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The Category of Node-and-Choice Preforms for Extensive-Form Games

Abstract. It would be useful to have a category of extensive-form games whose isomorphisms specify equivalences between games. Since working with entire games is too large a project for a single paper, I begin here with preforms, where a "preform" is a rooted tree together with choices and information sets. In particular, this paper first defines the category **Tree**, whose objects are "functioned trees", which are specially designed to be incorporated into preforms. I show that **Tree** is isomorphic to the full subcategory of **Grph** whose objects are converging arborescences. Then the paper defines the category **NCP**, whose objects are "node-and-choice preforms", each of which consists of a node set, a choice set, and an operator mapping node-choice pairs to nodes. I characterize the **NCP** isomorphisms, define a forgetful functor from **NCP** to **Tree**, and show that **Tree** is equivalent to the full subcategory of **NCP** whose objects are perfect-information preforms. The paper also shows that many game-theoretic entities can be derived from preforms, and that these entities are well-behaved with respect to **NCP** morphisms and isomorphisms.

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1. Introduction

Category theory has been used to systematize many subjects in mathematics and elsewhere. For example, **Grph** is the category of directed graphs. **Grph** morphisms can be used to state that one directed graph is embedded within another. Further, **Grph** isomorphisms can be used to state that two directed graphs are equivalent.

Similarly, it would be useful to have a category of extensive-form games whose morphisms would allow one to systematically compare extensive-form

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games. As yet, little has been done.[1,2] Lapitsky [13] and Jiménez [9] define categories of normal-form games. Machover and Terrington [15] define a category of simple voting games. Finally, Vannucci [26] defines categories of various kinds of games, but in its category of extensive-form games, every morphism merely maps a game to itself.

Building a category of extensive-form games with nontrivial morphisms is a large project because each extensive-form game has so many components: each is a rooted tree with choices, information sets, players, strategies, chance probabilities, and preferences. Accordingly, this paper takes a small necessary step: it builds a category of preforms, where a "preform" is a rooted tree with choices and information sets.

Since a preform incorporates a tree, this paper's first step is yet smaller: it develops a category of trees which are specially designed to be incorporated into preforms. These trees are called "functioned trees" because they consist of a set T of nodes together with an immediate-predecessor function p which specifies the immediate predecessor $p(t)$ of each nonroot node t. Section 2 defines not only functioned trees but also morphisms between them. Theorem 2.5 shows that the resulting category **Tree** is well-defined. Theorem 2.6 characterizes the **Tree** isomorphisms by the bijectivity of their node transformations. Finally, Theorem 2.8 shows that **Tree** is isomorphic to the full subcategory of **Grph** whose objects are those directed graphs that are converging arborescences.

A functioned tree determines several derivative entities. These include the tree's root node, its stage function, its (strict) precedence relation, and its weak precedence relation (which partially orders its set of nodes). Further, a functioned tree determines its set of decision nodes and its collection of plays (which are the maximal chains in its partially ordered set of nodes). The propositions of Section 2 develop these entities and show that they are well-behaved with respect to the morphisms and isomorphisms of **Tree**.

Section 3 then defines a "node-and-choice preform" to consist of a set T of nodes, a set C of choices, and an operator $\otimes$. The operator $\otimes$ is a new concept. It maps node-and-choice pairs to nodes. In particular, each node-and-choice pair in the operator's domain is mapped to the node that follows the pair's node by way of the pair's choice. Importantly, this operator determines both a functioned tree (T, p) and a collection $\mathcal{H}$ of information

[1] Extensive-form games are not readily comparable with the games defined in the theoretical computer-science literature. Categories of such games are developed in [1,8,17].

[2] In a different direction, [7] composes games by means of a category whose morphisms are game segments.

sets. Preform morphisms are then defined, and Theorem 3.6 shows that the resulting category **NCP** is well-defined. Theorem 3.7 characterizes the **NCP** isomorphisms by the bijectivity of their node and choice transformations. Theorem 3.9 establishes a forgetful functor from **NCP** to **Tree**, which serves to make Section 2's **Tree** results readily accessible. Finally, Theorem 3.13 shows that **Tree** is equivalent to the full subcategory of **NCP** whose objects are those preforms that have perfect information.

As already mentioned, a preform determines a collection $\mathcal{H}$ of information sets. More precisely, each choice determines an information set as the set of nodes from which the choice is feasible (several choices may determine the same information set). Section 3 explores this construction with examples and propositions. In addition, a preform determines a previous-choice function q which specifies the choice $q(t)$ that is previous to any nonroot node t. The propositions of Section 3 show that p and q together constitute the inverse of the operator $\otimes$, and that this inverse leads to useful characterizations of the **NCP** morphisms. Finally, the propositions of Section 3 show that $\otimes$, p, q, and $\mathcal{H}$ are well-behaved with respect to **NCP** morphisms and isomorphisms.

Section 4 sketches out future applications and extensions. In particular, Section 4.1 uses a collection of examples to suggest that this paper's abstract **NCP** framework nests as special cases the particular formulations of [4,11,18,22,27] (each of these particular formulations continues to have its own advantages and disadvantages).[3] Finally, Section 4.2 briefly discusses how we plan to build players, strategies, chance probabilities, and preferences on top of the preforms defined here.

2. The Category of Functioned Trees

2.1. Definition of Functioned Trees

By definition, a pair (T,p) is a *functioned tree* iff there exist $t^o \in T$ and $X \subseteq T$ such that

$$p \text{ is a nonempty function from } T{\smallsetminus}\{t^o\} \text{ onto } X, \text{ and} \tag{1a}$$

$$(\forall t \neq t^o)(\exists m \geq 1)\ p^m(t) = t^o. \tag{1b}$$

Call T the set of *nodes* t and call t^o the *root* node. Further, call p the *immediate-predecessor* function. (1a) states that every nonroot node t is

[3] Differential games, and the non-discrete games of [2,3], are beyond the scope of node-and-choice preforms.

assigned an immediate-predecessor $p(t)$. (1b) states that every nonroot node t is eventually preceded by the root node. Theorem 2.8 (Section 2.5) will show the formal sense in which functioned trees are equivalent to nontrivial and possibly infinite converging arborescences.

Here are some further remarks about definition (1). [i] Since (1a) implies that t^o is the only node outside the function's domain, a functioned tree determines its t^o. [ii] (1a) implies $p(t^o)$ is undefined, and (1b) precludes the existence of a $t \neq t^o$ such that $p(t) = t$. Hence $(\nexists t)\ p(t) = t$. [iii] The existence of X is not restrictive. Rather, (1a) defines X to be both the range and the codomain of p. Call X the set of *decision* nodes. [iv] Since p is nonempty by (1a), there exists $t \neq t^o$. Thus (1b) implies $t^o \in X$. In other words, the root node must be a decision node.

For example, define

$$T = \cup_n \{\mathsf{n}, \bar{\mathsf{n}}\} = \{\mathsf{0}, \bar{\mathsf{0}}, \mathsf{1}, \bar{\mathsf{1}}, \mathsf{2}, \bar{\mathsf{2}}, \dots\}, \text{ and} \tag{2}$$
$$p = \cup_n \{\, (\bar{\mathsf{n}}, \mathsf{n}),\ (\mathsf{n}+\mathsf{1}, \mathsf{n})\,\} = \{\, (\bar{\mathsf{0}}, \mathsf{0}),\ (\mathsf{1}, \mathsf{0}),\ (\bar{\mathsf{1}}, \mathsf{1}),\ (\mathsf{2}, \mathsf{1}), \dots\}.$$

where n denotes an arbitrary element of $\mathbb{N}_0$, and n denotes the corresponding node. By inspection,

$$p \text{ is a function from } T \smallsetminus \{\mathsf{0}\} \text{ onto } \{\mathsf{0}, \mathsf{1}, \mathsf{2}, \dots\}, \text{ and}$$
$$(\forall n)\ p^{n+1}(\bar{\mathsf{n}}) = \mathsf{0} \text{ and } p^{n+1}(\mathsf{n}+\mathsf{1}) = \mathsf{0}.$$

Hence (T, p) is a functioned tree with $t^o = \mathsf{0}$ and $X = \{\mathsf{0}, \mathsf{1}, \mathsf{2}, \dots\}$. See the centipede-like diagram of Figure 1.[4]

2.2. Entities Derived from Functioned Trees

Throughout this subsection, let (T, p) be a functioned tree.

By (1b), there exists a function $k{:}T \to \mathbb{N}_0$ such that

$$k(t^o) = 0, \text{ and} \tag{3a}$$
$$(\forall t \neq t^o)\ k(t) \geq 1 \text{ and } p^{k(t)}(t) = t^o. \tag{3b}$$

Because t^o is outside p's domain by (1a), $p(t^o)$ is undefined. Hence (3b) (uniquely) determines $k(t)$ for any $t \neq t^o$. Hence (3) (uniquely) determines k. In accord with the game-theory literature, call k the *stage*

[4] In this paper's figures, the immediate predecessor of a node appears above, or to the left of, the node. So in Figure 1, the arrows point up or to the left. Similarly in Figure 2, and in the left-hand diagram of Figure 5, the arrows representing p and p' point up. In many figures, such as Figure 3, the arrowheads are suppressed.

$$\begin{tikzcd}
0 & 1 \arrow[l] & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] & \cdots \\
\bar{0} \arrow[u] & \bar{1} \arrow[u] & \bar{2} \arrow[u] & \bar{3} \arrow[u] & \vdots \arrow[u, phantom] &
\end{tikzcd}$$

Figure 1. The functioned tree (2)

function, and call $k(t)$ the *stage* of node t. (Graph theorists might call $k(t)$ the "height" of node t.)

Define the *(strict) precedence relation* $\prec$ on T by

$$t^1 \prec t^2 \quad \text{iff} \quad (\exists m \geq 1)\; t^1 = p^m(t^2). \tag{4}$$

Say that t^1 *(strictly) precedes* t^2 iff $t^1 \prec t^2$. Equivalently, say that t^2 *(strictly) succeeds* t^1. Note that the range of p is the set of nodes that precede at least one node. Thus, since the range of p is X by definition (1a), X is the set of nodes that precede at least one node. Equivalently, X is the set of nodes that have at least one successor. Although this may suggest other names for X, the previous section called X the decision-node set, and I will continue to favour that term.[5]

Define the *weak precedence relation* $\preccurlyeq$ on T by

$$t^1 \preccurlyeq t^2 \quad \text{iff} \quad t^1 \prec t^2 \text{or } t^1 = t^2. \tag{5}$$

Notice that the term "precedence" without the modifier "weak" refers to strict precedence. The following proposition shows that $(T, \preccurlyeq)$ is a partially ordered set whenever (T, p) is an functioned tree. There is no converse because there are partially ordered sets that cannot be constructed from functioned trees. In particular, Alós-Ferrer and Ritzberger [2,3] define games over more general partially ordered sets. Apparently, their "discrete" partially ordered sets ([4, Section 3]) correspond to those partially ordered sets that can be derived from functioned trees.

[5] I avoid the term "nonterminal node" because I avoid the term "terminal node". I avoid the latter because it is natural to expect that the set of "terminal nodes" would be in one-to-one correspondence with the set of plays. This does not happen because there can be infinite plays that do not correspond to individual nodes. In general, Proposition 2.2(b) shows that infinite plays correspond to sequences of nodes rather than individual nodes. To illustrate this, the paragraph after the proposition discusses the centipede example (2), which has an infinite play.

PROPOSITION 2.1. *Suppose* (T,p) *is a functioned tree with its* $\prec$ *and* $\preccurlyeq$*. Then* $(T,\preccurlyeq)$ *is a partially ordered set, and* $\prec$ *is the asymmetric part of* $\preccurlyeq$*. (Proof: Lemma A.1(b,d).)*

Finally, let $\mathcal{Z}$ be the collection of maximal chains in $(T,\preccurlyeq)$, and call $Z\in\mathcal{Z}$ a *play*. In general, plays can be either finite or infinite. Accordingly, $\mathcal{Z}=\mathcal{Z}_{\mathsf{ft}}\cup\mathcal{Z}_{\mathsf{inft}}$, where

$$\mathcal{Z}_{\mathsf{ft}} := \{\text{finite maximal chains in}(T,\preccurlyeq)\}\text{ and} \tag{6a}$$

$$\mathcal{Z}_{\mathsf{inft}} := \{\text{infinite maximal chains in}(T,\preccurlyeq)\}. \tag{6b}$$

Part (a) of the following proposition shows that each finite play can be uniquely associated with a nondecision node. It does so by means of the maximization operator for $\preccurlyeq$. Meanwhile, part (b) shows that each infinite play can be uniquely associated with an infinite sequence of nodes (there is no single node associated with an infinite play). For this result, define the function E from $\mathcal{Z}_{\mathsf{inft}}$ into $T^{\mathbb{N}_1}$ by

$$E(Z) := (t^v)_{v\geq 1}, \tag{7}$$

where each t^v is the unique element t of Z for which $k(t)=v$.[6] Call E the *enumeration* operator.

PROPOSITION 2.2. *Suppose* (T,p) *is a functioned tree with its* t^o*,* X*,* k*,* $\prec$*,* $\preccurlyeq$*,* $\mathcal{Z}_{\mathsf{ft}}$*,* $\mathcal{Z}_{\mathsf{inft}}$*, and* E*. Then the following hold.*
(a) $\mathcal{Z}_{\mathsf{ft}}\ni Z\mapsto \max Z$ *is a bijection onto* $T\smallsetminus X$*. Its inverse is*

$$\{p^m(t)|k(t)\geq m\geq 1\}\cup\{t\}\leftarrow\!\shortmid\, t\in T\smallsetminus X.$$

(b) E *is a well-defined bijection from* $\mathcal{Z}_{\mathsf{inft}}$ *onto*

$$\mathcal{Y} := \{(t^v)_{v\geq 1}\mid t^o=p(t^1)\text{ and }(\forall v\geq 1)\, t^v=p(t^{v+1})\}.$$

Its inverse is

$$\{t^o\}\cup\{t^v|v\geq 1\}\leftarrow\!\shortmid\,(t^v)_{v\geq 1}\in\mathcal{Y}.$$

(Proof A.2.)

For example, consider the centipede example (2) of Figure 1. Here the stage function is defined by $(\forall n)\ k(\mathsf{n}) = n$ and $k(\bar{\mathsf{n}}) = n{+}1$, and the

[6]The sequence $E(Z)=(t^v)_{v\geq 1}$ is defined to start with a stage-1 node rather than the stage-0 node t^o. This is notationally convenient because the index $v=0$ would lead to the redundant and awkward equation $t^0=t^o$. Incidentally, the sequence $E(Z)=(t^v)_{v\geq v^*}$ could have been defined to start with any $v^*\geq 0$. I believe that a variant of Proposition 2.2(b) would still hold because the result is fundamentally concerned with the tails of the sequences.

(strict) precedence relation $\prec$ is $\{(\mathsf{m},\mathsf{n})|m<n\} \cup \{(\mathsf{m},\bar{\mathsf{n}})|m \leq n\}$. Proposition 2.2(a) implies that the maximization operator is a bijection from the finite-play collection

$$\mathcal{Z}_{\mathsf{ft}} = \{\{\mathsf{0},\bar{\mathsf{0}}\},\ \{\mathsf{0},\mathsf{1},\bar{\mathsf{1}}\},\ \{\mathsf{0},\mathsf{1},\mathsf{2},\bar{\mathsf{2}}\},\ \dots\}$$

onto the nondecision-node set $T \smallsetminus X = \{\bar{\mathsf{0}},\bar{\mathsf{1}},\bar{\mathsf{2}},\dots\}$. Proposition 2.2(b) implies that the enumeration operator E is a bijection from the (singleton) infinite-play collection

$$\mathcal{Z}_{\mathsf{inft}} = \{\{\mathsf{0},\mathsf{1},\mathsf{2},\dots\}\}$$

onto the (singleton) node-sequence collection $\mathcal{Y} = \{\ (\mathsf{1},\mathsf{2},\mathsf{3},\dots)\ \}$. In accord with footnote 6, the node sequence in $\mathcal{Y}$ begins with the stage-1 node $\mathsf{1}$ rather than the stage-0 node $\mathsf{0}$.

2.3. Functioned-Tree Morphisms

Let a *functioned-tree morphism* be a triple $\gamma = [(T,p),(T',p'),\tau]$ such that

$$(T,p) \text{ and } (T',p') \text{ are functioned trees,} \tag{8a}$$

$$\tau : T \to T', \text{ and} \tag{8b}$$

$$\{\ (\tau(t^\sharp),\tau(t))\ |\ (t^\sharp,t){\in}p\ \}\ \subseteq\ p'. \tag{8c}$$

The following proposition characterizes functioned-tree morphisms in terms of the category **Set**. The functions $\tau|_{T\smallsetminus\{t^o\}}{:}T\smallsetminus\{t^o\}{\to}T'\smallsetminus\{t'^o\}$ and $\tau|_X{:}X{\to}X'$ are well-defined by (9a) and the codomain definitions after (9b). These two functions as well as p and p' are displayed in the **Set** diagram of Figure 2. (9b) states that this diagram commutes.

Proposition 2.3. *A triple* $[(T,p),(T',p'),\tau]$ *is a functioned-tree morphism iff it satisfies (8a)–(8b),*

$$\tau(T\smallsetminus\{t^o\})\ \subseteq\ T'\smallsetminus\{t'^o\},\ \ \tau(X)\ \subseteq\ X',\ \textit{and} \tag{9a}$$

$$\begin{tikzcd}
X \subseteq T \arrow[r, "\tau|_X"] & X' \subseteq T' \\
T\smallsetminus\{t^o\} \arrow[u, "p"] \arrow[r, "\tau|_{T\smallsetminus\{t^o\}}"] & T'\smallsetminus\{t'^o\} \arrow[u, "p'"']
\end{tikzcd}$$

Figure 2. (9b) states that this diagram commutes

$$\tau|_X \circ p = p' \circ \tau|_{T \smallsetminus \{t^o\}}, \tag{9b}$$

where the codomain of $\tau|_{T \smallsetminus \{t^o\}}$ is defined to be $T' \smallsetminus \{t'^o\}$ and where the codomain of $\tau|_X$ is defined to be X'. (Proof A.3.)

(9) can be interpreted. (9a) states that the image of every nonroot node is a nonroot node, and that the image of every decision node is a decision node. (9b) states that the image of the predecessor of a nonroot node is the predecessor of the image of the nonroot node.

By Proposition 2.3, every morphism satisfies (9a) and (9b). Proposition 2.4 shows that a morphism has many other properties as well. Many of these properties are proved via property (b), which concerns iterations of the predecessor functions p and p'. Iterations of a predecessor function are not compositions within the category **Set** because the domain and codomain of a predecessor function are different. In particular, the root node is in the codomain but not the domain. Property (b) avoids this complication by assuming that $t^1 = p^m(t^2)$, which implicitly entails that $(\forall i < m)$ $p^i(t^2) \neq t^o$.

Proposition 2.4. *Suppose $[(T,p),(T',p'),\tau]$ is a morphism, where (T,p) determines t^o, k, $\prec$, $\preccurlyeq$, $\mathcal{Z}_{\mathsf{ft}}$, and $\mathcal{Z}_{\mathsf{inft}}$, and where (T',p') determines t'^o, k', $\prec'$, $\preccurlyeq'$, $\mathcal{Z}'_{\mathsf{ft}}$, and $\mathcal{Z}'_{\mathsf{inft}}$. Then the following hold.*

(a) $t'^o \preccurlyeq' \tau(t^o)$.

(b) If $m \geq 1$ *and* $t^1 = p^m(t^2)$*, then* $\tau(t^1) = (p')^m(\tau(t^2))$.

(c) $k'(\tau(t)) = k(t) + k'(\tau(t^o))$.

(d) If $t^1 \prec t^2$*, then* $\tau(t^1) \prec' \tau(t^2)$.

(e) If $t^1 \preccurlyeq t^2$*, then* $\tau(t^1) \preccurlyeq' \tau(t^2)$.

(f) If $S \subseteq T$ *is a chain, then* $\tau|_S$ *is injective and* $\tau(S)$ *is a chain.*[7]

(g) $(\forall Z \in \mathcal{Z}_{\mathsf{inft}})(\exists Z' \in \mathcal{Z}'_{\mathsf{inft}})\ \tau(Z) \subseteq Z'$.[7]

(h) $(\forall Z \in \mathcal{Z}_{\mathsf{ft}})(\exists Z' \in \mathcal{Z}'_{\mathsf{ft}} \cup \mathcal{Z}'_{\mathsf{inft}})\ \tau(Z) \subseteq Z'$.[7] *(Proof A.4.)*

2.4. The Category Tree

This paragraph and the following theorem define the category **Tree**, which is called the *category of functioned trees*. Let an object be a functioned tree (T,p). Let an arrow be a functioned-tree morphism $[(T,p),(T',p'),\tau]$. Let source, target, identity, and composition be

[7] The symbol τ is overloaded in Propositions 2.4(f)–(h), 2.7(h)–(j), 3.5, and 3.8(e). Specifically, if S is a set of nodes, then $\tau(S) := \{\tau(t)|t \in S\}$.

$$[(T,p),(T',p'),\tau]^{\mathsf{src}} = (T,p),$$
$$[(T,p),(T',p'),\tau]^{\mathsf{trg}} = (T',p'),$$
$$\mathsf{id}_{(T,p)} = [(T,p),(T,p),\mathsf{id}_T], \text{ and}$$
$$[(T',p'),(T'',p''),\tau']\circ[(T,p),(T',p'),\tau] = [(T,p),(T'',p''),\tau'\circ\tau],$$

where id_T is an identity in **Set**.

THEOREM 2.5. **Tree** *is a category. (Proof A.5.)*

The following theorem characterizes the isomorphisms in **Tree**. This characterization is then used by the subsequent proposition to establish the relationships between the entities derived from isomorphic trees.

THEOREM 2.6. *Suppose that $\gamma = [(T,p),(T',p'),\tau]$ is a morphism. Then γ is an isomorphism iff τ is a bijection. Further, if γ is an isomorphism, then $\gamma^{-1} = [(T',p'),(T,p),\tau^{-1}]$. (Proof A.8.)*

PROPOSITION 2.7. *Suppose $[(T,p),(T',p'),\tau]$ is an isomorphism, where (T,p) determines t^o, X, k, $\prec$, $\preccurlyeq$, $\mathcal{Z}_{\mathsf{ft}}$, $\mathcal{Z}_{\mathsf{inft}}$, and E, and where (T',p') determines t'^o, X', k', $\prec'$, $\preccurlyeq'$, $\mathcal{Z}'_{\mathsf{ft}}$, $\mathcal{Z}'_{\mathsf{inft}}$, and E'. Then the following hold.*

(a) $\tau|_X$ is a bijection from X onto X'.

(b) $\tau|_{T\smallsetminus\{t^o\}}$ is a bijection from $T\smallsetminus\{t^o\}$ onto $T'\smallsetminus\{t'^o\}$.

(c) $\tau(t^o) = t'^o$.

(d) $k'(\tau(t)) = k(t)$.

(e) $(\tau,\tau)|_p$ is a bijection from p onto p'.

(f) $(\tau,\tau)|_\prec$ is a bijection from $\prec$ onto $\prec'$.

(g) $(\tau,\tau)|_\preccurlyeq$ is a bijection from $\preccurlyeq$ onto $\preccurlyeq'$.

(h) $\tau|_{\mathcal{Z}_{\mathsf{ft}}}$ is a bijection from $\mathcal{Z}_{\mathsf{ft}}$ onto $\mathcal{Z}'_{\mathsf{ft}}$.[7]

(i) $\tau|_{\mathcal{Z}_{\mathsf{inft}}}$ is a bijection from $\mathcal{Z}_{\mathsf{inft}}$ onto $\mathcal{Z}'_{\mathsf{inft}}$.[7]

(j) $(\forall Z{\in}\mathcal{Z}_{\mathsf{inft}})(\forall v \geq 1)\ \tau(E[Z]^v) = E'[\tau(Z)]^v$.[7] *(Proof A.10.)*

2.5. Connection to Grph

This subsection shows that **Tree**, the category of functioned trees, is isomorphic to a full subcategory of **Grph**, the category of directed graphs.

By definition, let a *nontrivial converging arborescence* be a quadruple $(T,E,\mathsf{init},\mathsf{ter})$ such that

$$(T,E,\mathsf{init},\mathsf{ter}) \text{ is a nontrivial oriented tree, and} \tag{10a}$$

$$(\exists t^o)(\forall e)\ \mathsf{init}(e)\ \neq\ t^o \text{ and} \tag{10b}$$
$$\mathsf{ter}(e) \text{ is on the path linking } \mathsf{init}(e) \text{ and } t^o,$$

where [a] [6, p. 2, 6, 13, and 28] define "nontriviality", "path", "tree", and "orientation" for finite T, and where [b] [6, p. 203] extends these concepts for infinite T.

To be explicit about (10a), $(T, E, \mathsf{init}, \mathsf{ter})$ is an oriented tree iff [i] $(T, E, \mathsf{init}, \mathsf{ter})$ is a directed graph ([6, p. 28], [14, p. 48]), [ii] (T, E) is an (undirected) tree ([6, p. 13]), and [iii]

$$(\forall e)\ e = \{\mathsf{init}(e), \mathsf{ter}(e)\}. \tag{11}$$

In this context, nontriviality is equivalent to E being nonempty, which implies that T has at least two elements. In (10b), the (unique) path linking $\mathsf{init}(e) \neq t^o$ and t^o is well-defined by [6, Theorem 1.5.1]. Note that any two nodes in a tree are linked by a finite path, but that a tree does not necessarily have a finite maximum path length. Incidentally, (10) is equivalent to the definition of a converging arborescence in [25, p. 127], restricted to prohibit trivial graphs and extended to allow infinite graphs.

Grph, the category of directed graphs, is defined in [14, p. 48] and [5, p. 124]. Its objects are directed graphs $(T, E, \mathsf{init}, \mathsf{ter})$. Its arrows are directed-graph morphisms, which are quadruples

$$[(T, E, \mathsf{init}, \mathsf{ter}), (T', E', \mathsf{init}', \mathsf{ter}'), \tau, \varepsilon]$$

such that

$$(T, E, \mathsf{init}, \mathsf{ter}) \text{ and } (T', E', \mathsf{init}', \mathsf{ter}') \text{ are directed graphs,} \tag{12a}$$
$$\tau : T \to T',\ \varepsilon : E \to E', \tag{12b}$$
$$\mathsf{init}' \circ \varepsilon = \tau \circ \mathsf{init},\ \text{and } \mathsf{ter}' \circ \varepsilon = \tau \circ \mathsf{ter}. \tag{12c}$$

As noted in the previous paragraph, a nontrivial converging arborescence is a special kind of directed graph. Accordingly, let $\mathbf{Grph_{ca}}$ be the full subcategory of **Grph** whose objects are those directed graphs that are nontrivial converging arborescences.

