, or even S^2P^2: 'Smart Sammy the Polish Prodigy'" (from the foreword by D. Eisenbud, Mac Lane's Ph.D. student, in Mac Lane, 2005, p. xi).

[2]The IL case is obvious, while as to PL's relationship with CT see e.g. Estrada-González (2010, 2015b); James (1996); Mortensen (2003); Vasyukov (2000, 2017). We do not comment on these works here, at this point we only want to show that such publications exist. Interestingly, Goodman, whose work we shall discuss in more detail in sec. 2.3.3, writes that it was F.W. Lawvere, who "first suggested to us that there must be a logic dual to the intuitionistic in which contradictions would not be rejected" (Goodman, 1981, p. 119). Lawvere is the one who played a key role in the beginning of CT's relationship with logic and the foundations of mathematics, as Marquis notes: "it is ... in the early 1960s that the connection with logic and the foundations of mathematics was made and it was mostly the work of one person, namely Bill Lawvere" (Marquis, 2020, p. 71).

role among them is occupied by the so-called toposes.[3] The logic of toposes has its own specificity, which we will discuss in more detail below. The standard view is that the logic of toposes is IL, or more precisely the so-called intermediate logic. In special cases, the logic of a topos may also be classical logic (CL). Topos theory is an interesting and rich theory that goes far beyond logical applications. One of the creators of toposes, A. Grothendieck, laureate of the Fields Medal, considered topos theory to be a kind of generalization of topology, one of the most important branches of mathematics. Toposes, by linking algebra, geometry, and logic at a deep level, reveal their great unifying power, but they will be of interest to us especially in the context of logic. CT, as well as its part, the topos theory, already have a very extensive literature (see e.g. Awodey, 2010; Barr and Wells, 2005; Goldblatt, 2006; Johnstone, 2014, 2002; MacLane and Moerdijk, 1994; McLarty, 1995; Simmons, 2011; Smith, 2016). Relatively recently, some papers have suggested that the so-called co-toposes (also called complementary toposes) may provide a suitable semantics for PL, understood as dual to IL (see e.g. Mortensen, 1995, chapter 11 and Estrada-González, 2010, 2015a, though we shall discuss these and other papers as well as related topics in more detail later).

Our attitude

Our departure point is CT itself, its beauty and extensive development. It has originated from purely mathematical investigations in mathematics. CT has, however, also contributed enormously to the development of logic itself. It is well known that toposes are very closely connected with higher order intuitionistic logic, although this connection should be studied carefully, because, as Colin McLarty pointed out, “topos logic coincides with no intuitionist logic studied before toposes” (McLarty, 1995, p. vii).[4] It is very important to note that toposes were not designed to have this connection with IL. Although CT has interesting connections with philosophy

[3] We shall use this term, ‘toposes’, for the plural form of a ‘topos’, instead of sometimes used a word ‘topoi’. We do so in face of the convincing argument in this respect in the footnote 4 in (McLarty, 1990).

[4] As a comment let us copy here a part of the footnote 2 of our paper (Stopa, 2020): In short, higher-order intuitionistic logic of toposes agrees with traditional Heyting’s rules of inference for connectives and quantifiers, but the disjunction and existence properties, which are a traditional part of intuitionism, do not hold for toposes in general (see Mclarty, 1990, p. 154). Although toposes are intrinsically connected with (higher-order) intuitionistic logic, they were not simply designed to agree with it (see Mclarty, 1990, p. 152f).

and logic, its origins are purely mathematical. Let us mention a significant quote from Lambek and Scott: "Nothing could have been further from the minds of the founders of topos theory than the philosophy of intuitionism" (Lambek and Scott, 1994, p. 125). It is interesting in itself that in the definition of an elementary topos very little is assumed and many their unexpected and interesting properties are discovered on the way. There are of course toposes whose logic is just classical or one of the intermediate logics (ILs).[5]

Topos theory is sometimes considered as a generalization of Set Theory. However, as McLarty powerfully argues in his paper (McLarty, 1990), this is not the right way to think about toposes. He writes: "The view that toposes originated as generalized set theory is a figment of set theoretically educated common sense" (McLarty, 1990, p. 351, abstract). Let us quote a longer passage from this paper, as it is interesting also in other aspects (McLarty, 1990, p. 352):

> Category theory arose from a complicated array of practical problems in topology. Topos theory arose from Grothendieck's work in geometry, Tierney's interest in topology and Lawvere's interest in the foundations of physics. The two subjects are typical in this regard. An important mathematical concept will rarely arise from generalizing one earlier concept. More often it will arise from attempts to unify, explain, or deal with a mass of earlier concepts and problems. It becomes important because it makes things easier, so that an accurate historical treatment would begin at the hardest point.

Our attitude in the present work is somehow similar with respect to treating toposes autonomously. We want to study some aspects of toposes in the context of their connection with logic without assuming any prior philosophical or logical convictions. We want to investigate the properties of toposes as certain mathematical 'objects'. We want to allow ourselves to be guided by the mathematics, so to speak. Therefore our investigations of some algebraic aspects of both IL and PL from the categorical perspective do not indicate our philosophical positive atti-

[5]For simplicity, the abbreviation 'ILs' shall be used to denote intermediate logics, i.e. consistent superintuitionistic logics, which are intermediate between IL and classical logic, including those on boundaries. It can be understood less precisely as 'intuitionistic logics', but strictly speaking it means intermediate logics.

tude towards either intuitionistic ideas or inconsistencies in general. As IL is better known and most of mathematicians are much more familiar with it than with PL (if at all (s)he is familiar with either of them), our concern regarding especially PL might be considered odd and asks for more clarification about our motivations. As we mentioned above, our motivation is mathematical or more specifically algebraic. But if we were to give our philosophical attitude towards the inconsistencies it would be similar to that expressed by Perzanowski (Perzanowski, 1999, p. 23):

> Inconsistencies must be examined. Not prejudged. Nor worshipped as idols, as in the case of most Hegelians (excluding Graham Priest and other logical philosophers, I hope). Quite the contrary. We examine them in order to find a remedy. In search of the understanding about their sources, reasons and real consequences.

Some features of logic in toposes

Let us first emphasize that the logic of a given topos is not imposed from outside, but results from the structure of the arrows of a topos.[6] Every topos has its specific structure of arrows which is connected with logic. This is beyond our control, we can only discover it, describe it, and possibly use it somehow. Here we describe the standard approach to logic in toposes which is a common knowledge in textbooks. Later we shall argue that some arrows can be interpreted differently, or that they can play different roles in different structures which may lead to some new applications to logic, but first let us see just some examples of diversity and richness of the logical aspects of toposes. Let us note only that these logical aspects of toposes are just a portion of much wider logical aspects of CT in general.

First we observe that the logical operations can be described and even defined as so-called adjoints to some basic operations. This applies also to universal and existential quantifiers, which can be shown to be adjoints to the operation of substitution. We only mention very briefly these facts, but as Awodey comments "the profound analysis of quantifiers as adjoints . . . subsumes what is perhaps the most significant discovery of modern logic under a general notion that permeates mathematics" (see Awodey, 1996, p. 235). Adjunctions or adjoint functors are really one of

[6] In this part we use excerpts from section 7 of our paper (written in Polish) (Stopa, 2018, p. 39ff).

the greatest discoveries of CT, they are "the cornerstone of category theory" (Marquis, 2021), and as it turned out they really permeate mathematics, and as we have mentioned they also constitute the elementary logical operations. We only remark that the whole 'interaction' between syntax and semantics is also described by adjoint functors (see e.g. Heller, 2018, 2021; Lambek and Scott, 1994; Lawvere, 1969). There are many more results obtained so far in categorical approach to logic or the so-called categorical logic, such as the hierarchy of categorical doctrines, sheaf semantics, classifying topos of a theory, to mention just a few (see e.g. Bell, 2005, 2018; Marquis, 2021) and the bibliography therein). We do not intend to describe in detail all the achievements of categorical logic, but we wanted to name at least a few.

We now focus on toposes as semantics for propositional logic. Our current aim is just to show some of the examples illustrating richness of topos semantics, therefore our exposition is illustrative and we do not define the notions we use. The key role is played by the object called a subobject classifier (denoted as Ω) and arrows connected with it. Assuming that the appropriate notion of validity in a topos is known, we shall use the notation $\mathcal{E} \vDash \alpha$ to denote that the proposition α is valid in the topos $\mathcal{E}$. Topos semantics is connected with intermediate logics, that is consistent superintuitionistic logics. The strongest of such logics is classical logic. This can be written for any topos $\mathcal{E}$ as following inferences (written illustratively on a meta-level)

$$(\vdash_{\mathsf{IL}} \alpha) \to (\mathcal{E} \vDash \alpha) \to (\vdash_{\mathsf{CL}} \alpha),$$

where '$\vdash_{\mathsf{IL}}$', '$\vdash_{\mathsf{CL}}$' denote the deduction in IL and CL respectively (see Goldblatt, 2006, p. 143 and 186). In other words, all theorems of IL are tautologies of topos semantics, and all tautologies of topos semantics are CL's theorems. In particular, for the topos Set (objects of which are standard sets and morphisms are set-functions), Set-validity gives exactly the CL tautologies. More general, if a topos is *bivalent*, i.e. its only truth-values (defined as global elements of Ω, i.e. arrows from a terminal object, denoted as 1, to Ω) are *true* ($\top$) and *false* ($\bot$), then the set of propositions valid in it coincides with all tautologies of CL (Goldblatt, 2006, p. 143). There are also non-bivalent toposes which have the same feature, e.g. Set^2, which has four truth values.

Another collection of toposes whose tautologies are axiomatized by CL are the so-called classical toposes, for which by definition $[\top, \bot] : 1 + 1 \to \Omega$ is an isomorphism. It turns out that another characterization of a classical topos requires it to be a Boolean topos, i.e. for every object its subobjects form a Boolean algebra (for claims of this and of the previous sentence see Goldblatt, 2006, p. 156f and 159f). Some classical or Boolean toposes are bivalent but others are not. For example Set^2 turns out to be classical but is not bivalent. There is also an example of a topos that is bivalent although is not classical (this is the case for the topos of the so-called M_2-sets with appropriate arrows, where M_2 is the smallest monoid that is not a group, i.e. it is a two element algebra, which can be described as consisting of the numbers 0 and 1, under multiplication) (see Goldblatt, 2006, p. 100ff and 122f). An easy example of a topos that is neither bivalent nor Boolean is a topos $\mathrm{Set}^{\to}$ of set-functions (with appropriate arrows), which has three truth values. Let us note that these two notions, being bivalent and being classical or Boolean, although are different in general, coincide for the topos Set. This could be perhaps an example of a certain kind of preciseness present in CT, that distinguishes between notions that are otherwise identical. In general, it can be shown that any topos can be turned into a Boolean one (by the so-called process of sheafification), and moreover any Boolean topos can be turned into a bivalent one, such that there exists a logical morphism between the original Boolean one and the bivalent (this is done by the so-called filter-quotient construction) (for these results see MacLane and Moerdijk, 1994, p. 272, 274, 261).

A whole class of toposes can be given by noting that for any 'small' category $\mathcal{C}$, the so-called functor category $\mathrm{Set}^{\mathcal{C}}$ (of functors from $\mathcal{C}$ to Set as objects and natural transformations between them as arrows) is a topos (for this result see e.g. Goldblatt, 2006, p. 204), but a more general result is known where instead of a topos Set above, any topos can be taken (see MacLane, 1986, p. 405). All the above discussed toposes can be considered as $\mathrm{Set}^{\mathcal{C}}$ for appropriate $\mathcal{C}$, which at least partially shows how extensive this class of toposes is. Let us give one more example of a topos of this type which is related to a certain intermediate logic known in the literature as the LC logic, first studied by M. Dummett in (Dummett, 1959). This logic can be defined by adding to the IL-axioms the following classical tautology:

$$(\alpha \to \beta) \vee (\beta \to \alpha).$$

Now, for any infinite linearly ordered poset P the set of propositions valid in the topos Set^P is just the LC logic. For such P we can take e.g. ω with appropriate poset structure, which we shall denote in bold as $\boldsymbol{\omega}$. Thus we have (see Goldblatt, 2006, p. 228)

$$\mathsf{Set}^{\boldsymbol{\omega}} \models \alpha \quad \text{iff} \quad \vdash_{LC} \alpha.$$

Another important class of toposes is a class of bundles of sets over the base space, say I (for the construction of these toposes and their properties introduced below see e.g. Goldblatt, 2006, p. 88ff and 141). For any fixed set I the topos of bundles over this set, denoted as $\mathsf{Bn}(I)$, has the property that its truth values are in a bijective correspondence with the power set of I, which can be written as

$$\mathsf{Bn}(I)(1, \Omega) \cong \mathcal{P}(I).$$

This means, *inter alia*, that for infinite I the set of truth values of such a topos is infinite. At the same time it can be shown that topos validity in every such $\mathsf{Bn}(I)$ is equivalent to Boolean algebra validity of $(\mathcal{P}(I), \subseteq)$, i.e. it gives precisely the CL tautologies.

Another eminently important class of toposes are sheaves. *Nota bene*, "categories of sheaves were the original examples of toposes" (Barr and Wells, 2005, p. 65). Their importance for mathematics and connections with logic were studied extensively (see e.g. Fourman et al., 1977; Ghilardi and Zawadowski, 2002; MacLane and Moerdijk, 1994). We restrict ourselves to the sheaves of sets on a topological space, say X, which in a sense can be considered as bundles with some additional topological structures. We shall denote them as $Sh(X)$, and the topology of X, i.e. the set of all open subsets of X, by $\mathcal{O}(X)$. We note only one property that truth values in $Sh(X)$ are in a bijective correspondence with the set $\mathcal{O}(X)$ (cf. Goldblatt, 2006, p. 98, where this topos is denoted as $\mathbf{Top}(X)$, and MacLane and Moerdijk, 1994, p. 73, (5)), i.e.

$$Sh(X)(1, \Omega) \cong \mathcal{O}(X).$$

It follows that the topos validity of $Sh(X)$ is equivalent to Heyting algebra validity of the algebra of open subsets $(\mathcal{O}(X), \subseteq)$ (Goldblatt, 2006, p. 186).

The above examples are particular cases of the general property of any topos, namely that the topos validity is equivalent to the validity of a Heyting algebra of its truth values (Goldblatt, 2006, p. 186), i.e. for any topos $\mathcal{E}$ and any proposition α we have

$$\mathcal{E} \vDash \alpha \quad \text{iff} \quad \mathcal{E}(1, \Omega) \vDash \alpha,$$

where the Heyting algebra structure on $\mathcal{E}(1, \Omega)$ is here understood as an isomorphic translation of the Heyting algebra structure of $\mathsf{Sub}(1)$, with the poset structure defined by factorization of appropriate monic arrows. This however will be discussed by us later in our work.

Let us conclude this part with a quotation from Johnstone on the connections between CT and the famous proof of the independence of the Continuum Hypothesis from the axioms of set theory, which shows also some aspects of the historical development of topos theory (this comes from (Johnstone, 2014, p. 323f) but for important considerations about relations between topos theory, logic and set theory see also chapter VI in (MacLane and Moerdijk, 1994)):

> One of the most striking early applications of elementary topos theory was to give a categorical proof that the Continuum Hypothesis is independent of the axioms of set theory. This fact was first proved in 1963 by P. J. Cohen [156];[7] whilst categorical ideas are implicit in Cohen's work, the language of elementary topos theory was required to make the connection explicit—and indeed the desire to do this was one of the main driving forces behind the development of elementary topos theory by Lawvere and Tierney.

The purpose of our work

As we have at least partially seen, the connections of toposes with IL or any intermediate logic are many, diverse and moreover commonly known. From the algebraic point of view, these logics are connected with the so-called Heyting algebras. More precisely, given a Heyting algebra and an appropriate notion of validity the

[7] Johnstone refers here to (Cohen, 1966).

set of propositional formulas that are valid in it constitutes an intermediate logic, and conversely, given any intermediate logic its Lindenbaum-Tarski algebra is a Heyting algebra. We shall show that such an algebra can be considered as a certain generalization of a Boolean algebra. Moreover we shall see that this generalization can be performed differently, but in a parallel or symmetrical way, giving a structure dual to Heyting algebra which is a co-Heyting algebra. Therefore we have two symmetrical generalizations of a Boolean structure, i.e. Heyting and co-Heyting algebras, that are mutually dual (about these structures and dualities we write more in section 3.1). This algebraic state of affairs poses a question about its counterpart on the logical side in the context of algebraic logic. This leads us to PL, although as we shall see the world of PLs[8] is much more extensive than just the logics connected with co-Heyting algebras.

In the context of CT or topos theory in particular, there have appeared quite recently some proposals (see Mortensen, 1995, chapter 11; Estrada-González, 2010, 2015a,b; James, 1996; Mortensen, 2003) that suggest certain dualization of the logic of a topos, changing it from IL or intermediate logic into some kind of PL. The structures that emerge in this dualization process were labeled by these authors as complemented-toposes (or co-toposes in short). If this process turned out to be valid it would be highly fruitful as the connections of toposes with ILs are so manyfold.

We are now ready to formulate the purpose of this work, which is two-fold. First, we analyze and compare the two types of logics: intuitionistic (paracomplete) and paraconsistent. Second, we examine the validity of the notion of co-topos as presented e.g. in (Mortensen, 1995, chapter 11), especially we scrutinize their dualization process which supposedly, *inter alia*, transforms certain Heyting algebras of the topos into co-Heyting ones.

Regarding our first goal, we investigate the two types of non-classical logics, ILs and PLs. Heyting algebras and co-Heyting algebras are mutually dual, and the former are connected with ILs. However the connection of co-Heyting algebras with PLs is different, as we shall see. In order to be able to compare these two types of non-classical logics we want to better understand each of these groups of logics. We portray each of these types, their different philosophical connections, variety and

[8] We shall use the abbreviation 'PLs' for 'paraconsistent logics', i.e. simply as the plural form of PL. We shall define PLs in chapter 2, and PL means any of them in general.

special features. IL has a very profound philosophical roots. Therefore, we investigate these roots, focusing mainly on Brouwer and his philosophy of mathematics. As is well known, a crucial role in his approach is played by the principle of excluded middle, and its rejection as a principle that universally holds. This principle is thus of particular interest to us but we shall discuss also other aspects of his intuitionistic beliefs such as, for example, the nature of continuum. We also present some basic information about the syntactic and semantic aspects of intuitionistic logic itself.

On the other hand PLs are more diverse than ILs because many ways of evading explosive character of inconsistency have been proposed. We shall present some of the most important PLs, but we are mostly focused on the PLs that are connected with co-Heyting algebras, as this is our main concern in the context of topos theory that we shall investigate in further part of the dissertation. In particular, we define two types of logics related to co-Heying algebras, one of them can be called the assertional logic and the other the logic preserving degrees of truth. The former is not paraconsistent, while the latter is paraconsistent in general.

These considerations shall be undertaken in chapters 1 and 2.

Since ILs are closely related to Heyting algebras, and some PLs, the ones of particular interest to us, are related to co-Heyting algebras, at the beginning of chapter 3 we present both these structures quite thoroughly, emphasizing their mutual duality. In doing so, we also make a sort of comparison between these logics, analyzing their algebraic aspects. Regarding our second goal of this dissertation, we shall first present important for our investigations claims of the authors developing the (alleged) notion of co-toposes. As we shall see, they propose to denote or interpret certain arrow occurring in the definition of a topos, crucial for its logical aspects, the so-called generic subobject, as *false* instead of the standard *true*. This proposal opens a lot of questions. Is it just a denotation or is it an interpretation or something else? If it is just a denotation then this process changes only a label and it does not have any deeper importance as the entity in question is just what it is no matter how we denote it. If it is an interpretation then what does it mean, and can it be interpreted arbitrarily? We can go even further and ask: if it were possible to interpret this arrow as *false*, then what would be the consequences of this?

Moreover, there is (supposedly—we shall come back to that) another equivalent definition, that defines a topos without mentioning this arrow or the word 'true', how would then these authors propose to change this definition in order to obtain what they propose to get? We shall unfold these questions, make them clearer and try to answer at least some of them.

As we shall see, the crucial part in the co-topos proposal is that certain Heyting algebras become a co-Heyting ones. On a basis of the first of our goals this suggests that certain type of PLs should be possible to be developed in this context. However, one should be careful when claiming that such a dualization applies easily to all logical properties of a topos. One should be cautious especially about the elementary aspects of logic, that is about the quantifiers. We do not intend to examine all the aspects of the alleged notion of co-topos and its paraconsistent features, but through our investigations on the validity of a notion of co-topos we hope to contribute to better understanding of a topos theory itself and, *inter alia*, its connections with co-Heyting structures.

As will be discussed, the evaluation of the co-topos proposal is complex. It turns out that the abstract structure of a topos itself does indeed allow for the possibility of reversing the order on the corresponding Hom-sets, which thus acquire the structure of co-Heyting algebras that are, moreover, natural (in the strict sense, which we clarify in chapter 3). This allows for consideration of propositional PLs based on these structures in a topos. More important, however, is the question of examining the level of elementary logic, which remains for us an open problem for further research. Only these studies can allow for a comprehensive assessment of the validity of the concept of a co-topos.

Chapter 1

Brouwer's intuitionism and its formalization

In this chapter we present Brouwer's intuitionistic philosophy of mathematics and its formalization made mainly by Heyting. In the first section we briefly present Brouwer himself and a bit of a history of intuitionism. The next section deals with his views on mathematics. We focus, among other things, on the role of mathematical language, logic, the rejection of PEM, the notion of continuum, and some special features of Brouwer's philosophy of mathematics. Finally, we discuss the formalization of his views in the form of intuitionistic logic. We present its syntactic aspects and the so-called topological semantics. We also cite some basic theorems concerning this logic.

The literature on Brouwer's intuitionism is broad. Brouwer's original writings can be found in (Brouwer, 1975, 1976; Mancosu, 1998; van Heijenoort, 1967). The full bibliography of Brouwer's writings, made by D. van Dalen, can be found in (van Atten et al., 2008, p. 343–390) (the older version is available online and can be found in (van Dalen, 1997a)). There are a few very good entries on this subject in the Stanford Encyclopedia of Philosophy: (Iemhoff, 2016; van Atten, 2017a,b). Especially, at the end of (van Atten, 2017b), extensive Brouwer's bibliography can be found with some useful comments. Monographs on intuitionism or containing some exposition of it include: (Dummett, 2000; Heyting, 1971; Placek, 1999; Troelstra and van Dalen, 1988a,b; van Atten et al., 2008). A very good and concise presentation of the history of constructivism in the 20th century, with some exposition of intuitionistic logic and analysis, is given by (Troelstra, 2011) (we shall often refer to this paper in this chapter).

1.1 L. E. J. Brouwer and his predecessors

Luitzen Egbertus Jan Brouwer (1881–1966) was a Dutch mathematician and philosopher who is considered to be the founding father of intuitionism.[9] He earned his Ph.D. in 1907 from the University of Amsterdam. His doctoral thesis *On the Foundations of Mathematics* ('Over de Grondslagen der Wiskunde')[10] (English translation can be found in (Brouwer, 1975, p. 11–101)) was his first publication about the philosophy and foundations of mathematics. His first years after his dissertation (1909–1913) Brouwer spent very fruitfully working mainly in topology (as a part of classical mathematics). In this area, he is still known for his theory of dimension and his famous fixed point theorem (so called *Brouwer's fixed-point theorem*), which is a very important theorem in topology. During that time, in 1912, he was appointed full professor *extraordinarius* at the Univeristy of Amsterdam in mathematics. On that occasion he gave an inaugural lecture *Intuitionisme en Formalisme*, translated into English as *Intuitionism and Formalism* (see Brouwer, 1975, p. 123–138)). Next year he became full professor *ordinarius*, the position he held until 1951. At the end of this period, during the years 1946–1951, Brouwer gave several lectures on intuitionism at the University of Cambridge, which were published posthumously as (Brouwer, 1981). After retirement from academic work, in 1953, he still did a lecture tour through the USA and Canada (i.a. he visited MIT, Princeton, where he met Gödel, and Berkeley).

Quite naturally, Brouwer had predecessors in his constructive approach to mathematics. Constructivism emerged "in the final quarter of the 19th century, and may be regarded as a reaction to the rapidly increasing use of highly abstract concepts and methods of proof in mathematics, a trend exemplified by the works of R. Dedekind and G. Cantor" (Troelstra, 2011, p. 150). One of the first fathers of constructivist attitude was L. Kronecker (1823–1891). He famously considered natural numbers as given by God, and, in his opinion, "all other mathematical objects ought

[9] In this section we rely heavily on (van Atten, 2017b).

[10] We shall always refer to the English translations of the foreign works and only sometimes give the original title.

to be explained in terms of natural numbers (at least in algebra)" (Troelstra, 2011, p. 151). He rejected the notion of infinite set as well as the set of irrational numbers.

Another important Brouwer's predecessor was H. Poincaré (1854–1912). This French scholar emphasized the role of intuition in mathematics, especially the intuition of natural numbers. He considered the principle of induction for the natural numbers as a synthetic judgment a priori. Poincaré wanted also to develop mathematics without impredicative definitions,[11] which, according to him, were the source of set-theoretic paradoxes[12] (cf. Troelstra, 2011, p. 153f).

The others were the so-called French semi-intuitionists (in particular E. Borel, H. Lebesgue, R. Baire) together with the Russian mathematician N. Luzin. Their views have differed but they agreed that "even if mathematical objects exist independently of the human mind, mathematics can only deal with such objects if we can also attain them by mentally constructing them, i.e. have access to them by our intuition; in practice, this means that they should be explicitly definable" (Troelstra, 2011, p. 154). We shall not describe in more detail these authors, but we note only in connection with the name of French semi-intuitionists, that Brouwer himself, in his above mentioned inaugural lecture from 1912, qualified their point of view as intuitionism (to be exact, Brouwer mentioned intuitionism being "largely French") (see Brouwer, 1975, p. 124), but years later in his Cambridge lecture he was already saying about "Pre-intuitionist School, mainly led by Poincaré, Borel and Lebesgue" (Brouwer, 1981, p. 2).

Let us note finally H. Weyl (1885–1955). He rejected "the platonistic view of mathematics prevalent in Cantorian set theory and Dedekind's foundation of the natural number concept" (Troelstra, 2011, p. 156). He has also formulated, apparently independently of Poincaré and Russell, a program for predicative mathematics. Similarly to constructivistic approach, he wanted sets to be constructed from below and not singled out from some previously existing totality, however, he accepted classical logic. For some time Weyl was convinced about the correctness of

[11] In Carnap's words: "A definition is said to be 'impredicative' if it defines a concept in terms of a totality to which the concept belongs" (Carnap, 1931, p. 47f).

[12] His standard example for impredicative definition leading to paradox was *Richard's paradox*: "let E be the totality of all real numbers given by infinite decimal fractions, definable in finitely many words. E is clearly countable, so by a well known Cantor style diagonal argument we can define a real N not in E. But N has been defined in finitely many words! However, Poincaré, adopting Richard's conclusion, points out that the definition of N as element of E refers to E itself and is therefore impredicative" (Troelstra, 2011, p. 154). More on paradoxes can be found in (Łukowski, 2011).

Brouwer's intuitionism, but later he took a position outside either constructivism or formalism (Troelstra, 2011, p. 156).

Let us now examine the Brouwer's philosophy of mathematics, i.e. his intuitionism. He first presented his views in that matter in his doctoral dissertation (Brouwer, 1975, p. 11–101) and later *inter alia* in the aforementioned lecture *Intuitionism and Formalism* (see Brouwer, 1975, p. 123–138) and in his Cambridge lectures (Brouwer, 1981).

1.2 What is mathematics?

For Brouwer mathematics is the mind's free activity of exact thinking. Thus, it depends neither on any language, nor on formal manipulations of signs, nor on any realm of external objects, and moreover it is superior to logic. Therefore we see, and we shall see it somewhat clearer later, that Brouwer's philosophy of mathematics—intuitionism—differs from all the other standard philosophies: formalism, logicism, and platonism. Van Atten aptly describes Brouwer approach writing

> He thus strived to avoid the Scylla of platonism (with its epistemological problems) and the Charybdis of formalism (with its poverty of content). As, on Brouwer's view, there is no determinant of mathematical truth outside the activity of thinking, a proposition only becomes true when the subject has experienced its truth (by having carried out an appropriate mental construction); similarly, a proposition only becomes false when the subject has experienced its falsehood (by realizing that an appropriate mental construction is not possible). (van Atten, 2017b, sec. 3)

Brouwer thus says "outside human thought there are no mathematical truths" (Brouwer, 1981, p. 6). Brouwer's motivation for intuitionism came from his general philosophy and not from the desire to solve the paradoxes that were present in the foundations of mathematics at that time, although intuitionism offers some solutions to them (cf. van Atten, 2017b, sec. 4). When some consequences of his philosophy turned out to be classically invalid, he was ready to accept wholeheartedly this turn of events. As we shall see, some parts of intuitionistic mathematics are common with classical mathematics, but others indeed are not classically acceptable or, looking from different perspective, exceed standard mathematics (especially the conception of the continuum). Intuitionism demands that mathematical objects and arguments are constructive, but it also differs from most constructive theories which are just part of classical mathematics.

In his Cambridge lectures Brouwer has introduced, as he himself called them, two acts of intuitionism. These two acts “form the basis of Brouwer’s philosophy; from these two acts alone Brouwer creates the realm of intuitionistic mathematics” (Iemhoff, 2016, sec. 2.1.). About these acts Brouwer himself has commented that “the first seems to lead to destructive and sterilizing consequences, but then the second yields ample possibilities for new developments” (Brouwer, 1981, p. 4).

1.2.1 The first act, perception of time, and the role of mathematical language

The first act was expressed by Brouwer in a following way:

> Completely separating mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognizing that intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is this common substratum, this empty form, which is the basic intuition of mathematics. (Brouwer, 1981, p. 4f)

First, let us make some remarks concerning the role of the perception of a move of time. Van Atten notes: “as did Kant, Brouwer founds mathematics on a pure intuition of time (but Brouwer rejects pure intuition of space)” (van Atten, 2017b, sec. 4). We shall not consider in depth Brouwer’s considerations on that subject, only let us just give a longer quotation from his lecture *Intuitionism and Formalism* from 1912, which reveals moreover the creation of all finite ordinal numbers, the smallest infinite ordinal ω, and also shows some aspects of his conception of the continuum:

> This neo-intuitionism[13] considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely; this gives rise still further to the smallest infinite ordinal number ω. Finally this basal intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i.e., of the "between," which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units. (Brouwer, 1975, p. 127f)

The notion of continuum we shall consider later. Let us only note that a little later Brouwer says that "the intuitionist recognizes only the existence of denumerable sets" (Brouwer, 1975, p. 128). For him "the apriority of time does not only qualify the properties of arithmetic as synthetic a priori judgments, but it does the same for those of geometry" (Brouwer, 1975, p. 128).