The following theorem shows that **Tree** and $\mathbf{Grph_{ca}}$ are isomorphic ([14, p. 14]). A related result is [25, Theorem VI.1, p. 126], which implies that in a converging arborescence, every nonroot node is the initial node of exactly one edge. In terms of the following theorem, that result concerns the well-definition of H_0.

THEOREM 2.8. **Tree** *and* $\mathbf{Grph_{ca}}$ *are isomorphic. In particular, define* G *from* **Tree** *to* $\mathbf{Grph_{ca}}$ *by*

$$\begin{gathered}\mathsf{G}_0 \;:\; (T,p) \;\mapsto\; (T,E,\mathsf{init},\mathsf{ter}) \\ \textit{where } E = \{\{t^\sharp,t\} \mid (t^\sharp,t){\in}p\}, \\ \mathsf{init} = \{(\{t^\sharp,t\},t^\sharp) \mid (t^\sharp,t){\in}p\}, \textit{ and} \\ \mathsf{ter} = \{(\{t^\sharp,t\},t) \mid (t^\sharp,t){\in}p\}; \textit{ and}\end{gathered}$$

$$\begin{gathered}\mathsf{G}_1 \;:\; [(T,p),(T',p'),\tau] \;\mapsto\; [\mathsf{G}_0(T,p),\mathsf{G}_0(T',p'),\tau,\varepsilon] \\ \textit{where } \varepsilon = \{(\{t^\sharp,t\},\{\tau(t^\sharp),\tau(t)\}) \mid (t^\sharp,t){\in}p\}.\end{gathered}$$

Conversely, define H *from* $\mathbf{Grph_{ca}}$ *to* **Tree** *by*

$$\begin{gathered}\mathsf{H}_0 \;:\; (T,E,\mathsf{init},\mathsf{ter}) \;\mapsto\; (T,p) \\ \textit{where } p = \{\;(\mathsf{init}(e),\mathsf{ter}(e)) \mid e{\in}E\;\}, \;\textit{ and}\end{gathered}$$

$$\begin{gathered}\mathsf{H}_1 \;:\; [(T,E,\mathsf{init},\mathsf{ter}),(T',E',\mathsf{init}',\mathsf{ter}'),\tau,\varepsilon] \;\mapsto \\ [\mathsf{H}_0(T,E,\mathsf{init},\mathsf{ter}),\mathsf{H}_0(T',E',\mathsf{init}',\mathsf{ter}'),\tau].\end{gathered}$$

Then G *and* H *form an inverse pair of isomorphisms. (Proof B.7.)*

3. The Category of Node-and-Choice Preforms

3.1. Definition of Node-and-Choice Preforms

As in Section 2, let T be a set and call $t \in T$ a *node*. Further, let C be a set and call $c \in C$ a *choice*. A *(node-and-choice) preform Π* is a triple $(T,C,\otimes)$ such that

$$(\exists F \subseteq T{\times}C)(\exists t^o{\in}T) \quad \otimes \text{ is a bijection from } F \text{ onto } T{\setminus}\{t^o\}, \tag{13a}$$

$$(T,p) \text{ is a functioned tree where } p{:}T{\setminus}\{t^o\} \to F^{-1}(C) \quad \text{is defined by } p := \{(t^\sharp,t)|(\exists c)(t,c,t^\sharp){\in}\otimes\}, \text{ and} \tag{13b}$$

$$\mathcal{H} \text{ partitions } F^{-1}(C) \quad \text{where } \mathcal{H} := \{F^{-1}(c)|c\}. \tag{13c}$$

Call $\otimes$ the *node-and-choice operator*. Note that equation (13) derives F, t^o, p, and $\mathcal{H}$ from $(T,C,\otimes)$. Call F the *feasibility* correspondence. Call t^o the *root* node. Call p the *immediate-predecessor* function. Call $\mathcal{H}$ the collection of *information sets*, and let $H \in \mathcal{H}$ denote an information set. (This abstract concept of preform extends similar concepts in Streufert (2015b, 2015c).)

(13a) states that the operator $\otimes$ is a function from a subset of $T{\times}C$ into a subset of T. Thus it maps node-choice pairs to nodes. Let $(t,c,t^\sharp) \in \otimes$ be equivalent to $\otimes(t,c) = t^\sharp$, and let this also be equivalent to $t{\otimes}c = t^\sharp$. Call $t{\otimes}c$ the *result* of the node-choice pair (t,c).

Further, (13a) states that the range of $\otimes$ is $T{\smallsetminus}\{t^o\}$. This determines the root node t^o as the only node t that is not in the range of $\otimes$. Hence T has no superfluous elements: every node t other than the root node t^o is the result of some node-choice pair.

Further, (13a) states that the domain of $\otimes$ is $F \subseteq T{\times}C$. Thus

$$F = \{\ (t,c) \mid (\exists t^\sharp)(t,c,t^\sharp){\in}\otimes\ \}. \tag{14}$$

Since F is a subset of $T{\times}C$, F can be regarded as a (nonempty-valued) correspondence whose domain is some subset of T and whose range is some subset of C. In accord with this perspective, let the statement $(t,c) \in F$ be equivalent to the statement $c \in F(t)$. Thus by (14), $c \in F(t)$ iff $t{\otimes}c$ exists. Accordantly, $F(t)$ is called the set of choices that are *feasible* from the node t.

Now consider the range of F. This set consists of those choices c that are feasible from some node. By (13c) and the fact that a partition consists of nonempty sets, each inverse image $F^{-1}(c)$ is nonempty. Thus the range of F is all of C. Hence C has no superfluous elements: each choice c is feasible from at least one node.

Finally, note that the domain of F is $F^{-1}(C)$. This set consists of those nodes with at least one feasible choice. Accordantly, the elements of $F^{-1}(C)$ are called the *decision nodes*.

(13b) defines the function $p{:}T{\smallsetminus}\{t^o\}{\rightarrow}F^{-1}(C)$. Lemma C.1(a) shows that (13a) implies that p is well-defined and surjective. This function maps any $t^\sharp$ in the nonroot-node set $T{\smallsetminus}\{t^o\}$ to its immediate predecessor $p(t^\sharp)$ in the decision-node set $F^{-1}(C)$.

Substantively, (13b) assumes that (T,p) is a functioned tree, as defined by (1a)–(1b). Given (13a) and Lemma C.1(a), (1a) adds only that that the function p is nonempty. More significantly, (1b) adds that every nonroot node is eventually preceded by the root node.

Lemma C.1(b) shows that the root node t^o of the preform $(T,C,\otimes)$ equals the root node of the derived tree (T,p) (the root node of a tree is also denoted by t^o in Section 2). Further, Lemma C.1(c) shows that the decision-node set $F^{-1}(C)$ of the preform $(T,C,\otimes)$ equals decision-node set of the derived tree (T,p) (the decision-node set of a tree is denoted by X in Section 2).

Related to (13b), and for future reference, define $q{:}T{\smallsetminus}\{t^o\}{\rightarrow}C$ by

$$q = \{\ (t^\sharp,c) \mid (\exists t)(t,c,t^\sharp){\in}\otimes\ \}. \tag{15}$$

By Lemma C.2, q is well-defined and surjective. Call q the *previous-choice* function, and call $q(t^\sharp)$ the choice *previous* to $t^\sharp$. The function q resembles the function α defined by [16, p. 227]. Further, note that the definition of q in (15) closely resembles the definition of p in (13b). This resemblance is not coincidental: Proposition 3.1(b) shows that p is the first component of $\otimes^{-1}$, and that q is the second component of $\otimes^{-1}$.

PROPOSITION 3.1. *Suppose* $(T, C, \otimes)$ *satisfies (13a), derive* p *by (13b), and derive* q *by (15). Then, (a)* $\otimes = \{\ (p(t^\sharp), q(t^\sharp), t^\sharp) \mid t^\sharp \neq t^o\ \}$. *Further, (b)* $\otimes^{-1} = (p, q)$. *(Proof C.3.)*

(13c) defines the collection $\mathcal{H}$ of information sets H. This important construction will be discussed at length in Section 3.2.

In summary, many entities can be derived from a preform $\Pi = (T, C, \otimes)$. In particular, (13) and (15) define F, t^o, p, q, and $\mathcal{H}$. Further, T and p define a tree (T, p) from which more entities can be derived. In particular, (3)–(7) define k, $\prec$, $\preccurlyeq$, $\mathcal{Z}$, $\mathcal{Z}_{\mathsf{ft}}$, $\mathcal{Z}_{\mathsf{inft}}$, and E. Finally, as noted four paragraphs ago, Π's decision-node set $F^{-1}(C)$ equals (T, p)'s decision-node set X.

3.2. The Construction of Information Sets

(13c) defines the information-set collection $\mathcal{H}$ as $\{F^{-1}(c)|c\}$. This generalizes a similar construction by [20, p. 97].

For example, consider Figure 3a, which depicts a preform corresponding to Selten's horse game.[8] To be specific, the horse-like diagram depicts $(T, C, \otimes)$, where

$$T = \{0, 1, 2, 3, 4, 5, 6, 7, 8\},\ C = \{\mathsf{r}^\mathsf{S}, \mathsf{d}^\mathsf{S}, \mathsf{r}^\mathsf{G}, \mathsf{d}^\mathsf{G}, \mathsf{e}, \mathsf{f}\},\ \text{and}$$
$$\otimes = \{(0, \mathsf{r}^\mathsf{S}, 1), (0, \mathsf{d}^\mathsf{S}, 3), (1, \mathsf{r}^\mathsf{G}, 2), (1, \mathsf{d}^\mathsf{G}, 4),$$
$$(3, \mathsf{e}, 5), (3, \mathsf{f}, 6), (4, \mathsf{e}, 7), (4, \mathsf{f}, 8)\}.$$

Better, one can read T, C, and $\otimes$ directly from the diagram: Nodes and choices are as usual, and the eight triples in $\otimes$ are the eight node-choice-node segments in the diagram. Note that F consists of the eight node-choice

[8] To tell a story, suppose a student (S) must decide between the right choice (r^S) of doing her homework and the dumb choice (d^S) of not doing her homework. Knowing that the homework has been finished (node 1), a goat (G) must decide between the right choice (r^G) of taking a nap and the dumb choice (d^G) of eating the homework. Knowing that either the student played dumb (node 3) or that the student played right and the goat played dumb (node 4), the teacher must choose between excusing the student (e) and failing the student (f). The preform's three information sets correspond to the student, the goat, and the teacher.

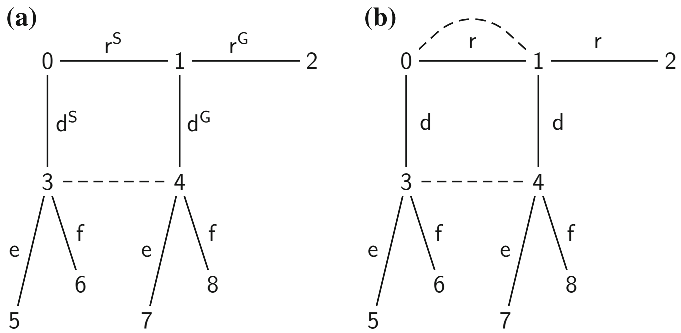

Figure 3. **a** A preform with three information sets. **b** A similar preform with only two information sets.

pairs $(0, \mathsf{r}^\mathsf{S})$, $(0, \mathsf{d}^\mathsf{S})$, $(1, \mathsf{r}^\mathsf{G})$, $(1, \mathsf{d}^\mathsf{G})$, $(3, \mathsf{e})$, $(3, \mathsf{f})$, $(4, \mathsf{e})$, $(4, \mathsf{f})$. Thus there are three information sets:

$$F^{-1}(\mathsf{r}^\mathsf{S}) = F^{-1}(\mathsf{d}^\mathsf{S}) = \{0\},$$
$$F^{-1}(\mathsf{r}^\mathsf{G}) = F^{-1}(\mathsf{d}^\mathsf{G}) = \{1\}, \text{ and}$$
$$F^{-1}(\mathsf{e}) = F^{-1}(\mathsf{f}) = \{3, 4\}.$$

In other words, $\mathcal{H} = \{\, \{0\}, \{1\}, \{3, 4\} \,\}$.

Figure 3b depicts a similar but different preform.[9] It differs from Figure 3a in that r replaces both r^S and r^G, and d replaces both d^S and d^G. Thus there are only four choices in total. Note that F consists of the eight node-choice pairs $(0, \mathsf{r})$, $(0, \mathsf{d})$, $(1, \mathsf{r})$, $(1, \mathsf{d})$, $(3, \mathsf{e})$, $(3, \mathsf{f})$, $(4, \mathsf{e})$, and $(4, \mathsf{f})$. Thus there are only two information sets:

$$F^{-1}(\mathsf{r}) = F^{-1}(\mathsf{d}) = \{0, 1\} \text{ and}$$
$$F^{-1}(\mathsf{e}) = F^{-1}(\mathsf{f}) = \{3, 4\}.$$

In other words, $\mathcal{H} = \{\, \{0, 1\}, \{3, 4\} \,\}$. Figure 3b depicts the first information set by the curved dashed line above the nodes 0 and 1. This new information set is absentminded in the sense of [19].

This way to construct information sets imposes a mild notational restriction. To see this restriction, recall again that (13c) specifies each information

[9] In terms of the story, there is now a "student-goat". This student-goat must decide between the right choice (r) and the dumb choice (d). And, when making this decision, she cannot remember whether or not she has made the right choice once before. The preform's two information sets correspond to the student-goat and the teacher.

set $H \in \mathcal{H}$ as the set $F^{-1}(c)$ of decision nodes from which a choice c is feasible. This implies that each choice determines exactly one information set (though several choices may determine the same information set). Thus each choice is associated with exactly one information set (though several choices can be associated with the same information set). Thus information sets cannot share choices. In other words, each information set must have its own choices. This notational restriction is conclusion (16b) of Proposition 3.2.

To illustrate this notational restriction, suppose that one wants to use a node-and-choice preform to express the rooted tree, choices, and information sets of Selten's horse game. That well-known game has three information sets with two choices each. So, since each information set must have its own choices, one must specify $3{\times}2 = 6$ choices. This is what Figure 3a does.

As a whole, Proposition 3.2 collects some general observations about the information sets constructed from a preform. It requires some introduction.

In the standard literature, [i] the collection $\mathcal{H}$ of information sets H partitions the set of decision nodes, and [ii] two nodes in the same information set H share the same set of feasible alternatives. Feature [i] is assured by (13c) itself since $F^{-1}(C)$ is the set of decision nodes. Feature [ii] is assured by Proposition 3.2(16a) below. Accordingly, both [i] and [ii] are exhibited by both of Figure 3's preforms.

Further, Proposition 3.2(16b) shows that the information sets constructed here have an additional property. As with any correspondence, the value $F(H)$ of the correspondence F at the set H is defined to be $\{\, c \,|\, (\exists t{\in}H)\, c{\in}F(t) \,\}$. Proposition 3.2(16a) implies that $(\forall H{\in}\mathcal{H})\ t \in H \Rightarrow F(t) = F(H)$. Hence each $F(H)$ is readily interpreted as the feasible-choice set of the information set H. Thus, Proposition 3.2(16b) states that each information set has its own choices. This is the notational restriction introduced four paragraphs ago. It is satisfied by both of Figure 3's preforms.

Proposition 3.2. *Suppose that* $(T, C, \otimes)$ *satisfies (13a) and (13c) with* F *and* $\mathcal{H}$. *Then the following hold.*

$$(\forall H{\in}\mathcal{H}) \quad \{t, t'\} \subseteq H \ \Rightarrow \ F(t) = F(t'). \tag{16a}$$

$$(\forall H{\in}\mathcal{H}, H'{\in}\mathcal{H}) \quad H \neq H' \ \Rightarrow \ F(H) {\cap} F(H') = \varnothing. \tag{16b}$$

(Proof C.4.)

Section 4.1 will provide four more examples of node-and-choice preforms. Among other things, these examples illustrate that information sets constructed via (13c) satisfy (16a) and (16b). All four examples express the rooted tree, choices, and information sets of Selten's horse game. Figure 3a does the same.

3.3. Preform Morphisms

A *(preform) morphism* is a quadruple $\alpha = [\mathit{\Pi}, \mathit{\Pi}', \tau, \delta]$ such that

$$\mathit{\Pi} \ = \ (T, C, \otimes) \text{ and } \mathit{\Pi}' \ = \ (T', C', \otimes') \text{ are preforms,} \tag{17a}$$

$$\tau : T \to T',\ \ \delta : C \to C',\ \text{ and} \tag{17b}$$

$$\{\ (\tau(t), \delta(c), \tau(t^\sharp)) \mid (t, c, t^\sharp) {\in} \otimes\ \} \ \subseteq \ \otimes'. \tag{17c}$$

Lemma C.6 shows that a quadruple $[\mathit{\Pi}, \mathit{\Pi}', \tau, \delta]$ is a morphism iff its satisfies (17a)–(17b),

$$(\forall (t,c){\in}F)\ (\tau(t), \delta(c)) \in F',\ \text{ and} \tag{18a}$$

$$(\forall (t,c){\in}F)\ \tau(t{\otimes}c) = \tau(c){\otimes}'\delta(c), \tag{18b}$$

where $\mathit{\Pi}$ determines F and $\mathit{\Pi}'$ determines F'. Proposition 3.3 extends this result to characterize preform morphisms in terms of the category **Set**. The functions $(\tau, \delta)|_F{:}F{\to}F'$ and $\tau|_{T\smallsetminus\{t^o\}}{:}T{\smallsetminus}\{t^o\}{\to}T'{\smallsetminus}\{t'^o\}$ are well-defined by (19a) and the proposition's definitions [c] and [d]. These two functions as well as the functions $\otimes$ and $\otimes'$ are displayed in the diagram of Figure 4. (19b) states that this diagram commutes.

Proposition 3.3. *A quadruple $[\mathit{\Pi}, \mathit{\Pi}', \tau, \delta]$ is a morphism iff it satisfies (17a)–(17b),*

$$(\tau, \delta)(F) \ \subseteq \ F',\ \ \tau(T{\smallsetminus}\{t^o\}) \ \subseteq \ T'{\smallsetminus}\{t'^o\},\ \ \textit{and} \tag{19a}$$

$$\tau|_{T\smallsetminus\{t^o\}}{\circ}{\otimes} = {\otimes}'{\circ}(\tau, \delta)|_F, \tag{19b}$$

where [a] $\mathit{\Pi}$ determines F and t^o, [b] $\mathit{\Pi}'$ determines F' and t'^o, [c] the function $(\tau, \delta)|_F{:}F{\to}F'$ is defined by $(\tau, \delta)(t, c) = (\tau(t), \delta(c))$, and [d] the codomain of $\tau|_{T\smallsetminus\{t^o\}}$ is defined to be $T'{\smallsetminus}\{t'^o\}$. (Proof C.7.)

(19) can be interpreted like (18) can be interpreted. (19a) states that the image of every feasible node-choice pair is a feasible node-choice pair, and that the image of every nonroot node is a nonroot node. (19b) states that

$$\begin{tikzcd}
F{\subseteq}T{\times}C \arrow[r, "(\tau,\delta)|_F"] \arrow[d, "\otimes"'] & F'{\subseteq}T'{\times}C' \arrow[d, "\otimes'"] \\
T{\smallsetminus}\{t^o\} \arrow[r, "\tau|_{T\smallsetminus\{t^o\}}"] & T'{\smallsetminus}\{t'^o\}
\end{tikzcd}$$

Figure 4. (19b) states that this diagram commutes

the image of the result of a feasible node-choice pair is the result of the image of that node-choice pair.

Alternatively, recall $\otimes^{-1} = (p, q)$ by Proposition 3.1(b). Lemma C.9 uses this identity to show that a quadruple $[\Pi, \Pi', \tau, \delta]$ is a morphism iff it satisfies (17a)–(17b),

$$\{\ (\tau(t^\sharp), \tau(t))\ |\ (t^\sharp, t) {\in} p\ \}\ \subseteq\ p',\ \text{and} \tag{20a}$$

$$\{\ (\tau(t^\sharp), \delta(c))\ |\ (t^\sharp, c) {\in} q\ \}\ \subseteq\ q', \tag{20b}$$

where Π determines p and q, and Π' determines p' and q'. Then Lemma C.10 shows that a quadruple is a morphism iff it satisfies (17a)–(17b),

$$(\forall t^\sharp \neq t^o)\ \tau(t^\sharp)\ \neq\ t'^o, \tag{21a}$$

$$(\forall t^\sharp \neq t^o)\ \tau(p(t^\sharp)) = p'(\tau(t^\sharp)),\ \text{and} \tag{21b}$$

$$(\forall t^\sharp \neq t^o)\ \delta(q(t^\sharp)) = q'(\tau(t^\sharp)). \tag{21c}$$

where Π determines t^o, p, and q, and Π' determines t'^o, p', and q'. Below, Proposition 3.4 extends this result to provide a second characterization of preform morphisms in terms of the category **Set**. The functions $\tau|_{T \smallsetminus \{t^o\}}$ and $\tau|_{F^{-1}(C)}$ are well-defined by (22a) and the proposition's definitions [c] and [d]. These two functions as well as the functions p, p', q, q', and δ are displayed in the diagrams of Figure 5. (22b)–(22c) state that both of these diagrams commute.

Proposition 3.4. *A quadruple* $[\Pi, \Pi', \tau, \delta]$ *is a morphism iff it satisfies (17a)–(17b),*

$$\tau(T \smallsetminus \{t^o\})\ \subseteq\ T' \smallsetminus \{t'^o\},\ \ \tau(F^{-1}(C))\ \subseteq\ (F')^{-1}(C'), \tag{22a}$$

$$\tau|_{F^{-1}(C)} {\circ} p = p' {\circ} \tau|_{T \smallsetminus \{t^o\}},\ \ \textit{and} \tag{22b}$$

$$\delta {\circ} q = q' {\circ} \tau|_{T \smallsetminus \{t^o\}}, \tag{22c}$$

$$\begin{tikzcd}
\begin{array}{c} F^{-1}(C) \\ \subseteq T \end{array} \arrow[r, "\tau|_{F^{-1}(C)}"] & \begin{array}{c} (F')^{-1}(C') \\ \subseteq T' \end{array} \\
T \smallsetminus \{t^o\} \arrow[u, "p"] \arrow[r, "\tau|_{T \smallsetminus \{t^o\}}"] & T' \smallsetminus \{t'^o\} \arrow[u, "p'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\delta"] & C' \\
T \smallsetminus \{t^o\} \arrow[u, "q"] \arrow[r, "\tau|_{T \smallsetminus \{t^o\}}"] & T' \smallsetminus \{t'^o\} \arrow[u, "q'"']
\end{tikzcd}$$

Figure 5. (22b)–(22c) state that these diagrams commute

where [a] Π determines t^o, F, p, and q, [b] Π' determines t'^o, F', p', and q', [c] the codomain of $\tau|_{T\smallsetminus\{t^o\}}$ is defined to be $T'\smallsetminus\{t'^o\}$, and [d] the codomain of $\tau|_{F^{-1}(C)}$ is defined to be $(F')^{-1}(C')$. (Proof C.11.)

(22) can be interpreted like (21) can be interpreted. (22a) states that the image of every nonroot node is a nonroot node, and that the image of every decision node is a decision node. (22b) states that the image of the predecessor of a nonroot node is the predecessor of the image of the nonroot node. (22c) states that the image of the previous choice of a nonroot node is the previous choice of the image of the nonroot node.

A morphism implies relationships between [a] the entities derived from the source preform and [b] the entities derived from the target preform. For example, a result about the two feasibility correspondences is in the first half of (19a), a result about the two nonroot-node sets in in the second half of (19a), a result about the two predecessor functions is in (20a), a result about the two previous-choice functions is in (20b), and a result about the two decision-node sets is in the second half of (22a). A result about the two information-set collections is in Proposition 3.5 below. Finally, results about the two trees are established categorically by Corollary 3.10 in the next subsection.

Proposition 3.5. *Assume $[\Pi, \Pi', \tau, \delta]$ is a morphism, derive $\mathcal{H}$ from Π, and derive $\mathcal{H}'$ from Π'. Then $(\forall H{\in}\mathcal{H})(\exists H'{\in}\mathcal{H}')\ \tau(H) \subseteq H'$.*[10] *(Proof C.12.)*

3.4. The Category NCP

This subsection defines the category **NCP**, which is called the *category of node-and-choice preforms*. Let an object be a node-and-choice preform $\Pi = (T, C, \otimes)$. Let an arrow be a preform morphism $\alpha = [\Pi, \Pi', \tau, \delta]$. Let source, target, identity, and composition be

$$\alpha^{\mathsf{src}} = [\Pi, \Pi', \tau, \delta]^{\mathsf{src}} = \Pi,$$
$$\alpha^{\mathsf{trg}} = [\Pi, \Pi', \tau, \delta]^{\mathsf{trg}} = \Pi',$$
$$\mathsf{id}_\Pi = [\Pi, \Pi, \mathsf{id}_T, \mathsf{id}_C], \text{ and}$$
$$\alpha'{\circ}\alpha = [\Pi', \Pi'', \tau', \delta']{\circ}[\Pi, \Pi', \tau, \delta] = [\Pi, \Pi'', \tau'{\circ}\tau, \delta'{\circ}\delta],$$

where id_T and id_C are identities in **Set**.

[10] The symbol τ is overloaded. See footnote 7.

THEOREM 3.6. **NCP** *is a category. (Proof C.13.)*

Theorem 2.6 characterized the isomorphisms in **Tree** by the bijectivity of τ. The following theorem provides an analogous characterization for the isomorphisms in **NCP**.

THEOREM 3.7. *Suppose that $\alpha = [\Pi, \Pi', \tau, \delta]$ is a morphism. Then α is an isomorphism iff τ and δ are bijections. Further, if α is an isomorphism, then $\alpha^{-1} = [\Pi', \Pi, \tau^{-1}, \delta^{-1}]$. (Proof C.16.)*

The following proposition uses Theorem 3.7 to provide some properties of **NCP** isomorphisms. Corollary 3.11 will categorically derive many more properties.

PROPOSITION 3.8. *Suppose $[\Pi, \Pi', \tau, \delta]$ is an isomorphism, where $\Pi = (T, C, \otimes)$ determines F, q, and $\mathcal{H}$, and where $\Pi' = (T', C', \otimes')$ determines F', q', and $\mathcal{H}'$. Then the following hold.*

(a) $\tau|_{F^{-1}(C)}$ is a bijection from $F^{-1}(C)$ onto $(F')^{-1}(C')$.

(b) $(\tau, \delta, \tau)|_{\otimes}$ is a bijection from $\otimes$ onto $\otimes'$.

(c) $(\tau, \delta)|_F$ is a bijection from F onto F'.

(d) $(\tau, \delta)|_q$ is a bijection from q onto q'.

(e) $\tau|_{\mathcal{H}}$ is a bijection from $\mathcal{H}$ onto $\mathcal{H}'$.[10] (Proof C.18.)

This paragraph makes two observations. First, by (13b), a preform incorporates a functioned tree. Second, since (20a) and (8c) are identical, a preform morphism incorporates a functioned-tree morphism. In accord with this second observation, the left-hand side of Figure 5 for preforms is identical to Figure 2 for functioned trees (recall that the preform's decision-node set $F^{-1}(C)$ equals the functioned tree's decision-node set X by Lemma C.1(c)).

Together, these two observations suggest that there is a forgetful functor from **NCP** to **Tree**. Theorem 3.9 establishes this result. Three corollaries follow immediately.

THEOREM 3.9. *Define F from **NCP** to **Tree** by*

$$\mathsf{F}_0 : \Pi \mapsto (T, p) \text{ and}$$

$$\mathsf{F}_1 : [\Pi, \Pi', \tau, \delta] \mapsto [\mathsf{F}_0(\Pi), \mathsf{F}_0(\Pi'), \tau],$$

where $\Pi = (T, C, \otimes)$ determines p by (13b). Then F is a well-defined functor. (Proof D.1.)

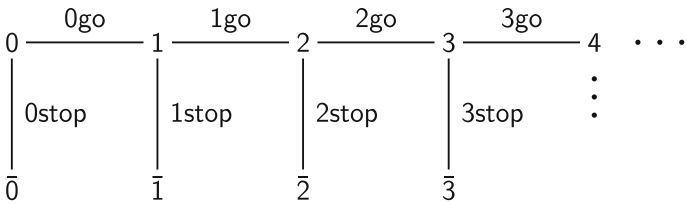

Figure 6. A perfect-information preform

COROLLARY 3.10. *Suppose* $[\Pi, \Pi', \tau, \delta]$ *is a morphism, where* Π *determines* (T, p) *and* Π' *determines* (T', p'). *Then* $[(T, p), (T', p'), \tau]$ *is a morphism in* **Tree**. *Hence it obeys properties (a)*[11]*–(h) of Proposition 2.4.*

COROLLARY 3.11. *Suppose* $[\Pi, \Pi', \tau, \delta]$ *is an isomorphism, where* Π *determines* (T, p) *and* Π' *determines* (T', p'). *Then* $[(T, p), (T', p'), \tau]$ *is an isomorphism in* **Tree**. *Hence it obeys properties (a)–(j) of Proposition 2.7.*

COROLLARY 3.12. *Recall the functor* G *from Theorem 2.8. Then* $\mathsf{G} \circ \mathsf{F}$ *is a well-defined functor from* **NCP** *to* $\mathbf{Grph_{ca}}$.

3.5. Perfect Information

A preform is said to have *perfect information* iff

$$(\forall c)\ F^{-1}(c) \text{ is a singleton.} \tag{23}$$

Thus, in light of (13c), a preform has perfect information iff each of its information sets is a singleton.

For example, Figure 6 depicts $\Pi = (T, C, \otimes)$, where T is defined in (2),

$$C = \{\mathsf{0stop},\ \mathsf{0go},\ \mathsf{1stop},\ \mathsf{1go},\ \ldots\}, \text{ and}$$
$$\otimes = \{(\mathsf{0}, \mathsf{0stop}, \bar{\mathsf{0}}),\ (\mathsf{0}, \mathsf{0go}, \mathsf{1}),\ (\mathsf{1}, \mathsf{1stop}, \bar{\mathsf{1}}),\ (\mathsf{1}, \mathsf{1go}, \mathsf{2}),\ \ldots\}.$$

Note $F = \{(\mathsf{0}, \mathsf{0stop}), (\mathsf{0}, \mathsf{0go}), (\mathsf{1}, \mathsf{1stop}), (\mathsf{1}, \mathsf{1go}), \ldots\}$. Thus, each F^{-1} $(\mathsf{nstop}) = \{\mathsf{n}\}$, and each $F^{-1}(\mathsf{ngo}) = \{\mathsf{n}\}$. Hence $(T, C, \otimes)$ has perfect information. In contrast, the horse-like preform of Figure 3a does not have perfect information because it has a non-singleton information set.

Several general observations can be made. [1] Perfect information implies that $\{F^{-1}(c)|c\}$ partitions $F^{-1}(C)$. Hence perfect information implies (13c). Therefore, a triple $(T, C, \otimes)$ is a perfect-information preform iff it satisfies (13a)–(13b) and (23). [2] Proposition 3.8(e) implies that isomorphisms preserve perfect information (nothing similar can be said for morphisms). [3]

---

[11] Property (a) may be less convenient than Proposition 3.8(a). The two are equivalent by Lemma C.1(c).

Lemma D.2 shows that a preform has perfect information iff its previous-choice function q is bijective.

Let $\mathbf{NCP_p}$ be the full subcategory of $\mathbf{NCP}$ whose objects are the preforms with perfect information. The following theorem shows that $\mathbf{NCP_p}$ and $\mathbf{Tree}$ are equivalent ([14, p. 18]). Incidentally, the natural isomorphisms ([14, p. 16]) used to establish Theorem 3.13(a) appear in the statement of Lemma D.6.

THEOREM 3.13. $\mathbf{NCP_p}$ *and* $\mathbf{Tree}$ *are equivalent. In particular, let* $\mathsf{F_p}$ *be the restriction of Theorem 3.9's functor* F *to* $\mathbf{NCP_p}$. *Conversely, define* E *from* $\mathbf{Tree}$ *to* $\mathbf{NCP_p}$ *by*

$$\mathsf{E_0} : (T, p) \mapsto (T, C, \otimes)$$

where $C = T \smallsetminus \{t^o\}$ *and* $\otimes = \{\ (t, t^\sharp, t^\sharp) \mid (t^\sharp, t) \in p\ \}$, *and by*

$$\mathsf{E_1} : [(T, p), (T', p'), \tau] \mapsto [\mathsf{E_0}(T, p), \mathsf{E_0}(T', p'), \tau, \tau|_{T \smallsetminus \{t^o\}}]$$

where $T \smallsetminus \{t^o\}$ *is the set of nonroot nodes of* (T, p), $T' \smallsetminus \{t'^o\}$ *is the set of nonroot nodes of* (T', p'), *and the codomain of* $\tau|_{T \smallsetminus \{t^o\}}$ *is* $T' \smallsetminus \{t'^o\}$. *Then (a)* $\mathsf{E} \circ \mathsf{F_p}$ *is naturally isomorphic to the identity functor for* $\mathbf{NCP_p}$ *, and (b)* $\mathsf{F_p} \circ \mathsf{E}$ *equals the identity functor for* $\mathbf{Tree}$. *(Proof D.8.)*

For example, consider again Figure 6's preform Π. Closely related is the "spartan" preform $\Pi^s = (T, C^s, \otimes^s)$, where T is again defined defined by (2), but

$$C^s = \{\bar{0}, 1, \bar{1}, 2,\ \ldots\ \} \text{ and}$$

$$\otimes^s = \{\ (0, \bar{0}, \bar{0}),\ (0, 1, 1),\ (1, \bar{1}, \bar{1}),\ (1, 2, 2),\ \ldots\ \}.$$

These two preforms share the same tree. Formally,

$$\mathsf{F_0}(\Pi) = \mathsf{F_0}(\Pi^s) = (T, p),$$

where (T, p) is defined by (2). The "expansion" of that tree is the spartan preform. Formally,

$$\mathsf{E_0}(T, p) = \Pi^s.$$

Thus [1] $\mathsf{F_0} \circ \mathsf{E_0}(T, p) = (T, p)$ and [2] $\mathsf{E_0} \circ \mathsf{F_0}(\Pi) = \Pi^s$. [1] agrees with Theorem 3.13(b). Since Π^s and Π are isomorphic, [2] implies that $\mathsf{E_0} \circ \mathsf{F_0}(\Pi)$ and Π are isomorphic but unequal. Their being isomorphic agrees with Theorem 3.13(a). Their being unequal shows that $\mathsf{F_p}$ and E do not form a pair of isomorphisms between $\mathbf{NCP_p}$ and $\mathbf{Tree}$.

$$\begin{array}{ccccc} & \text{Theorem } \mathbf{3.13} & & \text{Theorem } \mathbf{2.8} & \\ \mathbf{NCP_p} & \overset{\mathsf{F_p}}{\underset{\mathsf{E}}{\rightleftarrows}} & \mathbf{Tree} & \overset{\mathsf{G}}{\underset{\mathsf{H}}{\rightleftarrows}} & \mathbf{Grph_{ca}} \end{array}$$

Figure 7. $\mathbf{NCP_p}$, **Tree**, and $\mathbf{Grph_{ca}}$ are equivalent by Theorems 3.13 and 2.8

Finally, Corollary 3.14 follows from the two theorems shown in Figure 7. In addition, the figure almost shows Theorem 3.9's forgetful functor F, which goes from all of **NCP** to **Tree**.

Corollary 3.14. *$\mathbf{NCP_p}$ and $\mathbf{Grph_{ca}}$ are equivalent. In particular, define $\mathsf{F_p}$ and E as in Theorem 3.13, and define G and H as in Theorem 2.8. Then (a) $(\mathsf{E}\circ\mathsf{H})\circ(\mathsf{G}\circ\mathsf{F_p})$ is naturally isomorphic to the identity functor for $\mathbf{NCP_p}$, and (b) $(\mathsf{G}\circ\mathsf{F_p})\circ(\mathsf{E}\circ\mathsf{H})$ equals the identity functor for $\mathbf{Grph_{ca}}$. (Proof D.9.)*

4. Applications and Extensions

4.1. Applications

The intention behind developing categories for extensive-form games is to systematically compare the results which are obtained within different strands of the game-theory literature. Each of these strands has its own way of formulating games, and each these formulations has its own advantages and disadvantages.

For example, Figure 3a depicted a horse-like **NCP** preform. Figures 8 and 9 depict four more horse-like **NCP** preforms, all of which are isomorphic to the first. Each of these four additional preforms is formulated according to a particular strand of the game-theory literature. From the abstract perspective of **NCP**, each of these specialized formulations has its own particularly convenient means of specifying the node-and-choice operator $\otimes$.

Figure 8a uses a "choice" formulation in the sense that it expresses each node as a choice-sequence. In particular, Figure 8a depicts the **NCP** preform $(T, C, \otimes)$, where C consists of the six choices $\mathsf{r^S}$, $\mathsf{d^S}$, $\mathsf{r^G}$, $\mathsf{d^G}$, e, and f, where T consists of the nine choice-sequences $\{\}$, $(\mathsf{r^S})$, $(\mathsf{d^S})$, $(\mathsf{r^S},\mathsf{r^G})$, $(\mathsf{r^S},\mathsf{d^G})$, $(\mathsf{d^S},\mathsf{e})$, $(\mathsf{d^S},\mathsf{f})$, $(\mathsf{r^S},\mathsf{d^G},\mathsf{e})$, and $(\mathsf{r^S},\mathsf{d^G},\mathsf{f})$, and where $\otimes$ consists of the eight triples

$$\big(\{\},\mathsf{r^S},(\mathsf{r^S})\big),\ \big(\{\},\mathsf{d^S},(\mathsf{d^S})\big),\ \big((\mathsf{r^S}),\mathsf{r^G},(\mathsf{r^S},\mathsf{r^G})\big),$$
$$\big((\mathsf{r^S}),\mathsf{d^G},(\mathsf{r^S},\mathsf{d^G})\big),\ \big((\mathsf{d^S}),\mathsf{e},(\mathsf{d^S},\mathsf{e})\big),\ \big((\mathsf{d^S}),\mathsf{f},(\mathsf{d^S},\mathsf{f})\big),$$
$$\big((\mathsf{r^S},\mathsf{d^G}),\mathsf{e},(\mathsf{r^S},\mathsf{d^G},\mathsf{e})\big),\ \text{and}\ \big((\mathsf{r^S},\mathsf{d^G}),\mathsf{f},(\mathsf{r^S},\mathsf{d^G},\mathsf{f})\big).$$

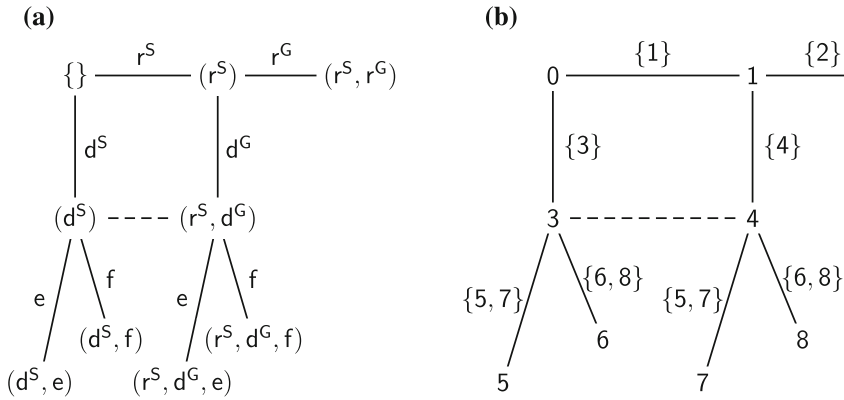

Figure 8. **a** A "choice" preform where $t \otimes c = t \oplus (c)$. **b** A "node" preform where $\{t \otimes c\} = p^{-1}(t) \cap c$

Better, one can read T, C, and $\otimes$ from the diagram directly: Nodes and choices are as usual, and the eight triples in $\otimes$ are the eight node-choice-node segments in the diagram. Notice that the node-and-choice operator $\otimes$ takes the form $t \otimes c = t \oplus (c)$, where $\oplus$ is the concatenation operator for sequences. For example, $(\mathsf{d}^\mathsf{S}) \otimes \mathsf{f} = (\mathsf{d}^\mathsf{S}) \oplus (\mathsf{f}) = (\mathsf{d}^\mathsf{S}, \mathsf{f})$. This formulation is popular. It appears in the textbook of Osborne and Rubinstein [18],[12] and there, choice-sequences are called "histories".

Figure 8b uses a "node" formulation in the sense that it expresses each choice as a node-set. In particular, Figure 8b depicts the **NCP** preform $(T, C, \otimes)$ where T consists of nine nodes such as $t = \mathsf{3}$ and $t^\sharp = \mathsf{6}$, where C consists of six node-sets such as $c = \{\mathsf{6}, \mathsf{8}\}$, and where $\otimes$ consists of the diagram's eight node-choice-node triples such as $(t, c, t^\sharp) = (\mathsf{3}, \{\mathsf{6}, \mathsf{8}\}, \mathsf{6})$. Notice that $\otimes$ obeys $\{t \otimes c\} = p^{-1}(t) \cap c$, where p is the preform's predecessor function. For example, $\{\mathsf{3} \otimes \{\mathsf{6}, \mathsf{8}\}\} = p^{-1}(\mathsf{3}) \cap \{\mathsf{6}, \mathsf{8}\} = \{\mathsf{5}, \mathsf{6}\} \cap \{\mathsf{6}, \mathsf{8}\} = \{\mathsf{6}\}$. This formulation appears in Alós-Ferrer and Ritzberger [4, p. 94], and is related to the classic formulation of Kuhn [11].

The two formulations of Figure 8 are "dual" in the sense that [a] choice preforms express each node in terms of choices and [b] node preforms express

[12] Figure 8a is meant to be emblematic of some, but not all, Osborne-Rubinstein structures. To be precise, let a "structure" be a rooted tree together with choices and information sets. Then Figure 8a is meant to be emblematic of those Osborne-Rubinstein structures that can be expressed as **NCP** preforms. In accord with Proposition 3.2(16b), this rules out Osborne-Rubinstein structures in which different information sets share the same choices. [Such a qualification is not needed for Figures 8b, 9a, and 9b because the node, choice-set, and outcome formulations all require that each information set has its own choices.]

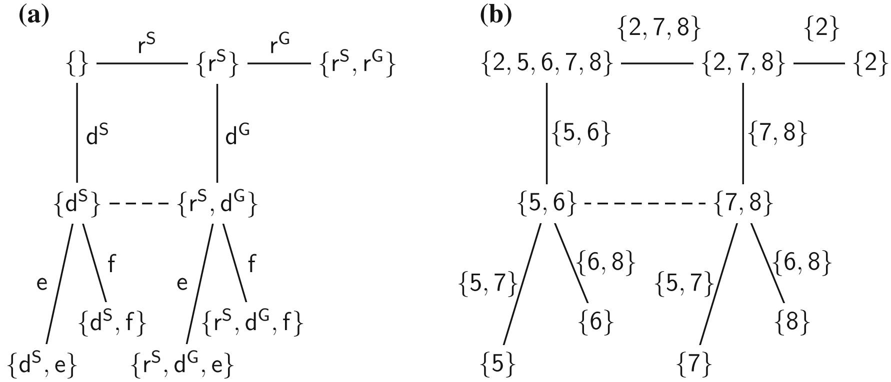

Figure 9. **a** A “choice-set” preform where $t{\otimes}c = t{\cup}\{c\}$. **b** An “outcome” preform where $t{\otimes}c = t{\cap}c$

each choice in terms of nodes. We plan to show categorically that either of these special formulations is effectively as general as all of **NCP**. The result for choice forms would be related to [10]'s non-categorical link between “OR trees” and “KS trees”.

Figure 9a uses a “choice-set” formulation. This is the same as the “choice” formulation of Figure 8a, except that it expresses each node as a choice-*set* rather than a choice-sequence. In particular, Figure 9a depicts the **NCP** preform $(T, C, \otimes)$, where C consists of six choices such as $c = \mathsf{f}$, where T consists of nine choice-sets such as $t = \{\mathsf{d}^\mathsf{S}\}$ and $t^\sharp = \{\mathsf{d}^\mathsf{S}, \mathsf{f}\}$, and where $\otimes$ consists of eight triples such as $(t, c, t^\sharp) = (\{\mathsf{d}^\mathsf{S}\}, \mathsf{f}, \{\mathsf{d}^\mathsf{S}, \mathsf{f}\})$. Notice that $\otimes$ obeys $t{\otimes}c = t{\cup}\{c\}$. For example, $\{\mathsf{d}^\mathsf{S}\}{\otimes}\mathsf{f} = \{\mathsf{d}^\mathsf{S}\}{\cup}\{\mathsf{f}\} = \{\mathsf{d}^\mathsf{S}, \mathsf{f}\}$. This formulation was introduced by Streufert [22], and was used to advantage by [21].

Figure 9b uses an “outcome” formulation in the sense that it expresses each node and each choice as an outcome-set. Routinely, outcomes are in one-to-one correspondence with plays, and in this example, the outcomes are $\mathsf{2}$, $\mathsf{5}$, $\mathsf{6}$, $\mathsf{7}$, and $\mathsf{8}$ (these labels correspond with the five nondecision nodes in Figure 8b). Accordantly, Figure 9b depicts the **NCP** preform $(T, C, \otimes)$, where T consists of nine nodes such as $t = \{\mathsf{5}, \mathsf{6}\}$ and $\mathsf{t}^\sharp = \{\mathsf{6}\}$, where C consists of six choices such as $c = \{\mathsf{6}, \mathsf{8}\}$, and where $\otimes$ consists of eight triples such as $(t, c, t^\sharp) = (\{\mathsf{5}, \mathsf{6}\}, \{\mathsf{6}, \mathsf{8}\}, \{\mathsf{6}\})$. Notice that $\otimes$ obeys $t{\otimes}c = t{\cap}c$. For example, $\{\mathsf{5}, \mathsf{6}\}{\otimes}\{\mathsf{6}, \mathsf{8}\} = \{\mathsf{5}, \mathsf{6}\}{\cap}\{\mathsf{6}, \mathsf{8}\} = \{\mathsf{6}\}$. This formulation was introduced by von Neumann and Morgenstern [27], and was extended to allow infinite plays by Alós-Ferrer and Ritzberger [4].

The two formulations of Figure 9 are “dual” in the sense that [a] the $\otimes$ of choice-set preforms uses a union, and [b] the $\otimes$ of outcome preforms uses an intersection. This dual pair is slightly less general than Figure 8's dual pair because only the formulations of the former pair can accommodate absentmindedness. Given this caveat, [4] essentially develops a non-categorical equivalence that links the node and the outcome formulations. Meanwhile, [23] develops a non-categorical equivalence that links the choice, the choice-set, and the outcome formulations. We plan to synthesize and strengthen these equivalences categorically.

In summary, the operator $\otimes$ provides a unified way of comparing apparently dissimilar formulations.

4.2. Extensions

As discussed in the introduction, this paper is the first step in a larger agenda to develop a category of extensive-form games. [24] takes the second step. It develops a category of node-and-choice forms, where a "form" is a preform from this paper, augmented with players. There, [i] a form assigns each information set (and concurrently each decision node and each choice) to a player, and [ii] a form morphism allows players to be renamed and merged. In addition, we are developing a third paper which augments forms with [a] pure player strategies and [b] player preferences over non-probabilistic outcomes (i.e. plays).

This paper, and the two papers just discussed, admit continuum feasible sets and infinite-horizon trees. They do so at relatively little cost because they rely on set theory and category theory alone. In contrast, measure-theoretic issues will arise when there are mixed player strategies with continuum supports, or chance moves with continuum supports, or player preferences over outcome lotteries with infinite supports. Accordingly, our next step will avoid these issues by considering only [a] mixed player strategies with finite support, [b] chance moves with finite support, and [c] player preferences over outcome lotteries with finite support. This intermediate step will likely entail a restriction to finite-horizon trees. The measure-theoretic issues can then be left for last.

Appendix A. Concerning Tree

A.1. Objects

LEMMA A.1. *Suppose (T,p) is a functioned tree with its t^o, k, $\prec$, $\preccurlyeq$, $\mathcal{Z}_{\mathsf{ft}}$, and $\mathcal{Z}_{\mathsf{inft}}$. Then the following hold.*
(a) $t^1 \prec t^2$ iff both $k(t^1) < k(t^2)$ and $t^1 = p^{k(t^2)-k(t^1)}(t^2)$.
(b) $\prec$ is the asymmetric part of $\preccurlyeq$.

(c) $t^1 \preccurlyeq t^2$ *iff both* $k(t^1) \leq k(t^2)$ *and* $t^1 = p^{k(t^2)-k(t^1)}(t^2)$*, where* p^0 *is the identity function.*
(d) $(T,\preccurlyeq)$ *is a partially ordered set.*
(e) If $S \subseteq T$ *is a chain,* $S \cup \{p^m(t)|t \in S, k(t) \geq m \geq 1\}$ *is a chain.*
(f) If $S \subseteq T$ *is an infinite chain,* $S \cup \{p^m(t)|t \in S, k(t) \geq m \geq 1\} \in \mathcal{Z}_{\mathsf{inft}}$*.*
(g) If $S \subseteq T$ *is a chain, there exists* $Z \in \mathcal{Z}_{\mathsf{ft}} \cup \mathcal{Z}_{\mathsf{inft}}$ *such that* $S \subseteq Z$*.*
(h) If $t \in Z \in \mathcal{Z}_{\mathsf{ft}} \cup \mathcal{Z}_{\mathsf{inft}}$ *and* $k(t) \geq m \geq 1$*, then* $p^m(t) \in Z$*.*

Proof. *(a).* The reverse direction follows immediately from the definition of $\prec$. To see the forward direction, suppose $t^1 \prec t^2$. Then by the definition of $\prec$, there exists an $m \geq 1$ such that $t^1 = p^m(t^2)$. Meanwhile by the definition of $k(t^1)$, I have $t^o = p^{k(t^1)}(t^1)$. Combining these two yields $t^o = p^{k(t^1)}(t^1) = p^{k(t^1)}(p^m(t^2)) = p^{k(t^1)+m}(t^2)$. Thus $k(t^2) = k(t^1)+m$ by the definition of $k(t^2)$. So $m = k(t^2)-k(t^1)$. This and the definition of m imply both $k(t^2) > k(t^1)$ and $t^1 = p^{k(t^2)-k(t^1)}(t^2)$.

(b). By the definition of $\preccurlyeq$, it suffices to prove that $\prec$ is asymmetric. This relation is asymmetric because if both $t^1 \prec t^2$ and $t^2 \prec t^1$ held, part (a) would imply both $k(t^1) < k(t^2)$ and $k(t^2) < k(t^1)$.

(c). By using the definition of $\preccurlyeq$ for the first equivalence, and by using part (a) for the second equivalence,

$$\begin{aligned} & t^1 \preccurlyeq t^2 \\ & \quad \Leftrightarrow \; t^1 \prec t^2 \text{ or } t^1 = t^2 \\ & \quad \Leftrightarrow \; [\ k(t^1) < k(t^2) \text{ and } t^1 = p^{k(t^2)-k(t^1)}(t^2)\ ] \text{ or} \\ & \quad\quad\;\; [\ k(t^1) = k(t^2) \text{ and } t^1 = p^{k(t^2)-k(t^1)}(t^2)\ ] \\ & \quad \Leftrightarrow \; k(t^1) \leq k(t^2) \text{ and } t^1 = p^{k(t^2)-k(t^1)}(t^2). \end{aligned}$$

(d). Reflexivity holds by the definition of $\preccurlyeq$. Transitivity holds by [1] the definition of $\preccurlyeq$ and [2] the transitivity of $\prec$, which follows immediately from its definition. To show antisymmetry, suppose $t^1 \preccurlyeq t^2$ and $t^2 \preccurlyeq t^1$. Then by two applications of part (c), $k(t^1) = k(t^2)$. Thus by $t^1 \preccurlyeq t^2$ and part (c) again, $t^1 = p^0(t^2) = t^2$.