Second, in the first act we can notice Brouwer's position on the relation between language (mainly mathematical) and mathematics. Mathematics is thus an activity of the mind and, as Brouwer is saying a bit further, "language plays no part other than that of an efficient, but never infallible or exact, technique for memorizing mathematical constructions, and for communicating them to others, so that mathematical language by itself can never create new mathematical systems" (Brouwer, 1981, p. 5). By all this we see that Brouwer was clearly opposing the formalist

[13] Brouwer calls it neo-intuitionism in comparison with the (old or pre-) intuitionism from XIX c. which was based also on the Kant's apriority of space. He notes a bit earlier that "the most serious blow for the Kantian theory was the discovery of non-euclidean geometry" and that this (old/pre-) intuitionism "has recovered by abandoning Kant's apriority of space but adhering the more resolutely to the apriority of time" becoming, what he calls, neo-intuitionism (see Brouwer, 1975, p. 127). His neo-intuitionism (at least in this his lecture) is what we consider as his (normal) intuitionism.

philosophy of mathematics. For Brouwer, mathematical objects exists by (mental) construction. It is far from enough to write certain non-contradictory axioms, to prove the existence of some objects, or models, fulfilling them.

Correspondingly, Brouwer rejects also the foundational role of language described by theoretical logic. Logic, in his framework, "systematizes certain patterns in the linguistic recordings of our activities of mathematical construction" (van Atten, 2017a, sec. 2.1). It is therefore subordinate to mathematics. Brouwer is thus evidently opposing the logicism. However, logic, when extracted form mathematical reasoning, can in turn also be studied by mathematicians, in this way becoming the metamathematics or, as Brouwer would call it, 'mathematics of the second order'.

Accordingly to what have been said, parts of standard logic may correctly describe inferences that are valid from intuitionist's point of view, but the question arises whether following the principles of classical logic is always accompanied by some languageless mathematical activity or construction, thus enabling us to use logic for discovering new mathematical truths. Brouwer states that

> briefly expressed, the answer is in the affirmative, as far as the principles of contradiction and syllogism are concerned, if one allows for the inevitable inadequacy of language as a mode of description and communication. But with regard to the principle of the excluded third, except in special cases, the answer is in the negative. (Brouwer, 1981, p. 5)

Now, it is not clear what Brouwer meant by "the principles of contradiction and syllogism." Van Dalen, the editor of *Brouwer's Cambridge lectures in intuitionism* suggests that "[i]t may safely be ventured that the inferences of intuitionistic logic qualify equally well as the principles of contradiction and syllogism"[14] (Brouwer, 1981, p. 95). We shall consider intuitionistic logic with greater attention later on, but let us now note only that with an appropriate choice of axioms, intuitionistic logic may be defined as a classical logic without the axiom of excluded middle.

This statement Brouwer has made in his Cambridge lecture, which took place long after the time when the intuitionistic logic was formulated by A. Heyting, but

[14]This is the last sentence of van Dalen's note, the previous are also worth mentioning: "Brouwer did not go beyond this cautious and rather dated appraisal of the use of logic. His reluctance to accept the conveniences of modern logic made him express himself here in colourful but rather uninformative terms."

of course Brouwer had conceived this idea before Heyting's work, at least already in 1908 when he published his paper *The Unreliability of the Logical Principles* (see Brouwer, 1975, p. 107–111). We can now see, at least in general, that intuitionistic logic is in some accordance with Brouwer's philosophy of mathematics, expressed e.g. in the above quote of Brouwer.

As van Atten notes about aforementioned article: "in that paper, Brouwer draws a consequence of his general view on logic that he had overlooked in his dissertation", namely that the principle of excluded middle (PEM) is not valid. Before, however, we get to PEM let us make few notes about truth and falsity in intuitionism as well as the meaning of logical symbols, which we will need in discussing PEM.

1.2.2 Truth, falsity and the meaning of logical symbols

As we have already mentioned at the beginning of this section, for intuitionists the truth and falsity of a mathematical statement isn't related to some objective state of the world (as would platonists say), but to our knowledge about it. The truth of a statement is related to an appropriate mental construction that proves it (the proposition is made true by construction). The falsity, on the other hand, is related to an appropriate mental construction that shows impossibility of any proof of that statement (cf. Troelstra and van Dalen, 1988a, p. 4). Therefore, a statement might become true or false in the course of time, while not being so before. In this sense, intuitionism essentially depends on time.

Let us now say a few words about understanding negation in intuitionism. The negation $\neg p$ of p is true if p is false (which means that it has been shown that from any hypothetical proof of p contradiction (falsum) can be derived), therefore $\neg p$ is understood as $p \to \perp$, where $\perp$ is an absurdity or contradiction. $\perp$ has no proof, and the notion of contradiction is to be regarded as a primitive (unexplained) notion. If $\neg p$ is true, we also say that p is refuted. Thus in this sense, negation always has some positive meaning, it conveys a positive ability to show impossibility of any proof. Naturally, it may be the case that neither can we show that p is true, nor that p is false. This last note is crucial for considerations about PEM, as we shall see below.

The above meaning of negation as well as the meaning of others logical symbols belong to the Brouwer-Heyting-Kolmogorov-interpretation (known also as BHK-interpretation) or Proof Interpretation of intuitionistic logic, and it is nowadays its standard explanation. The remaining logical symbols have the following explanation (cf. Troelstra and van Dalen, 1988a, p. 9; Iemhoff, 2016):

($\wedge$) A proof of $p \wedge q$ is given by presenting a proof of p and a proof of q.

($\vee$) A proof of $p \vee q$ is given by presenting either a proof of p or a proof of q (plus the stipulation that we want to regard the proof presented as evidence for $p \vee q$).

($\rightarrow$) A proof of $p \rightarrow q$ is a construction which permits us to transform any proof of p into a proof of q.

($\forall$) A proof of $\forall x p(x)$ is a construction which transforms every proof that d belongs to the domain D (where D is the intended range of the variable x) into a proof of $p(d)$.

($\exists$) A proof of $\exists x p(x)$ is given by providing $d \in D$ (where D is the intended range of the variable x), and a proof of $p(d)$.

This is of course a somewhat informal explanation, as the notion of construction (as well as the notions "presenting" and "transformation") are not formally defined, but it corresponds to Brouwer's intuitionistic ideas (an interesting analysis of the historical development of the intuitionistic understanding of the logical symbols, culminating in BHK-interpretation, is presented in (van Atten, 2017a)).

1.2.3 Principle of Excluded Middle

According to Brouwer the PEM does not universally hold (although at the beginning, in his doctoral dissertation from 1907 he still considered PEM "as correct but useless, interpreting $p \vee \neg p$ as $\neg p \rightarrow \neg p$" (van Atten, 2017b, ch. 4)). He compares the belief in the universal validity of it to such phenomena of the history of civilization as "belief in the rationality of π, or in the rotation of the firmament about the earth" (Brouwer, 1981, p. 7). As we have seen in the previous subsection, it is possible that neither can we show that some statement, say p, is true, nor that it is false. In this case Brouwer would say that the assertion that $p \vee \neg p$ "is devoid of sense" (cf. Brouwer, 1981, p. 6). This means that PEM does not hold in general in intuitionism. This is of course closely connected with intuitionistic understanding of the alternative (in order to prove $p \vee q$ we have to give a proof, either of p or of q, see above).

When we cannot assert that $p \vee \neg p$ is true, this however does not imply, as we have already seen, that $p \vee \neg p$ is false, i.e. that $\neg(p \vee \neg p)$ is true. As a matter of fact, it can be proven that $\neg\neg(p \vee \neg p)$, from which it follows that $\neg(p \vee \neg p)$ is false, which in turn is equivalent to $(p \vee \neg p) \rightarrow \perp$ being false. This last statement means that PEM, although in some cases might not be true, nevertheless never leads to contradiction.

We have just said that it can be proven in intuitionism that $\neg\neg(p \vee \neg p)$. At the same time, we have shown, and soon we shall deal with it in more detail, that not always can we assert that $p \vee \neg p$ is true. This fact, that from $\neg\neg$PEM it does not necessarily follow PEM is an example of the so-called strong double negation law ($\neg\neg p \rightarrow p$), which also does not hold in general in intuitionism. The close connection between the PEM and the strong double negation law[15] is reflected in intuitionistic logic by the fact that adding either of them to its axioms give us already the classical logic.

Let us go back to the examples of instances of PEM for which we cannot assert truth. They are called "weak counterexamples" of PEM. Sometimes it is distinguished between "strong negation" of some statement, which is the true negation connected with impossibility of proof, and "weak negation", which means only that for now neither we can show the truth nor the falsity of that statement. From this distinction come the names of weak and strong counterexamples.

Before we show an instance of a weak counterexample, let us first note, that for finite systems there are no undecidable problems, so we can always determine if a given statement is true or false (also in the intuitionistic meaning). We can quote Brouwer: "insofar as only finite discrete systems are introduced ... the principium tertii exclusi is reliable as a principle of reasoning" (Brouwer, 1975, p. 109), the same

[15] Let us note here that although we intuitively advocate for PEM, still at least in our language we use the expression "I'm not saying no", which not at all means "Yes". Therefore, it seems that it is not so obvious that for ordinary people the strong double negation law is always valid in everyday speech.

idea is presented in (Brouwer, 1981, p. 5f). Thus Brouwer considers PEM as a valid principle for finite systems but in an unjustified way adopted for infinite systems and thus for the whole mathematics[16].

For infinite systems, it is possible that the mathematical statement is neither true nor false. The simplest example, which is already the simplest instance of weak counterexample, is every yet unsolved mathematical problem (e.g. the Goldbach conjecture[17]). As we neither can show it to be true, nor show impossibility of any proof, at least until now, we can assert neither that it is true nor that it is false. Classically of course, we also know neither that e.g. Goldbach conjecture is true nor that it is false, but because of the different meanings of the terms "true", "false", and "or" (the logical alternative), we still do hold that 'Goldbach conjecture is true or it is false'.

As long as we have any unsolvable mathematical problem, we can use it, following Brouwer, to construct furthers weak counterexamples to other classically valid statements having the form of PEM in the following way.

Let P be some decidable property of natural numbers, for which $\forall_{n\in\mathbb{N}}\, P(n)$ is neither known to be true nor false (we base this exposition of weak counterexamples on (Iemhoff, 2016, sec. 3.4), (Brouwer, 1981, p. 6f) and (van Atten, 2017b, supplement "Weak Counterexamples")). Decidable property means that for each $n \in \mathbb{N}$ it can be decided whether or not n possesses the property P; e.g. $P(n)$ can mean "$2n+4$ is the sum of two primes", and then $\forall_{n\in\mathbb{N}}\, P(n)$ is the Goldbach conjecture. Now, we define a sequence of rational numbers a_n in the following way:

$$a_n = \begin{cases} (-2)^{-n} & \text{if } \ \forall_{k\leq n}\, P(k), \\ (-2)^{-n_0} & \text{if } \ \neg P(n_0) \wedge n_0 \leq n \wedge \forall_{k<n_0}\, P(k). \end{cases}$$

This is a brief definition of a sequence which elements a_n are equal $(-2)^{-n}$ as long as $P(n)$ is true, and for the smallest natural number for which $\neg P(n)$ is true, let it

[16] It is interesting to note Brouwer's comment in footnote regarding finite and infinite systems. After having noted about some investigation for finite systems that it can "in every case be made by a machine or by a trained animal, it does not require the basic intuition of mathematics, living in a human mind", he continues '[b]ut with respect to questions regarding infinite sets the basic intuition is indispensable; by disregarding this fact, Peano and Russell, Cantor and Bernstein have fallen into errors" (Brouwer, 1975, footnote 2, p. 109).

[17] The so-called "strong" or "binary" Goldbach conjecture states that all positive even integers greater or equal to 4 can be expressed as the sum of two primes.

be $n = n_0$, the sequence starts to be constant and equal $(-2)^{-n_0}$. $\{a_n\}$ is a Cauchy sequence[18] because for every n, any two members of the sequence after a_n lie within the distance $2 \cdot 2^{-n} = 2^{-n+1}$. Therefore the sequence $\{a_n\}$ converges and determines a real number, say a.

From the construction of $\{a_n\}$ it follows that we can assert $a = 0$ if and only if we know that $\forall_{n \in \mathbb{N}} P(n)$, which is equivalent to the validity of the Goldbach conjecture. It follows also that we can assert $a \neq 0$ if and only if we know that for some n it is $\neg P(n)$, because then, if n_0 were the smallest such n, the sequence $\{a_n\}$ for $n \geq n_0$ would be a constant sequence with value $(-2)^{-n_0}$, as we have already noticed. Let us call such a (hypothetical) number n_0 the *critical number*. As we neither can prove the Goldbach conjecture nor can we give a counterexample for it, we can no longer assert in intuitionism that for every real number x, the PEM in the form $x = 0 \vee x \neq 0$ holds (because the number a defined above is real and we do not know weather $a = 0$ or $a \neq 0$). Moreover, because we cannot demonstrate neither the absurdity of critical number being an odd nor an even number, we also cannot assert that for every real number x, it is $x \leq 0 \vee x \geq 0$. Among others, Brouwer constructed also weak counterexamples to the statements: (i) every set is either finite or infinite, (ii) any two straight lines in the Euclidean plane are either parallel, or coincide, or intersect, (iii) every infinite sequence of positive numbers either converges or diverges (see van Atten, 2017a, sec. 3.2).

As we have seen, the PEM does not always hold in intuitionism, at least currently. The question is if this will always be the case. At this point, we can understand Brouwer saying that "the question of the validity of the principium tertii exclusi is equivalent to the question whether unsolvable mathematical problems can exist" (Brouwer, 1975, p. 109). Brouwer, moreover, states that as long as the proposition: "every number is either finite or infinite", is unproved, it remains uncertain whether the problem: "Does the principium tertii exclusi hold in mathematics without exception?" is solvable (Brouwer, 1975, p. 110) and thus whether there are unsolvable mathematical problems.

In the same paper, however, Brouwer wrote that "there is not a shred of a proof for the conviction, which has sometimes been put forward [here he refers in a

[18] $\{b_n\}$ is a Cauchy sequence iff for any $\varepsilon > 0$ there is a natural number N such that for all $j, k > N$ it is $|b_j - b_k| < \varepsilon$.

footnote to D. Hilbert's paper] that there exist no unsolvable mathematical problems" (Brouwer, 1975, p. 109). As van Atten comments this passage in (van Atten, 2017a, sec. 2.5), Brouwer here "seems to overlook that, constructively, there is a difference between the claim that every mathematical problem is solvable and the weaker claim that there are no absolutely unsolvable problems." If we accept that the former is equivalent to PEM, then the latter is equivalent to ¬¬PEM, which, unlikely to the first one, is always true in intuitionism. This seems to be really only an oversight, because van Atten shows also Brouwer's note from about the same period, which has never been published, where Brouwer made a point that one cannot ever demonstrate that a proposition can never be decided. Apparently then, Brouwer might wanted to say that there is not a shred of a proof for the conviction that every mathematical problem is solvable, although he regarded that there are no absolutely unsolvable problems. As van Atten noted, "the explicit observation that $\neg\neg(p \vee \neg p)$ means that no absolutely unsolvable problem can be indicated was made [by Heyting in 1934]" (van Atten, 2017a, sec. 2.5).

We shall now consider the so-called strong counterexamples. We have already mentioned that because $\neg\neg(p \vee \neg p)$ is a tautology in intuitionistic propositional logic, it cannot be proved that PEM is false, i.e. it cannot be proved that $\neg(p \vee \neg p)$. However, some other statements that are sometimes considered to be of the form of a negation of PEM, like e.g. $\neg\forall x \in \mathbb{R} : (P(x) \vee \neg P(x))$ may, and have been proven by Brouwer. They are called strong counterexamples (of PEM), because they not only show that for some statement we cannot assert its truth (like in the case of weak counterexamples), but they refute certain statements, although strictly speaking they do not refute PEM in the form $p \vee \neg p$.

As his first strong counterexample, Brouwer proved the following statement for intuitionism:

$$\neg\forall x \in \mathbb{R} : (x \in \mathbb{Q} \vee x \notin \mathbb{Q}).$$

Later using the so-called creating subject method (about creating subject itself we shall write below) Brouwer proved for instance also (see van Atten, 2017a, sec. 3.3; or van Atten, 2017b, supplement "Strong Counterexamples"):

$$\neg\forall x \in \mathbb{R} : (\neg\neg x < 0 \rightarrow x < 0),$$

and

$$\neg\forall x \in \mathbb{R} : (x \neq 0 \rightarrow (x < 0 \vee x > 0)).$$

We do not intend here to go deeper into these issues, which are known in the literature. We only wanted to mention strong counterexamples as they can be considered as one of Brouwer's more important achievements.[19]

1.2.4 The second act, choice sequences and the continuum

As we have already mentioned, the mathematics based only on the first act of intuitionism would be very limited, "in particular, ... one might fear at this stage that in intuitionism there would be no place for analysis" (Brouwer, 1981, p. 7), but the second act of intuitionism opens "a much wider field of development, including analysis and often exceeding the frontiers of classical mathematics" (Brouwer, 1981, p. 8). We shall not study it in depth, but let us quote the following insightful passage (see Brouwer, 1981, p. 8):

> Admitting two ways of creating new mathematical entities: firstly in the shape of more or less freely proceeding infinite sequences of mathematical entities previously acquired (so that, for example, infinite decimal fractions having neither exact values[20], nor any guarantee of ever getting exact values are admitted); secondly in the shape of mathematical species, i.e. properties supposable for mathematical entities previously acquired, satisfying the condition that if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be 'equal' to it, definitions of equality having to satisfy the conditions of symmetry, reflexivity and transitivity.

Infinite sequences mentioned above, called also *choice sequences*, are the special feature of Brouwer intuitionism. Van Atten summarizes:

> The main use of choice sequences is the reconstruction of analysis; points

[19] In his preface to Brouwer's Cambridge lectures van Dalen writes: "I think that it is safe to agree that Brouwer's two most spectacular performances in his foundational work, given the basic notions (e.g. natural number, choice sequence), are *the continuity theorem* (involving continuity, bar induction, etc.) and *the strong counterexamples* (involving the creative subject)" (Brouwer, 1981, p. ix).

[20] Here, van Dalen, the editor, gives a valuable comment: "By 'infinite decimal fractions having no exact value' Brouwer clearly means 'decimal fractions not given by a law' " (Brouwer, 1981, p. 95).

> on the continuum (real numbers) are identified with choice sequences satisfying certain conditions. Choice sequences are collected together using a device called 'spread', which performs a function similar to that of the Cantorian set in classical analysis, and initially, Brouwer even uses the word 'Menge' ('set') for it. (van Atten, 2017b, sec. 4)

The presence of choice sequences distinguishes intuitionism from other constructive views on mathematics. Most of the constructive theories are in general part of classical mathematics, but intuitionistic mathematics, as we soon shall see, only partially overlap with classical mathematics (it does not have all of its theorems, but has also others that are not classically valid).

Before proceeding further, let us note only, that although two acts of intuitionism can be perceived in a more psychological way, which would also pose the problem of intersubjectivity of mathematics, Brouwer did not consider it this way. He introduced the idea of a *creating subject* (also known as *creative subject*), an idealized mind in which mathematics takes place. It is not subject to limitations of space, time, memory or even faulty arguments and thus also does not cause the problem of intersubjectivity (cf. Iemhoff, 2016, sec. 2.2).[21]

We have already given examples of sentences that are true for classical continuum but false for intuitionistic. We have also mentioned that the statement $\forall_{x \in \mathbb{R}} (x = 0 \vee x \neq 0)$, which is obviously true of real numbers in classical mathematics, is not true for intuitionistic continuum. Similar result can be shown for the law of trichotomy, intermediate value theorem and many others classically valid statements. In many cases however for such a statement another statement can be given which is valid in intuitionism and classically equivalent to the original one (cf. Iemhoff, 2016, sec. 3.4).

Probably the most known and important result showing the richness of the intuitionistic continuum is the *continuity theorem*. It can be stated as: every total function $[0, 1] \to \mathbb{R}$ is uniformly continuous. This implies *inter alia* that the intuitionistic continuum is not decomposable, meaning that there are no non-empty

[21] In our opinion, this is an important and at least puzzling issue. If we do not introduce a creating subject, intuitionism is hardly immune to accusation of being merely based on psychology and having serious problems with e.g. intersubjectivity or erroneous arguments. On the other hand, by introducing creating subject, intuitionism may be consider to become closer to platonism and the question then arises as to the relation between the mind of a given mathematician to the creating subject.

spreads[22] A and B such that $A \cup B = \mathbb{R}$ and $A \cap B = \emptyset$ (see van Atten, 2017b, supplement "Strong Counterexamples").

1.3 Formal intuitionistic logic

From considerations of a more historical-philosophical nature, we now turn to the formalization of intuitionism in the form of intuitionistic logic (IL). However, it is worth noting at the outset the problematic nature of this formalization itself. To quote Heyting:

> Intuitionistic mathematics is a mental activity, and for it every language, including the formalistic one, is only a tool for communication. It is in principle impossible to set up a system of formulas that would be equivalent to intuitionistic mathematics, for the possibilities of thought cannot be reduced to a finite number of rules set up in advance. (Heyting, 1930, p. 42; English translation in Mancosu, 1998, p. 311; we cite after van Atten, 2017a, sec. 4.4)

Let us add that although Heyting's achievements in the field of formalizing intuitionistic logic are of great importance to logic itself, in the context of Brouwer's own ideas, Heyting would many years later write:

> I regret that my name is known to-day mainly in connection with these papers [his three papers from 1930], which were very imperfect and contained many mistakes. They were of little help in the struggle to which I devoted my life, namely a better understanding and appreciation of Brouwer's ideas. They diverted the attention from the underlying ideas to the formal system itself. (Heyting, 1978, p. 15; we cite after van Atten, 2017a, sec. 4.4)

Our current goal, however, is to present the IL itself, regardless of its connections with Brouwer's philosophy.

[22]To understand this concept at least a little better let us quote Iemhoff: "A *spread* is the intuitionistic analogue of a set, and captures the idea of infinite objects as ever growing and never finished. A spread is essentially a countably branching tree labelled with natural numbers or other finite objects and containing only infinite paths" (Iemhoff, 2016, sec. 3.6).

1.3.1 Syntax

IL can be briefly described as classical logic (CL) with the omitted PEM or equivalently with the omitted strong double negation law (in particular, the languages of both logics and the rules of inference are the same). This means that after adding any of these laws to the axioms of IL (in Hilbert style), we obtain CL (this applies to both zero-order (sentences) and first-order (predicates) logic, respectively). For a precise list of axioms, see, e.g., (van Dalen, 2002, p. 9) or axioms from I to XI in (Goldblatt, 2006, p. 131). The deducibility relation, $\vdash$, is defined in a standard way (sometimes we add to this symbol a subscript denoting the logic within which the given deduction is considered). The formalization of IL was also performed in Gentzen style natural deduction. We do not deal with this formalization in more detail here; natural deduction for IL (and a discussion of the system itself) can be found, e.g., in (van Dalen, 1997b, 2002).

1.3.2 Semantics

The two-valued semantics of CL is not a suitable semantics for IL, in the sense of satisfying the completeness theorem. This semantics is obviously sound but not complete (if only because PEM is true in it, and it is not a theorem of IL).

The question naturally arises whether other many-valued semantics would not be appropriate (viz. sound and complete) for IL. Gödel in his paper (Gödel, 1932) showed that none of the n-valued semantics, for any n natural, satisfies this condition for propositional IL. Incidentally, he also showed in this paper that there is an infinite monotonically decreasing sequence of intermediate logics in strength between CL and IL (zero order).

There are many semantics that are sound and complete with respect to IL (one may consult, e.g., (Bezhanishvili and Holliday, 2019; Palmgren, 2009)). In the following, we only briefly discuss topological semantics.

Topological semantics

The topological aspects of IL were discovered independently by Marshall Stone (Stone, 1937) and Alfred Tarski (Tarski, 1938). Let $(X, \mathcal{O}(X))$ be a topological space.[23] Consider propositional IL first. Let P_0 denote a set of propositional variables (or sentence letters) and $\mathsf{Fm}(P_0)$ a set of formulas (which in this case are just all sentences). The role of logical values assigned to sentences will now be played by open sets (elements of $\mathcal{O}(X)$). Let us introduce the notation, $[\![p]\!]_X$, to denote the open set of a topological space X assigned to the sentence p (to be precise, we should write $[\![p]\!]_{(X,\mathcal{O}(X)),v}$ to denote not only the topology $\mathcal{O}(X)$ which can differ for the same set X, but also to denote the valuation $v : P_0 \to \mathcal{O}(X)$; usually, however, we omit even the subscript X). The role of truth is played by the entire set, so that $[\![\top]\!] = X$, and the role of falsity is played by the empty set $[\![\bot]\!] = \emptyset$. The assignment of open sets to compound sentences in terms of the assignment given to atomic sentences of P_0 is given by (in other words, this gives as the way logical operations work in this semantics)

$$
\begin{aligned}
[\![p \wedge q]\!] &= [\![p]\!] \cap [\![q]\!], \\
[\![p \vee q]\!] &= [\![p]\!] \cup [\![q]\!], \\
[\![p \to q]\!] &= Int([\![p]\!]^c \cup [\![q]\!]), \\
[\![\neg p]\!] &= Int([\![p]\!]^c),
\end{aligned}
\tag{1.1}
$$

where A^c is the complement of the set ($A^c \equiv X \setminus A$), and $Int(A)$ is the interior of the set A (i.e. the largest open subset of the set A). Note that we can equivalently define negation as $\neg p = p \to \bot$. Using the equations (1.1) we uniquely extend (inductively with respect to the complexity of the formula) the valuation $P_0 \to \mathcal{O}(X)$ to the mapping $\mathsf{Fm}(P_0) \to \mathcal{O}(X)$. Note that the above semantics is a generalization of the classical two-valued semantics, since the latter corresponds to a topological space consisting of a single point (then $\mathcal{O} = \{\emptyset, X\} \equiv \{0, 1\}$).

To extend the topological interpretation to first order IL, suppose that the cor-

[23] In a large part of this exposition of a topological semantics we rely heavily on (van Dalen, 2002, p. 23ff).

responding domain D is given and define:

$$\begin{aligned} [\![\exists x \varphi(x)]\!] &= \bigcup \{[\![\varphi(a)]\!] \mid a \in D\}, \\ [\![\forall x \varphi(x)]\!] &= Int \bigcap \{[\![\varphi(a)]\!] \mid a \in D\}. \end{aligned} \tag{1.2}$$

Let us introduce some standard definitions. We say that a formula α is *true in a topological space* $(X, \mathcal{O}(X))$, which is also denoted by $\vDash_X \alpha$, if $[\![cl(\alpha)]\!]_X = X$ for any assignment of open sets to atomic sentences, where $cl(\alpha)$ is the universal closure of α (for propositional logic '*cl*' of course can be omitted). We say that a formula α *is true*, which is denoted by $\vDash \alpha$, if α is true in every topological space. We say that a formula α *is a logical consequence of a set of formulas* Γ *in a topological space* X, which is denoted by $\Gamma \vDash_X \alpha$, if $Int \bigcap \{[\![\beta]\!]_X \mid \beta \in \Gamma\} \subseteq [\![\alpha]\!]_X$ for any assignment of open sets to atomic sentences. Finally, we say that a formula α *is a logical consequence of a set of formulas* Γ, which we denote by $\Gamma \vDash \alpha$, if $\Gamma \vDash_X \alpha$ for any topological space X.

We can now formulate a completeness theorem for IL and this semantics (which we give without proof and assume the above notation):

$$\Gamma \vdash \alpha \quad \text{iff} \quad \Gamma \vDash \alpha\,.$$

Let us see one example of using the above considerations to show the undeducibility of some sentences.

Example. Consider the topological space $(\mathbb{R}, \mathcal{O}(\mathbb{R}))$, with standard open sets of real numbers. In some valuation, let p be assigned an open set $\mathbb{R} \setminus \{0\}$. Then, $[\![\neg p]\!] = Int(\{0\}) = \emptyset$, so $[\![p \vee \neg p]\!] = \mathbb{R} \setminus \{0\} \neq \mathbb{R}$. Hence, by virtue of the completeness theorem, we get $\nvdash p \vee \neg p$. Similarly, in the situation under consideration, we obtain $[\![\neg\neg p \to p]\!] = \mathbb{R} \setminus \{0\} \neq \mathbb{R}$, and therefore $\nvdash \neg\neg p \to p$.

The algebra of open sets of a topological space is a special case of a Heyting algebra. We shall discuss this algebraic structure, as well as the dual structure of a co-Heyting algebra, in more detail in chapter 3 (especially in sec. 3.1). Among other things, we will show that both of these structures can be viewed as a generalization of a Boolean algebra, which is known to be the Lindenbaum-Tarski algebra of CL.

Finally, we discuss some basic properties of IL. In the following IL_0 denotes zero-order IL, IL_1 denotes first-order IL, and IL can be taken as any of these; analogously, we use the notation CL_0 and CL_1 with respect to CL. The following properties and a number of others can be found e.g. in (Rasiowa and Sikorski, 1963; van Dalen, 2002):

- For any proposition p:

$$\vdash_{\mathsf{CL}_0} p \quad \text{iff} \quad \vdash_{\mathsf{IL}_0} \neg\neg p,$$
$$\vdash_{\mathsf{CL}_0} \neg p \quad \text{iff} \quad \vdash_{\mathsf{IL}_0} \neg p.$$

- If p does not contain $\wedge$, nor $\neg$ (alternatively $\rightarrow$, nor $\vee$), then:

$$\vdash_{\mathsf{CL}_0} p \quad \text{iff} \quad \vdash_{\mathsf{IL}_0} p.$$

- The so-called disjunction property:

$$\vdash_{\mathsf{IL}} \alpha \vee \beta \quad \text{iff} \quad (\vdash_{\mathsf{IL}} \alpha \ \text{ or } \ \vdash_{\mathsf{IL}} \beta).$$

- The so-called existence property:

$$\vdash_{\mathsf{IL}_1} \exists x \alpha(x) \quad \text{iff} \quad \text{there exists a term } t \text{ such that} \vdash_{\mathsf{IL}_1} \alpha(t).$$

- IL_0, like CL_0, is decidable.
- IL_1, like CL_1, is undecidable.