(e). Let p^0 be the identity function, so that

$$S \cup \{p^m(t)|t \in S, k(t) \geq m \geq 1\} = \{p^m(t)|t \in S, k(t) \geq m \geq 0\}.$$

Then consider $p^{m^1}(t^1)$ and $p^{m^2}(t^2)$ such that $\{t^1,t^2\} \in S$, $k(t^1) \geq m^1 \geq 0$, and $k(t^2) \geq m^2 \geq 0$. Since S is a chain, assume without loss of generality that $t^1 \preccurlyeq t^2$. Thus by part (c), there is an $m \geq 0$ such that $t^1 = p^m(t^2)$. If $m^1+m > m^2$, $p^{m^1}(t^1) = p^{m^1+m}(t^2) \prec p^{m^2}(t^2)$. If $m^1+m = m^2$, $p^{m^1}(t^1) = p^{m^1+m}(t^2) = p^{m^2}(t^2)$. If $m^1+m < m^2$, $p^{m^1}(t^1) = p^{m^1+m}(t^2) \succ p^{m^2}(t^2)$.

(f). Suppose S is an infinite chain. Since S is a chain and since $\min S$ exists, I may number the elements of S so that $\min S = t^1 \prec t^2 \prec t^3 \ \ldots$. Thus by part (a), $(\forall n \geq 1)\ k(t^n) < k(t^{n+1})$. Hence $(\forall n \geq 1)\ k(t^n) \geq n-1$.

Now consider $\bar{S} := S\cup\{p^m(t)|t\in S, k(t) \geq m \geq 1\}$. By part (e), $\bar{S}$ is a chain. Further, it is infinite because S is infinite. Thus it remains to be shown that $\bar{S}$ is maximal. Accordingly, suppose that it were not maximal. Then there would be some $t' \notin \bar{S}$ such that $\bar{S}\cup\{t'\}$ is a chain.

This paragraph shows that $(\forall n \geq 1)\ k(t') \geq n$. Take any $n \geq 1$. Since $t' \notin \bar{S}$, and since t^n and all its predecessors are in $\bar{S}$, it must be that $t' \succ t^n$. Thus by part (a), $k(t') > k(t^n)$. Thus, since $k(t^n) \geq n{-}1$ by the second-previous paragraph, $k(t') \geq n$.

By the previous paragraph, $k(t') \notin \mathbb{N}_0$. This contradicts the definition of k.

(g). Suppose S is a chain. On the one hand, suppose S is infinite. Then part (f) shows that it is a subset of a member of $\mathcal{Z}_{\mathsf{inft}}$. On the other hand, suppose S is finite. Then $\max S$ exists, and two cases arise. These cases are defined in the first sentences of the next two paragraphs.

[1] Suppose that [a] $\max S$ does not have a successor or [b] $\max S$ has a successor that does not have a successor. In either [a] or [b], let t^* denote the node without a successor. Then $S\cup\{t^*\}$ is a chain. Thus by part (e), $\bar{S} = (S\cup\{t^*\})\cup\{p^m(t)|t\in S\cup\{t^*\}, k(t) \geq m \geq 1\}$ is a chain. If $\bar{S}$ were not maximal, there would be some $t' \notin \bar{S}$ such that $\bar{S}\cup\{t'\}$ is a chain. Since $\bar{S}$ contains all the predecessors of t^*, it must be that $t' \succ t^*$. But this contradicts the assumption that t^* does not have a successor.

[2] Suppose that $\max S$ has a successor and that every successor of $\max S$ has a successor. Then define S^1 by $S^1 = S\cup\{t^1\}$ where t^1 is some successor of $\max S$. Then, for every $n \geq 2$, define $S^n = S^{n-1}\cup\{t^2\}$ where t^n is some successor of t^{n-1}. Then $\cup_{n\geq 1} S^n$ is an infinite chain. Thus part (f) shows that it is a subset of a member of $\mathcal{Z}_{\mathsf{inft}}$.

(h). Suppose $t \in Z \in \mathcal{Z}_{\mathsf{ft}}\cup\mathcal{Z}_{\mathsf{inft}}$ and $k(t) \geq m \geq 1$. I argue

$$p^m(t) \in Z\cup\{p^{m'}(t')|t'\in Z, k(t') \geq m' \geq 1\} \ \subseteq\ Z.$$

The set membership holds because $t \in Z$ and $k(t) \geq m \geq 1$. The set inclusion holds because [1] $Z\cup\{p^{m'}(t')|t'\in Z, k(t') \geq m' \geq 1\}$ is a chain by part (e) and [2] Z is maximal by the assumption $Z \in \mathcal{Z}_{\mathsf{ft}}\cup\mathcal{Z}_{\mathsf{inft}}$. ■

Proof A.2. (for Proposition 2.2) *(a).* I must show that

$$\mathcal{Z}_{\mathsf{ft}} \ni Z \mapsto \max Z \tag{24}$$

is a bijection onto $T\smallsetminus X$, and that its inverse is

$$\{p^m(t)|k(t) \geq m \geq 1\}\cup\{t\} \leftarrow\!\shortmid t \in T\smallsetminus X. \tag{25}$$

These results follow from the next two paragraphs.

This paragraph argues that the function (24) followed by the function (25) is the identity function on $\mathcal{Z}_{\mathsf{ft}}$. Accordingly, take any $Z \in \mathcal{Z}_{\mathsf{ft}}$. The remainder of this paragraph argues

$$Z \mapsto \max Z \mapsto \tag{26}$$
$$\{\ p^m(\max Z)\ |\ k(\max Z) \geq m \geq 1\ \}\cup\{\ \max Z\ \} = Z,$$

where the two arrows apply the functions (24) and (25), respectively. By inspection, the first arrow applies the function (24). Before applying the function (25), it must be shown that $\max Z$ exists and is an element of $T \smallsetminus X$. First, $\max Z$ exists and is an element of T because $Z \in \mathcal{Z}_{\mathsf{ft}}$ by definition. Second, $\max Z$ is not an element of X, for if it were an element of X, [1] it would have a successor, thus [2] Z would not be a maximal chain, and thus [3] $Z \notin \mathcal{Z}_{\mathsf{ft}}$ in contradiction to the definition of Z. Accordingly, the second arrow in (26) applies the function (25) at $t = \max Z$. To continue, the $\subseteq$ direction of the equality in (26) holds by Lemma A.1(h) applied at $t = \max Z$. To see the $\supseteq$ direction, take any $t \in Z$. Because Z is a chain that contains $\max Z$, either $t \preccurlyeq \max Z$ or $\max Z \prec t$. The former implies that t is in the left-hand side. The latter contradicts the definition of the maximum operator.

This paragraph argues that the function (25) followed by the function (24) is the identity function on $T \smallsetminus X$. Accordingly, take any $t \in T \smallsetminus X$. The remainder of this paragraph argues

$$\begin{aligned} &t \mapsto \{p^m(t)|k(t) \geq m \geq 1\} \cup \{t\} \mapsto \\ &\max\{p^m(t)|k(t) \geq m \geq 1\} \cup \{t\} = t, \end{aligned} \tag{27}$$

where the two arrows apply the function (25) and (24), respectively. By inspection, the first arrow applies the function (25). Before applying the function (24), it must be shown that $S := \{p^m(t)|k(t) \geq m \geq 1\} \cup \{t\}$ is an element of $\mathcal{Z}_{\mathsf{ft}}$. Since S is a finite chain by inspection, I only need to show that S is maximal. Accordingly, suppose there were a $t' \notin S$ such that $S \cup \{t'\}$ was a chain. Because $t \in S$ and $S \cup \{t'\}$ is a chain, either $t' \preccurlyeq t$ or $t \prec t'$. The first case is impossible for it would imply that $t' \in S$, in contradiction to the definition of t'. The second case would imply [1] that t has a successor, and thus [2] that $t \in X$. This would contradict the definition of t. Accordingly, the second arrow in (27) applies the function (24) at $Z = S$. The equality is immediate.

(b). This paragraph shows that E is a well-defined function from $\mathcal{Z}_{\mathsf{inft}}$ into $T^{\mathbb{N}_1}$. Accordingly, take any $Z \in \mathcal{Z}_{\mathsf{inft}}$. It must be shown that

$$(\forall v \geq 1)(\exists! t{\in}Z)\ k(t) = v.$$

Take any $v \geq 1$. First, consider uniqueness. It must be shown that there are not two nodes in Z at stage v. This holds because distinct nodes in a chain have different stages by Lemma A.1(a). Second, consider existence. Let $S := \{t'{\in}Z|k(t'){\leq}v\}$. Since distinct nodes in a chain have different stages by Lemma A.1(a), S is finite. Thus since Z is infinite, $Z \smallsetminus S = \{t'{\in}Z|k(t'){>}v\}$ is nonempty. Take $t^* \in Z \smallsetminus S$ and let $t = p^{k(t^*)-v}(t^*)$. Note $t \in Z$ by Lemma A.1(h) at its t equal to t^* and its m equal to $k(t^*){-}v$. Further note that

$$t^o = p^{k(t^*)}(t^*) = p^v(p^{k(t^*)-v}(t^*)) = p^v(t),$$

where the first equality holds by the definition of $k(t^*)$, the second is a rearrangement, and the third holds by the definition of t. Thus $k(t) = v$ by the definition of $k(t)$.

This paragraph shows that E maps from $\mathcal{Z}_{\mathsf{inft}}$ into $\mathcal{Y} \subseteq T^{\mathbb{N}_1}$. Accordingly, take any $Z \in \mathcal{Z}_{\mathsf{inft}}$. By the previous paragraph, I may let $E(Z) = (t^v)_{v \geq 1}$. It must be shown that $t^o = p(t^1)$ and that $(\forall v \geq 1)\ t^v = p(t^{v+1})$. Since $k(t^1) = 1$ by the definition of E, $p(t^1) = t^o$ by the definition of k. Next take any $v \geq 1$. By the definition of E, [1] $\{t^v, t^{v+1}\} \subseteq Z$, [2] $k(t^v) = v$, and [3] $k(t^{v+1}) = v{+}1$. By [1], $t^v \prec t^{v+1}$ or $t^{v+1} \preccurlyeq t^v$. Thus $t^v \prec t^{v+1}$ because the alternative is impossible by [2], [3], and Lemma A.1(c). Finally, $t^v \prec t^{v+1}$ implies $t^v = p(t^{v+1})$ by [2], [3], and Lemma A.1(a).

The next two paragraphs prove that E is a bijection from $\mathcal{Z}_{\mathsf{inft}}$ onto $\mathcal{Y}$, and that its inverse is

$$\{t^o\} \cup \{t^v | v \geq 1\} \leftarrow\!\shortmid (t^v)_{v \geq 1} \in \mathcal{Y}. \tag{28}$$

This paragraph argues that E followed by the function (28) is the identity function on $\mathcal{Z}_{\mathsf{inft}}$. Accordingly, take any $Z \in \mathcal{Z}_{\mathsf{inft}}$. I argue

$$\begin{gathered} Z \mapsto E(Z) \mapsto \\ \{t^o\} \cup \{E(Z)^v | v \geq 1\} = Z, \end{gathered}$$

where the arrows apply the functions E and (28), respectively. The first arrow applies E by inspection. The second arrow applies (28) because $E(Z) \in \mathcal{Y}$ by the second-previous paragraph. To see the $\subseteq$ direction of the equality, take any $t \in \{t^o\} \cup \{E(Z)^v | v \geq 1\}$. If $t = t^o$, then $t \in Z$ because t^o belongs to every maximal chain and Z is a maximal chain. If $t = E(Z)^v$ from some $v \geq 1$, then $t \in Z$ by the definition of E. To see the $\supseteq$ direction of the equality, take any $t \in Z$. If $k(t) = 0$, then $t = t^o$. If $k(t) \geq 1$, then $t = E(Z)^{k(t)}$ by the definition of E.

This paragraph argues that the function (28) followed by E is the identity function on $\mathcal{Y}$. Accordingly, take any $(t^v)_{v \geq 1} \in \mathcal{Y}$. I argue

$$\begin{gathered} (t^v)_{v \geq 1} \mapsto \{t^o\} \cup \{t^v | v \geq 1\} \mapsto \\ E(\{t^o\} \cup \{t^v | v \geq 1\}) = (t^v)_{v \geq 1}, \end{gathered}$$

where the arrows apply the functions (28) and E, respectively. The first arrow applies (28) by inspection. Before applying E, it must be shown that $S := \{t^o\} \cup \{t^v | v \geq 1\}$ belongs to $\mathcal{Z}_{\mathsf{inft}}$. In other words, it must be shown that S is an infinite maximal chain. The definitions of $(t^v)_{v \geq 1}$ and $\mathcal{Y}$ assure that S is a chain and that S contains a node of every stage. This easily implies that S is infinite. It also implies that S is maximal because distinct nodes in a chain have different stages by Lemma A.1(a). Hence S belongs to $\mathcal{Z}_{\mathsf{inft}}$ and the second arrow applies E. The equality follows from the fact that $(\forall v \geq 1)\ k(t^v) = v$ by the definitions of $(t^v)_{v \geq 1}$ and $\mathcal{Y}$. ■

A.2. Arrows

Proof A.3. (for Proposition 2.3) Take any triple $[(T, p), (T', p'), \tau]$. By the definition of morphism, it suffices [a] to assume (8a)–(8b) and [b] to show that (8c) is equivalent to (9). Toward that end, assume (8a)–(8b).

Suppose (8c). To show the first half of (9a), take any $t^\sharp \in T\smallsetminus\{t^o\}$. Then by (8a) and (1a) for p, there exists t such that $(t^\sharp, t) \in p$. Hence by (8c), $(\tau(t^\sharp), \tau(t)) \in p'$. Thus by (8a) and (1a) for p', $\tau(t^\sharp) \in T'\smallsetminus\{t'^o\}$.

Similarly, to show the second half of (9a), take any $t \in X$. Then by (8a) and (1a) for p, there exists $t^\sharp$ such that $(t^\sharp, t) \in p$. Hence by (8c), $(\tau(t^\sharp), \tau(t)) \in p'$. Hence by (8a) and (1a) for p', $\tau(t) \in X'$.

Since (9a) has been established, the codomain definitions after (9b) guarantee that the equality of (9b) is well-defined in **Set**. Thus, to show that the equality holds, it suffices to show that

$$(\forall t^\sharp{\in}T\smallsetminus\{t^o\})\ \tau{\circ}p(t^\sharp) = p'{\circ}\tau(t^\sharp).$$

Take any $t^\sharp \in T\smallsetminus\{t^o\}$. By (8a) and (1a) for p, $(t^\sharp, p(t^\sharp)) \in p$. Thus by (8c), $(\tau(t^\sharp), \tau{\circ}p(t^\sharp)) \in p'$. By (8a) and (1a) for p', this is equivalent to $p'{\circ}\tau(t^\sharp) = \tau{\circ}p(t^\sharp)$.

Conversely, suppose (9). Take any $(t^\sharp, t) \in p$. By (8a) and (1a) for p, $t = p(t^\sharp)$. Thus $\tau(t) = \tau{\circ}p(t^\sharp) = p'{\circ}\tau(t^\sharp)$, where the first equality holds by the previous sentence and the second holds by (9b). By (8a) and (1a) for p', this is equivalent to $(\tau(t^\sharp), \tau(t)) \in p'$. ■

Proof A.4. (for Proposition 2.4)

(a). This is trivial. It holds because $\tau(t^o) \in T'$ and because $(\forall t')\ t'^o \preccurlyeq' t'$.

(b). Suppose $m \geq 1$ and $t^1 = p^m(t^2)$.

This paragraph shows by induction on i that

$$(\forall m \geq i \geq 1)\ \tau(p^i(t^2)) = (p')^i(\tau(t^2)). \tag{29}$$

The initial step ($i = 1$) holds by (9b) of Proposition 2.3, applied at $t^\sharp = t^2$ (note $t^2 \neq t^o$ because $p^m(t^2)$ exists and $m \geq 1$). To show the inductive step ($m \geq i > 1$), I argue

$$\begin{aligned}
\tau \circ p^i(t^2) &= \tau \circ p \circ p^{i-1}(t^2) \\
&= p' \circ \tau \circ p^{i-1}(t^2) \\
&= p' \circ (p')^{i-1} \circ \tau(t^2) \\
&= (p')^i \circ \tau(t^2).
\end{aligned}$$

The first equality is a rearrangement. The second equation holds by (9b) of Proposition 2.3, applied at $t^\sharp = p^{i-1}(t^2)$ (note $p^{i-1}(t^2) \neq t^o$ because $p^m(t^2)$ exists and $m \geq i$). The third equation holds by the inductive hypothesis, and the fourth is a rearrangement.

Finally, I argue

$$\tau(t^1) = \tau(p^m(t^2)) = (p')^m(\tau(t^2)).$$

The first equality holds by the assumption $t^1 = p^m(t^2)$, the second holds by (29) at $i = m$.

(c). By the definition of $k'(\tau(t))$, it suffices to show

$$\begin{aligned} t'^o &= (p')^{k'(\tau(t^o))}[\tau(t^o)] \\ &= (p')^{k'(\tau(t^o))}[(p')^{k(t)}(\tau(t))] \\ &= (p')^{k(t)+k'(\tau(t^o))}(\tau(t)). \end{aligned}$$

The first equality follows from the definition of $k'(\tau(t^o))$. The second equality holds because [a] $t^o = p^{k(t)}(t)$ by the definition of $k(t)$, and hence [b] $\tau(t^o) = (p')^{k(t)}(\tau(t))$ by part (b). The final equality is a rearrangement.

(d). Suppose $t^1 \prec t^2$. Then by the definition of $\prec$, there exists $m \geq 1$ such that $t^1 = p^m(t^2)$. Thus by part (b), $\tau(t^1) = (p')^m(\tau(t^2))$. Thus by the definition of $\prec'$, $\tau(t^1) \prec' \tau(t^2)$.

(e). Suppose $t^1 \preccurlyeq t^2$. Then by the definition of $\preccurlyeq$, either $t^1 = t^2$ or $t^1 \prec t^2$. In the case of equality, $\tau(t^1) = \tau(t^2)$. In the case of precedence, part (d) implies $\tau(t^1) \prec' \tau(t^2)$. Thus in either case, $\tau(t^1) \preccurlyeq' \tau(t^2)$.

(f). Suppose $S \subseteq T$ is a chain.

To show that $\tau|_S$ is injective, suppose t^1 and t^2 are distinct members of S. Since S is a chain, $t^1 \prec t^2$ without loss of generality. Hence $\tau(t^1) \prec' \tau(t^2)$ by part (d). Hence $\tau(t^1)$ and $\tau(t^2)$ are distinct.

To show that $\tau(S)$ is a chain, take any distinct t'^1 and t'^2 in $\tau(S)$. Since both are in $\tau(S)$, there exist distinct t^1 and t^2 in S such that $\tau(t^1) = t'^1$ and $\tau(t^2) = t'^2$. Thus since S is a chain, $t^1 \prec t^2$ without loss of generality. Hence $\tau(t^1) \prec' \tau(t^2)$ by part (d). Hence $t'^1 \prec' t'^2$ by the definition of t^1 and t^2.

(g). Take any $Z \in \mathcal{Z}_{\mathsf{inft}}$. Since Z is an infinite chain in T, part (f) implies that $\tau(Z)$ is an infinite chain in T'. Thus by Lemma A.1(f) applied to (T', p') at $S' = \tau(Z)$, there exists $Z' \in \mathcal{Z}'_{\mathsf{inft}}$ such that $\tau(Z) \subseteq Z'$.

(h). Take any $Z \in \mathcal{Z}_{\mathsf{ft}}$. Since Z is a chain in T, part (f) implies that $\tau(Z)$ is a chain in T'. Thus by Lemma A.1(g) applied to (T', p') at $S' = \tau(Z)$, there exists $Z' \in \mathcal{Z}'_{\mathsf{ft}} \cup \mathcal{Z}'_{\mathsf{inft}}$ such that $\tau(Z) \subseteq Z'$. ■

A.3. The Category

Proof A.5. (for Theorem 2.5) This paragraph notes that, for every functioned tree (T, p), the triple $[(T, p), (T, p), \mathsf{id}_T]$ is a morphism. (8a) and (8b) hold by inspection. (8c) holds with equality.

This paragraph demonstrates that, if $\gamma = [(T, p), (T', p'), \tau]$ and $\gamma' = [(T', p'), (T'', p''), \tau']$ are morphisms, then $\gamma' \circ \gamma = [(T, p), (T'', p''), \tau' \circ \tau]$ is a morphism. Toward that end, take any such γ and γ'. (8a) for $\gamma' \circ \gamma$ follows immediately from (8a) for γ and (8a) for γ'. For different reasons, (8b) for $\gamma' \circ \gamma$ follows immediately from (8b) for γ and (8b) for γ'. Finally, to show (8c) for $\gamma' \circ \gamma$, I argue

$$\begin{aligned} &\{(\tau' \circ \tau(t^\sharp), \tau' \circ \tau(t)) \mid (t^\sharp, t) \in p\} \\ &\quad = \{(\tau'(t'^\sharp), \tau'(t')) \mid (t'^\sharp, t') \in \{(\tau(t^\sharp), \tau(t)) | (t^\sharp, t) \in p\}\} \end{aligned}$$

$$\begin{aligned}&\subseteq \{(\tau'(t'^\sharp),\tau'(t')) \mid (t'^\sharp,t'){\in}p'\}\\&\subseteq p''.\end{aligned}$$

The equality is a rearrangement. The first inclusion holds by (8c) for γ, and the second inclusion holds by (8c) for γ'.

The first paragraph of this proof shows that the identity arrow $\mathsf{id}_{(T,p)}$ is well-defined for any functioned tree (T,p). The second paragraph shows that the composition $\gamma'{\circ}\gamma$ is well-defined for any morphisms γ and γ'. The unit and associative laws are immediate. Thus **Tree** is a category (e.g. [5, Section 1.3]). ■

LEMMA A.6. *Suppose $\gamma = [(T,p),(T',p'),\tau]$ is an isomorphism. Then (a) τ is bijective and (b) $\gamma^{-1} = [(T',p'),(T,p),\tau^{-1}]$.*

PROOF. Since $\gamma = [(T,p),(T',p'),\tau]$ is an isomorphism (e.g. [5, p. 12]), its inverse $\gamma^{-1} = [(T',p'),(T,p),\tau^*]$ exists. Thus

$$\begin{aligned}&[(T',p'),(T,p),\tau^*]{\circ}[(T,p),(T',p'),\tau] &(30a)\\&\quad = \gamma^{-1}{\circ}\gamma = \mathsf{id}_{(T,p)} = [(T,p),(T,p),\mathsf{id}_T], \text{ and}\\&[(T,p),(T',p'),\tau]{\circ}[(T',p'),(T,p),\tau^*] &(30b)\\&\quad = \gamma{\circ}\gamma^{-1} = \mathsf{id}_{(T',p')} = [(T',p'),(T',p'),\mathsf{id}_{T'}],\end{aligned}$$

where the first two equalities in both lines follow from the definition of the inverse $\gamma^{-1} = [(T',p'),(T,p),\tau^*]$, and where the third equality in both lines follows from the definition of id. The third component of (30a) implies that $\tau^*{\circ}\tau = \mathsf{id}_T$. The third component of (30b) implies that $\tau{\circ}\tau^* = \mathsf{id}_{T'}$. The last two sentences imply that τ is a bijection and that $\tau^* = \tau^{-1}$. Conclusion (a) follows from the previous sentence. Conclusion (b) holds because

$$\gamma^{-1} = [(T',p'),(T,p),\tau^*] = [(T',p'),(T,p),\tau^{-1}],$$

where the first equality holds by definition, and where the second equality holds because $\tau^* = \tau^{-1}$ by the second-previous sentence. ■

LEMMA A.7. *Suppose that $\gamma = [(T,p),(T',p'),\tau]$ is a morphism and that τ is bijective. Then γ is an isomorphism.*

PROOF. Define $\gamma^* = [(T',p'),(T,p),\tau^{-1}]$.

This paragraph shows that γ^* is a morphism. Specifically, it shows that

$$(T',p') \text{ and } (T,p) \text{ are functioned trees,} \tag{31a}$$

$$\tau^{-1} : T' \to T, \text{ and} \tag{31b}$$

$$\{\ (\tau^{-1}(t'^\sharp),\tau^{-1}(t')) \mid (t'^\sharp,t'){\in}p'\ \} \ \subseteq\ p. \tag{31c}$$

(31a) follows from (8a) for γ. (31b) follows from (8b) for γ and the bijectivity of τ. To show (31c), take any $(t'^\sharp,t') \in p'$. For notational ease, define $t^\sharp = \tau^{-1}(t'^\sharp)$ and $t = \tau^{-1}(t')$. Thus it suffices to show that $(t^\sharp,t) \in p$, or equivalently, that

$$p(t^\sharp) = t. \tag{32}$$

First, I argue

$$t^\sharp \;\neq\; t^o. \tag{33}$$

If $t^\sharp = t^o$ were true, [a] $t^\sharp \preccurlyeq t$ since t^o precedes every element of T, thus [b] $\tau(t^\sharp) \preccurlyeq' \tau(t)$ by γ being a morphism and Proposition 2.4(e), and thus [c] $t'^\sharp \preccurlyeq' t'$ by the definitions of $t^\sharp$ and t. This would contradict $t'^\sharp \succ t'$ which follows from the assumption that $p'(t'^\sharp) = t'$. Second, I argue

$$\tau{\circ}p(t^\sharp) = p'{\circ}\tau(t^\sharp) = p'(t'^\sharp) = t'. \tag{34}$$

The first equality follows from γ being a morphism, from (9b) of Proposition 2.3, and from (33). The second equality follows from the definition of $t^\sharp$. The third holds by assumption. Finally, I argue that (32) holds:

$$p(t^\sharp) = \tau^{-1}(t') = t.$$

The first equality follows from applying τ^{-1} to both sides of (34). The second equality is the definition of t.

Finally,

$$\gamma^*{\circ}\gamma = [(T',p'),(T,p),\tau^{-1}]{\circ}[(T,p),(T',p'),\tau] = \mathsf{id}_{(T,p)} \text{ and}$$
$$\gamma{\circ}\gamma^* = [(T,p),(T',p'),\tau]{\circ}[(T',p'),(T,p),\tau^{-1}] = \mathsf{id}_{(T',p')}.$$

Thus γ is an isomorphism (and $\gamma^{-1} = \gamma^*$). ■

Proof A.8. (for Theorem 2.6) Lemma A.6 establishes [a] the forward direction of the theorem's second sentence and [b] the theorem's third sentence. Lemma A.7 establishes the reverse direction of the theorem's second sentence. ■

Lemma A.9. *Suppose $\gamma = [(T,p),(T',p'),\tau]$ is an isomorphism, where (T,p) determines $\mathcal{Z} = \mathcal{Z}_{\mathsf{ft}}{\cup}\mathcal{Z}_{\mathsf{inft}}$ and where (T',p') determines $\mathcal{Z}' = \mathcal{Z}'_{\mathsf{ft}}{\cup}\mathcal{Z}'_{\mathsf{inft}}$. Then $(\forall Z{\in}\mathcal{Z})\ \tau(Z) \in \mathcal{Z}'$.*

Proof. By Theorem 2.6, τ is bijective and $\gamma^{-1} = [(T',p'),(T,p),\tau^{-1}]$. These facts will be used implicitly.

Take any $Z \in \mathcal{Z}$. Then by Proposition 2.4(f) applied to γ at $S = Z$, $\tau(Z)$ is a chain. Hence it remains to be shown that $\tau(Z)$ is maximal. Suppose not. Then there is $t' \notin \tau(Z)$ such that $\tau(Z){\cup}\{t'\}$ is a chain. By Proposition 2.4(f) applied to γ^{-1} at $S' = \tau(Z){\cup}\{t'\}$, $\tau^{-1}(\tau(Z){\cup}\{t'\}) = Z{\cup}\{\tau^{-1}(t')\}$ is a chain. Note $\tau^{-1}(t') \notin Z$ because $\tau^{-1}(t') \in Z$ would imply $t' \in \tau(Z)$ in contradiction to the definition of t'. The last two sentences contradict the maximality of Z. ■

Proof A.10. (for Proposition 2.7) Let γ denote $[(T,p),(T',p'),\tau]$. Theorem 2.6 implies

$$\tau \text{ is a bijection from } T \text{ onto } T', \text{ and} \tag{35a}$$
$$\gamma^{-1} = [(T',p'),(T,p),\tau^{-1}]. \tag{35b}$$

These facts will sometimes be used implicitly.

(a). By Proposition 2.3 (second half of (9a)), $\tau|_X$ is a well-defined function from X into X'. It is injective by (35a). To show it is surjective, take any $t' \in X'$. By (35b) and by Proposition 2.3 (second half of (9a)) for γ^{-1}, $\tau^{-1}(t') \in X$. Thus $\tau\circ\tau^{-1}(t') = t'$ is in the range of $\tau|_X$.

(b). By Proposition 2.3 (first half of (9a)), $\tau|_{T\smallsetminus\{t^o\}}$ is a well-defined function from $T\smallsetminus\{t^o\}$ into $T'\smallsetminus\{t'^o\}$. It is injective by (35a). To show it is surjective, take any $t' \in T'\smallsetminus\{t'^o\}$. By (35b) and by Proposition 2.3 (first half of (9a)) for γ^{-1}, $\tau^{-1}(t') \in T\smallsetminus\{t^o\}$. Thus $\tau\circ\tau^{-1}(t') = t'$ is in the range of $\tau|_{T\smallsetminus\{t^o\}}$.

(c). This follows immediately from (35a) and part (b).

(d). By part (c) and by the definition of $k'(t'^o)$, $k'(\tau(t^o)) = k'(t'^o) = 0$. Thus by Proposition 2.4(c), $k'(\tau(t)) = k(t) + k'(\tau(t^o)) = k(t)$.

(e). By (8c) for γ, $(\tau,\tau)|_p$ is a well-defined function from p into p'. It is injective by (35a). To show it is surjective, take any $(t'^\sharp, t') \in p'$. By (35b) and by (8c) for γ^{-1}, $(\tau^{-1},\tau^{-1})|_{p'}$ is a well-defined function from p' into p. Hence

$$(\tau^{-1}(t'^\sharp), \tau^{-1}(t')) \in p.$$

Thus $(\tau,\tau)(\tau^{-1}(t'^\sharp), \tau^{-1}(t')) = (t'^\sharp, t')$ is in the range of $(\tau,\tau)|_p$.

(f). Proposition 2.4(d) implies that $(\tau,\tau)|_\prec$ is a well-defined function from $\prec$ into $\prec'$. It is injective by (35a). To show it is surjective, take any $(t'^1, t'^2) \in\ \prec'$. By (35b) and by Proposition 2.4(d) for γ^{-1},

$$(\tau^{-1}(t'^1), \tau^{-1}(t'^2)) \in\ \prec .$$

Thus $(\tau,\tau)(\tau^{-1}(t'^1), \tau^{-1}(t'^2)) = (t'^1, t'^2)$ is in the range of $(\tau,\tau)|_\prec$.

(g). This proof is similar to that of the previous part. Replace $\prec$ with $\preccurlyeq$, and replace Proposition 2.4(d) with Proposition 2.4(e).

(h)–(i). Let $\mathcal{Z} = \mathcal{Z}_{\mathsf{ft}} \cup \mathcal{Z}_{\mathsf{inft}}$ and $\mathcal{Z}' = \mathcal{Z}'_{\mathsf{ft}} \cup \mathcal{Z}'_{\mathsf{inft}}$. Since τ is a bijection, the cardinality of S equals the cardinality of $\tau(S)$ for any set $S \subseteq T$. Thus it suffices for both parts (h) and (i) to show that $\tau|_{\mathcal{Z}}$ is a bijection from $\mathcal{Z}$ onto $\mathcal{Z}'$.

Lemma A.9 implies that $\tau|_{\mathcal{Z}}$ is a well-defined function from $\mathcal{Z}$ into $\mathcal{Z}'$. It is injective because τ is injective. To show that it is surjective, take any $Z' \in \mathcal{Z}'$. By Lemma A.9 applied to α^{-1}, $\tau^{-1}(Z') \in \mathcal{Z}$. Thus $\tau(\tau^{-1}(Z')) = Z'$ is in the range of $\tau|_{\mathcal{Z}}$.

(j). Take any $Z \in \mathcal{Z}_{\mathsf{inft}}$. The expression $E'[\tau(Z)]$ is well-defined because $\tau(Z) \in \mathcal{Z}'_{\mathsf{inft}}$ by part (i). Now take any $v \geq 1$. By the definition of E, $E[Z]^v$ is a stage-v member of Z. Thus by part (d), $\tau(E[Z]^v)$ is a stage-v member of $\tau(Z)$. Thus by the definition of E', $\tau(E[Z]^v)$ equals $E'[\tau(Z)]^v$. ■

Appendix B. Between Tree and $\mathbf{Grph_{ca}}$

This entire appendix is concerned with the proof of Theorem 2.8.

LEMMA B.1. *Suppose that* (T, p) *is a functioned tree. Then* $\mathsf{G}_0(T, p)$ *is a nontrivial converging arborescence, where* G_0 *is defined in Theorem 2.8.*

PROOF. Derive t^o from (T, p). Let $(T, E, \mathsf{init}, \mathsf{ter}) = \mathsf{G}_0(T, p)$. I will show that $(T, E, \mathsf{init}, \mathsf{ter})$ is a nontrivial converging arborescence in four steps.

Step 1 This step shows that (T, E) is a nontrivial graph. By remark (ii) in the paragraph following the definition (1) of a functioned tree, $(\nexists t)\ p(t) = t$. This, and the nonemptiness of p (1a), imply that E is a nonempty collection of two-element subsets of T. Thus by the definitions of [6, p. 2], (T, E) is a nontrivial graph.

Step 2 This step shows that (T, E) is a nontrivial tree. By Step 1 and [6, Theorem 1.5.1], it suffices to show that (T, E) is minimally connected. In other words, it suffices to show that [a] (T, E) is connected ([6, p. 10]) and [b] $(\forall e{\in}E)\ (T, E\smallsetminus\{e\})$ is not connected.

For [a], take any distinct t^A and t^B. If either t^A or t^B is t^o, (1b) and the definition of E imply the existence of a path linking t^A and t^B.

If neither t^A nor t^B is t^o, two applications of (1b) imply the existence of $m^A \geq 1$ and $m^B \geq 1$ such that $p^{m^A}(t^A) = t^o$ and $p^{m^B}(t^B) = t^o$. Let

$$m^* = \max\{m{\leq}\min\{m^A, m^B\} \mid p^{m^A-m}(t^A) = p^{m^B-m}(t^B)\},$$

where p^0 is the identity function. This m^* exists because the set in its definition contains 0 by the definitions of m^A and m^B. Let $t^* = p^{m^A-m^*}(t^A)$, which equals $p^{m^B-m^*}(t^B)$ by the definition of m^*. [It is true, but irrelevant, that $p^{m^*}(t^*) = t^o$.]

By the definition of E, there is a path[13]

$$(t^A, p(t^A), p^2(t^A), \ldots\ p^{m^A-m^*}(t^A) = t^*), \tag{36a}$$

and also a path

$$(t^B, p(t^B), p^2(t^B), \ldots\ p^{m^B-m^*}(t^B) = t^*). \tag{36b}$$

Suppose these two paths shared a node $t' \neq t^*$. Then $t' \neq t^*$ being on (36a) would imply the existence of $n \geq 1$ such that

$$p^{m^A-(m^*+n)}(t^A) = t' \tag{37a}$$

and $p^n(t') = t^*$, where p^0 is the identity function. Further, $p^n(t') = t^*$ and t' being on (36b) would imply

$$p^{m^B-(m^*+n)}(t^B) = t'. \tag{37b}$$

Since $n \geq 1$, (37a) and (37b) contradict the definition of m^*. Hence the sets (36a) and (36b) do not share a node other than t^*. Hence the union of the two paths (36a) and (36b) is a path. This path links t^A and t^B.

[13] As in [6, p. 6, note 3], I denote a path by a natural sequence of its nodes. Pairs of nodes that are adjacent in the sequence are adjacent in the sense of [6, p. 3].

For [b], take any edge $e \in E$ and suppose that $(T, E\smallsetminus\{e\})$ is connected. The following four paragraphs will derive a contradiction with (1b).

First, by the definition of E, there is some $(t^1, t^2) \in p$ such that $e = \{t^1, t^2\}$. Since $(t^1, t^2) \in p$, (1a) implies

$$t^1 \in T\smallsetminus\{t^o\} \text{ and } p(t^1) = t^2. \tag{38}$$

Second, since $(T, E\smallsetminus\{e\})$ is connected, there is a path without $e = \{t^1, t^2\}$ that links t^1 and t^2. Denote this path[13] by

$$(t^2, t^3, \dots\, t^n = t^1). \tag{39}$$

Since this path does not use $e = \{t^1, t^2\}$,

$$t^{n-1} \;\neq\; t^2. \tag{40}$$

Thus $n \;\geq\; 4$ and the path has at least three nodes.

This paragraph argues that

$$(\forall k{\in}\{2, 3, \dots\, n{-}1\}) \;\; t^k \in T\smallsetminus\{t^o\} \;\text{ and }\; p(t^k) = t^{k+1}. \tag{41}$$

In particular, I will make an inductive argument in which k is decreasing rather than increasing. Consider the initial step ($k = n{-}1$). By (39), $\{t^{n-1}, t^n\} \in E$. Thus by the definition of E, either $(t^{n-1}, t^n) \in p$ or $(t^n, t^{n-1}) \in p$. Thus, since p is a function by (1a), either $p(t^{n-1}) = t^n$ or $p(t^n) = p^{n-1}$. The second contingency is the first equality in the contradiction

$$t^2 \;\neq\; t^{n-1} = p(t^n) = p(t^1) = t^2,$$

where the inequality is (40), the second equality holds because $t^n = t^1$ by (39), and the third equality holds by (38). Thus $p(t^{n-1}) = p^n$. Further $t^{n-1} \in T\smallsetminus\{t^o\}$ since the domain of p is $T\smallsetminus\{t^o\}$ by (1a). Next consider the inductive step ($k{\in}\{2, 3, \dots\, n{-}2\}$). By the definition of t^k in (39), $\{t^k, t^{k+1}\} \in E$. Thus by the definition of E, either $(t^k, t^{k+1}) \in p$ or $(t^{k+1}, t^k) \in p$. Thus since p is a function by (1a), either $p(t^k) = t^{k+1}$ or $p(t^{k+1}) = t^k$. The second contingency is precluded by [a] the inductive hypothesis that $p(t^{k+1}) = t^{k+2}$ and [b] the fact that $t^k \;\neq\; t^{k+2}$ because the nodes of any path are distinct. Thus $p(t^k) = t^{k+1}$. Further $t^k \in T\smallsetminus\{t^o\}$ since the domain of p is $T\smallsetminus\{t^o\}$ by (1a).

This paragraph argues that

$$(\forall i{\in}\{0, 1, \dots\, n{-}1\}) \; p^i(t^1) = t^{1+i}, \tag{42}$$

where p^0 is the identity. The case $i = 0$ is trivial. The case $i = 1$ holds by (38). For any $i \in \{2, 3, \dots\, n{-}1\}$,

$$p^i(t^1) = p^{i-1}(t^2) = p^{i-2}(t^3) = \cdots = p(t^i) = t^{i+1},$$

where the first equality holds by (38) and the remaining $i{-}1$ equalities holds by (41).

Note that

$$p^{n-1}(t^1) = t^n = t^1,$$

where the first equality holds by (42) at $i = n{-}1$, and where the second equality holds by (39). Thus

$$(\forall m \geq 1)\ p^m(t^1) = p^{(m \bmod (n-1))}(t^1) = t^{(m \bmod (n-1))+1} \in T \smallsetminus \{t^o\},$$

where the first equality holds by the previous sentence, where the second equality holds by (42), and where the set membership holds by (38) [for $(m \bmod (n{-}1))+1 = 1$] and (41) [for $(m \bmod (n{-}1))+1 \in \{2,3,\ldots\ n{-}1\}$]. This contradicts (1b).

Step 3. This step shows (10a). In other words, it shows that $(T, E, \mathsf{init}, \mathsf{ter})$ is a nontrivial oriented tree. By Step 2, it suffices to show that $(T, E, \mathsf{init}, \mathsf{ter})$ is oriented ([6, p. 28]).

As a preliminary observation, this paragraph shows that p is asymmetric in the sense that

$$(\nexists \{t^A, t^B\} {\in} T)\ \{(t^A, t^B), (t^B, t^A)\}\ \subseteq\ p. \tag{43}$$

Suppose there were such t^A and t^B. Then (1a) implies that [a] neither t^A nor t^B equals t^o, [b] $p(t^A) = t^B$, and [c] $p^2(t^A) = p{\circ}p(t^A) = p(t^B) = t^A$. These three observations imply that $t^A \neq t^o$ and that there does not exist an $m \geq 1$ such that $p^m(t^A) = t^o$. This contradicts (1b).

It must be shown that [a] $\mathsf{init}{:}E{\to}T$ and $\mathsf{ter}{:}E{\to}T$ are well-defined functions and [b] $(\forall e{\in}E)\ \{\mathsf{init}(e), \mathsf{ter}(e)\} = e$. To show [a], take any $e \in E$. By the definition of E, there is a $(t^A, t^B) \in p$ such that $e = \{t^A, t^B\}$. Thus, by the definition of init, $(e, t^A) \in \mathsf{init}$. Further, by the definition of E and (43), there can be no more than one $(t^A, t^B) \in p$ such that $e = \{t^A, t^B\}$. Hence by the definition of init, there can be no more than one t such that $(e, t) \in \mathsf{init}$. A similar argument shows that ter is a well-defined function. To show [b], take any $e \in E$. By the definition of E, there is a $(t^A, t^B) \in p$ such that $e = \{t^A, t^B\}$. Thus by the definitions of init and ter, $\{\mathsf{init}(e), \mathsf{ter}(e)\} = \{t^A, t^B\} = e$.

Step 4. This step shows (10b). In other words, it shows that $(\forall e{\in}E)\ \mathsf{init}(e) \neq t^o$ and $\mathsf{ter}(e)$ is on the path linking $\mathsf{init}(e)$ and t^o.

Take any $e \in E$. By the definitions of E, init, and ter, there exists $(t^A, t^B) \in p$ such that [a] $e = \{t^A, t^B\}$, [b] $\mathsf{init}(e) = t^A$, and [c] $\mathsf{ter}(e) = t^B$. Note that $t^A \neq t^o$ by the definition of (t^A, t^B) and by (1a). Thus [b] implies $\mathsf{init}(e) \neq t^o$. Further, since $t^A \neq t^o$, (1b) implies the existence of an $m \geq 1$ such that the path linking t^A and t^o is

$$(t^A, p(t^A), p^2(t^A), \ldots\ p^m(t^A) = t^o).$$

Thus $p(t^A)$ is on the path linking p^A and t^o. Note $p(t^A) = t^B = \mathsf{ter}(e)$ by the definition of (t^A, t^B) and by [c]. Also note $p^A = \mathsf{init}(e)$ by [b]. Thus by substitution, the last three sentences imply that $\mathsf{ter}(e)$ is on the path linking $\mathsf{init}(e)$ and t^o. ■

Lemma B.2. *Suppose that* $[(T,p),(T',p'),\tau]$ *is a functioned-tree morphism. Then* $\mathsf{G}_1([(T,p),(T',p'),\tau])$ *is a directed-graph morphism, where* G_1 *is defined in Theorem 2.8. Further, its source and target are nontrivial converging arborescences.*

Proof. By the definitions of G_1 and G_0,

$$\begin{aligned}&\mathsf{G}_1([(T,p),(T',p'),\tau])\\ &\quad= [\mathsf{G}_0(T,p),\mathsf{G}_0(T',p'),\tau,\varepsilon]\\ &\quad= [(T,E,\mathsf{init},\mathsf{ter}),(T',E',\mathsf{init}',\mathsf{ter}'),\tau,\varepsilon],\end{aligned}$$

where

$$\varepsilon = \{(\{t^\sharp,t\},\{\tau(t^\sharp),\tau(t)\}) \mid (t^\sharp,t)\in p\}, \tag{44a}$$

$$E = \{\{t^\sharp,t\} \mid (t^\sharp,t)\in p\}, \tag{44b}$$

$$\mathsf{init} = \{(\{t^\sharp,t\},t^\sharp) \mid (t^\sharp,t)\in p\}, \tag{44c}$$

$$\mathsf{ter} = \{(\{t^\sharp,t\},t) \mid (t^\sharp,t)\in p\}, \tag{44d}$$

$$E' = \{\{t'^\sharp,t'\} \mid (t'^\sharp,t')\in p'\}, \tag{44e}$$

$$\mathsf{init}' = \{(\{t'^\sharp,t'\},t'^\sharp) \mid (t'^\sharp,t')\in p'\}, \text{ and} \tag{44f}$$

$$\mathsf{ter}' = \{(\{t'^\sharp,t'\},t') \mid (t'^\sharp,t')\in p'\}. \tag{44g}$$

It must be shown [a] that $[(T,E,\mathsf{init},\mathsf{ter}),(T',E',\mathsf{init}',\mathsf{ter}'),\tau,\varepsilon]$ satisfies (12a)–(12c), and [b] that both $(T,E,\mathsf{init},\mathsf{ter})$ and $(T',E',\mathsf{init}',\mathsf{ter}')$ are nontrivial converging arborescences. Both (12a) and [b] follow from (8a) and Lemma B.1.

The first half of (12b) follows from (8b). To show the second half of (12b), it must be shown [a] that ε is a function, [b] that its domain is E, and [c] that its range $\varepsilon(E)$ is a subset of E'. [a] holds by (44a) and the fact that τ is a function. [b] holds by (44a) and (44b). [c] holds by

$$\begin{aligned}\varepsilon(E) &= \{\{\tau(t^\sharp),\tau(t)\} \mid (t^\sharp,t)\in p\}\\ &= \{\{t'^\sharp,t'\} \mid (t'^\sharp,t') \in \{\,(\tau(t^\sharp),\tau(t)) \,|\, (t^\sharp,t)\in p\,\}\}\\ &\subseteq \{\{t'^\sharp,t'\} \mid (t'^\sharp,t') \in p'\}\\ &= E',\end{aligned}$$

where the first equality holds by (44a)–(44b), the second equality is a rearrangement, the set inclusion holds by (8c), and the third equality holds by (44e).

To see the first half of (12c), take any $e \in E$. By (44b), there is a $(t^\sharp,t) \in p$ such that $e = \{t^\sharp,t\}$. Thus by (44c), $\mathsf{init}(e) = t^\sharp$. Hence

$$\tau\circ\mathsf{init}(e) = \tau(t^\sharp).$$

Further, by (44a), $\varepsilon(e) = \{\tau(t^\sharp),\tau(t)\}$. Because $(t^\sharp,t) \in p$, (8c) implies $(\tau(t^\sharp),\tau(t)) \in p'$. Hence

$$\mathsf{init}'\circ\varepsilon(e) = \mathsf{init}'(\{\tau(t^\sharp),\tau(t)\}) = \tau(t^\sharp),$$

where the first equality holds by the second-previous sentence, and where the second equality holds by the previous sentence and (44f). By this paragraph's two centered equations, $\tau\circ\mathsf{init}(e) = \mathsf{init}'\circ\varepsilon(e)$.

Similarly, to see the second half of (12c), take any $e \in E$. By (44b), there is a $(t^\sharp, t) \in p$ such that $e = \{t^\sharp, t\}$. Thus by (44d), $\mathsf{ter}(e) = t$. Hence

$$\tau\circ\mathsf{ter}(e) = \tau(t).$$

Further, by (44a), $\varepsilon(e) = \{\tau(t^\sharp), \tau(t)\}$. Because $(t^\sharp, t) \in p$, (8c) implies $(\tau(t^\sharp), \tau(t)) \in p'$. Hence

$$\mathsf{ter}'\circ\varepsilon(e) = \mathsf{ter}'(\{\tau(t^\sharp), \tau(t)\}) = \tau(t),$$

where the first equality holds by the second-previous sentence, and where the second equality holds by the previous sentence and (44g). By this paragraph's two centered equations, $\tau\circ\mathsf{ter}(e) = \mathsf{ter}'\circ\varepsilon(e)$. ■

LEMMA B.3. *Theorem 2.8's* G *is a well-defined functor from* **Tree** *to* $\mathbf{Grph_{ca}}$.

PROOF. By Lemma B.1, G_0 maps objects of **Tree** to objects of $\mathbf{Grph_{ca}}$. By Lemma B.2, G_1 maps arrows of **Tree** to arrows of $\mathbf{Grph_{ca}}$. It remains to show [1] that G preserves sources and targets, [2] that G preserves identities, and [3] that G preserves compositions.

[1]. Take any **Tree** morphism $[(T, p), (T', p'), \tau]$. To show that G preserves sources, I argue

$$\begin{aligned}
&\mathsf{G}_1([(T, p), (T', p'), \tau])^{\mathsf{src}} \\
&\quad = [\mathsf{G}_0(T, p), \mathsf{G}_0(T', p'), \tau, \varepsilon]^{\mathsf{src}} \\
&\quad = \mathsf{G}_0(T, p) \\
&\quad = \mathsf{G}_0([(T, p), (T', p'), \tau]^{\mathsf{src}}),
\end{aligned}$$

where ε is defined in Theorem 2.8. The first equality holds by the definition of G_1, the second by the definition of src in $\mathbf{Grph_{ca}}$, and the third by the definition of src in **Tree**. A symmetric argument shows that G preserves targets.

[2]. Take any **Tree** object (T, p). As a preliminary step, note that the definition of G_0 implies

$$\mathsf{G}_0(T, p) = (T, E, \mathsf{init}, \mathsf{ter}), \tag{45}$$

where

$$E = \{\{t^\sharp, t\} \mid (t^\sharp, t) \in p\}, \tag{46}$$

and where init and ter are also defined in the definition of G_0 but not needed explicitly here. I argue that

$$\begin{aligned}
&\mathsf{G}_1(\mathsf{id}_{(T,p)}) \\
&\quad = \mathsf{G}_1([(T, p), (T, p), \mathsf{id}_T]) \\
&\quad = [(T, E, \mathsf{init}, \mathsf{ter}), (T, E, \mathsf{init}, \mathsf{ter}), \mathsf{id}_T, \varepsilon]
\end{aligned} \tag{47}$$

$$= [(T, E, \mathsf{init}, \mathsf{ter}), (T, E, \mathsf{init}, \mathsf{ter}), \mathsf{id}_T, \mathsf{id}_E]$$
$$= \mathsf{id}_{\mathsf{G}_0(T,p)},$$

where

$$\varepsilon = \{(\{t^\sharp, t\}, \{\mathsf{id}_T(t^\sharp), \mathsf{id}_T(t)\}) \mid (t^\sharp, t)\in p\}. \tag{48}$$

The first equality in (47) holds by the definition of id in **Tree**. The second equality in (47) holds by the definition of G_1 and by (45). For the third equality in (47), it suffices to show that $\varepsilon = \mathsf{id}_E$. This follows from inspecting (46) and (48). The fourth equality in (47) holds by the definition of id in $\mathbf{Grph_{ca}}$ and by (45).

[3]. Let $[(T, p), (T', p'), \tau]$ and $[(T', p'), (T'', p''), \tau']$ be any two **Tree** morphisms. As a preliminary step, note that the definition of G_0 implies

$$\begin{aligned} \mathsf{G}_0(T, p) &= (T, E, \mathsf{init}, \mathsf{ter}), \\ \mathsf{G}_0(T', p') &= (T', E', \mathsf{init}', \mathsf{ter}'), \text{ and} \\ \mathsf{G}_0(T'', p'') &= (T'', E'', \mathsf{init}'', \mathsf{ter}''), \end{aligned} \tag{49}$$

where

$$E = \{\{t^\sharp, t\} \mid (t^\sharp, t)\in p\}, \tag{50}$$

and where init, ter, E', init', ter', E'', init'', and ter'' are also derived from the definition of G_0 but not needed explicitly here.