Chapter 2

Paraconsistent logics

In this chapter, we introduce paraconsistent logics (PLs). First, we discuss some of the motivations that may inspire the study of these logics. Second, we define the basic notions needed to understand and discuss various PLs. Then, we present and discuss some examples of PLs: (i) one of the first PLs, namely the Jaśkowski's discursive logic; (ii) the whole system of countably many PLs, the so-called da Costa's C-system; (iii) the PLs which are dual (in a specific way) to ILs and are connected with co-Heyting algebras. In this last section, however, we also discuss other logics related to co-Heyting algebras, which are not paraconsistent although they are also dual in a certain way to ILs.

2.1 Some motivations

As we mentioned in the introduction, our main motivation to study PLs is their alleged connection to ILs and topos theory through appropriate dualities. As we shall see, some PLs are related to co-Heyting algebras, which are dual to Heyting algebras (see sec. 3.1). As is well known, the latter, on the other hand, are directly related to ILs and are commonly and intrinsically present in toposes.

However, as we will see in the next chapter, the presence of co-Heyting algebras in toposes turns out to be more natural and common than is standardly believed. In such a situation, a more in-depth understanding of PLs may prove valuable.

Yet PLs are also interesting in their own right, both from a purely formal point of view and in terms of their possible applications to other areas of science (see e.g.

Eva, 2015; Lambert, 2011; Tanaka et al., 2013). In this work, we do not explore possible applications for our formal research, all the more so it may be worth mentioning at least some examples of additional motivations in this regard, in the context of logics that can reliably deal with inconsistencies and their relationship to other research. Let us quote Abramsky who writes: "rich field of phenomena in logic and information, closely linked to key issues in the foundations of physics, arise at the borders of paradox" (Abramsky, 2018, p. 282).

More precisely, in connection with logics based on co-Heyting algebras let us quote Majid, who with great hope writes about some important issues in the context of his project:

> It has been argued by F. W. Lawvere[24] and his school that this intersection $\mathrm{d}(A) = A \cap \bar{A}$ is like the 'boundary' of the proposition, and, hence, that these co-Heyting algebras are the 'birth' of geometry.[25] . . . My long-term programme at the birth of physics is to develop this geometrical interpretation of co-intuitionistic logic further into the notion of metric spaces and ultimately into Riemannian or pseudo-Riemannian geometry. (Majid, 2012, p. 125f)

2.2 Definition of paraconsistent logic

Paraconsistency is, unlike intuitionism, not connected with a separate position in the philosophy of mathematics. One of its characteristics is its attitude to inconsistency. In CL the appearance of inconsistency has explosive consequences, from contradiction everything follows. Paraconsistent logic (PL) emerged as an attempt to protect inconsistencies from exploding, i.e. from implying anything. The fact that some logicians developing PL wanted to allow the appearance of contradictions[26] does not immediately mean that contradictions are welcome or even desirable. Our attitude to inconsistency and our motivation to study PL and possible paraconsis-

[24]Majid cites here (Lawvere, 1989), but one may also consult the more accessible one connected with this subject (Lawvere, 1991).

[25]By $\bar{A}$ Majid means in general an appropriate complement of A, in this case a co-Heyting negation. We define this notion in sec. 3.1.

[26]When writing in an informal way, the words "inconsistency" and "contradiction" can be considered synonymous. In formal writing, however, inconsistency will be shortly defined, and the contradiction can be understood as a sentence from which a *falsum* can be infered (α is a contradiction iff $\alpha \vdash \perp$).

tent aspects of toposes has already been presented in the introduction, we recall only that our primary motivation comes from mathematical inspiration related to a better understanding of topos theory.

There are quite a few introductions to PL (see e.g. Akama and da Costa, 2016; Carnielli et al., 2007; Priest, 2002; Priest et al., 2018). We do not need to present PL in its fullness. Based on the texts just mentioned, we only give a very general introduction to PL and describe some important examples in a little more detail. In particular, we focus on PLs which arise in the context of dualization of IL and which are based on algebraic structures dual to Heyting algebras, the so-called co-Heyting algebras. Our further research in the next chapter is precisely related to the presence of these structures in toposes, so these logics are of particular importance for our further considerations. If not stated otherwise we shall deal with propositional calculus (zero order).

Any logical consequence relation, which we shall denote as $\vdash$ (basically this will denote the proof theoretic deduction, but in this chapter it could be either proof theoretic or semantic), is called *explosive* if it satisfies the condition that for any sentences α, β we have $\{\alpha, \sim\alpha\} \vdash \beta$. Classical logic, but also most standard non-classical logics (among them intuitionistic logic), are all explosive (cf. Priest et al., 2018). Such logics are useless in any inconsistent context, as they lead to triviality (everything follows). A logic is called *paraconsistent* if its logical consequence relation is not explosive. This definition is weaker than sufficient[27], "but an elegant stronger definition is not at hand" (Priest, 2002, p. 288), and this one has become the standard one.

A theory T is called *inconsistent* if for some formula α, it is $T \vdash \alpha$ and $T \vdash \sim\alpha$. If T is not inconsistent, it is called *consistent*. T is *trivial* if $T \vdash \alpha$, for all α. Otherwise, T is *non-trivial*. We can thus say that paraconsistent logics are those which can be used as the basis for inconsistent but non-trivial theories. It is interesting to note that "a paraconsistent logic need not itself have an inconsistent set[28] of logical truths: most do not" (Priest, 2002, p. 288 (in footnote 3)).

[27] E.g. there is a logic (Johansson's minimal logic), which is paraconsistent according to this definition, but satisfies $\alpha, \sim\alpha \vdash \sim\beta$ (for any α and β), which is not really appropriate for the use (cf. Priest, 2002, p. 288).

[28] A set of propositions is inconsistent, if for some α, both α and $\sim\alpha$ belong to this set.

2.3 Some history and various types of paraconsistent logics

We shall concentrate on the twentieth century history of PL, although some paraconsistent traces can be noticed in an earlier history of human thought (much more extensive presentation of the history of PL can be found e.g. in Priest, 2002, pp. 292–296). Akama and da Costa state that they believe "the history of paraconsistent logic started in 1948" (Akama and da Costa, 2016, p. 9), when a Polish logician Stanisław Jaśkowski proposed a PL called discursive (or discussive) logic (see Jaśkowski, 1948, 1949 (for the English translations of these see Jaśkowski, 1999a,b; in what follows we shall refer to them)). Priest calls it "the first non-adjunctive paraconsistent logic" (Priest, 2002, p. 295) (because he claims that simply the earliest PLs were given by two Russians, Vasiliev and Orlov (Priest, 2002, p. 295)). *Adjunction*, in this context, is a rule of inference, which from $\vdash \alpha$, and $\vdash \beta$ gives $\vdash \alpha \wedge \beta$. "Discursive logic can avoid explosion by prohibiting adjunction" (Akama and da Costa, 2016, p. 9). We are going to take a closer look at this logic soon.

"An understanding of most paraconsistent logics can be obtained by looking at the strategies employed in virtue of which ECQ [*ex contradictione quodlibet*, i.e. from contradiction everything follows] fails" (Priest, 2002, p. 296). First we discuss Jaśkowki's discursive logic, which uses modified connectives. Secondly, we turn to da Costa's countably many PLs which use the strategy of non-truth-functionality (see below). Finally, we consider logics connected with co-Heyting algebras, thus in a certain way dual to ILs, some of which are paraconsistent.

2.3.1 Jaśkowski's discussive logic

The name of this logic, discursive or discussive, comes from the idea of an attempt to systematize in one logical system all the theses defended in some discussion. Arguing people hold sometimes contradictory opinions. However, at the same time, interlocutors do not accept all the existing sentences (theses). In this way, we may regard the discursive logic as one of the PLs. Jaśkowski's discursive logic, denoted as D_2, consists of propositional formulas built from language of classical logic enriched with the possibility operator $\Diamond$ of S5 modal logic. Let as see how Jaśkowski himself writes about the context of the logical system in question and about the connection with the modal operator of possibility. Such a system emerges e.g. when

> theses advanced by several participants in a discourse are combined into single system, or if one person's opinions are so pooled into one system although that person is not sure whether the terms occurring in his various theses are not slightly differentiated in their meanings. Let such a system which canot be said to include theses that express opinions in agreement with one another, be termed a *discussive system*. To bring out the nature of the theses of such a system it would be proper to precede each thesis by the reservation: 'in accordance with the opinion of one of the participants in the discussion' or 'for a certain admissible meaning of the terms used'. Hence the joining of a thesis to a discussive system has a different intuitive meaning than has assertion in an ordinary system. *Discussive assertion* includes an implicit reservation of the kind specified above, which — out of the logical operators so far introduced in this paper — has its equivalent in possibility $\Diamond$. (Jaśkowski, 1999a, p. 43)

However, if we interpret every formula α as 'it is possible that α', i.e. $\Diamond\alpha$, then the *modus ponens* rule fails, as

$$\Diamond(\alpha \rightarrow \beta) \rightarrow (\Diamond\alpha \rightarrow \Diamond\beta)$$

is not a theorem in S5. "This is why in the search for a 'logic of discussion' the prime task is to choose such a function which, when applied to discursive theses, would play the role analogous to that which in ordinary systems is played by implication" (Jaśkowski, 1999a, p. 44). Jaśkowski acknowledges that such a problem has a number of solutions. He chooses, however, to define *discussive implication* as

$$\alpha \rightarrow_d \beta := \Diamond\alpha \rightarrow \beta,$$

and notes that "it may be read: 'if it is possible that α, then β', or, if applied to a discourse, 'if anyone states that α, then β', or 'if, for a certain admissible meaning of the terms, α, then β' (Jaśkowski, 1999a, p. 44) (where we changed each "p" into "α", and each "q" into "β", as we use a different notation). Now, in a discussive system, knowing that $\alpha \rightarrow_d \beta$, and α, we may infer that β, based on the fact that

$$\Diamond(\Diamond\alpha \rightarrow \beta) \rightarrow (\Diamond\alpha \rightarrow \Diamond\beta)$$

is a theorem in S5. "Thus the rule of *modus ponens* may be applied to discussive theses if discussive implication is used instead of ordinary implication" (Jaśkowski, 1999a, p. 44).

In his next paper (Jaśkowski, 1999b, cf. especially p. 57), Jaśkowski defines the discussive conjunction $\wedge_d$ as

$$\alpha \wedge_d \beta := \alpha \wedge \Diamond\beta.$$

In terms of this symbol, discussive equivalence may be defined as

$$\alpha \leftrightarrow_d \beta := (\alpha \rightarrow_d \beta) \wedge_d (\beta \rightarrow_d \alpha).$$

Let us only note (cf. Jaśkowski, 1999b, p. 58), that the law of the inconsistency for the discussive conjunction is a theorem of D_2:

$$\vdash_{D_2} \neg(\alpha \wedge_d \neg\alpha),$$

and the system does not explode, as

$$\nvdash_{D_2} (\alpha \wedge_d \neg\alpha) \rightarrow_d \beta.$$

Moreover, as Perzanowski notes in his comment to this article's translation, "D_2 in fact contains the full positive part of the classical logic" (Jaśkowski, 1999b, p. 58).

Akama and da Costa comment that "[d]iscursive logics are considered weak as a paraconsistent logic, but they have some applications, e.g. logics for vagueness" (Akama and da Costa, 2016, p. 10). Being based on modal logic, "it is classified as the modal approach to paraconsistency" (Akama and da Costa, 2016, p. 9).

The above presentation of Jaśkowski's discussive logic is very fragmented. We just wanted to outline this approach. One can read more about this logic and possible variations on it in (Dunin-Kęplicz et al., 2018). A new axiomatizations of Jaśkowski's logic D_2 were also proposed in (da Costa and Doria, 1995; Vasyukov, 2001). For other references on Jaśkowski's logic see also references in the above literature.

2.3.2 Da Costa's C-system

Some years later PLs were again proposed in doctoral dissertations, first by F. G. Asenjo in Argentina in 1954, and then by Newton C. A. da Costa in Brazil in 1963. "Asenjo proposed the first many-valued paraconsistent logic" (Priest, 2002, p. 295). We shall concentrate for a while on da Costa's contribution. He constructed for the first time a whole hierarchical system of propositional PLs C_i $(1 \leq i \leq \omega)$ as well as their first-order and higher-order extensions. His approach "is based on the non-standard interpretation of negation" (Akama and da Costa, 2016, p. 9) and employs "non-truth-functional account of [it]" (Priest, 2002, p. 303), thanks to which he avoids explosion (triviality). We shall explain below more fully what non-truth-functionality means.

C_1 system

Let us first review C-system C_1. The language of C_1 has as the logical symbols: $\wedge$, $\vee, \rightarrow$, and $\neg$. $\leftrightarrow$ is defined as usual. It is useful to introduce a following shorthand notation

$$\alpha^\circ \equiv \neg(\alpha \wedge \neg\alpha),$$

which can be read as 'α is well-behaved' (cf. Akama and da Costa, 2016, p. 10). The Hilbert style definition of this system can be found in (Akama and da Costa, 2016, p. 11). It consists of the positive intuitionistic logic axioms extended by the following axiom schemata for negation:

$$(1) \quad \alpha \vee \neg\alpha, \tag{2.1}$$

$$(2) \quad \neg\neg\alpha \rightarrow \alpha, \tag{2.2}$$

$$(3) \quad \beta^\circ \rightarrow \big((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha)\big), \tag{2.3}$$

$$(4) \quad (\alpha^\circ \wedge \beta^\circ) \rightarrow (\alpha \wedge \beta)^\circ \wedge (\alpha \vee \beta)^\circ \wedge (\alpha \rightarrow \beta)^\circ. \tag{2.4}$$

The only rule of reference is the *modus ponens*. All the logical symbols, except the negation, have all properties of the classical positive logic (da Costa and Alves, 1977, p. 622). If we add the principle of non-contradiction we get classical propositional logic (cf. da Costa, 1974, theorem 3 on p. 499f). We note some schemata, among others, which are not valid in C_1 (cf. da Costa, 1974, theorem 2 on p. 499):

$$\nvdash_{C_1} \neg(\alpha \wedge \neg\alpha),$$
$$\nvdash_{C_1} \alpha \wedge \neg\alpha \to \beta.$$

Moreover, let us note that C_1 is consistent (da Costa, 1974, theorem 6 on p. 500).

Da Costa defines also *the strong negation* of α as

$$\neg^{\star}\alpha := \neg\alpha \wedge \alpha^{\circ}.$$

It has all properties of the classical negation (da Costa, 1974, theorem 5 on p. 500). We note only the following three:

$$\vdash_{C_1} \alpha \vee \neg^{\star}\alpha,$$
$$\vdash_{C_1} \neg^{\star}(\alpha \wedge \neg^{\star}\alpha),$$
$$\vdash_{C_1} \alpha \leftrightarrow \neg^{\star}\neg^{\star}\alpha.$$

A non-trivial logical system is said to be *finitely trivializable* if there is a formula (not a schema) such that, adjoining it to the system as a new axiom, makes the resulting system trivial. It may be shown that C_1 is finitely trivializable, as each formula of the type $\alpha \wedge \neg^{\star}\alpha$ trivializes C_1 (da Costa, 1974, theorem 7 on p. 500).

C_n and C_ω systems

The da Costa C_1 system can be extended to C_n for $1 < n \leq \omega$ (cf. the original paper (da Costa, 1974) or (da Costa and Alves, 1977), rather than (Akama and da Costa, 2016), as in the latter there are some typos in these definitions). Let now, for $1 \leq n$, α^n denote $\alpha^{\circ\ldots\circ}$ with $\circ$ taken n times, thus α^1 is exactly the same as α°. Let moreover, for $1 \leq n$, $\alpha^{(n)}$ denote $\alpha^1 \wedge \alpha^2 \wedge \ldots \wedge \alpha^n$.

The C_n system (for $1 < n < \omega$) is defined in the same way as C_1, except the axioms (axiom schemata) (3) and (4) in our notation (i.e. (2.3) and (2.4)), which are replaced by the following:

$$(3') \quad \beta^{(n)} \to \big((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)\big), \tag{2.5}$$

$$(4') \quad (\alpha^{(n)} \wedge \beta^{(n)}) \to (\alpha \wedge \beta)^{(n)} \wedge (\alpha \vee \beta)^{(n)} \wedge (\alpha \to \beta)^{(n)}. \tag{2.6}$$

C_ω is defined as C_1 with neither axiom schema (3), nor (4) (i.e. (2.3) and (2.4)) at all. Let moreover C_0 denote the classical propositional calculus (which we have also denoted as CL).

Let us recall some theorems about C_n and C_ω (for these and others see da Costa, 1974, p. 501):

- Every calculus belonging to the hierarchy C_n, for $0 \leq n < \omega$, is finitely trivializable. C_ω is not finitely trivializable.
- Every calculus of the hierarchy $C_0, C_1, C_2, \ldots, C_\omega$ is strictly stronger than those which follow it.
- C_n, for $0 \leq n \leq \omega$, are consistent.
- In C_n, for $1 \leq n \leq \omega$, we have:

$$\nvdash_{C_n} \neg(\alpha \wedge \neg\alpha),$$
$$\nvdash_{C_n} \alpha \wedge \neg\alpha \to \beta.$$

Semantics for C_n

Da Costa and Alves developed a two-valued semantics for C_n systems (see da Costa and Alves, 1977). The authors note that it seems to them, that their "proposed semantics for C_1 agrees with some views of the young Łukasiewicz" (da Costa and Alves, 1977, p. 622, see also therein for a reference to Łukasiewicz's bibligraphy).

Let us first concentrate on C_1. The proposed semantics will be sound and complete, however, it has a very characteristic feature that "[t]he value of a valuation v for an arbitrary formula is not in general determined by the values of v for the propositional variables" (da Costa and Alves, 1977, p. 624). Such a semantics can be called a non truth functional one. Because of this feature, the valuation is defined on te whole set of all formulas, which we denote as P. In what follows, the symbols $\Rightarrow$ and $\Leftrightarrow$ will be used as an abbreviation of the meta-linguistic notions of implication and equivalence.

A valuation of C_1 is a function $v : P \to \{0,1\}$ such that all the following conditions are fulfilled:

$$
\begin{array}{ll}
(v1) & v(\alpha \vee \beta) = 1 \quad \Leftrightarrow \quad v(\alpha) = 1 \text{ or } v(\beta) = 1, \\
(v2) & v(\alpha \wedge \beta) = 1 \quad \Leftrightarrow \quad v(\alpha) = v(\beta) = 1, \\
(v3) & v(\alpha \to \beta) = 1 \quad \Leftrightarrow \quad v(\alpha) = 0 \text{ or } v(\beta) = 1, \\
(v4) & v(\alpha) = 0 \quad \Rightarrow \quad v(\neg\alpha) = 1, \\
(v5) & v(\neg\neg\alpha) = 1 \quad \Rightarrow \quad v(\alpha) = 1, \\
(v6) & v(\beta^\circ) = v(\alpha \to \beta) = v(\alpha \to \neg\beta) = 1 \quad \Rightarrow \quad v(\alpha) = 0, \\
(v7) & v(\alpha^\circ) = v(\beta^\circ) = 1 \quad \Rightarrow \quad v((\alpha \wedge \beta)^\circ) = v((\alpha \vee \beta)^\circ) = v((\alpha \to \beta)^\circ) = 1.
\end{array}
$$

Let us note that the conditions ($v1$), ($v2$), and ($v3$) are identical to respective conditions for classical logic (CL). All the other conditions are not given by equivalence and concern the negation symbol either on its own (conditions ($v4$) and ($v5$)), or in connection with other logical symbols.

A formula α is said to be *valid*, which we denote also as $\vDash \alpha$, if for every valuation v, $v(\alpha) = 1$.

Thus defined semantics is sound and complete, i.e. we have (da Costa and Alves, 1977, p. 624)

$$\vDash \alpha \quad \Leftrightarrow \quad \vdash \alpha\,.$$

Also, more generally, for any set Γ of formulas it holds that

$$\Gamma \vDash \alpha \quad \Leftrightarrow \quad \Gamma \vdash \alpha\,.$$

Let us now note some of the properties of this semantics (we shall call α false if $v(\alpha) = 0$, and true otherwise). The condition ($v4$) implies that the negation of a false formula is true, however negation of a true proposition can be either true or false. Therefore, p and $\neg p$ cannot be both false, although they can be both true (in some valuation). Such a valuation for which there exists at least one formula α such that $v(\alpha) = v(\neg\alpha) = 1$ is called *singular* (otherwise v is said to be *normal*). There are singular valuations (as well as the normal ones). For the strong negation it can be proved that $v(\alpha) = 1 \Leftrightarrow v(\neg^{\star}\alpha) = 0$, as well as $v(\alpha) = 0 \Leftrightarrow v(\neg^{\star}\alpha) = 1$.

In connection with the semantics for the system C_n let us quote (we change A to α in accordance with our notation):

> The extension of the semantics of C_1 to the systems C_n, $1 \leq n < \omega$, is immediate. All definitions and theorems are the same, only with evident modifications as regards the strong negation (for example, $\neg^{\star}\alpha$ becomes $\neg^{(n)}\alpha$, which is an abbreviation of $\neg\alpha \wedge \alpha^{(n)}$), and the symbol $^{\circ}$ (for example, α° becomes $\alpha^{(n)}$). (da Costa and Alves, 1977, p. 628)

We should mention an important drawback of the C-systems. A following general comment on treating negation non-truth-functionally can be found in (Priest, 2002, p. 306): "It is a consequence of this that the substitutivity of provable equivalents breaks down in general. For example, even though α is logically equivalent to $\alpha \wedge \alpha$ there is no guarantee that the negations of these formulas have the same truth value in an interpretation." He then gives a reference to (Urbas, 1989) for a discussion of this issue for da Costa's logics. Indeed, Urbas proves, for an appropriate definition of intersubstitutivity of provable equivalents, that all the C_n systems, for $1 \leq n \leq \omega$ do not enjoy this property, and moreover that for $1 \leq n < \omega$ there is no extension of any C_n which enjoys this property but which is weaker than CL (see Urbas, 1989, theorems 1 and 10).

Our exposition of C-systems is of course very limited. We do not present here many other developments such as the calculi with equality, modal calculi built from C_n, the first-order and higher-order extensions of C-systems. One may consult especially (da Costa, 1974) and (da Costa and Alves, 1977), for other developments see e.g. (Osorio Galindo et al., 2016) and literature therein.

We note also that some papers have appeared on categorical semantics for da Costa paraconsistent logics (see Vasyukov, 2000, 2017). They may provide an interesting research area of applications of category theory to the semantics of da Costa logics. These studies, however, do not coincide with the purpose of this paper, therefore we do not analyze these papers, but only refer to them.

2.3.3 Co-Heyting PL and other dual to intuitionistic logics

We turn to PLs closely connected with algebraic structures. We shall limit ourselves to some aspects of the logics based on co-Heyting algebras. We assume here the Reader's familiarity with the co-Heyting structure and its duality to Heyting algebra, otherwise we recommend first to read section 3.1, where the main definitions, some intuitions, and basic knowledge are presented. This type of PLs is the most significant for the research undertaken in this paper. Let us recall that if not stated otherwise we are dealing with propositional calculus (zero order). We denote the set of all propositional variables by P_0, and the set of all propositions by P.

There are different ways to define logic using the appropriate algebraic structure. We do not intend here to make a full analysis of logics based on co-Heyting algebras. In the following, we just want to present some results and show that while some such logics are not paraconsistent, others have paraconsistent properties.

Let K be any non-empty class of co-Heyting algebras (or their reducts, depending on the language), and $\Gamma \cup \{\alpha\}$ be any finite subset of P. For any appropriate algebra $\mathcal{A}$, a valuation $v : P_0 \to A$, is extended uniquely to a function $\overline{v} : P \to A$ by the standard rules: $\overline{v}(\sim\alpha) = \neg\overline{v}(\alpha)$, etc. We can now define two types of logics:

- the logic $\mathcal{L}^{\top}_{\mathsf{K}}$, which consequence relation we denote as $\vdash^{\top}_{\mathsf{K}}$, is defined by (cf. Font, 2016, p. 110, def. 3.5)[29]

$$\Gamma \vdash^{\top}_{\mathsf{K}} \alpha \quad \text{iff} \quad \text{for every } \mathcal{A} \in \mathsf{K} \text{ and for every valuation } v : P_0 \to A,$$
$$\text{if } \overline{v}(\gamma) = 1_{\mathcal{A}} \text{ for all } \gamma \in \Gamma, \text{ then } \overline{v}(\alpha) = 1_{\mathcal{A}};$$

- the logic $\mathcal{L}^{\leq}_{\mathsf{K}}$, which consequence relation we denote as $\vdash^{\leq}_{\mathsf{K}}$, is the finitary (by assumption) logic defined by (cf. Font, 2016, p. 428, def. 7.26)

$$\Gamma \vdash^{\leq}_{\mathsf{K}} \alpha \quad \text{iff} \quad \text{for every } \mathcal{A} \in \mathsf{K}, \text{ every } a \in A, \text{ and for every valuation } v : P_0 \to A,$$
$$\text{if } a \leq_{\mathcal{A}} \overline{v}(\gamma) \text{ for all } \gamma \in \Gamma, \text{ then } a \leq_{\mathcal{A}} \overline{v}(\alpha),$$

where we assume that the partial order $\leq_{\mathcal{A}}$ (we may also drop the subscript)

[29] We would like to thank Prof. T. Kowalski who helped us to clarify some of these issues, in particular the main idea in the proof of the Proposition 1 below, and introduced us to the book written by J. Font.

on each algebra in K is the partial order corresponding to the lattice structure.

The logic $\mathcal{L}_{\mathsf{K}}^{\top}$ may be called the assertional logic of K, whereas $\mathcal{L}_{\mathsf{K}}^{\leq}$ may be called the logic of order of K or the finitary logic that preserves degrees of truth with respect to the class K.

Let us limit ourselves for a moment to a class K which has just one element (and we shall also write $\vdash_{\mathcal{A}}$ instead of $\vdash_{\{\mathcal{A}\}}$). The logic $\mathcal{L}_{\mathcal{A}}^{\leq}$, for an algebra that is a complete lattice (or at least complete meet-semilattice) can be equivalently defined by the following condition (cf. Font, 2016, p. 188)

$$\Gamma \vdash_{\mathcal{A}}^{\leq} \alpha \quad \text{iff} \quad \text{for every valuation } v : P_0 \to A, \ \bigwedge \overline{v}(\Gamma) \leq \overline{v}(\alpha), \tag{2.7}$$

where for $\Gamma = \emptyset$, formally we have $\bigwedge \emptyset = \max A$. Thus the set of theorems of $\mathcal{L}_{\mathcal{A}}^{\top}$ and of $\mathcal{L}_{\mathcal{A}}^{\leq}$ coincides (the same is true for $\mathcal{L}_{\mathsf{K}}^{\top}$ and $\mathcal{L}_{\mathsf{K}}^{\leq}$ for arbitrary K).

Let us now restrict ourselves to a logic with an alphabet whose logical symbols are $\vee, \wedge, \sim$ (at the beginning of this subsection we shall distinguish between the logical symbol of negation $\sim$, its Heyting algebra counterpart operation $\neg$, and its co-Heyting algebra counterpart operation $\llcorner$).

In a similar context (Mortensen, 1995, p. 103) and (James, 1996, p. 69f) define a *paraconsistent algebra* as a bounded distributive[30] lattice with a certain complement operation $\llcorner$, defined by the condition

$$a \vee b = 1 \quad \text{iff} \quad \llcorner a \leq b.$$

This operation is, in our terminology from sec. 3.1, exactly the same as the $\vee$-complement (the above condition is equivalent to the one in Definition 2 in sec. 3.1). Therefore, according to our discussion in sec. 3.1, for a lattice to be a paraconsistent algebra it is sufficient that it be a co-Heyting algebra (cf. also James, 1996, p. 70, where a co-Heyting algebra is named 'Brouwerian algebra', and denoted as BrA).

For any paraconsistent algebra we have, *inter alia*, the following properties (see

[30] In his definition James neither assumes distributivity, nor proves it, but he uses it e.g. in the proofs on p. 70f. As Mortensen assumes distributivity explicitly, we accept that it is a part of their definition of paraconsistent algebra.

e.g. Mortensen, 1995, p. 103)[31]:

$$\begin{gathered} a \vee \neg a = 1, \quad \neg\neg a \leq a, \\ \neg(a \wedge b) = \neg a \vee \neg b, \quad \neg(a \vee b) \leq \neg a \wedge \neg b, \\ a \wedge \neg a \neq 0 \text{ (in general), but always: } \neg(a \wedge \neg a) = 1. \end{gathered} \tag{2.8}$$

From these algebraic properties we can trivially obtain appropriate logical theorems. For example, if K is any (non-empty) class of such paraconsistent algebras, and α is any proposition, we get (the logics $\mathcal{L}_{\mathsf{K}}^{\top}$ and $\mathcal{L}_{\mathsf{K}}^{\leq}$ were defined for K being any non-empty class of co-Heyting algebras or their reducts; we can either expand these definitions for paraconsistent algebras or limit ourselves only to paraconsistent algebras that are reducts of co-Heyting algebras)

$$\vdash_{\mathsf{K}} \alpha \vee \sim\alpha, \quad \vdash_{\mathsf{K}} \sim(\alpha \wedge \sim\alpha), \quad \text{where } \vdash_{\mathsf{K}} \text{ is either } \vdash_{\mathsf{K}}^{\top} \text{ or } \vdash_{\mathsf{K}}^{\leq}. \tag{2.9}$$

Although the set of theorems of these logics coincide, their (possible) paraconsistent properties are different. Below we shall examine this set of theorems more closely for a certain class K, but for now let us consider the issue of the paraconsistent properties of these logics in general.

Let us first consider the logic $\mathcal{L}_{\mathsf{K}}^{\top}$. We can prove the following proposition

Proposition 1. *If K is any non-empty class of paraconsistent algebras or of $(\vee, \wedge, \sim)$ reducts of co-Heyting algebras, and α, β are any propositions, then the logic $\mathcal{L}_{\mathsf{K}}^{\top}$ considered above has the following property*

$$\alpha \wedge \sim\alpha \vdash_{\mathsf{K}}^{\top} \beta \quad (\textit{equivalently } \{\alpha, \sim\alpha\} \vdash_{\mathsf{K}}^{\top} \beta),$$

and thus it is explosive (i.e. not paraconsistent).

Proof. Take any algebra $\mathcal{A} \in \mathsf{K}$, and any valuation $v : P_0 \to A$. We assume moreover that $\overline{v}(\alpha \wedge \sim\alpha) = 1_{\mathcal{A}}$. From this we have (we drop the subscript '$\mathcal{A}$')

$$1 = \overline{v}(\alpha \wedge \sim\alpha) = \overline{v}(\alpha) \wedge \neg\overline{v}(\alpha).$$

[31] In the cited book, one equality is written as an inequality, however, the change was made in the errata.

Now, because for any elements $a, b \in A$ it is: $a \wedge b = 1$ iff $a = 1 = b$, we get

$$\overline{v}(\alpha) = 1, \quad \text{and} \quad \neg\overline{v}(\alpha) = 1.$$

From this we have $1 = \neg\overline{v}(\alpha) = \neg 1 = 0$. Now, because $0 \leq \overline{v}(\beta) \leq 1$, we thus get $\overline{v}(\beta) = 1$. The equivalency stated in the proposition (in parenthesis) is obvious, as $a \wedge b = 1$ iff $a = 1 = b$. $\square$

We can also get this result by noting that although for any algebra from the class under consideration $a \wedge \neg a \neq 0$ (in general), it always holds that $\neg(a \wedge \neg a) = 1$ (see the last line in (2.8)), and thus $a \wedge \neg a = 1$ implies that $0 = 1$, so it holds only for a trivial algebra (i.e. the algebra with a one-element universe).

Let us now turn to $\mathcal{L}^{\leq}_{\mathsf{K}}$. If K is any non-empty class of Boolean algebras (which are, *inter alia*, paraconsistent algebras or $(\vee, \wedge, \sim)$ reducts of co-Heyting algebras), then of course for any propositions α, β

$$\alpha \wedge {\sim}\alpha \vdash^{\leq}_{\mathsf{K}} \beta \quad (\text{equivalently } \{\alpha, {\sim}\alpha\} \vdash^{\leq}_{\mathsf{K}} \beta).$$

However, we have the following proposition

Proposition 2. *If K is any non-empty class of paraconsistent algebras or of $(\vee, \wedge, \sim)$ reducts of co-Heyting algebras, which has at least one algebra that is not Boolean, and p is any variable (i.e. $p \in P_0$), then there is a proposition β for which*

$$p \wedge {\sim}p \nvdash^{\leq}_{\mathsf{K}} \beta \quad (\textit{equivalently } \{p, {\sim}p\} \nvdash^{\leq}_{\mathsf{K}} \beta),$$

and thus the logic $\mathcal{L}^{\leq}_{\mathsf{K}}$ is paraconsistent (i.e. not explosive).

Proof. Let $\mathcal{A} \in \mathsf{K}$ be an algebra from the assumptions that is not Boolean. This means that there is $a \in A$ such that $a \wedge \neg a > 0$, and thus there is a valuation $v : P_0 \to A$ such that $\overline{v}(p \wedge {\sim}p) = a \wedge \neg a > 0$. But now, we can always find a proposition β (e.g. $\beta = {\sim}{\sim}(q \wedge {\sim}q)$, for any variable q) such that $\overline{v}(\beta) = 0$, and thus $a \wedge \neg a = \overline{v}(p \wedge {\sim}p) \nleq \overline{v}(\beta)$. The equivalency stated in the proposition (in parenthesis) is obvious, as $a \leq b \wedge c$ iff $a \leq b$ and $a \leq c$. $\square$

As this was one of the starting points in our considerations, let us note that the consequence relation considered in (Mortensen, 1995, p. 104, case (1) for theoremhood) is the same as ours $\vdash^{\leq}_{\mathcal{A}}$, for $\mathcal{A}$ a paraconsistent algebra (see (2.7)). Moreover, the authors have noted the paraconsistent nature of this logic (although it should be formally added that only if the given algebra is not a Boolean one).

We now turn to co-Heyting algebras and logics based on them, which have the additional binary logical connective $\dot{-}$. This connective corresponds to the co-Heyting algebra pseudo-difference operation denoted with the same sign (see Definitions 4 and 6 in sec. 3.1 and the subsequent considerations).

We start with referring some of the results obtained by Goodman in (Goodman, 1981). Incidentally, let us recall that, in the categorical context, it is significant, that as Goodman mentions, it was Lawvere himself who first suggested to him "that there must be a logic dual to the intuitionistic in which contradictions would not be rejected" (Goodman, 1981, p. 119).

We begin with Goodman's view on Heyting algebras and IL.[32] In order to understand his views let us remind that every Heyting algebra is isomorphic to a subalgebra of the algebra of open subsets of a topological space (see Rasiowa and Sikorski, 1963, p. 140), and for such an algebra of open sets the Heyting implication is defined by (3.9), i.e. $A \Rightarrow B := Int((X \setminus A) \cup B)$, which implies that the Heyting negation assumes the form $\neg A = Int(X \setminus A)$. Now, Goodman comments on this: "Thus we can always view a Heyting algebra as a lattice of sets with a relative pseudo-complement [i.e. the operation $\Rightarrow$] formed in such a way as to omit all borderline cases, that is, all boundary points" (Goodman, 1981, p. 120). After some further remarks he continues on the next page: "Intuitionistic logic is a weakening of classical logic in a conservative direction. When in doubt, it refuses to commit itself". Eventually he writes that the conservative character of IL can be made precise by quoting a well-known theorem of Glivenko (Goodman cites (Rasiowa and Sikorski, 1963, p. 409, Theorem IX.13.5)) that

$$\text{if} \quad \Gamma \vdash_{\mathsf{CL}} \sim\!\alpha, \quad \text{then} \quad \Gamma \vdash_{\mathsf{IL}} \sim\!\alpha,$$

where $\Gamma \cup \{\alpha\}$ is a set of IL sentences. Thus he writes: "the intuitionistic calculus

[32] In this subsection even when discussing Goodman's or other people's results we shall use basically our notation in order to be more consistent.

affirms less than does the classical calculus, but it denies everything that the classical calculus denies" (Goodman, 1981, p. 121). This corresponds well with the fact that

$$\vdash_{\mathsf{IL}} \alpha \to \sim\sim\alpha, \quad \text{but} \quad \nvdash_{\mathsf{IL}} \sim\sim\alpha \to \alpha.$$

He concludes this part by indicating that "Unsolved problems form one class of mathematical statements which might indicate the existence of mathematical borderline cases" (Goodman, 1981, p. 121).

We find it interesting to view IL and its algebraic aspects in an informal way by considering this logic as a more conservative in nature than CL and omitting all (possible) borderline cases. This is even more interesting when this way of looking at logic can be extended to logic defined using co-Heyting algebras, which Goodman does. He starts this part by indicating that another class of mathematical statements, connected with borderline cases "of a rather different character[,] is formed by the paradoxes" (Goodman, 1981, p. 121), and he mentions the Russell's paradox. Co-Heyting algebras, being dual to Heyting ones, have the dual properties, which with respect to topological spaces correspond to the properties of closed subsets. *Inter alia*, the pseudo-difference operation, $\dot{-}$, for a co-Heyting algebra of closed sets becomes $B \dot{-} A := Cl((X \setminus A) \cap B)$ (see (3.10)), which implies the following form of a co-Heyting negation: $\ulcorner A := Cl(X \setminus A)$. Goodman comments: "Thus we can always view a Brouwerian algebra [i.e. co-Heyting algebra] as a lattice of sets with a pseudo-difference formed in such a way as to include all borderline cases—that is, all boundary points" (Goodman, 1981, p. 121). From this point of view, it may be understandable that the paradoxes emerge since we have included all the borderline cases.

From an algebraic point of view we could say that the logic defined using co-Heyting algebras seems just as natural as the logic defined using Heyting algebras, i.e. the IL (cf. Goodman, 1981, p. 121). Goodman names this new calculus "the anti-intuitionistic propositional calculus" (Goodman, 1981, p. 121), but we shall use the name 'Dual to Intuitionistic Logic', abbreviated as DIL after (James, 1996, p. 73). As noted above, its language consists of binary propositional connectives: $\vee, \wedge, \dot{-}$, which correspond to meet, join, and pseudo-difference operations in a co-Heyting algebra (denoted alike). Goodman introduces also a zero-ary connective $\top$, which denotes truth, and corresponds to the top element of the algebra. We also have a unary negation connective, which is defined as $\sim\alpha := \top \dot{-} \alpha$ (thus it corresponds to the co-Heyting negation $\llcorner a := 1 \dot{-} a$).

Goodman does not precisely define the consequence relation for any (finite) set of premises. In the context of the $\mathcal{L}_{\mathsf{K}}^{\top}$ and $\mathcal{L}_{\mathsf{K}}^{\leq}$ logics defined above, Goodman takes as a class K the class of all co-Heyting algebras. He gives an exact definition of theorems, which coincides with our definition and, as we already know, is common to both of these logics. As part of the issue of theorems, he writes: "this characterization [of theorems] does not give a calculus distinct from the classical propositional calculus" (Goodman, 1981, p. 122). For Goodman proves the following theorem (see Goodman, 1981, p. 122, we use our notation and terminology):

Theorem 1. *Suppose α is a DIL-formula which is a tautology [i.e. a theorem of CL]. (That is, α becomes a theorem of the classical propositional calculus when $\beta \dot{-} \gamma$ is interpreted as $\beta \wedge \sim\gamma$.) Suppose $\mathcal{A}$ is any co-Heyting algebra and v is any map from the atomic sentence symbols into $\mathcal{A}$. Then $\overline{v}(\alpha) = 1$.*

Of course the theorem in the opposite direction is true as well: any DIL theorem is a classical tautology, when every occurrence of $\beta \dot{-} \gamma$ is interpreted as $\beta \wedge \sim\gamma$. This follows from the fact that every Boolean algebra is a co-Heyting algebra with $a \dot{-} b := a \wedge \llcorner b$, and the Boolean complement is a co-Heyting negation (see sec. 3.1).

This shows that indeed the set of DIL theorems is not distinct form the classical tautologies, however one should be careful here, because the logical connectives of DIL are different than the ones of CL (DIL has the connective $\dot{-}$, but (currently) has no implication—the issue of implication shall be discussed below).

Is DIL a PL? As long as we do not precisely define the consequence relations for any (finite) set of premises, this question is incorrectly posed and has no definite answer. However, since the reasoning in propositions 1 and 2 also apply to the present situation, we already know that for the class K under consideration, $\mathcal{L}_{\mathsf{K}}^{\top}$ is not paraconsistent and $\mathcal{L}_{\mathsf{K}}^{\leq}$ is paraconsistent.

In order to describe DIL theorems (for either of the two logics, and so in this context we can talk about DIL without specifying which logic we mean) more clearly and in closer connection with IL theorems, we introduce after (James, 1996, p. 73ff) certain dualization procedure below. Also, to easier distinguish between the DIL and IL languages, we start to use the signs of the algebraic operations also on the level of languages, as respective logical connectives. We know that any co-Heyting algebra is the lattice dual of some Heyting algebra and vice versa (see sec. 3.1 but also e.g. James, 1996, p. 72). First we observe that there is a bijection between the set of all propositions of the DIL language (which we shall denote as P_{DIL}) and the set of all propositions of the IL language (denoted as P_{IL}). The bijection can be given explicitly. For any $\alpha \in P_{\mathsf{IL}}$, it is convenient to define $\alpha^{op} \in P_{\mathsf{DIL}}$ recursively by the following conditions[33] (all α's without upper index are from P_{IL}, and all α^{op}'s are from P_{DIL})

(1) if α is an atomic sentence, then $\alpha^{op} = \alpha$,

(2) if $\alpha = \neg\alpha_1$, then $\alpha^{op} = \llcorner(\alpha_1^{op})$,

(3) if $\alpha = \alpha_1 \wedge \alpha_2$, then $\alpha^{op} = \alpha_1^{op} \vee \alpha_2^{op}$,

(4) if $\alpha = \alpha_1 \vee \alpha_2$, then $\alpha^{op} = \alpha_1^{op} \wedge \alpha_2^{op}$,

(5) if $\alpha = \alpha_1 \Rightarrow \alpha_2$, then $\alpha^{op} = \alpha_2^{op} \dot{-} \alpha_1^{op}$.

From the bijectivity of this operation any DIL proposition can be uniquely written in the form of α^{op} for certain $\alpha \in P_{\mathsf{IL}}$, thus we shall represent the set P_{DIL} as $\{\alpha_i^{op}, i \in I\}$, whereas $P_{\mathsf{IL}} = \{\alpha_i, i \in I\}$. We have also a bijection between the set of all valuations $\overline{v} : P_{\mathsf{IL}} \to A$, for any Heyting algebra $\mathcal{A}$ and the set of all valuations $\overline{v}^{op} : P_{\mathsf{DIL}} \to A$, for its dual, the co-Heyting algebra, which we denote as $\mathcal{A}'$. $\overline{v}$ and $\overline{v}^{op}$ coincide on the set of atomic propositions, and are extended uniquely to the whole P_{IL} and P_{DIL}, respectively, by the standard conditions: $\overline{v}(\neg\alpha) = \neg\overline{v}(\alpha)$, etc., and $\overline{v}^{op}(\llcorner\alpha^{op}) = \llcorner\overline{v}^{op}(\alpha^{op})$, etc. Due to the duality properties discussed at the end of sec. 3.1, we have that $\overline{v}(\alpha) = \overline{v}^{op}(\alpha^{op})$, for any α, and remembering then that the dual algebra has a reversed order, we have that $\overline{v}(\alpha) = 1$ iff $\overline{v}^{op}(\alpha^{op}) = 0'$, where $0'$ is the bottom element of $\mathcal{A}'$ which is the same as the top element, 1, of $\mathcal{A}$ (analogously, we shall denote the top element of $\mathcal{A}'$ as $1'$).

With these notations, let us remind that any proposition α^{op} of DIL is a DIL theorem, if for all valuations v^{op} on every co-Heyting algebra $\mathcal{A}'$ it is $\overline{v}^{op}(\alpha^{op}) = 1'$.

[33] This operation can be seen as arising from the * mapping of Czermak, and further developed by Urbas (see Czermak, 1977, p. 472 and Urbas, 1996, p. 444, respectively).

In this case we write $\vdash_{\mathsf{DIL}} \alpha^{op}$. Analogously we denote an IL theorem α as $\vdash_{\mathsf{IL}} \alpha$ (which means that for all valuations v on every Heyting algebra $\mathcal{A}$ it is $\overline{v}(\alpha) = 1$).

James gives the following two theorems that characterize the set of DIL theorems (see James, 1996, p. 75f) (we provide the proofs, based on his proofs, which may help to better understand DIL):

Theorem 2. *For any IL proposition α, if $\vdash_{\mathsf{IL}} \alpha$, then $\vdash_{\mathsf{DIL}} \llcorner\alpha^{op}$.*

Proof. Suppose $\vdash_{\mathsf{IL}} \alpha$. Then for any IL-valuation v we have $\overline{v}(\alpha) = 1$. Thus $\overline{v}(\neg\alpha) = 0$. By the fact that $\overline{v}(\alpha) = \overline{v}^{op}(\alpha^{op})$, we get that $\overline{v}^{op}(\llcorner\alpha^{op}) = 1'$. Since there is no co-Heyting algebra that is not the dual of a Heyting algebra, and no DIL-valuation that is not the dual of an IL-valuation, we have $\vdash_{\mathsf{DIL}} \llcorner\alpha^{op}$. □

Theorem 3. *For any IL proposition α, $\vdash_{\mathsf{IL}} \neg\alpha$ if and only if $\vdash_{\mathsf{DIL}} \alpha^{op}$.*

Proof. Suppose $\vdash_{\mathsf{IL}} \neg\alpha$. This means that for any IL-valuation v, it is $\overline{v}(\neg\alpha) = 1$. For any element a of a Heyting algebra we have $a \wedge \neg a = 0$, so if $\neg a = 1$, then $a = 0$. Thus we get that $\overline{v}(\alpha) = 0$. Therefore for any DIL-valuation v^{op} we have $\overline{v}^{op}(\alpha^{op}) = 1'$, so $\vdash_{\mathsf{DIL}} \alpha^{op}$.
The proof of the 'if' part of the theorem goes analogously to the proof of the previous theorem. □

We can now give a different proof, than the one given by Goodman, of his important Theorem 1 (we base our proof on the reasoning in (Priest, 2002, p. 315)).

Proof of Theorem 1. As any DIL-formula can be written in the form α^{op}, let us assume that a DIL-formula α^{op} is a tautology, i.e. $\vdash_{\mathsf{CL}} \alpha^{op}$ (in the sense specified in the formulation of the theorem). By the duality property (strictly speaking we use the so-called strong duality principle for Boolean algebras (see sec. 3.1, especially the discussion in the footnote 38 and literature given there), we get from this that α is a classical contradiction, i.e. we have $\vdash_{\mathsf{CL}} \sim\alpha$. Hence, by Glivenko's theorem,[34] we get $\vdash_{\mathsf{IL}} \neg\alpha$, and thus by theorem 3, we have $\vdash_{\mathsf{DIL}} \alpha^{op}$, which completes the proof. □

James gives also a theorem, which says that for any well-formed DIL-sentence α^{op} it is

$$\vdash_{\mathsf{DIL}} \llcorner(\alpha^{op} \wedge \llcorner\alpha^{op}),$$

[34]We mean the theorem which says that $\vdash_{\mathsf{CL}} \sim\alpha$ if and only if $\vdash_{\mathsf{IL}} \neg\alpha$ (see e.g. Rasiowa and Sikorski, 1963, p. 391, proposition 5.4).

but this follows already from the last equality shown above in (2.8), and was mentioned in (2.9) (for a smaller language).

The implication

So far we have only considered the logics without implication. Goodman comments on the fact of overlapping of DIL and CL theorems by writing that the divergence between these two logics "must lie not in the theorems but, somehow, in the deducibility relation" (Goodman, 1981, p. 122). It is well-known that in IL, as well as in CL, β is deducible from α if and only if $\alpha \Rightarrow \beta$ is a theorem (for CL $\Rightarrow\, = \,\rightarrow$). In any Heyting algebra (i.e. *inter alia* in any Boolean algebra as well) $a \Rightarrow b = 1$ if and only if $a \leq b$ (for Boolean algebras $a \Rightarrow b$ coincides with $-a \vee b$). Goodman notes, that in the DIL case "there is no analogue of implication" (Goodman, 1981, p. 122), and proves the following theorem (as before, we use our notation and terminology)

Theorem 4. *There does not exist a binary connective $\rightarrow$ definable in terms of $\wedge, \vee, \dot{-}, T$ such that for any co-Heyting algebra $\mathcal{A}$ and any map v from the atomic sentence symbols into $\mathcal{A}$, we have $\overline{v}(\alpha \rightarrow \beta) = 1$ if and only if $\overline{v}(\alpha) \leq \overline{v}(\beta)$.*

In order to axiomatize DIL, Goodman introduces a particular sequent calculus system in the Gentzen style with only one formula on the left, and proves certain completeness theorem for this system (see Goodman, 1981, p. 123). He does not cite Czermak's paper (Czermak, 1977), although it was Czermak who few years earlier analyzed "the Gentzen's calculus of sequents ... [with] the restriction to sequents whose antecedent contains at most one formula" (p. 471), and proved the following theorem: "Each classically valid formula without existential quantifier and implication is derivable in **DJ**" (p. 473), where by **DJ** ("dual-intuitionistic calculus **DJ**") he calls the system obtained from Gentzen's calculus by the above restriction.

Goodman also gives two "somewhat unreasonable possibilities" for implication, namely that one could take $\alpha \rightarrow \beta$ to be either $\urcorner\alpha \vee \beta$, or $\urcorner(\alpha \dot{-} \beta)$ (see Goodman, 1981, p. 124), and he analyzes some properties of the first of them. Let us observe that the two coincide for Boolean algebras and are then equivalent to material implication.

In connection with implication for our context of DIL, we add only that Priest claims that "given any algebraic structure with top ($\top$) and bottom ($\bot$) elements, the following conditions can always be used to define a conditional operator [we use our notation]:

$$\overline{v}(\alpha \rightarrow \beta) = \begin{cases} \top & \text{if } \overline{v}(\alpha) \leq \overline{v}(\beta), \\ \perp & \text{otherwise.} \end{cases} \tag{2.10}$$

Though this particular conditional is not suitable for robust paraconsistent purposes since it satisfies: $\alpha \rightarrow \beta,\ \neg(\alpha \rightarrow \beta) \vDash \gamma$" (Priest, 2002, p. 327f).

The DIL logics $\mathcal{L}_{\mathsf{K}}^{\top}$ and $\mathcal{L}_{\mathsf{K}}^{\leq}$ (for K being the class of all co Heyting algebras) are just two of many logics that can be considered to be dual to IL. We have considered first of all the logic based on co-Heyting algebras, as it is most closely related to the situation we will encounter in the context of toposes. Another approach to the dualization of IL comes from the modification of the Gentzen's calculus of sequents. We have noted already Czermak's sequent calculus $\mathbf{DJ}$. Another proposal in this respect is offered by Urbas in (Urbas, 1996). He considers the sequent system **LDJ**, which, unlike Czermak's system, has the same connectives as Gentzen's intuitionistic sequent system **LJ**, but, like Czermak's, is (at most) singular in the antecedent. Urbas introduces also its extension $\mathbf{LDJ}^{\dot{-}}$, which has the pseudo-difference operator $\dot{-}$ added. He proves several results and compares the different kinds of logics that can be considered to be dual to IL. We do not provide a comprehensive overview of the subject of logics dual to IL. Other literature on this and similar topics includes (Brunner and Carnielli, 2005; Goré, 2000; Kamide, 2003; Kamide and Wansing, 2010; Priest, 2009; Rauszer, 1974a,b; Shramko, 2005). The issue of the duality between intuitionism and paraconsistency has also been the subject of the Ph.D. thesis (Queiroz, 1998), however, this paper is not available in English, as far as we know.

We have not discussed the predicate calculus of PLs, but one may consult e.g. (Goodman, 1981, p. 124ff; Priest, 2002, p. 329ff; and Goré, 2000; Kamide, 2003; Kamide and Wansing, 2010; Urbas, 1996), where this subject is considered.

Chapter 3

Duality problem and its implications

In this chapter, we investigate certain properties of toposes, crucial for the whole work, and on this basis, we analyze the validity of the proposal to define the notion of the so-called co-toposes. For this purpose, we first discuss the basic algebraic structures, from the simpler ones up to the Heyting and co-Heyting algebras, which are essential for our considerations, with particular emphasis on the relationship between the reversal of the order in these structures and their algebraic properties. We also discuss the dualities between the corresponding structures. Then, we outline briefly the claims of the authors who propose to introduce the notion of co-topos. The intention to understand certain properties of toposes themselves, related to the attempt to evaluate their proposals, was one of the main motivations for the research undertaken in the present work. The analysis of these properties of toposes and the related study of the relationship between certain two definitions of a topos are the subject of the next section, which is central for the entire paper. We then examine the power objects and provide two very simple examples to help understand the issues discussed. The chapter ends with conclusions and further comments.

3.1 Heyting and co-Heyting algebras

In this section, we define basic notions that are essential for our further considerations and give some intuitions standing behind them as well as some of their properties. Especially, we introduce the structures of Heyting and co-Heyting algebras. These notions, especially the Heyting algebra, are well-known and extensively presented in the literature (see e.g. Balbes and Dwinger, 1974; Goldblatt, 2006; MacLane and Moerdijk, 1994; Rasiowa and Sikorski, 1963), therefore this exposition is only rudimentary. However, in our presentation we want to emphasize that from the algebraic point of view Heyting and co-Heyting algebras may emerge in a symmetrical way as certain generalizations of a Boolean algebra. In this way, they are easily understood as mutually dual algebraic structures.

Boolean algebra, as a complemented distributive lattice, has the complement operation (denoted as $-$) that satisfies two laws:

$$a \wedge (-a) = 0, \tag{3.1a}$$

$$a \vee (-a) = 1. \tag{3.1b}$$

We can now weaken these laws and get certain generalizations of a complement operation.[35] For this purpose let us look at some intuitions behind this notion. First, join and meet in any lattice can be considered as an abstract analogues of the union and intersection of subsets of a fixed set X. The complement is then considered as the complement of a subset, say A, which is at the same time the greatest subset of X which is disjoint from A, and the least subset of X whose union with A gives the whole X. If we use each of these two properties as a defining property of certain quasi-complement operations in a lattice, we get two notions that in general do not coincide. Let us first define them explicitly. In this exposition we shall use the following notation: the calligraphic font for the whole algebraic structure (e.g. $\mathcal{A}$), and a normal font for its universe, i.e. the set on which the operations are defined (e.g. A).

Definition 1. Suppose that a lattice $\mathcal{A}$ has the zero (the least) element, denoted

[35] In this exposition we rely heavily on (Rasiowa and Sikorski, 1963, p. 52ff).

as 0. An element $c \in A$ is said to be the $\wedge$-*complement* (or *pseudo-complement*) of an element $a \in A$ if c is the greatest element such that $a \wedge c = 0$ (i.e. if it is the greatest element in the set of all $x \in A$ such that $a \wedge x = 0$). The $\wedge$-complement of an element a shall be denoted, if it exists, as $\neg a$ (it should be noted though that the same notation is sometimes being used for the (Boolean) complement).

Definition 2. Suppose that a lattice $\mathcal{A}$ has the one (the greatest) element, denoted as 1. An element $c \in A$ is said to be the $\vee$-*complement* of an element $a \in A$ if c is the least element such that $a \vee c = 1$ (i.e. if it is the least element in the set of all $x \in A$ such that $a \vee x = 1$). The $\vee$-complement of an element a shall be denoted, if it exists, as $\invneg a$.

If an element $a \in A$ has simultaneously the $\wedge$-complement $\neg a$ and the $\vee$-complement $\invneg a$, then they are not, in general, equal, but if the lattice $\mathcal{A}$ is distributive, then it always holds that[36]

$$\neg a \leq \invneg a.$$

If however they do coincide, i.e. $\neg a = \invneg a$, then such an element is the (Boolean) complement of an element $a \in A$, i.e. it can be denoted as $-a$, and fulfills both equations (3.1). Conversely, for a distributive lattice, it follows that any (Boolean) complement of an element a is at the same time both its $\wedge$-complement and $\vee$-complement. In the literature sometimes the same notation for $\wedge$-complement and (Boolean) complement is used.

We can further generalize and define the following operations.

Definition 3. Suppose that $\mathcal{A}$ is a lattice and $a, b, c \in A$. c is said to be the $\wedge$-*complement* (or *pseudo-complement*) *of* a *relative to* b if c is the greatest element such that $a \wedge c \leq b$ (i.e. if it is the greatest element in the set of all $x \in A$ such that $a \wedge x \leq b$). The $\wedge$-complement of a relative to b shall be denoted, if it exists, as $a \Rightarrow b$.

[36] Note that in (James, 1996, p. 59) the inequality is erroneously in the opposite direction. Therefore let us explicitly calculate: $\neg a = \neg a \wedge 1 = \neg a \wedge (a \vee \invneg a) = (\neg a \wedge a) \vee (\neg a \wedge \invneg a) = 0 \vee (\neg a \wedge \invneg a) = \neg a \wedge \invneg a$ (see Rasiowa and Sikorski, 1963, p. 52; one can also cf. Goodman, 1981, p. 120).

We note that, by definition, for every $x \in A$ we have

$$x \leq a \Rightarrow b \quad \text{iff} \quad a \wedge x \leq b. \tag{3.2}$$

We note also that $\wedge$-complement of a is a special case of this last notion, namely it is a $\wedge$-complement of a relative to 0, i.e. we have

$$\neg a = a \Rightarrow 0. \tag{3.3}$$

Definition 4. Suppose that $\mathcal{A}$ is a lattice and $a, b, c \in A$. c is said to be the $\vee$-*complement of* a *relative to* b (or *pseudo-difference of* b *and* a) if c is the least element such that $a \vee c \geq b$ (i.e. if it is the least element in the set of all $x \in A$ such that $a \vee x \geq b$). The $\vee$-complement of a relative to b (the pseudo-difference of b and a) shall be denoted, if it exists, as $a \Leftarrow b$ or equivalently, especially when named as pseudo-difference, as $b \dot{-} a$.

We note that by definition, for every $x \in A$ we have

$$x \geq a \Leftarrow b \quad \text{iff} \quad a \vee x \geq b. \tag{3.4}$$

We note also that $\vee$-complement of a is the same as $\vee$-complement of a relative to 1 (pseudo-difference of 1 and a), i.e. we have

$$\ulcorner a = a \Leftarrow 1 \equiv 1 \dot{-} a. \tag{3.5}$$

Many properties of these operations can be found in the above mentioned literature (we follow mainly Rasiowa and Sikorski, 1963, p. 55ff). Let us only note a tiny portion of them. Since $a \wedge b \leq b$ from (3.2) we get that

$$b \leq a \Rightarrow b,$$

provided $a \Rightarrow b$ exists. The element $a \Rightarrow b$ does not always exist. For example, $a \Rightarrow a$ exists if and only if the lattice has the unit element 1. Then

$$a \Rightarrow a = 1, \quad \text{and} \quad 1 \Rightarrow b = b.$$

In this case, the order of a lattice can be characterized by the $\Rightarrow$ operation because we have

$$a \leq b \quad \text{iff} \quad a \Rightarrow b = 1.$$

Analogous dual properties for $\Leftarrow$ can be obtained by the dualization, because the operation $\Leftarrow$ is dual to $\Rightarrow$ (we shall clarify and expand these issues below). However, let us pause for a moment to comment on the subject of dualities that arise already in simpler structures.

Dualities

Let us start with a semilattice structure. Conceived algebraically, an abstract *semilattice* is an algebraic structure consisting of a set, say S, with a binary operation which is by definition associative, commutative, and idempotent. If there is in S the identity element with respect to this operation, we say that the semilattice is *bounded*. At the same time, from the order theoretic point of view, there is no semilattice structure per se, but rather either a join-semilattice, when every two-element subset (or equivalently every non-empty finite subset) has a join (i.e. the least upper bound or supremum), or a meet-semilattice when it has a meet (i.e. the greatest lower bound or infimum). A bounded join-semilattice has by definition the least element (also called minimum or bottom), and a bounded meet-semilattice has the greatest element (maximum or top). We note that in both approaches sometimes boundedness is already a part of the definition of an appropriate semi-lattice.

We emphasize the fact, that from an algebraic point of view, there is no difference between the (bounded) join-semilattice and (bounded, respectively) meet-semilattice. Johnstone mentions that abstractly the categories of join-semilattices and meet-semilattices "are of course the same category" (Johnstone, 2002, p. 472). He continues, "but their forgetful functors to **Poset** are different"; this is so because for a join-semilattice $(S, \vee)$ one defines a partial order on S by setting

$$a \leq b \quad \text{if} \quad a \vee b = b, \tag{3.6}$$

while for a meet-semilattice $(S, \wedge)$ the partial order is defined by

$$a \leq b \quad \text{if} \quad a \wedge b = a. \tag{3.7}$$

However, let us repeat that the semilattice operation, say $\star$, is *inter alia* commutative, so having $a \star b = b \star a = a$, the 'choice' of the direction of the order is the other side of the 'choice' of weather $\star$ is a join or meet operation, but these operations are algebraically the same (or indistinguishable).