I argue that

$$\begin{aligned} &\mathsf{G}_1([(T', p'), (T'', p''), \tau'] \circ [(T, p), (T', p'), \tau]) \\ &= \mathsf{G}_1([(T, p), (T'', p''), \tau'\circ\tau]) \\ &= [(T, E, \mathsf{init}, \mathsf{ter}), (T'', E'', \mathsf{init}'', \mathsf{ter}''), \tau'\circ\tau, \varepsilon^*] \\ &= [(T, E, \mathsf{init}, \mathsf{ter}), (T'', E'', \mathsf{init}'', \mathsf{ter}''), \tau'\circ\tau, \varepsilon'\circ\varepsilon] \\ &= [(T', E', \mathsf{init}', \mathsf{ter}'), (T'', E'', \mathsf{init}'', \mathsf{ter}''), \tau', \varepsilon'] \\ &\quad \circ [(T, E, \mathsf{init}, \mathsf{ter}), (T', E', \mathsf{init}', \mathsf{ter}'), \tau, \varepsilon] \\ &= \mathsf{G}_1([(T', p'), (T'', p''), \tau']) \circ \mathsf{G}_1([(T, p), (T', p'), \tau]), \end{aligned} \tag{51}$$

where

$$\varepsilon^* = \{(\{t^\sharp, t\}, \{\tau'\circ\tau(t^\sharp), \tau'\circ\tau(t)\}) \mid (t^\sharp, t)\in p\}, \tag{52a}$$
$$\varepsilon = \{(\{t^\sharp, t\}, \{\tau(t^\sharp), \tau(t)\}) \mid (t^\sharp, t)\in p\}, \text{ and} \tag{52b}$$
$$\varepsilon' = \{(\{t'^\sharp, t'\}, \{\tau'(t'^\sharp), \tau'(t')\}) \mid (t'^\sharp, t')\in p'\}. \tag{52c}$$

The first equality in (51) holds by the definition of $\circ$ in **Tree**. The second equality holds by the definition of G_1 and by (49). The third equality will be proved in the following paragraph. The fourth equality holds by the definition of $\circ$ in $\mathbf{Grph_{ca}}$. The fifth equality holds by two applications of the definition of G_1 and by (49).

For the third equality in (51), it suffices to show that $\varepsilon^* = \varepsilon'\circ\varepsilon$. Toward that end, take any e in E. By (50), there exists $(t^\sharp, t) \in p$ such that $e = \{t^\sharp, t\}$. Thus

(52b) implies $\varepsilon(e) = \{\tau(t^\sharp), \tau(t)\}$. Because $(t^\sharp, t) \in p$, (8c) implies $(\tau(t^\sharp), \tau(t)) \in p'$. Hence

$$\varepsilon'\circ\varepsilon(e) = \varepsilon'(\{\tau(t^\sharp), \tau(t)\}) = \{\tau'\circ\tau(t^\sharp), \tau'\circ\tau(t)\} = \varepsilon^*(e),$$

where the first equality holds by the second-previous sentence, where the second equality holds by the previous sentence and (52c), and where the third equality holds by [a] (52a), [b] $(t^\sharp, t) \in p$, and [c] $e = \{t^\sharp, t\}$. ■

LEMMA B.4. *Suppose* $(T, E, \mathsf{init}, \mathsf{ter})$ *is a nontrivial converging arborescence. Then* $\mathsf{H}_0(T, E, \mathsf{init}, \mathsf{ter})$ *is a functioned tree, where* H_0 *is defined in Theorem 2.8.*

PROOF. By the definition (10) of a nontrivial converging arborescence, there exists a t^o such that

$$(T, E, \mathsf{init}, \mathsf{ter}) \text{ is a nontrivial oriented tree and} \tag{53a}$$

$$(\forall e)\ \mathsf{init}(e) \neq t^o \text{ and } \mathsf{ter}(e) \text{ is on path linking} \mathsf{init}(e) \text{and } t^o. \tag{53b}$$

Let $(T, p) = \mathsf{H}_0(T, E, \mathsf{init}, \mathsf{ter})$. Then by the definition of H_0,

$$p = \{(\mathsf{init}(e), \mathsf{ter}(e)) \mid e \in E\}. \tag{53c}$$

I show that (T, p) is a functioned tree in four steps. During these steps, (53a) is often used implicitly, while (53b) and (53c) are used explicitly.

Step 1 This step proves the following lemma: "Take any $t \neq t^o$ and let $(t, t^1, t^2, \dots t^m = t^o)$ be the path linking t and t^o. Then $(t, t^1) \in p$."

To prove this lemma, take any $t \neq t^o$ and let $(t, t^1, t^2, \dots t^m = t^o)$ be the path linking t and t^o. Then $\{t, t^1\} \in E$. Thus by (11),

$$\big(\mathsf{init}(\{t, t^1\}),\ \mathsf{ter}(\{t, t^1\})\big) = (t, t^1) \text{ or}$$
$$\big(\mathsf{init}(\{t, t^1\}),\ \mathsf{ter}(\{t, t^1\})\big) = (t^1, t).$$

The latter, together with the second half of (53b) at $e = \{t, t^1\}$, would imply that t is on the path linking t^1 and t^o. But this is impossible because [a] the path linking t^1 and t^o is the subpath $(t^1, t^2, \dots t^n = t^o)$ and [b] t is not on this subpath since it is on the full path $(t, t^1, t^2, \dots t^n = t^o)$. Thus the former alternative holds. This equality leads to

$$(t, t^1) = \big(\mathsf{init}(\{t, t^1\}),\ \mathsf{ter}(\{t, t^1\})\big) \in p,$$

where the set membership follows from $\{t, t^1\} \in E$ and (53c).

Step 2 The following three paragraphs show $p{:}T \smallsetminus \{t^o\} \to T$.

This paragraph shows that $(\nexists t^1)\ (t^o, t^1) \in p$. Suppose there were such a t^1. By (53c), this would imply $\{t^o, t^1\} \in E$ and $\mathsf{init}(\{t^o, t^1\}) = t^o$. This would contradict the first half of (53b) at $e = \{t^o, t^1\}$.

This paragraph shows that $(\forall t \neq t^o)(\exists t^1)\ (t, t^1) \in p$. Take any $t \neq t^o$. Let $(t, t^1, t^2, \dots t^m = t^o)$ be the path linking t and t^o. Then $(t, t^1) \in p$ by the lemma of Step 1.

It remains to be shown that $(\forall t \neq t^o)(\nexists t^A \neq t^B)\ \{(t,t^A),(t,t^B)\} \subseteq p$. I will show this by contradiction. Toward this end, take any $t \neq t^o$ and suppose $t^A \neq t^B$ are such that $\{(t,t^A),(t,t^B)\} \subseteq p$. Then (53c) implies

$$\{t,t^A\} \in E \text{ and } (t,t^A) = \big(\mathsf{init}(\{t,t^A\}),\ \mathsf{ter}(\{t,t^A\})\big) \text{ and} \tag{54a}$$

$$\{t,t^B\} \in E \text{ and } (t,t^B) = \big(\mathsf{init}(\{t,t^B\}),\ \mathsf{ter}(\{t,t^B\})\big). \tag{54b}$$

Let $(t,t^1,t^2,\dots t^n = t^o)$ be the path linking t and t^o. Since $t \neq t^o$ by assumption, this path has at least two vertices and thus

$$t \neq t^1. \tag{55}$$

Further, the remainder of this paragraph shows that both t^A and t^B are on this path. Without loss of generality, consider t^A. By the first half of (54a), $\{t,t^A\} \in E$. Thus by the second half of (53b) at $e = \{t,t^A\}$, $\mathsf{ter}(\{t,t^A\})$ is on the path linking $\mathsf{init}(\{t,t^A\})$ and t^o. Thus by the second half of (54a), t^A is on the path linking t and t^o. Hence t^A is on $(t,t^1,t^2,\dots t^m = t^o)$.

Since $t^A \neq t^B$ by assumption, at least one of them is distinct from t^1. Without loss of generality, suppose $t^A \neq t^1$. Then consider the subpath $(t,t^1,t^2,\dots t^A)$. Since $t \neq t^1$ by (55), and since $t^1 \neq t^A$ by the second-previous sentence, this subpath has at least three nodes. This and the first half of (54a) imply the existence of a cycle ([6, p. 8]). This contradicts (T,E) being a tree ([6, p. 13]).

Step 3 This step shows that $(\forall t \neq t^o)(\exists m \geq 1)\ p^m(t) = t^o$.

Take any $t \neq t^o$. Let $(t,t^1,t^2,\dots t^m = t^o)$ be the path linking t and t^o. By the lemma of Step 1, $(t,t^1) \in p$. Thus since p is a function by Step 2,

$$p(t) = t^1. \tag{56}$$

Further, the remainder of this paragraph argues that

$$(\forall k \in \{1,2,\dots m{-}1\})\ p(t^k) = t^{k+1}. \tag{57}$$

Take any such k. Then $(t^k,t^{k+1},t^{k+2},\dots t^m = t^o)$ is the path linking t^k and t^o. Thus $(t^k,t^{k+1}) \in p$ by the lemma of Step 1. Thus $p(t^k) = t^{k+1}$ since p is a function by Step 2.

Finally, I argue

$$p^m(t) = p^{m-1}(t^1) = p^{m-2}(t^2) = \cdots = p(t^{m-1}) = t^m = t^o,$$

The first equality holds by (56). The last equality holds by the definition of t^m. The intervening equalities hold by (57).

Step 4 By Step 2, p is a function from $T \smallsetminus \{t^o\}$ onto X, where X is defined to be the range of p. Further, the nontriviality of (53a) implies that E is nonempty. Thus by (53c), p is nonempty. Hence (1a) has been established. (1b) was shown in Step 3. ∎

Lemma B.5. *Suppose* $[(T,E,\mathsf{init},\mathsf{ter}),(T',E',\mathsf{init},\mathsf{ter}),\tau,\varepsilon]$ *is a directed-graph morphism whose source and target are nontrivial converging arborescences. Then*

$\mathsf{H}_1([(T,E,\mathsf{init},\mathsf{ter}),(T',E',\mathsf{init},\mathsf{ter}),\tau,\varepsilon])$ *is a functioned-tree morphism, where* H_1 *is defined in Theorem 2.8.*

PROOF. By the definition of H_1,

$$\begin{aligned}&\mathsf{H}_1([(T,E,\mathsf{init},\mathsf{ter}),(T',E',\mathsf{init},\mathsf{ter}),\tau,\varepsilon])\\&\quad=[\mathsf{H}_0(T,E,\mathsf{init},\mathsf{ter}),\mathsf{H}_0(T',E',\mathsf{init}',\mathsf{ter}'),\tau].\end{aligned}$$

I will show $[\mathsf{H}_0(T,E,\mathsf{init},\mathsf{ter}),\mathsf{H}_0(T',E',\mathsf{init}',\mathsf{ter}'),\tau]$ satisfies (8a)–(8c). (8a) follows from Lemma B.4 and this lemma's assumption that $(T,E,\mathsf{init},\mathsf{ter})$ and $(T',E',\mathsf{init}',\mathsf{ter}')$ are nontrivial converging arborescences. (8b) is identical to the first half of (12b).

To show (8c), note that the definition of H_0 implies

$$[\mathsf{H}_0(T,E,\mathsf{init},\mathsf{ter}),\mathsf{H}_0(T',E',\mathsf{init}',\mathsf{ter}'),\tau]=[(T,p),(T',p'),\tau],$$

where

$$p=\{(\mathsf{init}(e),\mathsf{ter}(e))\mid e{\in}E\}\text{ and}\tag{58a}$$

$$p'=\{(\mathsf{init}'(e'),\mathsf{ter}'(e')\mid e'{\in}E'\}.\tag{58b}$$

I will show $[(T,p),(T',p'),\tau]$ satisfies (8c). Take any $(t^\sharp,t)\in p$. By (58a), there exists $e\in E$ such that

$$(t^\sharp,t)=(\mathsf{init}(e),\mathsf{ter}(e)).\tag{59}$$

By the first half of (12c) and the first component of (59),

$$\mathsf{init}'{\circ}\varepsilon(e)=\tau{\circ}\mathsf{init}(e)=\tau(t^\sharp).$$

Similarly, by the second half of (12c) and the second component of (59),

$$\mathsf{ter}'{\circ}\varepsilon(e)=\tau{\circ}\mathsf{ter}(e)=\tau(t).$$

Meanwhile, since $\varepsilon(e)\in E'$ by the second half of (12b), (58b) implies

$$(\mathsf{init}'{\circ}\varepsilon(e),\mathsf{ter}'{\circ}\varepsilon(e))\in p'.$$

By substitution, the last three sentences imply $(\tau(t^\sharp),\tau(t))\in p'$. ■

LEMMA B.6. *Theorem 2.8's* H *is a well-defined functor from* $\mathbf{Grph_{ca}}$ *to* $\mathbf{Tree}$.

PROOF. By Lemma B.4, H_0 maps objects of $\mathbf{Grph_{ca}}$ to objects of $\mathbf{Tree}$. By Lemma B.5, H_1 maps arrows of $\mathbf{Grph_{ca}}$ to arrows of $\mathbf{Tree}$. It remains to show [1] that H preserves sources and targets, [2] that H preserves identities, and [3] that H preserves compositions.

[1]. Take any $\mathbf{Grph_{ca}}$ arrow $[(T,E,\mathsf{init},\mathsf{ter}),(T',E',\mathsf{init}',\mathsf{ter}'),\tau,\varepsilon]$. To show that H preserves sources, note

$$\begin{aligned}&\mathsf{H}_1([(T,E,\mathsf{init},\mathsf{ter}),(T',E',\mathsf{init}',\mathsf{ter}'),\tau,\varepsilon])^{\mathsf{src}}\\&\quad=[\mathsf{H}_0(T,E,\mathsf{init},\mathsf{ter}),\mathsf{H}_0(T',E',\mathsf{init}',\mathsf{ter}'),\tau]^{\mathsf{src}}\\&\quad=\mathsf{H}_0(T,E,\mathsf{init},\mathsf{ter})\end{aligned}$$

$$= \mathsf{H}_0([(T, E, \mathsf{init}, \mathsf{ter}), (T', E', \mathsf{init}', \mathsf{ter}'), \tau, \varepsilon]^{\mathsf{src}}).$$

where the first equality holds by the definition of H_1, the second by the definition of src in **Tree**, and the third by the definition of src in $\mathbf{Grph_{ca}}$. A similar argument shows that H preserves targets.

[2]. Take any $\mathbf{Grph_{ca}}$ object $(T, E, \mathsf{init}, \mathsf{ter})$. Then

$$\begin{aligned}
&\mathsf{H}_1(\mathsf{id}_{(T,E,\mathsf{init},\mathsf{ter})}) \\
&\quad = \mathsf{H}_1([(T, E, \mathsf{init}, \mathsf{ter}), (T, E, \mathsf{init}, \mathsf{ter}), \mathsf{id}_T, \mathsf{id}_E]\ ) \\
&\quad = [\mathsf{H}_0(T, E, \mathsf{init}, \mathsf{ter}), \mathsf{H}_0(T, E, \mathsf{init}, \mathsf{ter}), \mathsf{id}_T] \\
&\quad = \mathsf{id}_{\mathsf{H}_0(T,E,\mathsf{init},\mathsf{ter})},
\end{aligned}$$

where the first equality holds by the definition of id in $\mathbf{Grph_{ca}}$, the second by the definition of H_1, and the third by the definition of id in **Tree**.

[3]. Take any two $\mathbf{Grph_{ca}}$ arrows $[(T, E, \mathsf{init}, \mathsf{ter}), (T', E', \mathsf{init}', \mathsf{ter}'),\ \tau, \varepsilon]$ and $[(T', E', \mathsf{init}', \mathsf{ter}'), (T'', E'', \mathsf{init}'', \mathsf{ter}''), \tau', \varepsilon']$. Then

$$\begin{aligned}
&\mathsf{H}_1([(T', E', \mathsf{init}', \mathsf{ter}'), (T'', E'', \mathsf{init}'', \mathsf{ter}''), \tau', \varepsilon'] \\
&\qquad\qquad \circ\ [(T, E, \mathsf{init}, \mathsf{ter}), (T', E', \mathsf{init}', \mathsf{ter}'), \tau, \varepsilon]) \\
&\quad = \mathsf{H}_1(\ [(T, E, \mathsf{init}, \mathsf{ter}), (T'', E'', \mathsf{init}'', \mathsf{ter}''), \tau'\circ\tau, \varepsilon'\circ\varepsilon]) \\
&\quad = [\mathsf{H}_0(T, E, \mathsf{init}, \mathsf{ter}), \mathsf{H}_0(T'', E'', \mathsf{init}'', \mathsf{ter}''), \tau'\circ\tau] \\
&\quad = [\mathsf{H}_0(T', E', \mathsf{init}', \mathsf{ter}'), \mathsf{H}_0(T'', E'', \mathsf{init}'', \mathsf{ter}''), \tau'] \\
&\qquad \circ [\mathsf{H}_0(T, E, \mathsf{init}, \mathsf{ter}), \mathsf{H}_0(T', E', \mathsf{init}', \mathsf{ter}'), \tau] \\
&\quad = \mathsf{H}_1([(T', E', \mathsf{init}', \mathsf{ter}'), (T'', E'', \mathsf{init}'', \mathsf{ter}''), \tau', \varepsilon']) \\
&\qquad \circ \mathsf{H}_1([(T, E, \mathsf{init}, \mathsf{ter}), (T', E', \mathsf{init}', \mathsf{ter}'), \tau, \varepsilon]),
\end{aligned}$$

where the first equality holds by the definition of $\circ$ in $\mathbf{Grph_{ca}}$, the second holds by the definition of H_1, the third holds by the definition of $\circ$ in **Tree**, and the fourth holds by the definition of H_1. ■

PROOF B.7. (for Theorem 2.8) Lemma B.3 shows that G is a well-defined operator from **Tree** to $\mathbf{Grph_{ca}}$. Conversely, Lemma B.6 shows that H is a well-defined functor from $\mathbf{Grph_{ca}}$ to **Tree**.

It remains to be shown [1] that $\mathsf{H}_0\circ\mathsf{G}_0$ maps any functioned tree to itself, [2] that $\mathsf{G}_0\circ\mathsf{H}_0$ maps any nontrivial converging arborescence to itself, [3] that $\mathsf{H}_1\circ\mathsf{G}_1$ maps any functioned-tree morphism to itself, and [4] that $\mathsf{G}_1\circ\mathsf{H}_1$ maps any directed-graph morphism, whose source and target are nontrivial converging arborescences, to itself.

[1]. Take any functioned tree (T, p). By the definitions of G_0 and H_0,

$$\begin{aligned}
\mathsf{H}_0\circ\mathsf{G}_0(T, p) &= (T, p^*),\ \text{ where} \\
p^* &= \{(\mathsf{init}(e), \mathsf{ter}(e)) \mid e\in E\}, && \text{(a)} \\
E &= \{\{t^\sharp, t\} \mid (t^\sharp, t)\in p\}, && \text{(b)}
\end{aligned}$$

$$\mathsf{init} = \{(\{t^\sharp,t\},t^\sharp) \mid (t^\sharp,t){\in}p\},\text{ and} \tag{c}$$

$$\mathsf{ter} = \{(\{t^\sharp,t\},t) \mid (t^\sharp,t){\in}p\}. \tag{d}$$

It must be shown that $p^* = p$.

This paragraph shows $p^* \subseteq p$. Take any $(t^{*\sharp},t^*) \in p^*$. By (a), there exists $e \in E$, such that $(t^{*\sharp},t^*) = (\mathsf{init}(e),\mathsf{ter}(e))$. By (b), there exists $(t^\sharp,t) \in p$ such that $e = \{t^\sharp,t\}$. I argue

$$(t^{*\sharp},t^*) = (\mathsf{init}(e),\mathsf{ter}(e)) = (\mathsf{init}(\{t^\sharp,t\}),\mathsf{ter}(\{t^\sharp,t\})) = (t^\sharp,t) \in p.$$

The first equality holds by the definition of e, and the second holds by the definition of $(t^\sharp,t)$. The third equality holds by (c) and (d) because $(t^\sharp,t) \in p$ by definition. The fourth equality holds because $(t^\sharp,t) \in p$ by definition.

This paragraph shows $p \subseteq p^*$. Take any $(t^\sharp,t) \in p$. By (b), (c), and (d), respectively,

$$\{t^\sharp,t\} \in E,$$
$$\mathsf{init}(\{t^\sharp,t\}) = t^\sharp,\text{ and}$$
$$\mathsf{ter}(\{t^\sharp,t\}) = t.$$

Thus by (a), $(t^\sharp,t) = (\mathsf{init}(\{t^\sharp,t\}),\mathsf{ter}(\{t^\sharp,t\}))$ belongs to p^*.

[2]. Take any nontrivial converging arborescence $(T,E,\mathsf{init},\mathsf{ter})$. Define E^*, init^*, and ter^* by equalities (b), (f), and (j) below. It suffices to show the remaining ten equalities.

$$\mathsf{G}_0{\circ}\mathsf{H}_0(T,E,\mathsf{init},\mathsf{ter}) = (T,E^*,\mathsf{init}^*,\mathsf{ter}^*), \tag{a}$$

$$\begin{aligned}
E^* &:= \{\{t^\sharp,t\} \mid (t^\sharp,t) \in \{(\mathsf{init}(e),\mathsf{ter}(e))|e{\in}E\}\} && \text{(b)}\\
&= \{\{\mathsf{init}(e),\mathsf{ter}(e)\} \mid e{\in}E\} && \text{(c)}\\
&= \{e \mid e{\in}E\} && \text{(d)}\\
&= E, && \text{(e)}
\end{aligned}$$

$$\begin{aligned}
\mathsf{init}^* &:= \{(\{t^\sharp,t\},t^\sharp) \mid (t^\sharp,t) \in \{(\mathsf{init}(e),\mathsf{ter}(e))|e{\in}E\}\} && \text{(f)}\\
&= \{(\{\mathsf{init}(e),\mathsf{ter}(e)\},\mathsf{init}(e)) \mid e{\in}E\} && \text{(g)}\\
&= \{(e,\mathsf{init}(e)) \mid e{\in}E\} && \text{(h)}\\
&= \mathsf{init},\text{ and} && \text{(i)}
\end{aligned}$$

$$\begin{aligned}
\mathsf{ter}^* &:= \{(\{t^\sharp,t\},t) \mid (t^\sharp,t) \in \{(\mathsf{init}(e),\mathsf{ter}(e))|e{\in}E\}\} && \text{(j)}\\
&= \{(\{\mathsf{init}(e),\mathsf{ter}(e)\},\mathsf{ter}(e)) \mid e{\in}E\} && \text{(k)}\\
&= \{(e,\mathsf{ter}(e)) \mid e{\in}E\} && \text{(l)}\\
&= \mathsf{ter}. && \text{(m)}
\end{aligned}$$

(a) follows from the definitions of H_0 and G_0. (c), (g), and (k) are rearrangements. (d), (h), and (l) hold by (11). (e) is trivial. (i) and (m) hold because E is the domain of init and ter.

[3]. Take any functioned-tree morphism $[(T,p),(T',p'),\tau]$. I argue

$$\begin{aligned}
&\mathsf{H}_1{\circ}\mathsf{G}_1([(T,p),(T',p'),\tau])\\
&\quad= \mathsf{H}_1([\mathsf{G}_0(T,p),\mathsf{G}_0(T',p'),\tau,\varepsilon])\\
&\quad= [\mathsf{H}_0{\circ}\mathsf{G}_0(T,p),\mathsf{H}_0{\circ}\mathsf{G}_0(T',p'),\tau]\\
&\quad= [(T,p),(T',p'),\tau],
\end{aligned}$$

$$\text{where } \varepsilon = \{\ (\{t^\sharp,t\},\{\tau(t^\sharp),\tau(t)\})\mid (t^\sharp,t){\in}p\ \}.$$

The first equality follows from the definition of G_1. The second equality follows from the definition of H_1. The third equality follows from (8a) and part [1].

[4]. Let $[(T,E,\mathsf{init},\mathsf{ter}),(T',E',\mathsf{init}',\mathsf{ter}'),\tau,\varepsilon]$ be a directed-graph morphism whose source and target are nontrivial converging arborescences. As a preliminary observation, note that the definition of H_0 implies

$$\mathsf{H}_0(T,E,\mathsf{init},\mathsf{ter}) = (T,p) \tag{60a}$$

$$\text{where } p = \{(\mathsf{init}(e),\mathsf{ter}(e))\mid e{\in}E\}\ \text{ and}$$

$$\mathsf{H}_0(T',E',\mathsf{init}',\mathsf{ter}') = (T',p') \tag{60b}$$

$$\text{where } p' = \{(\mathsf{init}'(e'),\mathsf{ter}'(e'))\mid e'{\in}E'\}.$$

Define

$$\varepsilon^* = \{\ (\{t^\sharp,t\},\{\tau(t^\sharp),\tau(t)\})\mid (t^\sharp,t){\in}p\ \}.$$

In the remainder of this paragraph I argue

$$\begin{aligned}
\varepsilon^* &= \{(\{\mathsf{init}(e),\mathsf{ter}(e)\},\ \{\tau{\circ}\mathsf{init}(e),\tau{\circ}\mathsf{ter}(e)\})\mid e{\in}E\} &\quad (61)\\
&= \{(\{\mathsf{init}(e),\mathsf{ter}(e)\},\ \{\mathsf{init}'{\circ}\varepsilon(e),\mathsf{ter}'{\circ}\varepsilon(e)\})\mid e{\in}E\}\\
&= \{(e,\ \{\mathsf{init}'{\circ}\varepsilon(e),\mathsf{ter}'{\circ}\varepsilon(e)\})\mid e{\in}E\}\\
&= \{(e,\ \varepsilon(e))\mid e{\in}E\}\\
&= \varepsilon.
\end{aligned}$$

The first equation holds by substituting the definition of p into the definition of ε^*. The second equation holds by (12c). The third equation holds by (11) for $(T,E,\mathsf{init},\mathsf{ter})$. The fourth equation holds by [a] (11) for $(T',E',\mathsf{init}',\mathsf{ter}')$ because [b] $\varepsilon(E)\ \subseteq\ E'$ by the second half of (12b). The fifth equation holds because the domain of ε is E by the second half of (12b).