This means that from the order theoretic point of view, every join-semilattice is a meet-semilattice in the inverse order, and vice versa. Because of this fact we call these structures (mutually) dual.

Let us note however an apparent 'inconsistency' between an abstract algebraic approach and the more concrete order theoretic. From the algebraic point of view we have only an abstract semilattice operation. Let us take two semilattices $(S, \star)$, and $(T, \bullet)$, as well as a function $f : S \to T$ such that $f(a \star b) = f(a) \bullet f(b)$ (if the semilattices are bounded we assume moreover that f preserves the identity element). Such a function is called a *homomorphism* between these semilattices. However, if from the order theoretic point of view one of these semilattices 'turned out' to be a join-semilattice and the other a meet-semilattice, then in this approach f could not be said to be a homomorphism, but an anti-homomorphism. We add only that if f were a bijection, we would get, in this case, that abstract algebraic isomorphic semilattices may not be isomorphic from the order theoretic point of view, but only anti-isomorphic; or to put it the other way, an anti-isomorphic, from the order theoretic point of view, semilattices are isomorphic in the abstract algebraic approach.

The situation is different with richer structures that have more operations. We shall start with lattices. Let us recall that a lattice as a partially ordered set is both a join- and a meet-semilattice with respect to the same partial order structure. From an algebraic approach, on the other hand, the lattice is a structure $(L, \vee, \wedge)$, consisting of a non-empty set L and two binary operations that are assumed to be commutative, associative and such that the following, the so-called absorption, laws hold for all elements $a, b \in L$:

$$a \vee (a \wedge b) = a, \tag{3.8a}$$

$$a \wedge (a \vee b) = a. \tag{3.8b}$$

From this definition it follows that each of these operations are also idempotent,[37] and thus both $(L, \vee)$ and $(L, \wedge)$ are semilattices. The partial order can be defined equivalently by using either of these operations, as in (3.6) or (3.7). It is a simple and well-known fact, that the reversal of the order is equivalent to the interchanging of the operations meet and join. The lattice homomorphism is a function that preserves each of the two operations (as well as the top and bottom element if the lattice is bounded).

We do not want to discuss the details of the abstract (universal algebra) approach to the algebraic structures and their connections. Let us only observe that if the lattice operations are distinguished as 'the first' and 'the second', or join and meet, as it is the case in the standard approach, this does not mean that every lattice, say $(L, \vee, \wedge)$, and its dual, $(L, \vee', \wedge')$ with $\vee' = \wedge$ and $\wedge' = \vee$, are isomorphic. At the same time however, from the algebraic point of view, the defining properties of these two operations are exactly the same or symmetrical (so to speak, these operations are algebraically the same or indistinguishable). To put it differently, if one gives us a set L with two binary, commutative, and associative operations, say $\star$ and $\bullet$, such that for every elements $a, b \in L$ it is $a \star (a \bullet b) = a$, and $a \bullet (a \star b) = a$, then the choice of which operation is join and which is meet, is arbitrary and it is equivalent to choosing the direction of an order. We note only that the same applies also to distributive lattices and Boolean algebras. For the latter, the conditions for the complement operation, i.e. (3.1), could be written in the following manner in terms of $\star$, and $\bullet$ without settling which is join or meet:

$$a \star (-a) = e_{\bullet},$$

$$a \bullet (-a) = e_{\star},$$

where $e_{\bullet}$ is a neutral element for the operation $\bullet$, and analogously for the other operation.

Due to the symmetrical character of the join and meet operations in a lattice, the so-called *duality principle for lattices* holds: "every theorem proved for $\vee$ and $\wedge$

[37] E.g. applying first (3.8b), and then using (3.8a) with appropriate b, we get $a \vee a = a \vee (a \wedge (a \vee b)) = a$.

remains true if $\vee$ and $\wedge$ are replaced by $\wedge$ and $\vee$, respectively" (see Rasiowa and Sikorski, 1963, p. 36, where we have only changed the notation for join and meet; the Reader interested in more detailed analysis of the duality principles for algebraic structures may consult e.g. Balbes and Dwinger, 1974, p. 12). We should add that if the proven expression contains explicitly also any sign of the set $\{0, 1, \leq, \geq\}$, then apart from replacing $\vee$ and $\wedge$ by $\wedge$ and $\vee$, respectively, we have to replace also 0 by 1, 1 by 0, $\leq$ by $\geq$, and $\geq$ by $\leq$.[38]

Heyting and co-Heyting algebras

We are ready to define the Heyting and co-Heyting algebras. In the literature different names of these structures are present, e.g. Heyting algebras may be called pseudo-Boolean algebras and co-Heyting ones—Brouwerian algebras (see below).

Definition 5. A lattice $\mathcal{A}$ is said to be a *Heyting algebra* if it has a bottom element (also called "zero" and denoted by 0) and there exists $\wedge$-complement (pseudo-complement) of a relative to b, $a \Rightarrow b$, for all $a, b \in A$.

In the context of Heyting algebras the $\Rightarrow$ operation is also called a Heyting implication. In order to have the most important information in one place, we copy here (3.2), i.e. recall that by definition we have for every $x \in A$

$$x \leq a \Rightarrow b \quad \text{iff} \quad a \wedge x \leq b. \tag{3.2}$$

The pseudo-complement operation in a Heyting algebra exists for every element as it was defined by means of Heyting implication in the following manner

$$\neg a = a \Rightarrow 0. \tag{3.3}$$

In the context of Heyting algebras the pseudo-complement operation is also called a Heyting negation.

[38] For an interested Reader we note only that using the notions defined for a precise statement of the duality principles (cf. e.g. Balbes and Dwinger, 1974, p. 12 or McKinsey and Tarski, 1946, p. 159f), some of the above considerations, and some new ones, can be stated as follows: the class of all lattices, $\{(L, \vee, \wedge)\}$, is π-weakly dual (or weakly dual under the permutation π), for $\pi = (2, 1)$, therefore it satisfies the weak duality principle with respect to $\pi = (2, 1)$; this class is not $(2, 1)$-strongly dual; the class of all Boolean algebras, considered either as structures $(L, \vee, \wedge)$ or $(L, \vee, \wedge, -)$ is π-strongly dual (or strongly dual under the permutation π), for $\pi = (2, 1)$ or $\pi = (2, 1, 3)$, respectively, therefore this class satisfies the strong duality principle with respect to this π.

Moreover every Heyting algebra has also the one (top) element because

$$a \Rightarrow a = 1.$$

One of the consequences of the existence of all relative pseudo-complements is that every Heyting algebra is distributive (see e.g. Rasiowa and Sikorski, 1963, p. 59).[39] Some basic properties of the operations $\Rightarrow$ and $\neg$ in Heyting algebras are collected e.g. in (Rasiowa and Sikorski, 1963, p. 59ff), although the Reader should be careful with the different notation that is used there.

Thus a Heyting algebra may be presented in short as the following algebraic structure: $\mathcal{A} = (A, \vee, \wedge, \Rightarrow, \neg, 0, 1)$.

To give some examples let us first note that obviously every Boolean algebra is a Heyting algebra with

$$a \Rightarrow b = -a \vee b,$$

and thus we have also $\neg a = -a$. This example, by the connection of Boolean algebras with classical logic, shows some of the reasons for the notation of the relative pseudo-complement and its name—'Heyting implication', as well as for the notation and name of the Heyting negation.

As we have already mentioned in chapter 1, another example of a Heyting algebra, a very important one, is given by the algebra of open subsets of a topological space. Take any topological space, say X, and let $\mathcal{O}(X)$ be the set of all the open subsets of X. The operations on $\mathcal{O}(X)$ are defined as follows: join and meet are just the union and intersection of sets, correspondingly; for Heyting implication we have

$$A \Rightarrow B := Int\big((X \setminus A) \cup B\big), \tag{3.9}$$

where Int is the interior operation and $X \setminus A$ is the set difference of X and A, i.e. the set complement of A. The bottom element is the empty set, and the top is the

[39] Precisely, to show that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, it is sufficient that just the following relative pseudo-complement exists: $a \Rightarrow ((a \wedge b) \vee (a \wedge c))$. We do not settle the issue weather the existence of these relative pseudo-complements for any a, b, c implies the existence of all the relative pseudo-complements.

whole set X. The Heyting negation is thus given by

$$\neg A := Int(X \setminus A).$$

This last example is typical for Heyting algebras, as it can be shown that every Heyting algebra is isomorphic to a subalgebra of the algebra of open subsets of a topological space (see Rasiowa and Sikorski, 1963, p. 140, where Heyting algebras are called pseudo-Boolean).

Now let us move on to co-Heyting algebra, which is just the dual notion of Heyting algebra, but let us present the full definition.

Definition 6. A lattice $\mathcal{A}$ is said to be a *co-Heyting algebra* if it has a top element (also called "one" and denoted by 1) and there exists $\vee$-complement of a relative to b (pseudo-difference of b and a), $a \Leftarrow b$, for all $a, b \in A$.

In the context of co-Heyting algebras the $\Leftarrow$ operation is also called a co-implication. Let us collect some important facts. By definition, we have for every $x \in A$

$$x \geq a \Leftarrow b \quad \text{iff} \quad a \vee x \geq b. \tag{3.4}$$

The $\vee$-complement operation in a co-Heyting algebra exists for every element as it was defined by means of co-implication in the following manner

$$\neg a = a \Leftarrow 1. \tag{cf. 3.5}$$

In the context of co-Heyting algebras the $\vee$-complement operation is also called a co-Heyting negation.

Every co-Heyting algebra has the zero (bottom) element because $a \Leftarrow a = 0$ and every such algebra is also distributive.[40] Some basic properties of the operations $\Leftarrow$ and $\neg$ in co-Heyting algebras can be obtained from the above mentioned properties of Heyting operations by the process of dualization. We shall discuss the issue of dualization shortly, but for this purpose it is enough to consult our footnote

[40]The explicit proof of the last assertion can be obtained by dualization of the proof of 11.1 in (Rasiowa and Sikorski, 1963, p. 55), i.e. by literally exchanging every occurrence of each sign of the sequence $(\Rightarrow, \cap, \cup, \leqslant)$ by the sign of the same place in the following sequence $(\Leftarrow, \vee, \wedge, \geq)$ (where we have used our notation), and whenever there is a reference to equation '(1)' in the proof, reference to (3.4) should be used instead.

40 with the addition that the sign that denotes there the unit element should be changed to 0, and the sign that denotes there the zero should be changed to 1. Some of the properties of co-Heyting algebras and their operations are also given in (McKinsey and Tarski, 1946, p. 124f), where $+, \cdot, \neg$ denotes in our notation the operations $\vee, \wedge, \urcorner$, respectively, and $b \dot{-} a = a \Leftarrow b$, as was already mentioned above.

Thus a co-Heyting algebra may be presented in short as the following algebraic structure: $\mathcal{A} = (A, \vee, \wedge, \Leftarrow, \urcorner, 0, 1)$.

To give some examples let us first note that obviously every Boolean algebra is also a co-Heyting algebra with

$$a \Leftarrow b = -a \wedge b,$$

and thus we have also $\urcorner a = -a$.

Recall that in any co-Heyting algebra we have introduced two notations: $a \Leftarrow b$ and $b \dot{-} a$ (which denote the same element, i.e. $b \dot{-} a \equiv a \Leftarrow b$), and especially in the latter case this element is called the pseudo-difference of b and a. We decided to generally use the notation $a \Leftarrow b$ due to its similarity with the Heyting implication, as these notions are mutually dual. However, for the purpose of interpretation it is helpful to use for a moment the other notation. In any co-Heyting algebra we have, *inter alia*, that $b \geq b \dot{-} a$, $b \dot{-} a = 0$ iff $b \leq a$ (in particular $b \dot{-} b = 0$), and $b \dot{-} 0 = b$. Moreover, as we have just seen, in Boolean algebra treated as a co-Heyting algebra we have moreover that $b \dot{-} a = b \wedge -a$, and thus $b \dot{-} a$ may be considered as a lattice-theoretical generalization of the notion of difference of sets (cf. Rasiowa and Sikorski, 1963, p. 58). This explains also the name 'pseudo-difference'.

Analogously to the case of Heyting algebras, now for its dual, the co-Heyting algebras, we have a very important example given by the algebra of certain subsets of a topological space, this time it concerns the closed subsets. Take any topological space, say X, and let $\overline{\mathcal{O}}(X)$ be the set of all the closed subsets of X. The operations on $\overline{\mathcal{O}}(X)$ are defined as follows: join and meet are just the union and intersection of sets, correspondingly; for co-Heyting implication we have

$$A \Leftarrow B \equiv B \dot{-} A := Cl\big((X \setminus A) \cap B\big), \tag{3.10}$$

where Cl is the closure operation and $X \setminus A$, as previously, is the set difference of X and A, i.e. the set complement of A. As previously, the bottom element is the empty set, and the top is the whole set X. The co-Heyting negation is thus given by

$$\neg A := Cl(X \setminus A).$$

This last example is typical for co-Heyting algebras, as it can be shown that every co-Heyting algebra is isomorphic to a subalgebra of the algebra of closed subsets of a topological space (see McKinsey and Tarski, 1946, p. 134, Theorem 1.19, where co-Heyting algebras are called Brouwerian algebras; for the last assertion cf. Definition 1.1 on p. 124 therein with our Definition 6).

Dualities—continuation

Let us continue for a moment our considerations about algebraic structures in the abstract approach. Here we just want to observe that the above considerations that do not distinguish between the two operations of a lattice, because they are defined in a fully symmetrical way, is still possible in a certain way with respect to Heyting and co-Heyting algebras. Let us first note that the reversal of the order in a Heyting algebra $\mathcal{A} = (A, \vee, \wedge, \Rightarrow, \neg, 0, 1)$ changes it into the following co-Heyting algebra $\mathcal{A}' = (A' = A, \vee' = \wedge, \wedge' = \vee, \Leftarrow' = \Rightarrow, \neg' = \neg, 0' = 1, 1' = 0)$. This is easily seen if we recall that the reversal of the order in a lattice exchanges the join and meet operations, and thus for the case of Heyting implication and co-implication we only need to compare the defining conditions, i.e. (3.2) and (3.4). Similarly, we get the result for the negations by comparing (3.3) and (3.5). In order to conceive Heyting algebras in a purely algebraic approach we note that the Heyting implication can be defined equationally, without mentioning at all the order structure on a lattice (see MacLane and Moerdijk, 1994, Prop. 3 on p. 54). Therefore, similarly to our considerations with respect to lattices, let us only observe that it is obviously not the case that every Heyting algebra, say $\mathcal{A} = (A, \vee, \wedge, \Rightarrow, \neg, 0, 1)$, and its dual, the co-Heyting algebra $\mathcal{A}'$ defined as above, are isomorphic. At the same time however, from the algebraic point of view, the defining properties of their operations are fully symmetrical. To put it differently, if one gives us a set A with two binary, commutative, and associative operations, say $\star$, and $\bullet$, such that for every elements

$a, b \in A$ it is $a \star (a \bullet b) = a$, and $a \bullet (a \star b) = a$, and such that there exist neutral elements for each of them, let us denote them as $e_\star$ and $e_\bullet$ (so far this gives us the bounded lattice structure), and moreover there is a third binary operation, let us denote it as $\Uparrow$, which satisfies the following identities for any $a, b, c \in A$ (cf. equations (12), (13), and (14) in MacLane and Moerdijk, 1994, p. 54)

$$a \Uparrow a = e_\star,$$
$$a \star (a \Uparrow b) = a \star b, \quad b \star (a \Uparrow b) = b,$$
$$a \Uparrow (b \star c) = (a \Uparrow b) \star (a \Uparrow c),$$

then the choice weather this structure is a Heyting algebra (if $\star = \wedge$, $\bullet = \vee$, $e_\star = 1$, $e_\bullet = 0$, and $\Uparrow = \Rightarrow$) or a co-Heyting algebra (with dual operations and elements, i.e. if $\star = \vee$, $\bullet = \wedge$, $e_\star = 0$, $e_\bullet = 1$, and $\Uparrow = \Leftarrow$) is arbitrary, and it is the other side of the choice of the direction of the order.

Our goal in this section was primarily to introduce the basic concepts and show the relationship between certain properties and the order defined on a given algebraic structure. We have also discussed relevant dualities related to these issues. Finally, we emphasize that although our approach to the algebraic structures discussed in this section was based on standard Set Theory, there is also a corresponding approach to at least some of these issues within Category Theory. This approach is not directly needed by us in this paper, therefore we only mention it and refer to literature, such as e.g. (Wood, 2004; Marquis, 2009, chapter 6; Awodey, 2010, for Heyting algebras see sec. 6.3).

3.2 Introduction to the alleged co-toposes and motivations

We do not intend to fully present the approach of some authors that write about the so-called co-toposes (complemented toposes), nor to discuss the content of all the papers concerning this subject.[41] Their work however was an important motivation for our investigations, therefore it seems useful to describe their approach in a little more detail.

[41] In this section we use parts of our papers (Stopa, 2020) and (Stopa, 2018) (the latter after our translation from Polish).

As far as we know, the first definition of a co-topos appeared in (Mortensen, 1995, chapter 11)—this is suggested in (James, 1996, p. 80)—written together with Peter Lavers. After that publication there appeared subsequent papers dealing with co-toposes, see e.g. (Estrada-González, 2010, 2015a,b; James, 1996; Mortensen, 2003). Let us first see what the motivations for co-toposes are. The logic of toposes is known to be intuitionistic logic (IL), although one should be careful with such a simplification because, as we have noted already in the introduction, Colin McLarty pointed out: "topos logic coincides with no intuitionist logic studied before toposes" (see McLarty, 1995, p. vii).[42] In one of the approaches, a topos is considered as a generalization of a topological space. As we already know, the algebra of open sets (of any topological space) is the Heyting algebra and therefore it forms a semantics for IL (cf. Stone, 1937; Tarski, 1938 on topological interpretation of IL). However, a topology can be equivalently specified by the family of closed sets. Incidentally, it is closure operation and closed sets, rather than interior operation and open sets, that were first analyzed: McKinsey and Tarski first considered closure algebra as an algebra of topology (see McKinsey and Tarski, 1944, 1946) and the first general and explicit definition of a sheaf on a space was described by Leray in terms of the closed sets of that space (cf. MacLane and Moerdijk, 1994, p. 1).

In this context we can better understand Mortensen's motivation when he writes (see Mortensen, 1995, p. 102):

> Specifying a topological space by its closed sets is as natural as specifying it by its open sets. So it would seem odd that topos theory should be associated with open sets rather than closed sets. Yet this is what would be the case if open set logic were the natural propositional logic of toposes. At any rate, there should be a simple 'topological' transformation of the theory of toposes, which stands to closed sets and their logic, as topos theory does to open sets and intuitionism. Furthermore, the logic of closed sets is paraconsistent.

Mortensen gives the following definition of a complement-classifier (see Mortensen, 1995, p. 104f, we keep exactly the same wording, changing only F to *false*, a to A, and b to X in order to standardize the notation in this work):

[42] We have commented on this in footnote 4.

> A *complement-classifier* for a category $\mathcal{E}$ with terminal object 1, is an object Ω together with an arrow $false : 1 \to \Omega$ satisfying the condition that for every monic arrow $f : A \rightarrowtail X$ there exists a unique arrow $\overline{\chi}_f$ such that
>
> $$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, "!"'] & X \arrow[d, "\overline{\chi}_f"] \\ 1 \arrow[r, "false"'] & \Omega \end{tikzcd}$$
>
> is a pullback. $\overline{\chi}_f$ is the *complement-character* of f.

Then it is stated what a *complement-topos* is (we quote from Mortensen, 1995, p. 105):

> An (elementary) *complement-topos* is a category with initial and terminal objects, pullbacks, pushouts, exponentiation, and a complement classifier.

In what follows, we shall use the term complement-topos (or in short co-topos) meaning the notion as it is described above by Mortensen (the same notion is also used in e.g. Estrada-González, 2010, 2015a). As an example of a different definition of a co-topos see e.g. (Angot-Pellissier, 2015), where "cotopos" is considered as 'a closed co-Cartesian category with quotient classifier" (p. 189), which seems to be a different notion, although the author suggests he is considering the same notion and makes reference to unpublished work by James and Mortensen.

Let us first show some of the consequences of such a definition (cf. e.g. Estrada-González, 2010; Mortensen, 1995), assuming for a moment its validity. Having the complement-classifier $false : 1 \to \Omega$ (which will be denoted also as $\bot$), they define, by analogy with the standard approach,

$$true \equiv \top = \overline{\chi}_{0_1}\,, \tag{3.11}$$

where 0_1 is the only, and always existing, arrow from the initial object, 0, to the terminal object, 1, which is a monomorphism. The logical connectives are also defined by analogy with the standard approach[43], but we have to take into account that now $\overline{\chi}_f$ is the complement-characteristic arrow. We have therefore (we use the

[43] I assume the Reader's familiarity with the standard definition of logical connectives in a topos, which can be found in e.g. (Goldblatt, 2006, p. 139).

same symbols for denoting the logical connectives and for corresponding arrows that represent the logical operations on truth-values):

$$\begin{aligned}
\neg &:= \overline{\chi}_{\top}\,, \\
\vee &:= \overline{\chi}_{\langle \bot, \bot \rangle}\,, \\
\wedge &:= \overline{\chi}_{\mathrm{Im}[\langle \bot_\Omega, id_\Omega \rangle, \langle id_\Omega, \bot_\Omega \rangle]}\,, \\
\dot{-} &:= \overline{\chi}_{e}\,,
\end{aligned}$$

where $\langle f, g \rangle$ is the product arrow of f and g (with respect to the projections π_1, π_2 on its first and second factor, respectively), $[f, g]$ is the co-product arrow of f and g (with respect to the standard injections), $\mathrm{Im} f$ is an image of f, i.e. the monic of the epi-monic factorization (which, in a topos, exists for any arrow), $\bot_\Omega = \bot \circ !_\Omega$ ($!_\Omega$ is the only arrow $\Omega \to 1$), and e is the equalizer of $\vee$ and π_1.[44] Therefore, in the process of such a dualization: (i) conjunction and disjunction interchange, and (ii) in place of implication we get the pseudo-difference.

As a result $(\mathcal{E}(X, \Omega), \sqsubseteq)$ changes the order, so it becomes a co-Heyting algebra.[45] $(\mathsf{Sub}(X), \subseteq)$ remains a Heyting algebra, as it depends only on the factorization of the appropriate arrows and thus is independent of the (complement-) classifier. In this way, $(\mathsf{Sub}(X), \subseteq)$ and $(\mathcal{E}(X, \Omega), \sqsubseteq)$ are no longer isomorphic

[44] Mortensen and Lavers define conjunction without writing "Im" in front of square brackets (see Mortensen, 1995, p. 106). They do not give an explicit definition of an arrow of the form $[f, g]$, but it is commonly assumed that this is just a co-product arrow of the arrows f and g (these authors also repeatedly cite (Goldblatt, 2006), who likewise defines this notation). If this is also their understanding of this notation, then their definition of conjunction is incorrect. We shall describe it for toposes, but in this respect, their approach to co-toposes is analogous to the one in toposes. Not only in a general topos one has to take the image of an arrow (i.e. the monic of the epi-monic factorization), but even for a topos Set, for which the authors give comparisons, the co-product arrow itself, $[\langle \top_\Omega, id_\Omega \rangle, \langle id_\Omega, \top_\Omega \rangle]$, is not a monic arrow (in short, if only because it is a function from a 4 element set $2+2$ and takes only 3 different values). But the (anti)-characteristic morphisms are defined only for monic arrows. The same error seems to be present in (Mortensen, 2003, p. 261). Worse yet, Estrada-González in (Estrada-González, 2010) has the same error in presenting both the topos and the co-topos cases (see Estrada-González, 2010, p. 29 and 34), however, he introduces further errors in both of these places when he writes that the product arrow is taken for the arrows *true* and id_Ω (we give just one example), for which one cannot even construct the product arrow as their domains are different (the terminal object and Ω, respectively). In his next paper (Estrada-González, 2015a), he has the image 'Im', but still suggests constructing the same erroneous product arrows (see Estrada-González, 2015a, p. 388 for toposes, and p. 404 for co-toposes). The same situation is in (Estrada-González, 2015b, p. 288). At first, we have not noticed these errors and copied them from someplace, so they are also present in our paper (Stopa, 2020, p. 116), although this concerns only the product arrows and not the image of the co-product.

[45] We sometimes refer to a set or poset as an algebra without giving a complete record of the relevant operations defining it. This is an obvious abbreviation used when it is clear what operations are involved.

Heyting algebras. The arrow $\perp$, the one distinguished (up to isomorphism) by the complement-classifier, which is a generic subobject, is now the lowest element of $\big(\mathcal{E}(1,\Omega),\sqsubseteq\big)$.

Let us recall briefly how the topos semantics for propositional logic is defined. The essential part is common to the standard and the current approach. We denote as previously the set of all propositional variables by P_0, and the set of all propositions by P. As is well known, the global elements of Ω, i.e. the arrows $1 \to \Omega$, are truth-values. Thus the collection of all the truth-values of a given topos can be written as $\mathcal{E}(1,\Omega)$. By *$\mathcal{E}$-evaluation* we call any function $v : P_0 \to \mathcal{E}(1,\Omega)$.

Using the action of logical operations as arrows $\Omega \to \Omega$ (for $\neg$) or $\Omega \times \Omega \to \Omega$ (for binary logical symbols), we extend uniquely (analogically to the standard procedure in logical semantics, i.e. inductively with respect to the complexity of the formula) $\mathcal{E}$-evaluation v to the valuation $\overline{v} : P \to \mathcal{E}(1,\Omega)$, for any proposition of the whole set P, by the following standard conditions:

$$\overline{v}(\neg\alpha) = \neg \circ \overline{v}(\alpha), \qquad \begin{tikzcd} 1 \arrow[r, "\overline{v}(\alpha)"] \arrow[dr, "\overline{v}(\neg\alpha)"'] & \Omega \arrow[d, "\neg"] \\ & \Omega \end{tikzcd} ,$$

$$\overline{v}(\alpha \wedge \beta) = \wedge \circ \langle \overline{v}(\alpha), \overline{v}(\beta) \rangle , \qquad \begin{tikzcd} & 1 \arrow[dl, bend right, "\overline{v}(\alpha)"'] \arrow[dr, bend left, "\overline{v}(\beta)"] \arrow[d, dashed, "\langle\overline{v}(\alpha),\overline{v}(\beta)\rangle"] \arrow[dd, bend right=60, "\overline{v}(\alpha\wedge\beta)"'] & \\ \Omega & \Omega \times \Omega \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[d, "\wedge"] & \Omega \\ & \Omega & \end{tikzcd} .$$

Analogously for other binary logical operations. We now define topos validity in the standard way. We say that the proposition α is *$\mathcal{E}$-valid*, denoted $\mathcal{E} \vDash \alpha$, if for every $\mathcal{E}$-valuation v, $\overline{v}(\alpha) = \top : 1 \to \Omega$. The link between topos semantics and algebraic semantics of the algebra of truth-values $\mathsf{Hom}(1,\Omega)$ is direct, namely (see Goldblatt, 2006, p. 186)

$$\mathcal{E} \vDash \alpha \quad \text{iff} \quad \big(\mathcal{E}(1,\Omega),\sqsubseteq\big) \vDash \alpha. \tag{3.12}$$

In standard topos theory the algebra (denoted briefly as a poset) $(\mathcal{E}(1,\Omega),\sqsubseteq)$ is isomorphic to $\big(\mathsf{Sub}(1),\subseteq\big)$. But now in the co-toposes, as we have already mentioned, the order of $\big(\mathcal{E}(X,\Omega),\sqsubseteq\big)$ is reversed, and thus these algebras become co-Heyting. By that means, based on (3.12), we obtain a different set of tautologies ($\mathcal{E}$-valid sentences).

Let us analyze one example, for which we use the subscript "S" for the notation of standard toposes and no subscript for the present case of co-toposes. Because the definition of a co-topos, in comparison with the one of a topos, assumes the same properties for certain arrows, but gives only different names (or interpretations) to them, we have that e.g. the arrow $\perp$ defined as χ_{0_1} in a standard topos (i.e. $\perp_S$, in our current notation), is the same as $\top$ defined in (3.11), and thus we have $\perp_S = \top$. The situation is analogous for the other arrows. Now, for any (standard) topos $\mathcal{E}$ we have:

$$\mathcal{E} \models_S \sim(\alpha \wedge \sim\alpha)\ .$$

In IL we have: if $\sim\beta = 1$, then $\beta = 0$ (but not *vice versa*). Thus for all $\mathcal{E}$-evaluations we have trivially (assuming, for convenience, that α is an atomic sentence and valuations v_S and v coincide (on atomic sentences))

$$\begin{aligned}
\overline{v}_S(\alpha \wedge \sim\alpha) &= \perp_S \\
&= \wedge_S \circ \langle v_S(\alpha), \neg_S \circ v_S(\alpha)\rangle \\
&= \vee \circ \langle v(\alpha), \neg \circ v(\alpha)\rangle \\
&= \overline{v}(\alpha \vee \sim\alpha) \\
&= \top\,.
\end{aligned}$$

This means that for any co-topos we would have $\mathcal{E} \models \alpha \vee \sim\alpha$. This result, as well as the whole analysis, is essentially just the application of the Theorem 3 in sec. 2.3, and its proof, for that particular sentence in the case of co-topos semantics.

At the level of definition, a co-topos differs from a topos only by a complement-classifier. Thus this latter notion deserves a more thorough examination. The definition of the complement-classifier compared to the one of the subobject classifier raises serious doubts as to its correctness. Let us put these two definitions side by side:

A *complement-classifier* for a category $\mathcal{E}$ with a terminal object 1, is an object Ω together with an arrow $false : 1 \to \Omega$ satisfying the condition that for every monic arrow $f : A \rightarrowtail X$ there exists a unique arrow $\overline{\chi}_f : X \to \Omega$ such that

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, "!"'] & X \arrow[d, "\overline{\chi}_f"] \\ 1 \arrow[r, "false"'] & \Omega \end{tikzcd}$$

is a pullback.

A *subobject classifier* for a category $\mathcal{E}$ with a terminal object 1, is an object Ω together with an arrow $true : 1 \to \Omega$ satisfying the condition that for every monic arrow $f : A \rightarrowtail X$ there exists a unique arrow $\chi_f : X \to \Omega$ such that

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, "!"'] & X \arrow[d, "\chi_f"] \\ 1 \arrow[r, "true"'] & \Omega \end{tikzcd}$$

is a pullback.

It is evident that the two definitions use different labels for the corresponding 'mathematical objects' having identical properties. However, nothing important in mathematics can depend on notation itself. If anything valuable is occurring here, then it involves a different level than mathematics itself or it should be possible to translate these issues into the level of mathematics. Soon we shall analyze in detail the notion of subobject classifier, including the role of the distinguished arrow *true* in it. This will allow us to better understand this notion and the intentions of the authors proposing to introduce the notion of a complement-classifier and, in a further step, a co-topos. Thanks to this, we will also be able to better evaluate their approach.

In this context of choosing between labeling this distinguished arrow as *true* or *false*, and thus possibly also in the context of this different level than mathematics itself, Estrada-González in his paper (Estrada-González, 2015a)[46] writes about "Skolemization of the (equational) formula describing the (bare) subobject classifier" (Estrada-González, 2015a, p. 381, but see also p. 401, and 414). By "(bare) subobject classifier" he means a subobject classifier with the distinguished arrow denoted neither as *true*, nor as *false*, but just using some meaningless label (he uses ν) (cf. Estrada-González, 2015a, p. 380). We are not going to present his approach in detail here, nor are we going to discuss at this point the relationship between this

[46]Incidentally, this paper has so many (mathematical) typos that one is wondering if anyone responsible for the publication had read it. Therefore, we warn the potential Reader to be vigilant.

approach and our further considerations. We just wanted to mention his approach, which was part of the motivation behind our work. We shall come back to these issues briefly after presenting our results.

The following questions arise from the above considerations, to which we will try to find answers: (i) what are the connections, if any, between naming the arrow as *true* or *false* and its other mathematical properties, or to put it another way, what is the mathematical meaning, if any, of giving a given name to that arrow? (ii) to what extent the properties of this arrow are determined by the entire structure of the (co-)topos? (iii) given that there is another definition of a topos that does not refer at all to the subobject classifier or the arrow it distinguishes (see MacLane and Moerdijk, 1994, p. 161f) (in the next section we quote this definition and analyze it), what would the proposed change in the definition of a topos, which is to constitute the definition of a co-topos, amount to in this case?

The first two of these questions seem to concern issues important to the topos theory itself, regardless of the co-topos proposal. Therefore, we hope that our further considerations may contribute not only to clarifying the issue of a possible co-topos but also to a better understanding of the structure of a topos itself.

3.3 Interpretation of a generic subobject

Let us consider two definitions of a topos. The first one is given on p. 161f in (MacLane and Moerdijk, 1994), and the second on p. 163 therein (we only slightly change the text or notation, and skip some comments). We copy them in full detail as we shall work with them and compare both of them.

Definition 7. A topos $\mathcal{E}$ is a category with all finite limits, equipped with an object Ω, with a function P which assigns to each object B of $\mathcal{E}$ an object PB of $\mathcal{E}$, and, for each object A of $\mathcal{E}$, with two isomorphisms, each natural in A

$$\mathsf{Sub}_{\mathcal{E}}(A) \cong \mathsf{Hom}_{\mathcal{E}}(A, \Omega), \tag{3.13}$$

$$\mathsf{Hom}_{\mathcal{E}}(B \times A, \Omega) \cong \mathsf{Hom}_{\mathcal{E}}(A, PB). \tag{3.14}$$

Definition 8. A topos is a category $\mathcal{E}$ with

(i) A pullback for every diagram $X \longrightarrow B \longleftarrow Y$;

(ii) A terminal object 1;

(iii) An object Ω and a monic arrow $true : 1 \rightarrowtail \Omega$ such that for any monic $m : S \rightarrowtail B$ there is a unique arrow $\chi_m : B \to \Omega$ in $\mathcal{E}$ for which the following square is a pullback:

$$\begin{tikzcd} S \arrow[r, "!"] \arrow[d, tail, "m"'] & 1 \arrow[d, tail, "true"] \\ B \arrow[r, "\chi_m"'] & \Omega . \end{tikzcd}$$

(iv) An object PB and an arrow $\in_B$: $B \times PB \to \Omega$, to each object B, such that for every arrow $f : B \times A \to \Omega$ there is a unique arrow $g : A \to PB$ for which the following diagram commutes:

$$\begin{tikzcd} B \times A \arrow[r, "f"] \arrow[d, "id_B \times g"'] & \Omega \arrow[d, equal] \\ B \times PB \arrow[r, "\in_B"'] & \Omega . \end{tikzcd}$$

It is important for our study to see the connection between these two definitions. It is well known that having all finite limits (the finite completeness) is equivalent to having a terminal object and a pullback for any 'corner' (cf. e.g. Smith, 2016, theorem 52 on p. 81), so the beginning of the first definition is equivalent to parts (i) and (ii) of the second. Let us then examine now the consequences of the natural isomorphism (3.13) in the first definition.[47]

Sub can be understood in general as a contravariant functor between some category with pullbacks and the category Set. However, here we shall consider Sub as a covariant functor between a topos $\mathcal{E}^{op}$, and Set. It is defined in the following way:

$$\begin{aligned} \mathsf{Sub} : \mathcal{E}^{op} &\longrightarrow \mathsf{Set} \\ X &\longmapsto \mathsf{Sub}(X) \\ (g : X \to Y) &\longmapsto \big(\mathsf{Sub}(g) : \mathsf{Sub}(Y) \ni m \mapsto m' \in \mathsf{Sub}(X)\big), \end{aligned}$$

[47]Substantial part of this section is taken from our previous work (Stopa, 2020).

where for $m : S \rightarrowtail Y$, m' is defined by pulling m back along g, i.e.

$$\begin{tikzcd} S' \arrow[r] \arrow[d, tail, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, tail, "m"] \\ X \arrow[r, "g"'] & Y\,. \end{tikzcd} \tag{3.15}$$

For $\mathcal{E} = \mathsf{Set}$ we have $S' = g^{-1}[S]$, so this works as taking the inverse image, and thus $\mathsf{Sub}(g)$ is denoted, also in a general case, as g^{-1}. It may be denoted as well as g^*, especially in a more general case of a pullback functor (or a *change-of-base* functor).

Let us recall that representability of some functor means that it is naturally isomorphic to some Hom-functor. Thus, the natural isomorphism of (3.13) can now be stated equivalently by saying that the functor Sub is representable, with Ω being the representing object. Let us denote the natural isomorphism going from the functor Sub to $\mathsf{Hom}(-,\Omega)$ as θ. It is an isomorphism, so there exists its inverse, θ^{-1}, which is also a natural isomorphism. We can picture θ^{-1} as

$$\begin{tikzcd} \mathcal{E}^{op} \arrow[r, bend left=50, "\mathsf{Hom}(-{,}\Omega)", ""{name=U, below}] \arrow[r, bend right=50, "\mathsf{Sub}"', ""{name=D, above}] & \mathsf{Set}\,. \arrow[from=U, to=D, Rightarrow, "\theta^{-1}", "\wr"'] \end{tikzcd} \tag{3.16}$$

θ^{-1} is one of the natural transformations between $\mathsf{Hom}(-,\Omega)$ and Sub. Let us denote the set of all such natural transformations as $Nat\big(\mathsf{Hom}(-,\Omega),\mathsf{Sub}\big)$. Now, from the Yoneda lemma we know that

$$Nat\big(\mathsf{Hom}(-,\Omega),\mathsf{Sub}\big) \cong \mathsf{Sub}(\Omega)\,,$$

and that every natural transformation α from $\mathsf{Hom}(-,\Omega)$ to Sub is completely determined by an element of $\mathsf{Sub}(\Omega)$ (i.e. by a subobject of Ω), which is $\alpha_\Omega(id_\Omega)$. Namely, for any object X of $\mathcal{E}$, a component α_X is an arrow between sets $\mathsf{Hom}(X,\Omega)$ and $\mathsf{Sub}(X)$, and its action on any arrow $g : X \to \Omega$ is given by

$$\alpha_X(g) = \mathsf{Sub}(g)\big(\alpha_\Omega(id_\Omega)\big)\,,$$

where $\mathsf{Sub}(g)$ is an arrow in Set between $\mathsf{Sub}(\Omega)$ and $\mathsf{Sub}(X)$, which takes a monic, say $\Omega_0 \rightarrowtail \Omega$, and by pulling it back along g gives another monic, some $A \rightarrowtail X$, given by the pullback

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_0 \arrow[d, tail] \\ X \arrow[r, "g"'] & \Omega . \end{tikzcd}$$

$\alpha_\Omega(id_\Omega)$ is a subobject of Ω. However, it can be shown that if α is a natural isomorphism (and not just any natural transformation), then $\alpha_\Omega(id_\Omega)$ is not any subobject of Ω, but precisely a global element of it, i.e. an arrow $1 \to \Omega$ (see e.g. MacLane and Moerdijk, 1994, part of the proof on p. 33f).

Now, let us repeat that from the representability of Sub (see (3.13)) we know that among all natural transformations $\{\alpha\}$ there is at least one that is actually a natural isomorphism, which we have denoted as θ^{-1} (see (3.16)). We now know that such θ^{-1} is completely determined by $\theta^{-1}_\Omega(id_\Omega)$, a global element of Ω, which we shall denote as $\eta : 1 \to \Omega$. In this way, for any object X we have an isomorphism (a bijection) θ^{-1}_X between $\mathsf{Hom}(X, \Omega)$ and $\mathsf{Sub}(X)$ given by a pullback (to have standard notation, we may now denote the arrow from $\mathsf{Sub}(X)$ as f, and the corresponding arrow from $\mathsf{Hom}(X, \Omega)$ as χ_f)

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, tail, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\eta"] \\ X \arrow[r, "\chi_f"'] & \Omega . \end{tikzcd}$$

This is precisely the bijection between characteristic morphisms and subobjects as is assumed in the definition of a subobject classifier, however, for the time being, we know nothing about the arrow $\eta : 1 \to \Omega$ except what was stated above. Therefore, we prefer to keep the general label for it, say η, instead of the name *true*, which is present in all definitions of a subobject classifier. We shall come later to the role this arrow plays in any topos.

We thus propose to define a subobject classifier in a usual way except that the distinguished arrow (also called a universal element of the functor Sub or a generic subobject) should be labeled η instead of *true*, in order to emphasize that it is any arrow that satisfies the required conditions and its role (as *true* or a different arrow) is to be discovered. The usual definition, by labeling the arrow as *true* may suggest that this arrow has to fulfill some other requirements. We shall address this issue later on. For the convenience of the Reader we copy here the full definition.

Definition 9. If $\mathcal{C}$ is a category with finite limits, then a *subobject classifier* for $\mathcal{C}$ is an object Ω together with an arrow $\eta : 1 \to \Omega$ such that for every monic $f : A \rightarrowtail X$ there is a unique arrow $\chi_f : X \to \Omega$ which makes the following diagram a pullback

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, tail, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\eta"] \\ X \arrow[r, "\chi_f"'] & \Omega . \end{tikzcd}$$

We have thus seen that assuming natural isomorphism, as in (3.13), we get a subobject classifier. These two are in fact equivalent as the following proposition states (cf. Proposition 1 on p. 33 in MacLane and Moerdijk, 1994):

Proposition 3. *A category $\mathcal{C}$ with finite limits and small Hom-sets has a subobject classifier if and only if there is an object Ω and an isomorphism*

$$\theta_X : \mathsf{Sub}_{\mathcal{C}}(X) \cong \mathsf{Hom}_{\mathcal{C}}(X, \Omega), \quad \textit{natural for } X \in \mathcal{C}. \tag{3.17}$$

When this holds, $\mathcal{C}$ is well-powered.

We have already proved the "if" part above. The whole proof in (MacLane and Moerdijk, 1994, p. 33f) is carried out for a subobject classifier defined with *true* instead of our η, but this proof shows that for the authors of the book the name *true* does not mean anything more (at least at this point) than our η (nothing more is proved, but at the same time the arrow is denoted as *true*). On the basis of this proposition we say that having a subobject classifier is equivalent to functor Sub being representable, and thus requirement of (3.13) is equivalent to having a subobject classifier. A category $\mathcal{C}$ is said to be *well-powered* when $\mathsf{Sub}_{\mathcal{C}}(X)$ is isomorphic to a small set for all X. In the above proposition, since Hom-sets are all small, the bijection (3.17) entails that $\mathcal{C}$ is well-powered.

In order to further explore the natural isomorphism (3.13), let us first consider the set $\mathsf{Sub}(A)$ (in any topos, for any object A). It is well known that any such set can be considered as a partially ordered set with the order defined by factorization (see e.g. Goldblatt, 2006, p. 76f).[48]

Furthermore, $\mathsf{Sub}(A)$ can be considered as a lattice, or even a Heyting algebra (see e.g. Proposition. 3 in MacLane and Moerdijk, 1994, p. 186f about the lattice structure, and Theorem 1 (External) in MacLane and Moerdijk, 1994, p. 201 about the whole Heyting algebra structure). We do not present here the details of these constructions as they are a common knowledge in topos theory. We note only that this structure is natural in A, in the sense that the pullback along any morphism $h : A \to B$ (as in (3.15)) induces a map $h^{-1} \equiv \mathsf{Sub}(h)$ (it may be considered as a pullback functor) which is a homomorphism of Heyting algebras (and, *inter alia*, of posets).

Let us make a trivial remark, which will help us to understand our further considerations. There is a freedom to define the poset (or algebra) structure on sets $\mathsf{Sub}(A)$ in any appropriate way, but the one described above has the following properties: it generalizes the structure of subsets inclusion for ordinary sets, it is based on the operations inherently present in any topos such like pullbacks or coproducts, and moreover is natural in the sense that it is preserved under a pullback functor (as we have mentioned above). Let us moreover emphasize that the Heyting algebra structure of any $\mathsf{Sub}(A)$ is independent of the subobject classifier and of any interpretation of the arrow $\eta : 1 \to \Omega$.

We now want to explore a little more closely the natural isomorphism (3.17). Let us repeat that this natural isomorphism is completely determined through the Yoneda lemma, by $\eta : 1 \to \Omega$, so fixing any such appropriate η (i.e. a subobject classifier) determines the whole correspondence (natural and iso) between subobjects of any object and their characteristic morphisms from that object to Ω. By the very definition, as discussed above, the characteristic morphism id_Ω corresponds to $\eta : 1 \to \Omega$ (as a subobject of Ω). But $\eta : 1 \to \Omega$ can be considered also as a characteristic morphism which classifies certain subobject of a terminal object. An easy exercise of computing a pullback of η back along η itself shows that η as a

[48]In short, for two subobjects $[f]$ and $[g]$, taken as appropriate equivalence classes of monics with codomain A, the order is defined by $[f] \subseteq [g]$ iff $f \subseteq g$ iff f factors through g. Given $f : X \rightarrowtail A$ and $g : Y \rightarrowtail A$, we say that *f factors through g* if there is $h : X \to Y$, such that $f = g \circ h$ (such h is in fact a monic too). The definition of the order can be shown to be independent of the choice of the representatives of the equivalence classes.

characteristic morphism corresponds to the maximal subobject of 1 (top element in the poset or algebra of $\mathsf{Sub}(1)$), i.e. id_1.

To see what is the characteristic morphism for the maximal subobject of any object, say X, which is id_X, we can consider the following diagram ($!_X$ denotes the only arrow from an object X to a terminal one):

$$\begin{tikzcd}
X \arrow[r, "!_X"] \arrow[d, tail, "id_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[r, "!_1"] \arrow[d, tail, "id_1"] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\eta"] \\
X \arrow[r, "!_X"'] & 1 \arrow[r, "\eta=\chi_{id_1}"'] & \Omega\,.
\end{tikzcd}$$

The right square is a pullback because $\eta : 1 \to \Omega$ is a subobject classifier, and the left one is constructed by pulling id_1 back along $!_X$, which gives us id_X (e.g. because $\mathsf{Sub}(!_X)$ preserves the Heyting algebra structure, and thus, *inter alia*, the top element goes to the other top element). By the pullback lemma (cf. e.g. Goldblatt, 2006, p. 67), the outer rectangle is a pullback, and thus $\chi_{id_X} = \eta\circ!_X$.

Let us summarize these simple observations:

	$\mathsf{Sub}(X)$	$\mathsf{Hom}(X,\Omega)$
for $X = \Omega$	$\eta : 1 \to \Omega$	id_Ω
for $X = 1$	id_1	$\eta : 1 \to \Omega$
for any X	id_X	$\eta\circ!_X$

(3.18)

So far in our explorations of the natural isomorphism (3.13), which from now on, in accordance with (3.16) and (3.17), we shall be denoting as θ, we have seen how it is equivalent to having a subobject classifier, and we have shown some particular examples of the correspondence between certain subobjects and Hom–arrows being their characteristic morphisms. Moreover we have seen how each set $\mathsf{Sub}(A)$ has a Heyting algebra structure that generalizes the one in Set, is natural in A, and is defined by operations inherently present in every topos.

Assuming, as in (3.13), that θ_A is an isomorphism between sets $\mathsf{Sub}(A)$ and $\mathsf{Hom}(A,\Omega)$, which is natural in A, we underline that it does not imply anything about the poset or algebraic structure on sets $\mathsf{Hom}(A,\Omega)$. This isomorphism may be extended into an isomorphism of appropriate structures, like isomorphism of posets or Heyting algebras, but, at least in principle, there is a freedom to define such structures in any other, correct way. As this seems to be crucial to our considerations let us now explore some possible options in this subject and their consequences.

First, we consider the poset structure. The standard option is the one mentioned above, namely to transfer isomorphically the structure form $\mathsf{Sub}(A)$ to $\mathsf{Hom}(A, \Omega)$ (for any A) by the natural isomorphism between them as sets. In this approach we extend the isomorphism of sets into an isomorphism of appropriate structures. This is the standard way of doing this, therefore we shall not discuss this as it is textbook knowledge.

However, we want to consider now another option, namely to reverse the order on Hom–sets, i.e. to transfer the structure form $\mathsf{Sub}(A)$ to $\mathsf{Hom}(A, \Omega)$ (for any A) anti-isomorphically. Why such a choice could be of any importance and worth our attention? As we shall see, this is closely connected with the role played by the η arrow. In the standard approach, when the structure is transferred isomorphically, η is the maximal element in the poset or Heyting algebra structure of $\mathsf{Hom}(1, \Omega)$, because, as it was shown above, it always corresponds to the maximal subobject of a terminal object, i.e. to id_1. However, sometimes certain specific choices of a subobject classifiers may suggest that η could play a different role than a maximal element. For example, when working in the universe of set theory and taking at first the simplest topos Set, any two element set with any of its global elements is a subobject classifier. If we take

$$\eta : 1 = \{\star\} \ni \star \mapsto 0 \in \Omega = \{0, 1\},$$

then the subobjects are classified by characteristic morphisms which are, from the set theory point of view, anti-characteristic functions on sets. Thus the maximal subobject (the whole set) is classified by a function identically equal to zero. If one would like to stick to the order in which $0 < 1$, then this suggests that such a function should be the smallest in appropriate set of functions (on particular set). This particular situation makes us reflect on the possible roles that η could play, and therefore to ask about different possible choices of the structure on sets $\mathsf{Hom}(A, \Omega)$. This is also closely connected with the approach present in the already mentioned papers on co-toposes (see e.g. Estrada-González, 2010; Mortensen, 1995), where it is suggested to assume that η plays the role of a *false* arrow. At this moment we are not commenting on such an approach, we just wanted to give some reasons why it might be interesting and worth to examine such an option.

Let us consider a general case of any topos $\mathcal{E}$. We write precisely, what it means that the isomorphism (3.13) is natural. We assume that (3.13) holds (which we showed to be equivalent to say that $\mathcal{E}$ has a subobject classifier). We take the notation as in (3.17) but with A instead of X. The naturality condition means that for any $h : B \to A$ the following diagram commutes:

$$\begin{tikzcd} \mathsf{Sub}(A) \arrow[r, "\mathsf{Sub}(h) \equiv h^{-1}"] \arrow[d, "\theta_A"', "\wr"] & \mathsf{Sub}(B) \arrow[d, "\theta_B", "\wr"'] \\ \mathsf{Hom}(A,\Omega) \arrow[r, "\mathsf{Hom}(h,\Omega) = \_\circ h"'] & \mathsf{Hom}(B,\Omega)\,. \end{tikzcd} \tag{3.19}$$

The corners of this diagram are just sets. We know already that h^{-1} is a homomorphism of posets. On $\mathsf{Hom}(A,\Omega)$ we do not have yet any defined structure. Nothing prevents us to define the structure on $\mathsf{Hom}(A,\Omega)$ in an anti-isomorphic way in comparison with the structure on $\mathsf{Sub}(A)$. Thus defined structure is also natural in A, in the sense that, the morphism $\mathsf{Hom}(h,\Omega) = \_\circ h$ becomes also in this case a homomorphism of posets. This is easily seen from the diagram (3.19) with θ_A (for any A) being isomorphism of sets but anti-isomorphism of posets, and h^{-1} being homomorphism of posets. We can check this directly as follows.

Let us therefore assume that a natural isomorphism (of sets) θ_A in (3.17) is an anti-isomorphism of posets (for any A):

$$\theta_A : \big(\mathsf{Sub}(A), \subseteq\,\big) \cong \big(\mathsf{Hom}(A,\Omega), \sqsupseteq\,\big),$$

i.e.

$$g \subseteq f \quad \iff \quad \theta_A(f) \sqsubseteq \theta_A(g). \tag{3.20}$$

We shall show that $\mathsf{Hom}(h,\Omega)$ is (still) a homomorphism of posets. We know that h^{-1} is a homomorphism of posets, so we get

$$g \subseteq f \quad \Longrightarrow \quad h^{-1}(g) \subseteq h^{-1}(f). \tag{3.21}$$

Now, from (3.21) and (3.20) (taken with $A = B$ and appropriate subobjects) we get

$$g \subseteq f \quad \Longrightarrow \quad \theta_B(h^{-1}(f)) \sqsubseteq \theta_B(h^{-1}(g)),$$

but from (3.19) we have that $\theta_B \circ h^{-1} = \mathsf{Hom}(h, \Omega) \circ \theta_A$, and so we get

$$g \subseteq f \quad \Longrightarrow \quad \mathsf{Hom}(h, \Omega)(\theta_A(f)) \sqsubseteq \mathsf{Hom}(h, \Omega)(\theta_A(g)). \tag{3.22}$$

Finally, from (3.20) and (3.22) we get

$$\theta_A(f) \sqsubseteq \theta_A(g) \quad \Longrightarrow \quad \mathsf{Hom}(h, \Omega)(\theta_A(f)) \sqsubseteq \mathsf{Hom}(h, \Omega)(\theta_A(g)),$$

which means that $\mathsf{Hom}(h, \Omega)$ is also (together with h^{-1}) a homomorphism of posets, while θ_A is an anti-isomorphism of posets. Thus, let us repeat, also in this case, the poset structure on $\mathsf{Hom}(A, \Omega)$ is natural in A, and obviously the poset structure does not change the sets to be naturally isomorphic to $\mathsf{Set}(A)$ as stated by (3.13).

Turning to the Heyting algebra structure, we shall follow the reasoning in (MacLane and Moerdijk, 1994, p. 188), however, we have to be very cautious because there it is assumed from the very beginning that $\eta = \mathit{true}$ (see the definition of a topos on p. 163), which from the algebraic point of view means, as we shall also see, that η it the top element of the structure on $\mathsf{Hom}(1, \Omega)$.

Let us go back to a general situation. We take any topos $\mathcal{E}$, with $\eta : 1 \to \Omega$ a subobject classifier, and we assume that we have already defined a Heyting algebra structure on each $\mathsf{Sub}(A)$ as was mentioned above. Let us repeat that this structure is natural in A in the sense that the pullback along any morphism $h : B \to A$ induces a map h^{-1} of Heyting algebras (i.e. preserving the Heyting algebra structure).

Let us first focus on the meet operation. Let S, T be any two subobjects of the object A. Since h^{-1} preserves the meet operation (the intersection of subobjects) we have

$$h^{-1}(S \cap T) = h^{-1}(S) \cap h^{-1}(T).$$

To be more precise, and have more consistency with the subsequent notation, we write

$$h^{-1}(S \cap_A T) = h^{-1}(S) \cap_B h^{-1}(T). \tag{3.23}$$

This is equivalent to say that the meet operation on subobjects

$$\cap_A : \mathsf{Sub}(A) \times \mathsf{Sub}(A) \to \mathsf{Sub}(A)$$

is natural in A. We can check it by drawing the naturality square for it:

$$\begin{tikzcd}
\mathsf{Sub}(A) \times \mathsf{Sub}(A) \arrow[r, "\cap_A"] \arrow[d, "h^{-1} \times h^{-1}"'] & \mathsf{Sub}(A) \arrow[d, "h^{-1}"] \\
\mathsf{Sub}(B) \times \mathsf{Sub}(B) \arrow[r, "\cap_B"'] & \mathsf{Sub}(B)\,.
\end{tikzcd}$$

If we take any pair of subobjects of A, say (S, T) in the top-left corner, then going right and then down gives us $h^{-1}(S \cap_A T)$, while going first down and then right gives $h^{-1}(S) \cap_B h^{-1}(T)$, so the commutativity of the diagram means exactly (3.23).

Now comes the key point. Under the natural isomorphism (3.13) we obtain certain operation on sets $\mathsf{Hom}(A, \Omega)$, which we denote for the time being as $\lozenge_A$, making the following diagram commute:

$$\begin{tikzcd}
\mathsf{Sub}(A) \times \mathsf{Sub}(A) \arrow[r, "\cap_A"] \arrow[d, "\theta_A \times \theta_A"', "\wr"] & \mathsf{Sub}(A) \arrow[dd, "\theta_A", "\wr"'] \\
\mathsf{Hom}(A, \Omega) \times \mathsf{Hom}(A, \Omega) \arrow[d, equal, "\wr"] & \\
\mathsf{Hom}(A, \Omega \times \Omega) \arrow[r, "\lozenge_A"'] & \mathsf{Hom}(A, \Omega)\,.
\end{tikzcd} \tag{3.24}$$

We underline the fact that from this definition it does not necessarily follow that $\lozenge_A$ is the meet operation on $\mathsf{Hom}(A, \Omega)$, despite having already fixed Ω and the natural iso θ. Let us check some of the properties this operation has. First of all, it is natural in A. This fact is quite evident by its definition, and by the fact that $\cap_A$ is natural in A. If however one would like to check it directly, we note that the naturality of $\lozenge_A$ means that for any $h : B \to A$, the following diagram should commute

$$\begin{tikzcd}
\mathsf{Hom}(A, \Omega \times \Omega) \arrow[r, "\Diamond_A"] \arrow[d, "\mathsf{Hom}(h, \Omega\times\Omega) = _\circ h"'] & \mathsf{Hom}(A, \Omega) \arrow[d, "\mathsf{Hom}(h, \Omega) = _\circ h"] \\
\mathsf{Hom}(B, \Omega \times \Omega) \arrow[r, "\Diamond_B"'] & \mathsf{Hom}(B, \Omega)
\end{tikzcd} \quad . \tag{3.25}$$

It is indeed the case as one can easily check by chasing the following diagram (where we omit the labels of the iso arrows), knowing also that $\cap_A$ is natural in A (so the roof commutes)

$$\begin{tikzcd}
 & \mathsf{Sub}(B) \times \mathsf{Sub}(B) \arrow[rr, "\cap_B"] \arrow[dd, equal, "\wr"'] & & \mathsf{Sub}(B) \arrow[dd, equal, "\wr"] \\
\mathsf{Sub}(A) \times \mathsf{Sub}(A) \arrow[ur, "h^{-1} \times h^{-1}"] \arrow[rr, "\cap_A"] \arrow[dd, equal, "\wr"'] & & \mathsf{Sub}(A) \arrow[ur, "h^{-1}"'] \arrow[dd, equal, "\wr"] & \\
 & \mathsf{Hom}(B, \Omega \times \Omega) \arrow[rr, "\Diamond_B"] & & \mathsf{Hom}(B, \Omega) \\
\mathsf{Hom}(A, \Omega \times \Omega) \arrow[ur, "_\circ h"] \arrow[rr, "\Diamond_A"'] & & \mathsf{Hom}(A, \Omega) \arrow[ur, "_\circ h"'] &
\end{tikzcd} \quad .$$

Since $\Diamond_A$ is natural in A, by the Yoneda lemma it follows that it is fully determined by its action on the identity $id_{\Omega\times\Omega}$. We can see it by taking $A = \Omega \times \Omega$ in (3.25) and applying the functions to the element $id_{\Omega\times\Omega}$ of $\mathsf{Hom}(\Omega \times \Omega, \Omega \times \Omega)$ in the upper-left corner obtaining

$$\begin{tikzcd}
id_{\Omega\times\Omega} \arrow[r, mapsto, "\Diamond_{\Omega\times\Omega}"] \arrow[d, mapsto, "_\circ h"'] & \Diamond_{\Omega\times\Omega}(id_{\Omega\times\Omega}) \arrow[d, mapsto, "_\circ h"] \\
h \arrow[r, mapsto, "\Diamond_B"'] & \Diamond_B(h) = \Diamond_{\Omega\times\Omega}(id_{\Omega\times\Omega}) \circ h\,,
\end{tikzcd}$$

where $h : B \to \Omega \times \Omega$ (any such arrow). From this, because B is any object (and thus we can freely rename it), we get the desired action of $\Diamond_A$ on any arrow $h : A \to \Omega \times \Omega$ expressed by means of

$$\Diamond_{\Omega\times\Omega}(id_{\Omega\times\Omega}) \equiv \Diamond : \Omega \times \Omega \to \Omega, \tag{3.26}$$

via composition, i.e.

$$\Diamond_A : \mathsf{Hom}(A, \Omega\times\Omega) \ni (h : A \to \Omega\times\Omega) \longmapsto \Diamond_A(h) = \Diamond \circ h \in \mathsf{Hom}(A, \Omega). \tag{3.27}$$

This means that if the subobjects S and T of A have characteristic maps $s, t : A \to \Omega$ (accordingly), and thus $\langle s, t\rangle : A \to \Omega \times \Omega$, then the subobject $S \cap_A T$ of A has characteristic map

$$A \xrightarrow{\langle s,t\rangle} \Omega\times\Omega \xrightarrow{\Diamond} \Omega,$$

which we may write shortly as $s \Diamond t$.