I argue

$$\begin{aligned}
&\mathsf{G}_1{\circ}\mathsf{H}_1([(T,E,\mathsf{init},\mathsf{ter}),(T',E',\mathsf{init}',\mathsf{ter}'),\tau,\varepsilon])\\
&\quad= \mathsf{G}_1([\mathsf{H}_0(T,E,\mathsf{init},\mathsf{ter}),\mathsf{H}_0(T',E',\mathsf{init}',\mathsf{ter}'),\tau])
\end{aligned}$$

$$\begin{aligned}
&= \mathsf{G}_1([(T,p),(T',p'),\tau]) \\
&= [\mathsf{G}_0(T,p),\mathsf{G}_0(T',p'),\tau,\varepsilon^*] \\
&= [\mathsf{G}_0(T,p),\mathsf{G}_0(T',p'),\tau,\varepsilon] \\
&= [\mathsf{G}_0{\circ}\mathsf{H}_0(T,E,\mathsf{init},\mathsf{ter}),\mathsf{G}_0{\circ}\mathsf{H}_0(T',E',\mathsf{init}',\mathsf{ter}'),\tau,\varepsilon] \\
&= [(T,E,\mathsf{init},\mathsf{ter}),(T',E',\mathsf{init}',\mathsf{ter}'),\tau,\varepsilon].
\end{aligned}$$

The first equality follows from the definitions of H_1, the second from (60), and the third from the definition of G_1. The fourth equality follows from (61), the fifth from (60), and the sixth from part [2]. ■

Appendix C. Concerning NCP

C.1. Objects

Lemma C.1. *Suppose that $(T,C,\otimes)$ satisfies (13a) with its F and t^o. Define p as in (13b). Then (a) p is a well-defined function from $T\smallsetminus\{t^o\}$ onto $F^{-1}(C)$. Consequently, if (T,p) is a functioned tree as (13b) requires, then (b) t^o is the root node of (T,p), and (c) $F^{-1}(C)$ is the decision-node set X of (T,p).*

Proof. *(a).* Since $\otimes$ is a bijection onto $T\smallsetminus\{t^o\}$ by (13a), p is a function with domain $T\smallsetminus\{t^o\}$. Further,

$$\begin{aligned}
p(T\smallsetminus\{t^o\}) &= \{t \mid (\exists t^\sharp)\ (t^\sharp,t){\in}p\} \\
&= \{t \mid (\exists t^\sharp)(\exists c)\ (t,c,t^\sharp){\in}\otimes\} \\
&= \{t \mid (\exists c)\ (t,c){\in}F\} \\
&= F^{-1}(C),
\end{aligned}$$

where the first equality holds by the previous sentence, the second equality follows from the definition of p, the third equality follows from (13a), and the fourth equality is a rearrangement.

(b)–(c). Suppose (T,p) is a functioned tree. Then by (1a), p is a function, from T less the root node of (T,p), onto the decision-node set X of (T,p). Thus by part (a), t^o is the root node of (T,p), and $F^{-1}(C)$ is the decision-node set X of (T,p). ■

Lemma C.2. *Suppose $(T,C,\otimes)$ satisfies (13a) and (13c) with its t^o. Define q as in (15). Then q is a function from $T\smallsetminus\{t^o\}$ onto C.*

Proof. Since $\otimes$ is a bijection onto $T\smallsetminus\{t^o\}$ by (13a), q is a function with domain $T\smallsetminus\{t^o\}$. Further, I argue

$$\begin{aligned}
q(T\smallsetminus\{t^o\}) &= \{c \mid (\exists t^\sharp)\ (t^\sharp,c){\in}q\} \\
&= \{c \mid (\exists t^\sharp)(\exists t)\ (t,c,t^\sharp){\in}\otimes\} \\
&= \{c \mid (\exists t)\ (t,c){\in}F\} \\
&= C,
\end{aligned}$$

where F is derived from $(T, C, \otimes)$. The first equality holds by the previous sentence, the second equality holds by the definition of q, and the third equality holds by the definition of F. The $\subseteq$ half of the fourth equality holds simply because every c is in C. The converse holds because each $F^{-1}(c)$ is nonempty by (13c). ∎

Proof C.3. (for Proposition 3.1) (a) To show the $\subseteq$ direction, take any $(t, c, t^\sharp) \in \otimes$. Then [1] $t^\sharp \neq t^o$ by (13a), [2] $t = p(t^\sharp)$ by the definition of p, and [3] $c = q(t^\sharp)$ by the definition of q. Conclusions [2] and [3] imply $(t, c, t^\sharp) = (p(t^\sharp), q(t^\sharp), t^\sharp)$. Thus conclusion [1] implies that $(t, c, t^\sharp)$ belongs to $\{\ (p(t^\sharp), q(t^\sharp), t^\sharp))\ |\ t^\sharp \neq t^o\ \}$.

To show the $\supseteq$ direction, take any $t^\sharp \neq t^o$. Then by (13a) there exists (t, c) such that $(t, c, t^\sharp) \in \otimes$. By the definition of p, $t = p(t^\sharp)$. By the definition of q, $c = q(t^\sharp)$. Therefore by the last three sentences, $(p(t), q(t), t^\sharp) \in \otimes$.

(b). Part (a) suffices because (13a) assumes that $\otimes$ is a bijection when viewed as a function from the first two components of its constituent triples to the third component of its constituent triples. ∎

Proof C.4. (for Proposition 3.2) To show the contrapositive of (16a), suppose $F(t) \neq F(t')$. Without loss of generality, suppose that $c^* \in F(t)$ but $c^* \notin F(t')$. Then $t \in F^{-1}(c^*)$ but $t' \notin F^{-1}(c^*)$. Thus, since $\mathcal{H} = \{F^{-1}(c)|c\}$ is a partition by (13c), there cannot be an $H \in \mathcal{H}$ that contains both t and t'.

To show the contrapositive of (16b), suppose $H \in \mathcal{H}$ and $H' \in \mathcal{H}$ satisfy $F(H) \cap F(H') \neq \varnothing$. Then there exists c^*, t, and t' such that

$$c^* \in F(t),\ t \in H, \tag{62a}$$

$$c^* \in F(t'),\ \text{and } t' \in H'. \tag{62b}$$

Since $\mathcal{H} = \{F^{-1}(c)|c\}$ is a partition by (13c), and since $t \in F^{-1}(c^*)$ by the first half of (62a), the second half of (62a) implies that $F^{-1}(c^*) = H$. By similar reasoning with (62b), $F^{-1}(c^*) = H'$. By the last two sentences, $H = H'$. ∎

C.2. Arrows

Lemma C.5. *Suppose a quadruple $[\Pi, \Pi', \tau, \delta]$ satisfies (17a)–(17b). Then (17c) iff (18) iff (19).*

Proof. *(17c)⇒(18)*. Assume (17c). Take any $(t, c) \in F$. Then by (13a) for Π, $(t, c, t{\otimes}c) \in \otimes$. Thus by (17c),

$$(\tau(t), \delta(c), \tau(t{\otimes}c)) \in \otimes'.$$

This set membership is equivalent to (18b). Further, by (13a) for Π', this set membership implies $(\tau(t), \delta(c)) \in F'$, which is (18a).

(17c)⇐(18). Assume (18). Take any $(t, c, t^\sharp) \in \otimes$. Then $(t, c) \in F$ by the definition of F. Thus $\tau(t){\otimes}'\delta(c) = \tau(t{\otimes}c)$ by (18b). Thus since $t{\otimes}c = t^\sharp$ by the definition of $(t, c, t^\sharp)$, I have $\tau(t){\otimes}'\delta(c) = \tau(t^\sharp)$. Thus $(\tau(t), \delta(c), \tau(t^\sharp)) \in \otimes'$.

(18)⇒(19). Assume (18). (18a) implies the first half of (19a). For the second half of (19a), take any $t^\sharp \in T \smallsetminus \{t^o\}$. Since $\otimes$ is onto $T \smallsetminus \{t^o\}$ by (13a), there exists $(t, c) \in F$ such that $t^\sharp = t{\otimes}c$. Thus $\tau(t^\sharp) = \tau(t{\otimes}c) = \tau(t){\otimes}'\delta(c)$, where the second equality holds by (18b). Thus since $\otimes'$ is onto $T' \smallsetminus \{t'^o\}$ by (13a), $\tau(t^\sharp) \in T' \smallsetminus \{t'^o\}$.

(19b) is an equation in the category **Set**. The left-hand side is well-defined because the codomain of $\otimes$ is $T\smallsetminus\{t^o\}$ by (13a). The right-hand side is well-defined because the domain of $\otimes'$ is F' by (13a), and because the codomain of $(\tau,\delta)|_F$ is F' by the proposition's definition [c]. The codomains of the two sides are both equal to $T'\smallsetminus\{t'^o\}$ by the proposition's definition [d] and by (13a) for Π'. Finally, the domains of the two sides are both equal to F by (13a) for Π. The previous four sentences have established that (19b) is a well-defined equation. By the second-previous sentence and (18b), the equation is true.

(18)⇐(19). The first half of (19a) implies (18a), and (19b) implies (18b). ■

Lemma C.6. *A quadruple* $[\Pi,\Pi',\tau,\delta]$ *is a morphism iff it satisfies (17a)–(17b) and (18).* [Proof: By definition, a quadruple is a morphism iff it satisfies (17a)–(17c). Thus the result follows immediately from Lemma C.5((17c)⇔(18)).]

Proof C.7. (for Proposition 3.3) By definition, a quadruple is a morphism iff it satisfies (17a)–(17c). Thus the result follows immediately from Lemma C.5((17c)⇔(19)). ■

Lemma C.8. *Suppose a quadruple* $[\Pi,\Pi',\tau,\delta]$ *satisfies (17a)–(17b). Then (17c) iff (20) iff (21) iff (22).*

Proof. *(17c)⇒(20).* Assume (17c). For (20a), I argue

$$\begin{aligned}
&\{(\tau(t^\sharp),\tau(t)) \mid (t^\sharp,t)\in p\}\\
&\quad= \{(\tau(t^\sharp),\tau(t)) \mid (\exists c)\ (t,c,t^\sharp)\in\otimes\}\\
&\quad\subseteq \{(\tau(t^\sharp),\tau(t)) \mid (\exists c)\ (\tau(t),\delta(c),\tau(t^\sharp))\in\otimes'\}\\
&\quad\subseteq \{(t'^\sharp,t') \mid (\exists c')\ (t',c',t'^\sharp)\in\otimes'\}\\
&\quad= \{(t'^\sharp,t') \mid (t'^\sharp,t')\in p'\}\\
&\quad= p',
\end{aligned}$$

where the first equality holds by the definition of p, the first set inclusion holds by (17c), the second set inclusion holds by (17b), and the second equality holds by the definition of p'. Similarly for (20b), I argue

$$\begin{aligned}
&\{(\tau(t^\sharp),\delta(c)) \mid (t^\sharp,c)\in q\}\\
&\quad= \{(\tau(t^\sharp),\delta(c)) \mid (\exists t)\ (t,c,t^\sharp)\in\otimes\}\\
&\quad\subseteq \{(\tau(t^\sharp),\delta(c)) \mid (\exists t)\ (\tau(t),\delta(c),\tau(t^\sharp))\in\otimes'\}\\
&\quad\subseteq \{(t'^\sharp,c') \mid (\exists t')\ (t',c',t'^\sharp)\in\otimes'\}\\
&\quad= \{(t'^\sharp,c') \mid (t'^\sharp,c')\in q'\}\\
&\quad= q',
\end{aligned}$$

where the first equality holds by the definition of q, the first set inclusion holds by (17c), the second set inclusion holds by (17b), and the second equality holds by the definition of q'.

(20)⇒(21). Assume (20). To show (21a), take any $t^\sharp \neq t^o$. Since p is a function from $T \smallsetminus \{t^o\}$ by (17a) and (13b), there exists t such that $(t^\sharp, t) \in p$. Thus by (20a), $(\tau(t^\sharp), \tau(t)) \in p'$. Thus since p' is a function from $T' \smallsetminus \{t'^o\}$ by (17a) and (13b), $\tau(t^\sharp) \neq t'^o$.

I show (20a) implies (21b) by

$$\begin{aligned}
&\{(\tau(t^\sharp), \tau(t)) \mid (t^\sharp, t) \in p\} \subseteq p' \\
&\Rightarrow \{(\tau(t^\sharp), \tau(t)) \mid t = p(t^\sharp), t^\sharp \neq t^o\} \subseteq p' \\
&\Rightarrow \{(\tau(t^\sharp), \tau(p(t^\sharp)) \mid t^\sharp \neq t^o\} \subseteq p' \\
&\Rightarrow (\forall t^\sharp \neq t^o)\ \tau(p(t^\sharp)) = p'(\tau(t^\sharp)),
\end{aligned}$$

where the first implication holds because p is a function from $T \smallsetminus \{t^o\}$ by (17a) and (13b), and where the next two implications are rearrangements.

Similarly, I show (20b) implies (21c) by

$$\begin{aligned}
&\{(\tau(t^\sharp), \delta(c)) \mid (t^\sharp, c) \in q\} \subseteq q' \\
&\Rightarrow \{(\tau(t^\sharp), \delta(c)) \mid c = q(t^\sharp),\ t^\sharp \neq t^o\} \subseteq q' \\
&\Rightarrow \{(\tau(t^\sharp), \delta(q(t^\sharp)) \mid t^\sharp \neq t^o\} \subseteq q' \\
&\Rightarrow (\forall t^\sharp \neq t^o)\ \delta(q(t^\sharp)) = q'(\tau(t^\sharp)),
\end{aligned}$$

where the first implication holds because q is a function from $T \smallsetminus \{t^o\}$ by (17a) and Lemma C.2, and where the next two implications are rearrangements.

(17c)⇐(21). Assume (21). I argue

$$\begin{aligned}
&\{(\tau(t), \delta(c), \tau(t^\sharp)) \mid (t, c, t^\sharp) \in \otimes\} \\
&\quad = \{(\tau(t), \delta(c), \tau(t^\sharp)) \mid t = p(t^\sharp),\ c = q(t^\sharp), t^\sharp \neq t^o\} \\
&\quad = \{(\ \tau(p(t^\sharp)),\ \tau(q(t^\sharp)),\ \tau(t^\sharp)\ ) \mid t^\sharp \neq t^o\} \\
&\quad = \{(\ p'(\tau(t^\sharp)),\ q'(\tau(t^\sharp)),\ \tau(t^\sharp)\ ) \mid t^\sharp \neq t^o\} \\
&\quad \subseteq \{(p'(\tau(t^\sharp)),\ q'(\tau(t^\sharp)),\ \tau(t^\sharp)\ ) \mid \tau(t^\sharp) \neq t'^o\} \\
&\quad \subseteq \{(p'(t'^\sharp),\ q'(t'^\sharp),\ t'^\sharp\ ) \mid t'^\sharp \neq t'^o\} \\
&\quad = \otimes'.
\end{aligned}$$

The first equality holds by Proposition 3.1(a) for Π. The second equality is a rearrangement. The third equality holds by (21b)–(21c). The first set inclusion holds by (21a). The second set inclusion holds by the first half of (17b). The fourth equality holds by Proposition 3.1(a) for Π'.

(21)⇒(22). Assume (21). (21a) implies the first half of (22a). For the second half of (22a), take any $t' \in \tau(F^{-1}(C))$. Since p is a function from $T \smallsetminus \{t^o\}$ onto $F^{-1}(C)$ by Lemma C.1(a), there exists a $t^\sharp \in T \smallsetminus \{t^o\}$ such that $t' = \tau(p(t^\sharp))$. Thus by (21b), $t' = p'(\tau(t^\sharp))$. Thus since p' is into $(F')^{-1}(C')$ by the definition of p' (at (13b)), $t' \in (F')^{-1}(C')$.

(22b) is an equation in **Set**. The left-hand side is well-defined because the codomain of p is $F^{-1}(C)$ by definition of p (at (13b)). The right-hand side is well-defined because the domain of p' is $T'\smallsetminus\{t'^o\}$ by the definition of p', and because the codomain of $\tau|_{T\smallsetminus\{t^o\}}$ is $T'\smallsetminus\{t'^o\}$ by the proposition's definition [c]. The codomains of the two sides are both equal to $(F')^{-1}(C')$ by the proposition's definition [d] and by the definition of p'. Finally, the domains of the two sides are both equal to $T\smallsetminus\{t^o\}$ by the definition of p. The previous four sentences have established that (22b) is a well-defined equation. By the second-previous sentence and (21b), the equation is true.

(22c) is another equation in **Set**. The left-hand side is well-defined because the domain of δ is C by the second half of (17b), and because the codomain of q is C by definition of q (at (15)). The right-hand side is well-defined because the domain of q' is $T'\smallsetminus\{t'^o\}$ by the definition of q', and because the codomain of $\tau|_{T\smallsetminus\{t^o\}}$ is $T'\smallsetminus\{t'^o\}$ by the proposition's definition [c]. The codomains of the two sides are both equal to C' by the second half of (17b) and the definition of q'. The domains of the two sides are both equal to $T\smallsetminus\{t^o\}$ by the definition of q. The previous four sentences have established that (22c) is a well-defined equation. By the second-previous sentence and (21c), the equation is true.

(21)⇐(22). The first half of (22a) implies (21a), (22b) implies (21b), and (22c) implies (21c). ■

Lemma C.9. *A quadruple $[\Pi, \Pi', \tau, \delta]$ is a morphism iff it satisfies (17a)–(17b) and (20).* [Proof: By definition, a quadruple is a morphism iff it satisfies (17a)–(17c). Thus the result follows immediately from Lemma C.8((17c)⇔(20)).]

Lemma C.10. *A quadruple $[\Pi, \Pi', \tau, \delta]$ is a morphism iff it satisfies (17a)–(17b) and (21).* [Proof: By definition, a quadruple is a morphism iff it satisfies (17a)–(17c). Thus the result follows immediately from Lemma C.8((17c)⇔(21)).]

Proof C.11. (for Proposition 3.4) By definition, a quadruple is a morphism iff it satisfies (17a)–(17c). Thus the result follows immediately from Lemma C.8((17c)⇔ (22)). ■

Proof C.12. (for Proposition 3.5) Derive F from Π and F' from Π'. Then take any $H \in \mathcal{H}$. By (13c) for Π, there exists c such that $H = F^{-1}(c)$. Let $H' = (F')^{-1}(\delta(c))$. Note $H' \in \mathcal{H}'$ by (13c) for Π'. Thus it suffices to argue

$$\begin{aligned}
\tau(H) &= \{\tau(t) \mid t{\in}H\} \\
&= \{t' \mid (\exists t)\ t' = \tau(t) \text{ and } t{\in}H\} \\
&= \{t' \mid (\exists t)\ t' = \tau(t) \text{ and } t{\in}F^{-1}(c)\} \\
&= \{t' \mid (\exists t)\ t' = \tau(t) \text{ and } (t,c){\in}F\} \\
&\subseteq \{t' \mid (\exists t)\ t' = \tau(t) \text{ and } (\tau(t),\delta(c)){\in}F'\} \\
&= \{t' \mid (\exists t)\ t' = \tau(t) \text{ and } (t',\delta(c)){\in}F'\} \\
&\subseteq \{t' \mid (t',\delta(c)){\in}F'\}
\end{aligned}$$

$$= (F')^{-1}(\delta(c))$$
$$= H'.$$

The first and second equalities are rearrangements, the third follows from the definition of c, and the fourth is a rearrangement. The first inclusion follows from the first half of (19a) in Proposition 3.3. The fifth equality is a rearrangement. The second inclusion follows from the first half of (17b). The sixth equality is a rearrangement, and the final equality follows from the definition of H'. ■

C.3. The Category

Proof C.13. (for Theorem 3.6) This paragraph notes that, for every preform $\Pi = (T, C, \otimes)$, the quadruple $[\Pi, \Pi, \mathsf{id}_T, \mathsf{id}_C]$ is a morphism. (17a) and (17b) hold by inspection. (17c) holds with equality.

This paragraph shows that, if $\alpha = [\Pi, \Pi', \tau, \delta]$ and $\alpha' = [\Pi', \Pi'', \tau', \delta']$ are morphisms, then $\alpha' \circ \alpha = [\Pi, \Pi'', \tau' \circ \tau, \delta' \circ \delta]$ is a morphism. Toward that end, take any such α and α'. Let $\Pi = (T, C, \otimes)$, $\Pi' = (T', C', \otimes')$, and $\Pi'' = (T'', C'', \otimes'')$. First, (17a) for α and (17a) for α' imply (17a) for $\alpha' \circ \alpha$. Second, note that $\tau{:}T \to T'$ by the first half of (17b) for α, and that $\tau'{:}T' \to T''$ by the first half of (17b) for α'. Hence $\tau' \circ \tau{:}T \to T''$, which is the first half of (17b) for $\alpha' \circ \alpha$. A parallel argument shows $\delta' \circ \delta{:}C \to C''$, which is the second half of (17b) for $\alpha' \circ \alpha$. Finally, to show that (17c) holds for $\alpha' \circ \alpha$, I argue

$$\begin{aligned}
&\{(\tau' \circ \tau(t), \delta' \circ \delta(c), \tau' \circ \tau(t^\sharp)) \mid (t, c, t^\sharp) \in \otimes\} \\
&\quad = \{(\tau'(t'), \delta'(c'), \tau'(t'^\sharp)) \mid (t', c', t'^\sharp) \in \{(\tau(t), \delta(c), \tau(t^\sharp)) | (t, c, t^\sharp) \in \otimes\}\} \\
&\quad \subseteq \{(\tau'(t'), \delta'(c'), \tau'(t'^\sharp)) \mid (t', c', t'^\sharp) \in \otimes'\} \\
&\quad \subseteq \otimes''.
\end{aligned}$$

The equality is a rearrangement. The first inclusion holds by (17c) for α, and the second inclusion holds by (17c) for α'.

The first paragraph of this proof shows that the identity arrow id_Π is well-defined for any preform Π. The second paragraph shows that the composition $\alpha' \circ \alpha$ is well-defined for any arrows α and α'. The unit and associative laws are immediate. Thus **NCP** is a category. ■

Lemma C.14. *Suppose $\alpha = [\Pi, \Pi', \tau, \delta]$ is an isomorphism. Then (a) τ and δ are bijective, and (b) $\alpha^{-1} = [\Pi', \Pi, \tau^{-1}, \delta^{-1}]$.*

Proof. Let $\Pi = (T, C, \otimes)$ and let $\Pi' = (T', C', \otimes')$. Because $\alpha = [\Pi, \Pi', \tau, \delta]$ is an isomorphism (e.g. [5, p. 12]), its inverse $\alpha^{-1} = [\Pi', \Pi, \tau^*, \delta^*]$ exists. Thus

$$\begin{gathered}
[\Pi', \Pi, \tau^*, \delta^*] \circ [\Pi, \Pi', \tau, \delta] = \\
\alpha^{-1} \circ \alpha = \mathsf{id}_\Pi = [\Pi, \Pi, \mathsf{id}_T, \mathsf{id}_C], \text{ and}
\end{gathered} \tag{63a}$$

$$\begin{gathered}
[\Pi, \Pi', \tau, \delta] \circ [\Pi', \Pi, \tau^*, \delta^*] = \\
\alpha \circ \alpha^{-1} = \mathsf{id}_{\Pi'} = [\Pi', \Pi', \mathsf{id}_{T'}, \mathsf{id}_{C'}],
\end{gathered} \tag{63b}$$

where the first two equalities in both lines follows from the definition of the inverse α^{-1}, and the third equality in both lines follows from the definition of id. The third component of (63a) implies that $\tau^*\circ\tau = \mathsf{id}_T$. The third component of (63b) implies that $\tau\circ\tau^* = \mathsf{id}_{T'}$. The last two sentences imply that τ is a bijection from T onto T' and that

$$\tau^* = \tau^{-1}. \tag{64}$$

Similarly, the fourth components of (63a) and (63b) imply that δ is a bijection from C onto C' and that

$$\delta^* = \delta^{-1}. \tag{65}$$

Conclusion (a) holds by the last two sentences. Conclusion (b) holds by

$$\alpha^{-1} = [\mathit{\Pi}^*, \mathit{\Pi}^{**}, \tau^*, \delta^*] = [\mathit{\Pi}', \mathit{\Pi}, \tau^{-1}, \delta^{-1}],$$

where the first equality holds by definition, and the second equality follows from (64)–(65). ■

Lemma C.15. *Suppose that $\alpha = [\mathit{\Pi}, \mathit{\Pi}', \tau, \delta]$ is a morphism and that τ and δ are bijective. Then α is an isomorphism.*

Proof. Define $\alpha^* = [\mathit{\Pi}', \mathit{\Pi}, \tau^{-1}, \delta^{-1}]$. This and the next two paragraphs show that α^* is a morphism. Specifically, they show that

$$\mathit{\Pi}' \text{ and } \mathit{\Pi} \text{ are preforms,} \tag{66a}$$

$$\tau^{-1}\!: T' \to T,\ \ \delta^{-1}\!: C' \to C, \text{ and} \tag{66b}$$

$$\{\ (\tau^{-1}(t'), \delta^{-1}(c'), \tau^{-1}(t'^{\sharp})) \mid (t', c', t'^{\sharp}) \in \otimes'\ \} \ \subseteq\ \otimes, \tag{66c}$$

where $\mathit{\Pi} = (T, C, \otimes)$ and $\mathit{\Pi}' = (T', C', \otimes')$. (66a) follows from (17a) for α. (66b) follows from (17b) for α and the bijectivity of τ and δ. To show (66c), suppose $(t', c', t'^{\sharp}) \in \otimes'$. For notational ease, define $t = \tau^{-1}(t')$ and $t^{\sharp} = \tau^{-1}(t'^{\sharp})$. Further derive t^o, p, and q from $(T, C, \otimes)$, and t'^o, p', and q' from $(T', C', \otimes')$.

This paragraph argues [1] that (T, p) and (T', p') are functioned trees and [2] that $\gamma = [(T, p), (T', p'), \tau]$ is a functioned-tree morphism.[14] For [1], note that $\mathit{\Pi}$ and $\mathit{\Pi}'$ are preforms by (17a) for α. Thus (T, p) and (T', p') are functioned trees by two applications of (13b). For [2], it must be shown that γ satisfies (8). (8a) holds by [1]. (8b) holds by the first half of (17b) for α. (8c) is identical to (20a), which holds by Lemma C.9 and α being a morphism.