The arrow $\Diamond$ defined in (3.26) is a certain internal operation on Ω. At this point, Mac Lane and Moerdijk, having *true* instead of η, and denoting our $\Diamond$ as $\bigwedge$, state that $\bigwedge$ is called "the internal meet operation; it makes $(\Omega, \bigwedge, \mathit{true} : 1 \to \Omega)$ into an internal meet semilattice object in the topos" (MacLane and Moerdijk, 1994, p. 188). In our approach however, as we consider a more general case and allow that the natural isomorphism of sets, θ, may not be an isomorphism of appropriate structures of sets, we want to permit at least an anti-isomorphic correspondence between $\mathsf{Sub}(A)$ and $\mathsf{Hom}(A, \Omega)$ (for any A), and therefore we now check some of the properties of the operation $\Diamond$ in this case.

We recall after (MacLane and Moerdijk, 1994, p. 198), that a lattice object L (for its definition cf. e.g. the same page of the cited textbook) has a zero and one (or bottom element $\perp$ and a top element $\top$), when there are arrows

$$\top : 1 \to L, \qquad \perp : 1 \to L,$$

which for any arrow $x : L \to L$ satisfy the identities ($\bigwedge$ is the lattice meet and $\bigvee$ is the join)

$$x \bigvee \perp = x, \qquad x \bigwedge \top = x. \tag{3.28}$$

This means that both of the following diagrams commute

$$\begin{tikzcd} L \arrow[r, equal, "\sim"] \arrow[rrr, bend right, "id_L"'] & L\times 1 \arrow[r, "id_L\times\perp"] & L\times L \arrow[r, "\bigvee"] & L, \end{tikzcd} \tag{3.29a}$$

$$\begin{tikzcd} L \arrow[r, equal, "\sim"] \arrow[rrr, bend right, "id_L"'] & L \times 1 \arrow[r, "id_L \times \top"] & L \times L \arrow[r, "\wedge"] & L. \end{tikzcd} \tag{3.29b}$$

In order to see that (3.29) gives (3.28) (and we shall use some of these detailed considerations later), we note that the first equation of (3.28) is an abbreviation for saying that the following diagram commutes

$$\begin{tikzcd} L \arrow[r, equal, "\sim"] \arrow[rrr, bend right, "x"'] & L \times 1 \arrow[r, "x \times \perp"] & L \times L \arrow[r, "\vee"] & L. \end{tikzcd}$$

We get this from (3.29a) by composing first with $x : L \to L$ in the following manner

$$\begin{tikzcd} L \arrow[r, "x"] & L \arrow[r, equal, "\sim"] \arrow[rrr, bend right, "id_L"'] & L \times 1 \arrow[r, "id_L \times \perp"] & L \times L \arrow[r, "\vee"] & L. \end{tikzcd}$$

The iso arrow between L and $L \times 1$ is $\langle id_L, !_L \rangle$, and we have

$$\begin{aligned} (id_L \times \perp) \circ \langle id_L, !_L \rangle \circ x &= (id_L \times \perp) \circ \langle id_L \circ x, !_L \circ x \rangle = (id_L \times \perp) \circ \langle x, !_L \rangle \\ &= \langle id_L \circ x, \perp \circ !_L \rangle = \langle x, \perp \circ !_L \rangle . \end{aligned}$$

This is precisely what we get by composing the iso arrow between L and $L \times 1$ with the arrow $x \times \perp$. Analogous considerations apply to $\top$.

In our case, we consider the diagram

$$\begin{tikzcd} \Omega \arrow[r, equal, "\sim"] \arrow[rrr, bend right, "id_\Omega"'] & \Omega \times 1 \arrow[r, "id_\Omega \times \eta"] & \Omega \times \Omega \arrow[r, "\Diamond"] & \Omega. \end{tikzcd} \tag{3.30}$$

In order to check if it commutes we use the fact that the operation $\Diamond$ on characteristic morphisms corresponds to the meet operation on corresponding subobjects, and for the moment we denote as $\overline{k}$ a corresponding subobject of a characteristic morphism k (i.e. $k = \chi_{\overline{k}}$). We have

$$\Diamond \circ (id_\Omega \times \eta) \circ \langle id_\Omega, !_\Omega \rangle = \Diamond \circ \langle id_\Omega, \eta \circ !_\Omega \rangle = \chi_{\overline{id_\Omega} \cap_\Omega \overline{(\eta \circ !_\Omega)}} = \chi_{\eta \cap_\Omega id_\Omega} = \chi_\eta = id_\Omega,$$

where we have used three times the information from the table (3.18). This shows that the diagram (3.30) indeed commutes. We may write this fact in an abbreviated form as

$$x \lozenge \eta = x, \qquad \text{for any } x : \Omega \to \Omega.$$

From this we get an important result that η is a neutral element with respect to the operation $\lozenge$. Thus, having in mind that $\lozenge$ corresponds to the meet operation on subobjects via natural isomorphism θ of sets, we have that if θ is moreover an isomorphism of appropriate structures then η is a top element and $\lozenge$ is a lattice meet ($\eta = \top$, $\lozenge = \bigwedge$), i.e. $(\Omega, \lozenge, \eta)$ is then an internal bounded meet semilattice object in the topos. If, however, θ is an anti-isomorphism of appropriate structures, then η is a bottom element and $\lozenge$ is a lattice join ($\eta = \bot$, $\lozenge = \bigvee$), i.e. $(\Omega, \lozenge, \eta)$ is then an internal bounded join semilattice object in the topos.

This concerns the internal semilattice structure on the Ω object. Later we shall consider some further internal structure on Ω, but now we go back to the structure on Hom-sets. As we have already seen, the operation $\lozenge$ on Ω determines uniquely the action of $\lozenge_A$ on $\mathsf{Hom}(A, \Omega)$ for any A (cf. (3.27)). Analogously to the situation with Ω, let us repeat that fixing the natural isomorphism θ_A of sets does not yet determine what kind of operation $\lozenge_A$ is. From now on, we focus only on the two possible options, the one where θ is also an isomorphism of appropriate structures, and the other where it is an anti-isomorphism of these structures. The former case is standard and well known. In the latter case the poset structure on Hom-sets is reversed, and the meet operation on subobjects corresponds to the join operation on characteristic morphisms ($\lozenge_A$ becomes $\bigvee_A$), which however is still a natural operation in the sense that (3.25) commutes. This means also that $\mathsf{Hom}(h, \Omega) = \_ \circ h$ preserves the join semilattice structure on $\mathsf{Hom}(A, \Omega)$ (cf. analogously (3.23) and the discussion in the subsequent paragraph, which, in short, shows how in such cases naturality of a certain operation is equivalent to the preservation of that operation). We can write down this fact in a simplified form, in both cases of θ, for any two elements of $\mathsf{Hom}(A, \Omega)$, which are always of the form χ_f, χ_g, for some $f, g \in \mathsf{Sub}(A)$, as follows

$$\mathsf{Hom}(h, \Omega)(\chi_f \lozenge_A \chi_g) = \big(\mathsf{Hom}(h, \Omega)(\chi_f)\big) \lozenge_B \big(\mathsf{Hom}(h, \Omega)(\chi_g)\big).$$

So far we have discussed the poset structures and the meet/join semilattice structures. The analogous considerations apply to the whole lattice structure as well as to the (co-)Heyting algebra structure. Namely, in the case that θ is an isomorphism of appropriate structures we get the standard result, but in the case that it is an anti-isomorphism of the appropriate structures, which is also an admissible possibility as we have already discussed, the result is somewhat new. The hitherto results may be summed up by the three theorems below.

The first theorem below summarizes the algebraic structure on sets of subobjects and its natural character (cf. MacLane and Moerdijk, 1994, p. 201, Theorem 1 (External), but we have used our notation and the conditional character depending on the definition of the operations):

Theorem 5. *For any object A in a topos, if we define the poset structure and other operations as in the above considerations, then the poset $\mathsf{Sub}(A)$ of subobjects of A has the structure of a Heyting algebra. This structure is natural in A in the sense that the pullback along any morphism $h : B \to A$ induces a map $h^{-1} : \mathsf{Sub}(A) \to \mathsf{Sub}(B)$, which is a homomorphism of Heyting algebras.*

The next theorem states the standard situation of algebraic structures on Hom-sets and the internal structure of Ω known from textbooks (cf. e.g. MacLane and Moerdijk, 1994, p. 201, Theorem 1 (Internal), but we have limited the thesis, used our notation and emphasized the conditional character depending on the way the algebraic structure is transferred):

Theorem 6. *If, in addition to the assumption of the previous theorem, we assume that the natural isomorphism between sets θ_A, for any A, as in* (3.13) *(for the assumed direction of that arrow cf.* (3.19)*), is also an isomorphism of the appropriate structures of $\mathsf{Sub}(A)$ and $\mathsf{Hom}(A, \Omega)$, then each $\mathsf{Hom}(A, \Omega)$, for any A, becomes also a Heyting algebra with the operations transferred from $\mathsf{Sub}(A)$ as in* (3.24)*. Every such operation (the meet, the join, and the Heyting implication (relative pseudo-complementation)) is natural in A, and $\mathsf{Hom}(h, \Omega) = \_ \circ h$, for any $h : B \to A$, preserves every such operation, as well as the Heyting zero (bottom) element, which means that $\mathsf{Hom}(h, \Omega)$ becomes a homomorphism of Heyting algebras. From the naturality of these operations, by the Yoneda lemma, we get that each such operation is fully determined by its action on the identity, $id_{\Omega\times\Omega}$, via composition* *(cf.* (3.26) *and* (3.27)*). In this way we get then that* Ω *is a Heyting algebra object (internal Heyting algebra).*

The last theorem presents our key results obtained so far:

Theorem 7. *If, in addition to the assumption of the theorem 5, we assume that the natural isomorphism between sets* θ_A*, for any* A*, as in* (3.13) *is an anti-isomorphism of the appropriate structures of* $\mathsf{Sub}(A)$ *and* $\mathsf{Hom}(A,\Omega)$*, then each* $\mathsf{Hom}(A,\Omega)$*, for any* A*, becomes a co-Heyting algebra, with its zero (bottom) element corresponding to the one (top) element of* $\mathsf{Sub}(A)$ *(i.e. to the* id_A*), its meet operation corresponding to the join on* $\mathsf{Sub}(A)$ *(i.e.* $\cup_A$*), its join operation corresponding to the meet on* $\mathsf{Sub}(A)$ *(i.e.* $\cap_A$*), and its pseudo-difference corresponding to the Heyting implication on* $\mathsf{Sub}(A)$ *(for the exemplary case of the co-Heyting join operation cf.* (3.24) *with* $\Diamond_A$ *being the co-Heyting join in this case). Every such co-Heyting operation is still natural in* A*, and* $\mathsf{Hom}(h,\Omega) = \_ \circ h$*, for any* $h : B \to A$*, preserves every such operation, as well as the co-Heyting zero element, which means that* $\mathsf{Hom}(h,\Omega)$ *becomes a homomorphism of co-Heyting algebras. From the naturality of these operations, by the Yoneda lemma, we get that each such operation is fully determined by its action on the identity,* $id_{\Omega\times\Omega}$*, via composition (cf.* (3.26) *and* (3.27)*). In this way we get then that* Ω *is a co-Heyting algebra object (internal co-Heyting algebra).*

Different internal algebras and partial orders

It is a standard situation that on the same set, e.g. $\mathsf{Hom}(A,\Omega)$, we can define different algebraic structures. It is well known that if one has a Heyting algebra structure on a set, then just by reversing the order one obtains a co-Heyting algebra. Yet it is not trivial that the co-Heyting structure on $\mathsf{Hom}(A,\Omega)$ is also natural in A, with $\mathsf{Hom}(h,\Omega) = \_ \circ h$ being a homomorphism of co-Heyting algebras. One can however wonder if there should not be just one internal structure on Ω. We answer that the two internal structures, Heyting and co-Heyting, may be considered as being connected with different internal partial orders.

Let us recall (see e.g. MacLane and Moerdijk, 1994, p. 199) that for an internal lattice L, we define a subobject $\leq_L$ of $L \times L$ as the equalizer

$$\leq_L \overset{e}{\rightarrowtail} L \times L \underset{\pi_1}{\overset{\wedge}{\rightrightarrows}} L,$$

where $\bigwedge$ is a lattice meet, and π_1 is the projection on the first factor. Then $(L, \leq_L)$ is an internal partial order (or internal poset). This comes from the fact that the order relation for a poset can be defined by

$$x \leq y \quad \text{iff} \quad x \bigwedge y = x.$$

In the case of a Heyting structure on $\mathsf{Hom}(A, \Omega)$, we have already shown that $(\Omega, \Diamond, \eta)$ is then an internal bounded meet semilattice object. Thus the internal partial order with respect to which Ω is a Heyting algebra object is the following

$$\leq_\Omega \overset{e}{\rightarrowtail} \Omega \times \Omega \underset{\pi_1}{\overset{\Diamond}{\rightrightarrows}} \Omega.$$

In the case of a co-Heyting structure on $\mathsf{Hom}(A, \Omega)$, we have also shown that $(\Omega, \Diamond, \eta)$ is an internal bounded join semilattice object. Thus if we want to point out the internal partial order with respect to which Ω is a co-Heyting algebra object, we either have to use the other arrow that is now the lattice meet, or, if we want to stick to the $\Diamond$ arrow, we could remind ourselves that the order relation on a poset can equivalently be defined by

$$x \leq y \quad \text{iff} \quad x \bigvee y = y,$$

and so we get

$$\preceq_\Omega \overset{e}{\rightarrowtail} \Omega \times \Omega \underset{\pi_2}{\overset{\Diamond}{\rightrightarrows}} \Omega.$$

This partial order is the reverse of the previous one.

To sum up, we observe that for both cases of internal Heyting and co-Heyting structures, if we consider a Heyting algebra to be a relatively pseudo-complemented bounded lattice and a co-Heyting algebra to be its dual with respect to the change of the order, then Ω as a Heyting algebra object or a co-Heyting algebra object has exactly the same arrows, but they play different roles in these structures (*inter alia* respectively: η is one or zero, $\Diamond$ is meet or join). In this way, we can understand how the same object Ω with appropriate arrows may be an internal Heyting algebra or an internal co-Heyting algebra, not being, in general, a bi-Heyting. Thus the situation with the Heyting and co-Heyting internal structures of Ω is similar to the presence of one of these structures on a set if and only if the other is present (in the sense of the possibility of reversing the order and obtaining the other structure).

3.4 Power objects

So far we have discussed the first isomorphism defining the topos in the Definition 7, i.e. the (3.13). Let us now examine the second natural isomorphism in this definition, i.e. (3.14) (for any fixed B, naturally in A), which for convenience we copy here:

$$\mathsf{Hom}_{\mathcal{E}}(B \times A, \Omega) \cong \mathsf{Hom}_{\mathcal{E}}(A, PB). \tag{3.14}$$

This natural isomorphism, which we shall denote as σ,[49] and the two functors that are naturally isomorphic can be depicted as follows

$$\begin{tikzcd} \mathcal{E}^{op} \arrow[r, bend left=50, "\mathsf{Hom}(-, PB)"] \arrow[r, bend right=50, "\mathsf{Hom}(B\times -, \Omega)"'] \arrow[r, phantom, "\wr \Downarrow \sigma"] & \mathsf{Set}\,. \end{tikzcd}$$

The first functor, $\mathsf{Hom}\,(-,\ PB)$, is a standard Hom-functor, and the second is a composition of the $(B \times -)$ functor and a Hom-functor. It works in the following way

$$\begin{aligned} \mathsf{Hom}\,(B \times -,\ \Omega) : \mathcal{E}^{op} &\longrightarrow \mathsf{Set} \\ A &\longmapsto \mathsf{Hom}(B \times A, \Omega) \\ f &\longmapsto \mathsf{Hom}(B \times f, \Omega) = \_ \circ (id_B \times f). \end{aligned}$$

As before, we apply the Yoneda lemma, and denoting the arrow from $\mathsf{Hom}(B \times PB, \Omega)$ as

$$\zeta_B \equiv \sigma_{PB}(id_{PB}) : B \times PB \to \Omega,$$

we find that the whole σ is fully determined by ζ_B. Namely, for any A, and any

[49] As this is an isomorphism we could denote it as σ^{-1} instead of σ, which would be more consistent with the previous notation, but having this freedom, for some reason, we prefer to do it the way we did.

$g \in \mathsf{Hom}(A, PB)$ we have

$$\sigma_A(g) = \mathsf{Hom}(B \times g, \Omega)(\zeta_B) = \zeta_B \circ (id_B \times g). \tag{3.31}$$

As σ is a natural isomorphism, we know that the correspondence between $\mathsf{Hom}(A, PB)$ and $\mathsf{Hom}(B \times A, \Omega)$, for any A, is bijective. Thus for every arrow $f : B \times A \to \Omega$ there is a unique arrow $g : A \to PB$, such that $\sigma_A(g) = f$ for which (3.31) holds, which means that the following diagram commutes

$$\begin{tikzcd} B \times A \arrow[d, "id_B \times g"'] \arrow[dr, "f"] & \\ B \times PB \arrow[r, "\zeta_B"'] & \Omega . \end{tikzcd}$$

In this way, starting from (3.14), we have derived part (iv) of the Definition 8 (with ζ_B instead of $\in_B$; we comment on that difference in notation below). Let us note, that the object PB with the arrow ζ_B, for any B, is by definition the exponential of Ω by B, i.e. $PB = \Omega^B$, and ζ_B is the evaluation arrow.

In (MacLane and Moerdijk, 1994, p. 163) the arrow $\sigma_{PB}(id_{PB})$ is denoted as $\in_B$ (see also our Definition 8 above). The reason for this notation is connected with the issue of denoting the arrow $\theta_\Omega^{-1}(id_\Omega) \equiv \eta$ as *true*. To be consistent, since we had changed the notation *true* into η, we changed also the notation $\in_B$ into some other, less meaningful, i.e. ζ_B.

Now, let us proceed with the algebraic structure on $\mathsf{Hom}(A, PB)$. We have already discussed in detail the issue of transferring the algebraic structure from $\mathsf{Sub}(A)$ to $\mathsf{Hom}(A, \Omega)$. Almost the same considerations apply now to the natural isomorphism (3.14). The isomorphism between $\mathsf{Hom}(B \times A, \Omega)$ and $\mathsf{Hom}(A, PB)$, which is natural in A,[50] does not have to be the isomorphism of the appropriate structures on these sets. As previously, we may follow the exposition in (MacLane and Moerdijk, 1994, p. 188ff), having in mind that η does not have to be the top element, i.e. *true*, and thus the structure can be transferred at least in one of the two ways, that is isomorphically and anti-isomorphically (from now on, we limit

[50]Incidentally, these sets can also be isomorphic naturally in B, but first P, as a function on objects, has to be properly extended to the functor $P : \mathcal{E}^{op} \to \mathcal{E}$ (see MacLane and Moerdijk, 1994, p. 162f).

ourselves to one of these two options). Thus, we get that $\mathsf{Hom}(A, PB)$ is either a Heyting or co-Heyting algebra.

Let us show only one diagram connected with the meet operation on subobjects:

$$\begin{tikzcd}
\mathsf{Sub}(B \times A) \times \mathsf{Sub}(B \times A) \arrow[r, "\cap_{B\times A}"] \arrow[d, "\theta_{B\times A}\times\theta_{B\times A}"', "\wr"] & \mathsf{Sub}(B \times A) \arrow[d, "\theta_{B\times A}", "\wr"'] \\
\mathsf{Hom}(B \times A, \Omega) \times \mathsf{Hom}(B \times A, \Omega) \arrow[d, "\sigma_A^{-1}\times\sigma_A^{-1}"', "\wr"] & \mathsf{Hom}(B \times A, \Omega) \arrow[dd, "\sigma_A^{-1}", "\wr"'] \\
\mathsf{Hom}(A, PB) \times \mathsf{Hom}(A, PB) \arrow[d, equal, "\wr"] & \\
\mathsf{Hom}(A, PB \times PB) \arrow[r, "\Diamond_A^{(PB)}"'] & \mathsf{Hom}(A, PB)\,.
\end{tikzcd}$$

Depending on the poset structure on $\mathsf{Hom}(A, PB)$ the operation $\Diamond_A^{(PB)}$ becomes either lattice meet or join.

Moreover, since the operation $\cap_{B\times A}$ is natural in A (see MacLane and Moerdijk, 1994, p. 188), we have that $\Diamond_A^{(PB)}$ is also natural in A. Thus just like in the previous situation, we get that $\Diamond_A^{(PB)}$ is, by Yoneda lemma, fully determined by means of

$$\Diamond_{PB\times PB}^{(PB)}(id_{PB\times PB}) \equiv \Diamond^{(PB)} : PB \times PB \to PB,$$

via composition. Following the analogous steps as in the case of the internal structure of Ω, we get that PB, for any B, is either an internal Heyting or co-Heyting algebra (Heyting or co-Heyting object).

3.5 Simple examples

Let us repeat that our hitherto argument says that reversal of the order on the level of Hom-sets or power objects with respect to the sets of subobjects is always possible. The following examples show only that in certain cases such a reversal of the order may be “expected” or regarded as “suitable”. This state of affairs is strictly connected with Set Theory. For Category Theory what matters is the structure of the arrows, and to put it simple, anything isomorphic is equally good. The basic notions such as every limit or co-limit (that is, *inter alia*, terminal and initial objects, products and co-products, pullbacks and pushouts) are defined only up to (a unique) isomorphism. The same applies to a subobject classifier, which is also defined up to (a unique) isomorphism, but different isomorphic subobject classifiers may be chosen for particular cases.

We have already mentioned the trivial case of the topos Set, where as a generic subobject we can take

$$\eta : 1 = \{\star\} \ni \star \mapsto 0 \in \Omega = \{0, 1\}.$$

This choice together with the standard order where $0 < 1$ suggests that the order on the set of characteristic functions (which in this case are precisely anti-characteristic functions), i.e. the order on a set $\mathsf{Hom}(A, \Omega)$, should be reversed with respect to the order on $\mathsf{Sub}(A)$ (otherwise the function identically equal to zero would be the top element). This case however is very special, as the topos Set is Boolean, and so the reversal of the order is done on the Boolean algebra $\mathsf{Hom}(A, \Omega)$, and thus does not change the type of the algebra, although interchanges the join with the meet operations and top with bottom.

Let us consider now the topos $\mathsf{Set}^{\to}$, which is the category of set functions and appropriate arrows between them (see e.g. Goldblatt, 2006, p. 34, 86ff). It is neither Boolean, nor bivalent. The subobject classifier is usually chosen in the following way (cf. Goldblatt, 2006, p. 87):

$$\Omega = t : \left\{0, \tfrac{1}{2}, 1\right\} \to \{0, 1\}, \quad \text{with} \quad t(0) = 0,\ t\left(\tfrac{1}{2}\right) = t(1) = 1,$$

and $\eta \equiv \top : 1 \to \Omega$ is defined as the pair $\langle t', \top_{\mathsf{Set}} \rangle$ as in

$$\begin{tikzcd} \{0\} \arrow[r, "t'"] \arrow[d, "id_{\{0\}}"'] & \{0, \tfrac{1}{2}, 1\} \arrow[d, "t"] \\ \{0\} \arrow[r, "\top_{\mathsf{Set}}"'] & \{0, 1\}, \end{tikzcd}$$

where $\top_{\mathsf{Set}}(0) = t'(0) = 1$, and thus making the diagram commute.

This topos has three truth values. The other two are $\top_{\frac{1}{2}} \equiv \langle t'', \top_{\mathsf{Set}} \rangle$, with $t''(0) = \frac{1}{2}$, and $\perp \equiv \langle t''', \perp_{\mathsf{Set}} \rangle$, with $\perp_{\mathsf{Set}}(0) = t'''(0) = 0$. The only nontrivial automorphism for this Ω is $\langle a, id_{\{0,1\}}\rangle$ as in

$$\begin{array}{ccc} \{0,\frac{1}{2},1\} & \xrightarrow{a} & \{0,\frac{1}{2},1\} \\ {\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ \{0,1\} & \xrightarrow[id_{\{0,1\}}]{} & \{0,1\}\,, \end{array}$$

with $a(0) = 0$, $a(\frac{1}{2}) = 1$, $a(1) = \frac{1}{2}$, thus it only exchanges 1 with $\frac{1}{2}$ in the domain of Ω. This means that we can take as a subobject classifier the same Ω but with $\top_{\frac{1}{2}}$ instead of $\top$. However, this also means that we cannot take as a subobject classifier the same Ω with $\perp$. In this case one may ask weather there is any situation that would make the reversal of the order to be "expected" or more "suitable". We can easily achieve this by taking the following subobject classifier (of course all the subobject classifiers are isomorphic with each other):

$$\Omega' = f : \{0, \tfrac{1}{2}, 1\} \to \{0,1\}, \quad \text{with} \quad f(0) = f\left(\tfrac{1}{2}\right) = 0,\ f(1) = 1,$$

and $\eta' : 1 \to \Omega'$ defined as the pair $\langle f', \perp_{\mathsf{Set}}\rangle$ as in

$$\begin{array}{ccc} \{0\} & \xrightarrow{f'} & \{0,\frac{1}{2},1\} \\ {\scriptstyle id_{\{0\}}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \{0\} & \xrightarrow[\perp_{\mathsf{Set}}]{} & \{0,1\}\,, \end{array}$$

where $\perp_{\mathsf{Set}}(0) = f'(0) = 0$, and thus making the diagram commute. Let us observe that as functions $\perp \equiv \langle t''', \perp_{\mathsf{Set}}\rangle$ is the same as $\langle f', \perp_{\mathsf{Set}}\rangle$, but the first is an arrow $1 \to \Omega$, while the second is $1 \to \Omega'$.

3.6 Conclusions and comments

We have analyzed the relationship between the two definitions of a topos (Definitions 7 and 8).[51] It is now clear that if the two definitions are to be equivalent, then in the second one, in the context of a subobject classifier, there should be a neutral label denoting the generic subobject, e.g. η, or if the label *true* appears there instead, it carries no additional assumptions.

[51] In this section we use parts of our paper (Stopa, 2020).

Furthermore, we have shown that the natural isomorphism (3.13) leaves some freedom in defining the algebraic structure on $\mathsf{Hom}(A, \Omega)$ (for any A), even if we want it to be natural in A. To put it differently, this natural isomorphism does not imply that the only way to define an algebraic structures on $\mathsf{Hom}(A, \Omega)$, natural in A, is by the isomorphism of the algebraic structure on $\mathsf{Sub}(A)$. If one would like to stipulate the structure of a Heyting algebra on $\mathsf{Hom}(A, \Omega)$, then one must additionally assume that η is the top element of the algebra defined on $\mathsf{Hom}(1, \Omega)$. On the other hand, if one assumes that η is the bottom element of this algebra, then one obtains the co-Heyting algebras on $\mathsf{Hom}(A, \Omega)$, which are also natural in A, and the natural isomorphism will be an anti-isomorphism between the algebras on $\mathsf{Sub}(A)$ and $\mathsf{Hom}(A, \Omega)$. In this way, we obtained a connection between labeling η as *true* or *false* and the role of this arrow in the $\mathsf{Hom}(1, \Omega)$ algebra, and through naturality also in any $\mathsf{Hom}(A, \Omega)$ (where one has to compose this arrow first with $!_A : A \to 1$).

Let us repeat once again, as this is one of the most important clarifications resulting from our considerations: if the labeling of η as *true* in the definition of a topos is not supposed to be any additional requirement as to the property of this arrow, then the algebraic structure on $\mathsf{Hom}(A, \Omega)$ is not uniquely determined by the definition of a topos. However, if the labeling of η as *true* is meant to imply a demand that this arrow be a top element in the algebra on $\mathsf{Hom}(1, \Omega)$, and thus the structure of Heyting algebra is transferred isomorphically via natural isomorphism from $\mathsf{Sub}(A)$ to $\mathsf{Hom}(A, \Omega)$, then the Definition 8 of a topos with such an understanding of the name *true* is not strictly equivalent to the Definition 7, since the latter has no such demand in it as to the role of the arrow η (understood in this context as a global element of Ω, which uniquely determines the whole natural transformation by the Yoneda lemma (see the beginning of the sec. 3.3)).

The Theorem 1 (Internal) in (MacLane and Moerdijk, 1994, p. 201) may suggest that the generic subobject always plays the role of the top element in $\mathsf{Hom}(1, \Omega)$, and that Ω is always an internal Heyting algebra. In the face of these results, the opinion that "To dualize, simply rename T [True] with F [False], and relabel the classifier arrow chi$_{\mathsf{f}}$ as chi-bar$_{\mathsf{f}}$" (Mortensen, 2003, p. 259) seems wrong, since the above mentioned theorem suggests that no matter how we label the generic subobject, it always plays the role of a top element in $\mathsf{Hom}(1, \Omega)$.

This theorem is not written in a form stating its assumptions explicitly. If these assumptions are satisfied, then it certainly implies that the generic subobject plays the role of the top element in $\mathsf{Hom}(1, \Omega)$, and that Ω is an internal Heyting algebra. However, if these assumptions are not satisfied, then obviously the theorem need not hold. Therefore, it was necessary first to show that the theorem relies on the assumption that η is a top element of $\mathsf{Hom}(1, \Omega)$, an assumption that might be present in Definition 8 (if denoting η as *true* in this definition is meant to assume that property, although as we have noted, it is not clear what is meant by this notation), but which is not present in or follows from Definition 7. Only in the light of these results can it be admitted that the claim that $\eta = \mathit{false}$ is permissible, at least if understood merely as a claim that this arrow is a bottom element in $\mathsf{Hom}(1, \Omega)$. As far as we know, the above considerations are not described in the literature.

However, we strongly emphasize that it seems to us that extending the natural isomorphism (3.13) to an anti-isomorphism of appropriate algebras, instead to an isomorphism of these structures, is in the context of broader applications of the topos theory a 'risky and questionable move'. In particular, this can lead to serious problems when first-order and higher-order logics are concerned, and also perhaps when it comes to the question of implication even for zero-order logics. A number of key properties of toposes (in particular for logical applications) that have been proved when taking $\eta = \mathit{true}$ may no longer hold if $\eta = \mathit{false}$. Therefore, further consequences of such an assumption must be carefully examined. However, from a purely algebraic point of view, the very Definition 7 of a topos permits both cases. The present work demonstrates this possibility, without prejudging its further consequences.

Conclusions

Intuitionistic logic (IL) and paraconsistent logics (PLs) are well known and have been studied independently of category theory (CT). However, the connections of CT with various logics are extensive. Among many categories that have their own internal logic, toposes play a special role through their relationship with intuitionistic higher order logic.

As we have already mentioned, the algebraic counterpart of IL (exactly its Lindenbaum-Tarski algebra) is a Heyting algebra. Such structures appear naturally in toposes. However, several papers have appeared in recent times, most notably (Mortensen, 1995, chapter 11; Estrada-González, 2010; James, 1996; Mortensen, 2003), which suggest that by making certain changes in the definition or the interpretation of a topos, one can obtain the so-called complementary-topos (co-topos for short), in which in place of some Heyting algebras, co-Heyting algebras, dual to them, appear. In this way, these authors suggest, the logic of co-toposes becomes paraconsistent. This proposal, however, seems problematic for a number of reasons.

This raises two issues that we have addressed in this work: 1) the relation between IL and PL; 2) the validity of the notion of co-topos and the possible relation of co-toposes to toposes.

In connection with the first goal, we have presented both types of logics in a little more detail (basically only their zero order, since further analyzes also concerned only propositional logic in toposes). Since IL is historically related to Brouwer's intuitionism in his philosophy of mathematics, we have also presented his basic views on this matter (chapter 1). Within PLs, we have discussed primarily, though not exclusively, those logics that are related to the structures of co-Heyting algebras (chapter 2). Among other things, we have shown that a certain type of logics based on co-Heyting algebras do not have paraconsistent properties, but it is possible to define another type of logics based on these algebras that are paraconsistent in general. We discussed the relationship between ILs and PLs primarily through the analysis of the properties of Heyting and co-Heyting algebras which constitute the algebraic aspects of ILs and some of the PLs.

At the beginning of chapter 3, we showed how both types of algebras can be derived as a kind of symmetric generalization of a Boolean algebra. This manifests itself in the fact that these algebras are dual to each other, in the sense that when the order is reversed, one becomes the other. We have also shown that, from a purely algebraic point of view, a set with the corresponding operations satisfying certain properties can be viewed as either a Heyting algebra, or a co-Heyting one, depending on the interpretation of these operations or, equivalently, depending on the order defined on this set. This duality plays an important role in further considerations.

In connection with the second goal, we first presented some of the more important proposals made in the aforementioned papers on co-toposes. These works have provided an important motivation for our investigations. In these papers, let us recall, a co-topos is defined as a Cartesian closed category with a complement-classifier (instead of a subobject classifier). A complement-classifier, in turn, is defined identically, with respect to assumed properties, to a subobject classifier having only different labels, most notably *false* instead of *true* (and moreover a different label for characteristic morphism now called anti-characteristic morphism).[52] The important question is whether these labels carry any additional meaning, or whether they are meaningless and merely make it easier to refer to certain "objects" possessing the given properties. This question applies to both the notion of co-topos and topos. If these were only signs without any meaning, then these two concepts were essentially indistinguishable, while if they were meaningful, the question arises as to whether the change that the concept of co-topos brings does not lead to contradiction or at least some problems. This state of affairs has necessitated a clarification of the relevant concepts and consideration of the correctness of the definition of a co-topos.

Consider for the moment the concept of a topos. Our analysis in chapter 3

[52] Both definitions have been presented side by side on p. 81.

showed that if the two definitions of a topos given there (definitions 7 and 8) are to be equivalent, then the label *true*, which occurs in the latter, carries no additional meaning. The generic subobject to which this label applies, and which we have therefore begun to denote by neutral η, is always related to the top element of the algebra defined on $\mathsf{Sub}(1)$ (we assume that the definition of Heyting algebras associated with subobjects is not in doubt), precisely it follows that $\eta = \chi_{id_1}$.

The correspondence between subobjects and characteristic morphisms, i.e. between $\mathsf{Sub}(A)$ and $\mathsf{Hom}(A, \Omega)$ (for any A), is not only bijective, but there is the so-called natural isomorphism between them:

$$\theta_A : \mathsf{Sub}(A) \cong \mathsf{Hom}(A, \Omega), \quad \text{natural in } A.$$

This is the case both in the context of Definition 7, where this natural isomorphism is directly assumed, and in the context of Definition 8, since the existence of a subobject classifier entails such a natural isomorphism (cf. proposition 3 in sec. 3.3).

On $\mathsf{Sub}(A)$, the structure of Heyting algebra is defined in a standard way, using only the "operations" present in the topos, such as factorization of subobjects, pullbacks, etc. The Heyting algebra structure on $\mathsf{Sub}(A)$ is accepted without reservation also by authors of the papers on co-toposes. However, the question of algebraic structure on $\mathsf{Hom}(A, \Omega)$ arises. The natural isomorphism θ_A is more than just a bijection between sets, but it does not imply an isomorphism of the corresponding algebraic structures. This fact is often overlooked.

Thus, it turns out that despite the presence of a natural isomorphism, there remains some freedom to define the algebraic structure on $\mathsf{Hom}(A, \Omega)$. The standard approach assumes that on these sets we also have a Heyting algebra structure isomorphic to that on $\mathsf{Sub}(A)$. Then it can be proved that Ω is an internal Heyting algebra and on $\mathsf{Hom}(A, \Omega)$ we have trivially the same Heyting algebra operations which are present on $\mathsf{Sub}(A)$. Usually, these facts are given as a theorem arising directly from the topos definition. However, it is worth emphasizing that there is a certain additional non-trivial choice at the beginning of such a state of affairs, as long as we assume in this context only the presence of a natural isomorphism or the existence of a subobject classifier. This choice can be enforced by demanding that the generic subobject be the top element of this algebra. If this is understood by denoting this arrow by *true*, as in Definition 8, then the topos so understood does indeed have the relevant properties, such as the Heyting structure on $\mathsf{Hom}(A, \Omega)$, Ω being an internal Heyting algebra, and others.

In the context of the theorem saying that $\mathsf{Hom}(A, \Omega)$ has a structure of a Heyting algebra, and Ω is a Heyting algebra object (cf. Theorem 1 (Internal) in MacLane and Moerdijk, 1994, p. 201), it seemed that the co-topos proposal was illegitimate. It is only by showing that this theorem is based on the additional assumption that $\eta = \text{top} = \mathit{true}$, that it opens up the possibility of assuming that $\eta = \text{bottom} = \mathit{false}$, and thus allows for considering the category understood as a co-topos. We have also shown that then the algebraic structure on $\mathsf{Hom}(A, \Omega)$ will be a co-Heyting algebra, which will also be natural in A, and Ω will be an internal co-Heyting algebra. Then θ_A, without ceasing to be the natural isomorphism, is the anti-isomorphism of the corresponding algebraic structures.

Thus, if denoting a generic subobject by *true* (in short $\eta = \mathit{true}$) means that this arrow should be the top element in the algebra defined on $\mathsf{Hom}(A, \Omega)$, then Definitions 7 and 8 are not equivalent. In this situation, Definition 7 is broader and allows η to be either *true* (topos) or *false* (co-topos). On the other hand, if denoting η as *true*, e.g. in the Definition 8, does not imply a choice of a particular place of η in the order on $\mathsf{Hom}(A, \Omega)$, then these two definitions are equivalent, allow both options, and do not explicitly define the algebraic structure on $\mathsf{Hom}(A, \Omega)$. As far as we know, the above considerations are not described as such in literature.

It is worth emphasizing that since η through the natural isomorphism θ_1 is related to id_1 and thus to the top element of $\mathsf{Sub}(1)$ ($\eta = \chi_{id_1}$), interpreting it as *true* and defining the order on $\mathsf{Hom}(1, \Omega)$ so that η plays the role of the top element in it, seems more appropriate. Formally, from a purely algebraic point of view, reversing the order on Hom-sets is permissible, but the consequences of such a choice should be carefully examined, as most of the theorems concerning various properties of toposes have been proved with the assumption that $\eta = \mathit{true}$. In particular, it seems crucial to investigate the relevant properties related to quantifiers, especially Beck-Chevalley conditions, which in the context of logical properties of toposes can be briefly understood as meaning that substitution commutes with quantification.

In addition to the aforementioned open problems for further investigations, another research problem may be the further analysis of toposes in which co-Heyting structures appear not by inversion of Heyting algebras, but exist side by side with them, forming the so-called bi-Heyting algebras. Such structures are interesting from the point of view of logics based on them, since they combine properties of intermediate logics and logics dual to them. Since Lawvere's paper (Lawvere, 1991) it is known that a vast class of toposes exhibit the co-Heyting structure of subobjects (and therefore bi-Heyting, as subobjects always have the structure of Heyting algebra). In this paper he pointed out that "In any presheaf topos (and more generally any essential subtopos of a presheaf topos), the lattice of all subobjects of any given object is ... [an] example of a co-Heyting algebra" (Lawvere, 1991, p. 280). Reyes and Zolfaghari in (Reyes and Zolfaghari, 1996) pursued this line of reasoning. They proved the above Lawvere's assertion and introduced a new approach to the modal operators based on bi-Heyting structures.

It seems, therefore, that the relationship of intuitionistic logic and paraconsistent logic in the categorical approach is a research project which, although it has been developed for some time, still hides much to be discovered.

Bibliography

Abramsky, S., 2018. Contextuality: At the Borders of Paradox. In Landry, E., editor, *Categories for the Working Philosopher*, pages 262–285. Oxford University Press. See also arXiv:2011.04899.

Abramsky, S. and Coecke, B., 2008. Categorical quantum mechanics. See also arXiv:0808.1023.

Akama, S. and da Costa, N. C. A., 2016. Why Paraconsistent Logics? In Akama, S., editor, *Towards Paraconsistent Engineering*, pages 7–24. Springer International Publishing, Cham. <https://doi.org/10.1007/978-3-319-40418-9_2>.

Angot-Pellissier, R., 2015. The Relation Between Logic, Set Theory and Topos Theory as it Is Used by Alain Badiou. In Koslow, A. and Buchsbaum, A., editors, *The Road to Universal Logic, vol. II*, Studies in Universal Logic, pages 181–200. Birkhäuser, Springer, Cham.

Awodey, S., 1996. Structure in Mathematics and Logic: A Categorical Perspective. *Philosophia Mathematica*, 4(3):209–237.

Awodey, S., 2010. *Category Theory*. Oxford Logic Guides. OUP Oxford.

Balbes, R. and Dwinger, P., 1974. *Distributive Lattices*. University of Missouri Press.

Barr, M. and Wells, C., 2005. Toposes, Triples and Theories. *Reprints in Theory and Applications of Categories*, (12):1–287. This is a reprint of the book originally published in 1985. <http://www.tac.mta.ca/tac/reprints/articles/12/tr12.pdf>.

Bell, J. L., 2005. The Development of Categorical Logic. In Gabbay, D. and Guenthner, F., editors, *Handbook of Philosophical Logic*, volume 12, pages 279–361. Springer.

Bell, J. L., 2018. Categorical Logic and Model Theory. In Landry, E., editor, *Categories for the Working Philosopher*. Oxford University Press.

Bezhanishvili, G. and Holliday, W. H., 2019. A semantic hierarchy for intuitionistic logic. *Indagationes Mathematicae*, 30(3):403–469. <https://doi.org/10.1016/j.indag.2019.01.001>.

Brouwer, L. E. J., 1975. *Collected Works. I: Philosophy and Foundations of Mathematics*. Collected Works. North-Holland Publishing Company. Edited by A. Heyting.

Brouwer, L. E. J., 1976. *Collected Works. II: Geometry, Analysis, Topology and Mechanics*. North-Holland Publishing Company. Edited by H. Freudenthal.

Brouwer, L. E. J., 1981. *Brouwer's Cambridge Lectures on Intuitionism*. Cambridge University Press. Edited by D. van Dalen.

Brunner, A. B. and Carnielli, W. A., 2005. Anti-intuitionism and paraconsistency. *Journal of Applied Logic*, 3(1):161–184. <https://doi.org/10.1016/j.jal.2004.07.016>.

Carnap, R., 1931. The Logicist Foundations of Mathematics. In Benacerraf, P. and Putnam, H., editors, *Philosophy of Mathematics: Selected Readings*, pages 41–52. Cambridge University Press, 1983.

Carnielli, W., Coniglio, M. E., and Marcos, J., 2007. Logics of Formal Inconsistency. In Gabbay, D. and Guenthner, F., editors, *Handbook of Philosophical Logic*, pages 1–93. Springer Netherlands, Dordrecht. <https://doi.org/10.1007/978-1-4020-6324-4_1>.

Cohen, P. J., 1966. *Set Theory and the Continuum Hypothesis*. New York: W. A. Benjamin.

Czermak, J., 1977. A Remark on Gentzen's Calculus of Sequents. *Notre Dame Journal of Formal Logic*, 18(3):471–474. <https://doi.org/10.1305/ndjfl/1093888021>.

da Costa, N. C. A., 1974. On the Theory of Inconsistent Formal Systems. *Notre Dame Journal of Formal Logic*, 15(4):497–510. <http://dx.doi.org/10.1305/ndjfl/1093891487>.

da Costa, N. C. A. and Alves, E. H., 1977. A Semantical Analysis of the Calculi C_n. *Notre Dame Journal of Formal Logic*, 18(4):621–630. <http://dx.doi.org/10.1305/ndjfl/1093888132>.

da Costa, N. C. A. and Doria, F. A., 1995. On Jaśkowski's Discussive Logics. *Studia Logica: An International Journal for Symbolic Logic*, 54(1):33–60. <http://www.jstor.org/stable/20015762>.

Döring, A. and Isham, C., 2011. "What is a Thing?": Topos Theory in the Foundations of Physics. In Coecke, B., editor, *New Structures for Physics*, volume 813 of *Lecture Notes in Physics, Berlin Springer Verlag*, pages 753–937. See also arXiv:0803.0417.

Dummett, M., 1959. A Propositional Calculus with Denumerable Matrix. *The Journal of Symbolic Logic*, 24(2):97–106.

Dummett, M., 2000. *Elements of Intuitionism*. Oxford Logic Guides. Clarendon Press, 2nd, revised edition.

Dunin-Kęplicz, B., Powała, A., and Szałas, A., 2018. Variations on Jaśkowski's Discursive Logic. In Garrido, Á. and Wybraniec-Skardowska, U., editors, *The Lvov-Warsaw School. Past and Present*, pages 485–497. Springer International Publishing, Cham. <https://doi.org/10.1007/978-3-319-65430-0_34>.

Eilenberg, S. and Mac Lane, S., 1945. General theory of natural equivalences. *Transactions of the American Mathematical Society*, 58:231–294.

Estrada-González, L., 2010. Complement-Topoi and Dual Intuitionistic Logic. *Australasian Journal of Logic*, 9:26–44.

Estrada-González, L., 2015a. The Evil Twin: The Basics of Complement-Toposes. In Beziau, J.-Y., Chakraborty, M., and Dutta, S., editors, *New Directions in Paraconsistent Logic*, pages 375–425, New Delhi. Springer India.

Estrada-González, L., 2015b. From (Paraconsistent) Topos Logic to Universal (Topos) Logic. In Koslow, A. and Buchsbaum, A., editors, *The Road to Universal Logic, vol. II*, Studies in Universal Logic, pages 263–295. Birkhäuser, Springer, Cham. <https://doi.org/10.1007/978-3-319-15368-1_12>.

Eva, B., 2015. Towards a Paraconsistent Quantum Set Theory. In Heunen, C., Selinger, P., and Vicary, J., editors, *Proceedings, 12th International Workshop on Quantum Physics and Logic (QPL 2015): Oxford, UK, July 15-17, 2015*, pages 158–169. <https://doi.org/10.4204/EPTCS.195>. See also arXiv:1511.01571.

Feferman, S., Dawson, J., Kleene, S., Moore, G., Solovay, R., and van Heijenoort, J., editors, 1986. *Kurt Gödel: Collected Works: Volume I: Publications 1929-1936*. Collected Works of Kurt Godel. Oxford University Press.

Font, J., 2016. *Abstract Algebraic Logic: An Introductory Textbook*. Studies in logic and the foundations of mathematics. College Publications.

Fourman, M., Mulvey, C., and Scott, D., editors, 1977. *Applications of Sheaves: Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra and Analysis, Durham, July 9-21, 1977*, Lecture Notes in Mathematics. Springer Berlin Heidelberg.

Ghilardi, S. and Zawadowski, M., 2002. *Sheaves, Games, and Model Completions: A Categorical Approach to Nonclassical Propositional Logics*. Trends in Logic. Springer Netherlands.

Gödel, K., 1932. Zum intuitionistischen Aussagenkalkül. *Anzeiger der Akademie der Wissenschaften in Wien*, 69:65–66. English translation in (Feferman et al., 1986, p. 222-225).

Goldblatt, R., 2006. *Topoi: The Categorial Analysis of Logic*. Dover Books on Mathematics. Dover Publications.

Goodman, N. D., 1981. The Logic of Contradiction. *Zeitschrift fur mathematische Logik und Grundlagen der Mathematik*, 27:119–126. <https://doi.org/10.1002/malq.19810270803>.

Goré, R., 2000. Dual Intuitionistic Logic Revisited. In Dyckhoff, R., editor, *Automated Reasoning with Analytic Tableaux and Related Methods*, pages 252–267, Berlin, Heidelberg. Springer Berlin Heidelberg.

Heller, M., 2018. Syntax-Semantics Interaction in Mathematics. *Studia Semiotyczne*, 32(2):87–105. <https://doi.org/10.26333/sts.xxxii2.06>.

Heller, M., 2021. The Geometrization of Meaning. <https://arxiv.org/abs/2104.13288>.

Heller, M. and Król, J., 2017. Infinitesimal Structure of Singularities. *Universe*, 3:16. <https://www.mdpi.com/2218-1997/3/1/16>.

Heyting, A., 1930. Die formalen Regeln der intuitionistischen Logik I. *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, pages 42–56.

Heyting, A., 1971. *Intuitionism. An Introduction*. Amsterdam: North-Holland Pub. Co., 3rd edition.

Heyting, A., 1978. History of the Foundations of Mathematics. *Nieuw Archief voor Wiskunde*, 26(3):1–21.

Iemhoff, R., 2016. Intuitionism in the Philosophy of Mathematics. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2016 edition.

Isham, C. J. and Butterfield, J., 1999. Some Possible Roles for Topos Theory in Quantum Theory and Quantum Gravity. See also arXiv:gr-qc/9910005.

James, W., 1996. *Closed Set Logic in Categories*. PhD thesis, Department of Philosophy, The University of Adelaide.

Jaśkowski, S., 1948. Rachunek zdań dla systemów dedukcyjnych sprzecznych. *Studia Soc. Scient. Torunensis, sec. A*, I(5):57–77.

Jaśkowski, S., 1949. O koniunkcji dyskusyjnej w rachunku zdań dla systemów dedukcyjnych sprzecznych. *Studia Soc. Scient. Torunensis, sec. A*, I(8):171–172.

Jaśkowski, S., 1999a. A Propositional Calculus for Inconsistent Deductive Systems. *Logic and Logical Philosophy*, 7:35–56. <http://dx.doi.org/10.12775/LLP.1999.003>.

Jaśkowski, S., 1999b. On the Discussive Conjunction in the Propositional Calculus for Inconsistent Deductive Systems. *Logic and Logical Philosophy*, 7:57–59. <http://dx.doi.org/10.12775/LLP.1999.004>.

Johnstone, P., 2014. *Topos Theory*. Dover Books on Mathematics. Dover Publications.

Johnstone, P. T., 2002. *Sketches of an Elephant: A Topos Theory Compendium*. Oxford, England: Oxford University Press UK.

Kamide, N., 2003. A note on dual-intuitionistic logic. *Mathematical Logic Quarterly*, 49(5):519. <https://doi.org/10.1002/malq.200310055>.

Kamide, N. and Wansing, H., 2010. Symmetric and Dual Paraconsistent Logics. *Logic and Logical Philosophy*, 19(1-2):7–30. <https://doi.org/10.12775/LLP.2010.002>.

Lambek, J. and Scott, P., 1994. *Introduction to Higher-Order Categorical Logic*. Cambridge Studies in Advanced Mathematics. Cambridge University Press.

Lambert, D., 2011. Does Paraconsistent Logic Play Some Role in Physics. In Brożek, B., Mączka, J., and Grygiel, W. P., editors, *Philosophy in Science*, pages 33–44. Copernicus Center.

Landry, E., 2018. *Categories for the Working Philosopher*. Oxford University Press.

Lawvere, F. W., 1969. Adjointness in Foundations. *Dialectica*, 23(3-4):281–296. See also a reprint with author's commentary at *Reprints in Theory and Applications of Categories*, No. 16 (2006), p. 1–16 <http://www.tac.mta.ca/tac/reprints/articles/16/tr16.pdf>.

Lawvere, F. W., 1989. Intrinsic boundary in certain mathematical toposes exemplify 'logical' operators not passively preserved by substitution. Preprint, University of Buffalo.

Lawvere, F. W., 1991. Intrinsic co-Heyting boundaries and the Leibniz rule in certain toposes. In Carboni, A., Pedicchio, M. C., and Rosolini, G., editors, *Category Theory*, pages 279–281, Berlin, Heidelberg. Springer Berlin Heidelberg. <https://github.com/mattearnshaw/lawvere/blob/master/pdfs/1991-intrinsic-co-heyting-boundaries-and-the-leibniz-rule-in-certain-toposes.pdf>.

Łukowski, P., 2011. *Paradoxes*. Trends in Logic. Dordrecht and New York: Springer.

Mac Lane, S., 2005. *Saunders Mac Lane: A Mathematical Autobiography*. A K Peters Series. Taylor & Francis.

MacLane, S., 1986. *Mathematics: Form and Function*. Springer New York.

MacLane, S. and Moerdijk, I., 1994. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Universitext. Springer New York.

Majid, S., 2012. Quantum spacetime and physical reality. In Majid, S., editor, *On Space and Time*, Canto Classics, pages 56–140. Cambridge University Press.

Mancosu, P., 1998. *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s*. Oxford University Press.

Marquis, J. P., 2009. *From a Geometrical Point of View. A Study of the History and Philosophy of Category Theory*. Logic, Epistemology, and the Unity of Science. Springer, Netherlands.

Marquis, J.-P., 2020. Abstract logical structuralism. *Philosophical Problems in Science (Zagadnienia Filozoficzne w Nauce)*, (69):67–110. <`https://zfn.edu.pl/index.php/zfn/article/view/518`>.

Marquis, J. P., 2021. Category Theory. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, fall 2021 edition.

McKinsey, J. C. C. and Tarski, A., 1944. The Algebra of Topology. *Annals of Mathematics, Second Series*, 45:141–191.

McKinsey, J. C. C. and Tarski, A., 1946. On Closed Elements in Closure Algebras. *Annals of Mathematics, Ser. 2*, 47:122–162.

Mclarty, C., 1990. Book Review: John Bell, "Toposes and Local Set Theories, an Introduction". *Notre Dame Journal of Formal Logic*, 31:150–161.

McLarty, C., 1990. The Uses and Abuses of the History of Topos Theory. *British Journal for the Philosophy of Science*, 41(3):351–375.

McLarty, C., 1995. *Elementary Categories, Elementary Toposes*. Oxford University Press.

Mortensen, C., 1995. *Inconsistent Mathematics*. Kluwer Academic Publishers.

Mortensen, C., 2003. Closed Set Logic. In Brady, R. T., editor, *Relevant Logics and Their Rivals*, volume II, pages 254–262.

Osorio Galindo, M., Borja Macías, V., and Arrazola Ramírez, J. R. E., 2016. Revisiting da Costa Logic. *Journal of Applied Logic*, 16:111–127. <https://doi.org/10.1016/j.jal.2016.02.004>.

Palmgren, E., 2009. Semantics of intuitionistic propositional logic. Lecture Notes for Applied Logic, Fall 2009. <http://www2.math.uu.se/~palmgren/tillog/heyting3.pdf>.

Perzanowski, J., 1999. Fifty Years of Parainconsistent Logics. *Logic and Logical Philosophy*, 7:21–24. <https://doi.org/10.12775/LLP.1999.001>.

Pierce, B., C, B., Pierce, B., Garey, M., Meyer, A., and Benjamin, I., 1991. *Basic Category Theory for Computer Scientists*. Foundations of computing series: research reports and notes. MIT Press.

Placek, T., 1999. *Mathematical Intuitionism and Intersubjectivity: A Critical Exposition of Arguments for Intuitionism*. Synthese Library. Springer Netherlands.

Priest, G., 2002. Paraconsistent Logic. In Gabbay, D. M. and Guenthner, F., editors, *Handbook of Philosophical Logic*, pages 287–393. Springer Netherlands, Dordrecht. <https://doi.org/10.1007/978-94-017-0460-1_4>.

Priest, G., 2009. Dualising Intuitionictic Negation. *Principia: An International Journal of Epistemology*, 13(2):165–184. <https://doi.org/10.5007/1808-1711.2009v13n2p165>.

Priest, G., Tanaka, K., and Weber, Z., 2018. Paraconsistent Logic. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, summer 2018 edition. <https://plato.stanford.edu/archives/sum2018/entries/logic-paraconsistent/>.

Queiroz, G. S., 1998. *Sobre a Dualidade entre Intuicionismo e Paraconsistncia (in Portuguese; English translation of the title: On the Duality Between Intuitionism and Paraconsistency)*. PhD thesis, Department of Philosophy, State University of Campinas.

Rasiowa, H. and Sikorski, R., 1963. *The Mathematics of Metamathematics*. Monografie Matematyczne. Warszawa, Państwowe Wydawn. Naukowe.

Rauszer, C., 1974a. A Formalization of the Propositional Calculus of H-B Logic. *Studia Logica*, 33(1):23–34. <`https://doi.org/10.1007/BF02120864`>.

Rauszer, C., 1974b. Semi-Boolean algebras and their applications to intuitionistic logic with dual operations. *Fundamenta Mathematicae*, 83:219–249. <`https://doi.org/10.4064/fm-83-3-219-249`>.

Reyes, G. E. and Zolfaghari, H., 1996. Bi-Heyting Algebras, Toposes and Modalities. *Journal of Philosophical Logic*, 25(1):25–43. <`http://dx.doi.org/10.1007/BF00357841`>.

Shramko, Y., 2005. Dual Intuitionistic Logic and a Variety of Negations: The Logic of Scientific Research. *Studia Logica*, 80(2-3):347–367. <`https://doi.org/10.1007/s11225-005-8474-7`>.

Simmons, H., 2011. *An Introduction to Category Theory*. Cambridge University Press. See also <`http://www.cs.man.ac.uk/~hsimmons/zCATS.pdf`>.

Smith, P., 2016. *Category Theory: A Gentle Introduction*. University of Cambridge, notes. <`http://www.logicmatters.net/resources/pdfs/GentleIntro.pdf`>.

Stone, M. H., 1937. Topological Representation of Distributive Lattices and Brouwerian Logics. *Casopis pro Pestovani Matematiky a Fysiky*, 67:1–25.

Stopa, M., 2018. Teoria kategorii i niektóre jej logiczne aspekty (Category theory and some of its logical aspects). *Philosophical Problems in Science (Zagadnienia Filozoficzne w Nauce)*, (64):7–58. <`https://zfn.edu.pl/index.php/zfn/article/view/409`>.

Stopa, M., 2020. On the validity of the definition of a complement-classifier. *Philosophical Problems in Science (Zagadnienia Filozoficzne w Nauce)*, (69):111–128. <`https://zfn.edu.pl/index.php/zfn/article/view/520`>.

Tanaka, K., Berto, F., Mares, E., and Paoli, F., editors, 2013. *Paraconsistency: Logic and Applications*. Dordrecht, Netherland: Springer.

Tarski, A., 1938. Der Aussagenkalkül und die Topologie. *Fundamenta Mathematicae*, 31:103–134. The English translation can be found in (Tarski, 1956, p. 421–454).

Tarski, A., 1956. *Logic, Semantics, Metamathematics: Papers from 1923 to 1938*. Oxford, Clarendon Press. Translated by Woodger, J. H. A second edition by another publisher is also available.

Troelstra, A. S., 2011. History of constructivism in the 20th century. In Kennedy, J. and Kossak, R., editors, *Set Theory, Arithmetic, and Foundations of Mathematics: Theorems, Philosophies*, Lecture Notes in Logic, 36, page 150–179. Cambridge University Press.

Troelstra, A. S. and van Dalen, D., 1988a. *Constructivism in Mathematics I*, volume 121 of *Studies in Logic and the Foundations of Mathematics*. Elsevier Science.

Troelstra, A. S. and van Dalen, D., 1988b. *Constructivism in Mathematics II*, volume 123 of *Studies in Logic and the Foundations of Mathematics*. Elsevier Science.

Urbas, I., 1989. Paraconsistency and the C-Systems of da Costa. *Notre Dame Journal of Formal Logic*, 30(4):583–597. <https://doi.org/10.1305/ndjfl/1093635241>.

Urbas, I., 1996. Dual-Intuitionistic Logic. *Notre Dame Journal of Formal Logic*, 37(3):440–451. <https://doi.org/10.1305/ndjfl/1039886520>.

van Atten, M., 2017a. The Development of Intuitionistic Logic. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2017 edition.

van Atten, M., 2017b. Luitzen Egbertus Jan Brouwer. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2017 edition.

van Atten, M., Boldini, P., Bourdeau, M., and Heinzmann, G., 2008. *One Hundred Years of Intuitionism: The Cerisy Conference*. Birkhäuser Basel.

van Dalen, D., 1997a. *A bibliography of L.E.J. Brouwer*, volume 175 (preprint). Utrecht Logic Group Preprint Series. <https://dspace.library.uu.nl/handle/1874/26868>.

van Dalen, D., 1997b. *Logic and Structure*. Berlin, Springer Verlag, revised in 2008, 3 edition.

van Dalen, D., 2002. Intuitionistic Logic. In Gabbay, D. and Günthner, F., editors, *Handbook of Philosophical Logic*, volume 5, pages 1–114. Kluwer, 2nd edition.

van Heijenoort, J., 1967. *From Frege to Gödel*. Cambridge: Harvard University Press.

Vasyukov, V. L., 2000. Paraconsistency in Categories. In Batens, D., Mortensen, C., Priest, G., and van Bendegem, J.-P., editors, *Frontiers of Paraconsistent Logic*, pages 263–278. Research Studies Press Ltd., Baldock, Hartfordshire, England.

Vasyukov, V. L., 2001. A New Axiomatization of Jaśkowski's Discussive Logic. *Logic and Logical Philosophy*, 9:35–46. <https://doi.org/10.12775/LLP.2001.003>.

Vasyukov, V. L., 2017. Potoses: Categorical Paraconsistent Universum for Paraconsistent Logic and Mathematics. *Logical Investigations*, 23(2):76–95. <https://doi.org/10.21146/2074-1472-2017-23-2-76-95>.

Wood, R. J., 2004. Ordered Sets via Adjunctions. In Pedicchio, M. C. and Tholen, W., editors, *Categorical Foundations*, pages 5–47. Cambridge University Press.