This paragraph shows that $t^{\sharp} \neq t^o$. Derive $\preccurlyeq$ from (T, p) and $\preccurlyeq'$ from (T', p') (this is possible by [1]). Now suppose $t^{\sharp} = t^o$ were true. Then [a] $t^{\sharp} \preccurlyeq t$ since t^o precedes every element of T, thus [b] $\tau(t^{\sharp}) \preccurlyeq' \tau(t)$ by γ being a morphism (by [2]) and Proposition 2.4(e), and thus [c] $t'^{\sharp} \preccurlyeq' t'$ by the definitions of $t^{\sharp}$ and t. This

[14] This argument could have been made categorically via Corollary 3.10 if the text had arranged the results in a different order. In fact, claims [1] and [2] in this sentence correspond to paragraphs [1] and [2] in Proof D.1 for Theorem 3.9.

contradicts $t'^{\sharp} \succ t'$ which follows from $p'(t'^{\sharp}) = t'$, which in turn follows from the assumption that $(t', c', t'^{\sharp}) \in \otimes'$.

Since the range of $\otimes$ is $T \smallsetminus \{t^o\}$, and since $t^{\sharp} \neq t^o$ by the previous paragraph, Proposition 3.1(b) implies that

$$p(t^{\sharp}) \otimes q(t^{\sharp}) = t^{\sharp}. \tag{67}$$

Note that

$$\tau{\circ}p(t^{\sharp}) \otimes' \delta{\circ}q(t^{\sharp}) = \tau(t^{\sharp}) = t'^{\sharp},$$

where the first equality holds by (67) and (17c) for α, and the second equality holds by the definition of $t^{\sharp}$. Because of the previous equality, because $t' {\otimes'} c' = t'^{\sharp}$ by assumption, and because $\otimes'$ is a bijection by (17a) for α and (13a) for Π',

$$\tau{\circ}p(t^{\sharp}) = t' \text{ and } \delta{\circ}q(t^{\sharp}) = c'.$$

Hence

$$p(t^{\sharp}) = \tau^{-1}(t') \text{ and } q(t^{\sharp}) = \delta^{-1}(c').$$

Now take (67) and replace its three terms by means of [a] the two equalities in the last sentence and [b] the definition of $t^{\sharp}$. The result is

$$\tau^{-1}(t') \otimes \delta^{-1}(c') = \tau^{-1}(t'^{\sharp}),$$

as required by (66c).

Finally,

$$\alpha^*{\circ}\alpha = [\Pi', \Pi, \tau^{-1}, \delta^{-1}]{\circ}[\Pi, \Pi', \tau, \delta] = [\Pi, \Pi, \mathsf{id}_T, \mathsf{id}_C] = \mathsf{id}_\Pi \text{ and}$$
$$\alpha{\circ}\alpha^* = [\Pi, \Pi', \tau, \delta]{\circ}[\Pi', \Pi, \tau^{-1}, \delta^{-1}] = [\Pi', \Pi', \mathsf{id}_{T'}, \mathsf{id}_{C'}] = \mathsf{id}_{\Pi'}.$$

Thus α is an isomorphism (and $\alpha^{-1} = \alpha^*$). ∎

Proof C.16. (for Theorem 3.7) Lemma C.14 establishes [a] the forward direction of the theorem's second sentence and [b] the theorem's third sentence. Lemma C.15 establishes the reverse direction of the theorem's second sentence. ∎

Lemma C.17. *Suppose $\alpha = [\Pi, \Pi', \tau, \delta]$ is an isomorphism, where $\Pi = (T, C, \otimes)$ determines F, q, and $\mathcal{H}$, and where $\Pi' = (T', C', \otimes')$ determines F', q', and $\mathcal{H}'$. Then the following hold.*
(a) $\tau|_{F^{-1}(C)}$ is a bijection from $F^{-1}(C)$ onto $(F')^{-1}(C')$.
(b) $(\tau, \delta, \tau)|_\otimes$ is a bijection from $\otimes$ onto $\otimes'$.
(c) $(\tau, \delta)|_F$ is a bijection from F onto F'.
(d) $(\tau, \delta)|_q$ is a bijection from q onto q'.
(e) $(\forall H{\in}\mathcal{H})\ \tau(H) \in \mathcal{H}'$.

Proof. Theorem 3.7 implies

$$\tau \text{ is a bijection from } T \text{ onto } T', \tag{68a}$$
$$\delta \text{ is a bijection from } C \text{ onto } C', \text{ and} \tag{68b}$$
$$\alpha^{-1} = [\Pi', \Pi, \tau^{-1}, \delta^{-1}]. \tag{68c}$$

(a). By Proposition 3.4 (second half of (22a)) for α, $\tau|_{F^{-1}(C)}$ is a well-defined function from $F^{-1}(C)$ into $(F')^{-1}(C')$. It is injective by (68a). To show it is surjective, take any $t' \in (F')^{-1}(C')$. By (68c), and by Proposition 3.4 (second half of (22a)) for α^{-1}, $\tau^{-1}(t') \in F^{-1}(C)$. Thus $\tau(\tau^{-1}(t')) = t'$ is in the range of $\tau|_{F^{-1}(C)}$.

(b). By (17c) for α, $(\tau,\delta,\tau)|_{\otimes}$ is a well-defined function from $\otimes$ into $\otimes'$. It is injective by (68a)–(68b). To show it is surjective, take any $(t',c',t'^{\sharp}) \in \otimes'$. By (68c), and by (17c) for α^{-1},

$$(\tau^{-1}(t'), \delta^{-1}(c'), \tau^{-1}(t'^{\sharp})) \in \otimes.$$

Thus $(\tau,\delta,\tau)(\tau^{-1}(t'), \delta^{-1}(c'), \tau^{-1}(t'^{\sharp})) = (t',c',t'^{\sharp})$ is in the range of $(\tau,\delta,\tau)|_{\otimes}$.

(c). By (18a) (via Lemma C.6) for α, $(\tau,\delta)|_F$ is a well-defined function from F into F'. It is injective by (68a)–(68b). To show it is surjective, take any $(t',c') \in F'$. By (68c), and by (18a) (via Lemma C.6) for α^{-1},

$$(\tau^{-1}(t'), \delta^{-1}(c')) \in F.$$

Thus $(\tau,\delta)(\tau^{-1}(t'), \delta^{-1}(c')) = (t',c')$ is in the range of $(\tau,\delta)|_F$.

(d). By (20b) (via Lemma C.9) for α, $(\tau,\delta)|_q$ is a well-defined function from q into q'. It is injective by (68a)–(68b). To show it is surjective, take any $(t'^{\sharp},c') \in q'$. By (68c), and by (20b) (via Lemma C.9) for α^{-1},

$$(\tau^{-1}(t'^{\sharp}), \delta^{-1}(c')) \in q.$$

Thus $(\tau,\delta)(\tau^{-1}(t'^{\sharp}), \delta^{-1}(c')) = (t'^{\sharp},c')$ is in the range of $(\tau,\delta)|_q$.

(e). Take any $H \in \mathcal{H}$. By the definition of $\mathcal{H}$, there exists c such that $H = F^{-1}(c)$. Note that

$$\begin{aligned} H &= F^{-1}(c) \\ &= \{t \mid (t,c){\in}F\} \\ &= \{t \mid (\exists (t',c'){\in}F')\ t = \tau^{-1}(t') \text{ and } c = \delta^{-1}(c')\} \\ &= \{t \mid (\exists t')\ (t',\delta(c)){\in}F' \text{ and } t = \tau^{-1}(t')\} \\ &= \{\tau^{-1}(t') \mid (t',\delta(c)){\in}F'\} \\ &= \{\tau^{-1}(t') \mid t' \in (F')^{-1}(\delta(c))\} \\ &= \tau^{-1}(\,(F')^{-1}(\delta(c))\,), \end{aligned} \tag{69}$$

where the first equation holds by the definition of c, the third equation holds by part (c), and the remaining equations are rearrangements. Because τ is a bijection, (69) implies $\tau(H) = (F')^{-1}(\delta(c))$. Thus $\tau(H) \in \mathcal{H}'$ by the definition of $\mathcal{H}'$. ■

Proof C.18. (for Proposition 3.8)

(a)–(d). Lemma C.17(a)–(d).

(e). By Lemma C.17(e), $\tau|_{\mathcal{H}}$ is a well-defined function from $\mathcal{H}$ into $\mathcal{H}'$. Thus it remains to show that $\tau|_{\mathcal{H}}$ is bijective.

Let $\alpha = [\Pi, \Pi', \tau, \delta]$. By Theorem 3.7, [a] τ is bijective, [b] δ is bijective, and [c] $\alpha^{-1} = [\Pi', \Pi, \tau^{-1}, \delta^{-1}]$. Thus $\tau|_{\mathcal{H}}$ is injective by [a]. To show that $\tau|_{\mathcal{H}}$ is surjective, take any $H' \in \mathcal{H}'$. By [c], and by Lemma C.17(e) applied to α^{-1}, $\tau^{-1}(H') \in \mathcal{H}$. Thus $\tau(\tau^{-1}(H')) = H'$ is in the range of $\tau|_{\mathcal{H}}$. ■

Appendix D. Between NCP and Tree

D.1. The Forgetful Functor F

Proof D.1. (for Theorem 3.9) It must be shown [1] that F_0 takes each preform to a functioned tree, [2] that F_1 takes each preform morphism to a functioned-tree morphism, [3] that F preserves identity, [4] that F preserves sources and targets, and [5] that F preserves composition.

[1] Take any preform Π. Then $\mathsf{F}_0(\Pi)$ is a functioned tree by (13b).

[2] Take any preform morphism $\alpha = [\Pi, \Pi', \tau, \delta]$. Then $\mathsf{F}_1(\alpha) = [\mathsf{F}_0(\Pi), \mathsf{F}_0(\Pi'), \tau]$. (8a) holds by (17a) for α and by two applications of step [1]. (8b) holds by the first half of (17b) for α. (8c) is identical to (20a), which holds by Lemma C.9 and α being a morphism.

[3] Take any preform $(T, C, \otimes)$. Let $\mathsf{F}_0(T, C, \otimes) = (T, p)$. I argue

$$\begin{aligned}
\mathsf{F}_1(\mathsf{id}_{(T,C,\otimes)}) &= \mathsf{F}_1([(T,C,\otimes), (T,C,\otimes), \mathsf{id}_T, \mathsf{id}_C]) \\
&= [\mathsf{F}_0(T,C,\otimes), \mathsf{F}_0(T,C,\otimes), \mathsf{id}_T] \\
&= [(T,p), (T,p), \mathsf{id}_T] \\
&= \mathsf{id}_{(T,p)} \\
&= \mathsf{id}_{\mathsf{F}_0(T,C,\otimes)},
\end{aligned}$$

where the first equality holds by the definition of identity in **NCP**, the second holds by the definition of F_1, the third holds by the definition of (T,p), the fourth holds by the definition of the identity in **Tree**, and the fifth holds by the definition of (T,p).

[4] Take any preform morphism $[\Pi, \Pi', \tau, \delta]$. For sources,

$$\begin{aligned}
&\mathsf{F}_1([\Pi, \Pi', \tau, \delta])^{\mathsf{src}} \\
&\quad = [\mathsf{F}_0(\Pi), \mathsf{F}_0(\Pi'), \tau]^{\mathsf{src}} \\
&\quad = \mathsf{F}_0(\Pi) \\
&\quad = \mathsf{F}_0([\Pi, \Pi', \tau, \delta]^{\mathsf{src}}),
\end{aligned}$$

where the first equality holds by the definition of F_1, the second by the definition of src in **Tree**, and the third by the definition of src in **NCP**. A symmetric argument shows that targets are preserved.

[5] Take any two preform morphisms $[\Pi, \Pi', \tau, \delta]$ and $[\Pi', \Pi'', \tau', \delta']$. I argue

$$\begin{aligned}
&\mathsf{F}_1([\Pi', \Pi'', \tau', \delta']\circ[\Pi, \Pi', \tau, \delta]) \\
&\quad = \mathsf{F}_1([\Pi, \Pi'', \tau'\circ\tau, \delta'\circ\delta])
\end{aligned}$$

$$\begin{aligned} &= [\mathsf{F}_0(\Pi), \mathsf{F}_0(\Pi''), \tau'\circ\tau] \\ &= [\mathsf{F}_0(\Pi'), \mathsf{F}_0(\Pi''), \tau']\circ[\mathsf{F}_0(\Pi), \mathsf{F}_0(\Pi'), \tau] \\ &= \mathsf{F}_1[\Pi', \Pi'', \tau', \delta']\circ\mathsf{F}_1[\Pi, \Pi', \tau, \delta], \end{aligned}$$

where the first equality holds by the definition of $\circ$ in **NCP**, the second by the definition of F_1, the third by the definition of $\circ$ in **Tree**, and the fourth by the definition of F_1. ■

D.2. Perfect Information

LEMMA D.2. *Suppose $(T, C, \otimes)$ is a preform. Then $(T, C, \otimes)$ has perfect information iff its q is bijective.*

PROOF. Derive F from $(T, C, \otimes)$. Note that

$$\begin{aligned} q^{-1}(c) &= \{t^\sharp \mid (\exists t)\ (t, c, t^\sharp)\in\otimes\} \\ &= \{t^\sharp \mid (\exists t)\ (t, c)\in F \text{ and } (t, c, t^\sharp)\in\otimes\} \\ &= \{t^\sharp \mid (\exists t)\ (t, c)\in F \text{ and } t\otimes c = t^\sharp)\} \\ &= \{t\otimes c \mid (t, c)\in F\} \\ &= \{t\otimes c \mid t\in F^{-1}(c)\}, \end{aligned} \tag{70}$$

where the first equality follows from the definition of q, the second follows from the definition of F, the third holds because $\otimes$ is a function by (13a), and the fourth and the fifth are rearrangements.

Suppose $(T, C, \otimes)$ has perfect information. By Lemma C.2, q is a function onto C. Thus it remains to show that each $q^{-1}(c)$ is a singleton. This follows from (70) because perfect information means that each $F^{-1}(c)$ is a singleton.

Conversely, suppose q is bijective. By the definition of perfect information, it suffices to show that each $F^{-1}(c)$ is a singleton. Take any c. Because q is bijective and because q is onto C by Lemma C.2, $q^{-1}(c)$ is a singleton. Thus, by (70), $\{\ t\otimes c \mid t\in F^{-1}(c)\ \}$ is a singleton. Thus, since $\otimes$ is injective by (13a), $F^{-1}(c)$ is a singleton. ■

LEMMA D.3. *Suppose that (T, p) is a functioned tree. Then $\mathsf{E}_0(T, p)$ is a perfect-information preform, where E_0 is defined in Theorem 3.13.*

PROOF. Let (T, p) determine its t^o. By the definition of E_0, $\mathsf{E}_0(T, p) = (T, C, \otimes)$ where

$$C = T\smallsetminus\{t^o\} \text{ and} \tag{71a}$$

$$\otimes = \{(t, t^\sharp, t^\sharp) \mid (t^\sharp, t)\in p\}. \tag{71b}$$

I will show that $(T, C, \otimes)$ satisfies (13a)–(13b) and (23). This suffices because (23) is both the definition of perfect information and a sufficient condition for (13c).

For (13a), define $F = \{\ (t, t^\sharp) \mid (t^\sharp, t)\in p\ \}$. Note

$$F\ \subseteq\ T\times(T\smallsetminus\{t^o\}) = T\times C,$$

where the set inclusion holds because $p{:}T{\smallsetminus}\{t^o\}{\rightarrow}T$ by (1a), and the equality holds by (71a). It remains to show that $\otimes$ is a bijection from F onto $T{\smallsetminus}\{t^o\}$. Two observations suffice. First, (71b) and the definition of F together imply that $\otimes$ is a function from F. Second, since $p{:}T{\smallsetminus}\{t^o\}{\rightarrow}T$ by (1a), (71b) implies that the converse of $\otimes$ is a function from $T{\smallsetminus}\{t^o\}$.

For (13b), this paragraph shows that p is the set that (13b) derives from $(T, C, \otimes)$. In other words, it shows

$$\begin{aligned} p &= \{(t^\sharp, t) \mid (t^\sharp, t){\in}p \text{ and } t^\sharp{\in}T{\smallsetminus}\{t^o\}\} \\ &= \{\ (t^\sharp, t) \mid (t, t^\sharp, t^\sharp){\in}\otimes \text{ and } t^\sharp{\in}T{\smallsetminus}\{t^o\}\} \\ &= \{(t^\sharp, t) \mid (\exists t'{\in}T{\smallsetminus}\{t^o\})\ (t, t', t^\sharp){\in}\otimes\} \\ &= \{(t^\sharp, t) \mid (\exists c)\ (t, c, t^\sharp){\in}\otimes\}. \end{aligned} \tag{72}$$

The first equality holds because p is a function from $T{\smallsetminus}\{t^o\}$ by (1a), and the second equality holds by (71b). The $\subseteq$ direction of the third equality holds by inspection, and the reverse direction holds by (71b). The fourth equality holds by (71a).

Further, Lemma C.1(a), (13a), and (72) imply $p{:}T{\smallsetminus}\{t^o\}{\rightarrow}F^{-1}(C)$. The previous paragraph, the previous sentence, and the lemma's assumption that (T, p) is a functioned tree together imply (13b).

For (23), take any c. Then $F^{-1}(c) = \{t|(t, c){\in}F\} = \{t|(c, t){\in}p\} = \{p(c)\}$, where the first equality is a rearrangement, the second follows from the definition of F, and the last equality holds because p is a function. ■

LEMMA D.4. *Suppose that $\gamma = [(T, p), (T', p'), \tau]$ is a functioned-tree morphism. Then $\mathsf{E}_1(\gamma)$ is a preform morphism, where E_1 is defined in Theorem 3.13. Further, its source and target have perfect information.*

PROOF. The definition of E_1 defines the codomain of $\tau|_{T\smallsetminus\{t^o\}}$ to be $T'{\smallsetminus}\{t'^o\}$, where t^o is the root of (T, p) and t'^o is the root of (T', p'). This codomain is well-defined because $\tau(T{\smallsetminus}\{t^o\}) \subseteq T'{\smallsetminus}\{t'^o\}$ by Proposition 2.3 (first half of (9a)).

By the definition of E,

$$\begin{aligned} \mathsf{E}_1(\gamma) &= \mathsf{E}_1([(T, p), (T', p'), \tau]) \\ &= [\mathsf{E}_0(T, p), \mathsf{E}_0(T', p'), \tau, \tau|_{T\smallsetminus\{t^o\}}] \\ &= [(T, C, \otimes), (T', C', \otimes'), \tau, \tau|_{T\smallsetminus\{t^o\}}], \end{aligned}$$

$$\text{where}\quad C = T{\smallsetminus}\{t^o\}, \tag{73a}$$

$$\otimes = \{(t, t^\sharp, t^\sharp) \mid (t^\sharp, t){\in}p\}, \tag{73b}$$

$$C' = T'{\smallsetminus}\{t'^o\}, \text{ and} \tag{73c}$$

$$\otimes' = \{(t', t'^\sharp, t'^\sharp) \mid (t'^\sharp, t'){\in}p'\}. \tag{73d}$$

It must be shown [a] that the above quadruple satisfies (17a)–(17c) and [b] that its source and target have perfect information. (17a) and [b] follow from Lemma D.3. The first half of (17b) follows from (8b). To see the second half of (17b), recall that

$\tau|_{T\smallsetminus\{t^o\}}$ maps from $T\smallsetminus\{t^o\}$ to $T'\smallsetminus\{t'^o\}$ by the first paragraph of this proof. Thus it maps from C to C' by (73a) and (73c). Finally, for (17c), I argue

$$\begin{aligned}
&\{(\tau(t),\tau|_{T\smallsetminus\{t^o\}}(c),\tau(t^\sharp))\mid(t,c,t^\sharp)\in\otimes\}\\
&=\{(\tau(t),\tau|_{T\smallsetminus\{t^o\}}(t^\sharp),\tau(t^\sharp))\mid(t^\sharp,t)\in p\}\\
&=\{(\tau(t),\tau(t^\sharp),\tau(t^\sharp))\mid(t^\sharp,t)\in p\}\\
&\subseteq\{(\tau(t),\tau(t^\sharp),\tau(t^\sharp))\mid(\tau(t^\sharp),\tau(t))\in p'\}\\
&\subseteq\{(t',t'^\sharp,t'^\sharp)\mid(t'^\sharp,t')\in p'\}\\
&=\otimes'.
\end{aligned}$$

The first equality holds by (73b). The second equality holds because the domain of p is $T\smallsetminus\{t^o\}$ by (8a) and (1a). The first set inclusion holds by (8c). The second set inclusion holds by (8b). The final equality is (73d). ■

Lemma D.5. E *is a well-defined functor from* **Tree** *to* $\mathbf{NCP_p}$, *where* E *is defined in Theorem 3.13.*

Proof. By Lemma D.3, E_0 maps objects of **Tree** to objects of $\mathbf{NCP_p}$. By Lemma D.4, E_1 maps arrows of **Tree** to arrows of $\mathbf{NCP_p}$. Thus it remains to show [1] that E preserves identities, [2] that E preserves sources and targets, and [3] that E preserves compositions.

[1]. Take any functioned tree (T,p) with its t^o. Note

$$\mathsf{E}_0(T,p)=(T,C,\otimes), \tag{74a}$$
$$\text{where } C=T\smallsetminus\{t^o\} \text{ and} \tag{74b}$$
$$\otimes=\{(t,t^\sharp,t^\sharp)|(t^\sharp,t)\in p\}. \tag{74c}$$

by the definition of E_0. I argue

$$\begin{aligned}
&\mathsf{E}_1(\mathsf{id}_{(T,p)})\\
&\quad=\mathsf{E}_1([(T,p),(T,p),\mathsf{id}_T])\\
&\quad=[\mathsf{E}_0(T,p),\mathsf{E}_0(T,p),\mathsf{id}_T,\mathsf{id}_T|_{T\smallsetminus\{t^o\}}]\\
&\quad=[\mathsf{E}_0(T,p),\mathsf{E}_0(T,p),\mathsf{id}_T,\mathsf{id}_{T\smallsetminus\{t^o\}}]\\
&\quad=[(T,C,\otimes),(T,C,\otimes),\mathsf{id}_T,\mathsf{id}_C]\\
&\quad=\mathsf{id}_{(T,C,\otimes)}\\
&\quad=\mathsf{id}_{\mathsf{E}_0(T,p)}.
\end{aligned}$$

The first equation holds by the definition of id in **Tree**. The second holds by the definition of E_1. The third holds since the final clause in the definition of E_1 implies that the codomain of $\mathsf{id}_T|_{T\smallsetminus\{t^o\}}$ is $T\smallsetminus\{t^o\}$. The fourth holds by (74a) and (74b). The fifth holds by the definition of id in $\mathbf{NCP_p}$. The sixth holds by (74a).

[2]. Take any functioned-tree morphism $[(T,p),(T',p'),\tau]$. Let (T,p) determine t^o. For sources,

$$\begin{aligned}&\mathsf{E}_1([(T,p),(T',p'),\tau])^{\mathsf{src}}\\&\quad=\ [\mathsf{E}_0(T,p),\mathsf{E}_0(T',p'),\tau,\tau|_{T\smallsetminus\{t^o\}}]^{\mathsf{src}}\\&\quad=\ \mathsf{E}_0(T,p)\\&\quad=\ \mathsf{E}_0([(T,p),(T',p'),\tau]^{\mathsf{src}}),\end{aligned}$$

where the first equality holds by the definition of E_1, the second by the definition of src in $\mathbf{NCP_p}$, and the third by the definition of src in **Tree**. A symmetric argument shows that targets are preserved.

[3]. Take any two functioned-tree morphisms $[(T,p),(T',p'),\tau]$ and $[(T',p'),(T'',p''),\tau']$. Let (T,p) determine t^o, let (T',p') determine t'^o, and let (T'',p'') determine t''^o. To align with the last clause in the definition of E_1, let the codomain of $\tau|_{T\smallsetminus\{t^o\}}$ be $T'\smallsetminus\{t'^o\}$, let the codomain of $\tau'|_{T'\smallsetminus\{t'^o\}}$ be $T''\smallsetminus\{t''^o\}$, and let the codomain of $(\tau'\circ\tau)|_{T\smallsetminus\{t^o\}}$ be $T''\smallsetminus\{t''^o\}$. I then argue

$$\begin{aligned}&\mathsf{E}_1(\ [(T',p'),(T'',p''),\tau']\circ[(T,p),(T',p'),\tau]\ )\\&\quad=\mathsf{E}_1(\ [(T,p),(T'',p''),\tau'\circ\tau]\ )\\&\quad=[\mathsf{E}_0(T,p),\mathsf{E}_0(T'',p''),\tau'\circ\tau,(\tau'\circ\tau)|_{T\smallsetminus\{t^o\}}]\\&\quad=[\mathsf{E}_0(T,p),\mathsf{E}_0(T'',p''),\tau'\circ\tau,\tau'|_{T'\smallsetminus\{t'^o\}}\circ\tau|_{T\smallsetminus\{t^o\}}]\\&\quad=[\mathsf{E}_0(T',p'),\mathsf{E}_0(T'',p''),\tau',\tau'|_{T'\smallsetminus\{t'^o\}}]\circ[\mathsf{E}_0(T,p),\mathsf{E}_0(T',p'),\tau,\tau|_{T\smallsetminus\{t^o\}}]\\&\quad=\mathsf{E}_1([(T',p'),(T'',p''),\tau'])\circ\mathsf{E}_1([(T,p),(T',p'),\tau]).\end{aligned}$$

The first equality holds by the definition of $\circ$ in **Tree**. The second holds by the definition of E_1. The third is a rearrangement. The fourth holds by the definition of $\circ$ in $\mathbf{NCP_p}$. The fifth holds by two applications of the definition of E_1. ∎

LEMMA D.6. *Define the functors $\mathsf{F_p}$ and E as in Theorem 3.13. Then $\mathsf{E}\circ\mathsf{F_p}$ is naturally isomorphic to the identity functor for $\mathbf{NCP_p}$. In particular, for every object Π in $\mathbf{NCP_p}$, define the quadruple*

$$\eta_\Pi=[\Pi,\mathsf{E}_0\circ\mathsf{F}_{\mathsf{p}0}(\Pi),\mathsf{id}_T,q^{-1}],$$

where q is the previous-choice function of $\Pi=(T,C,\otimes)$. Then (a) for every object Π in $\mathbf{NCP_p}$, η_Π is an isomorphism in $\mathbf{NCP_p}$. Further (b) for every arrow $\alpha=[\Pi,\Pi',\tau,\delta]$ in $\mathbf{NCP_p}$,

$$(\mathsf{E}_1\circ\mathsf{F}_{\mathsf{p}1})(\alpha)\circ\eta_\Pi=\eta_{\Pi'}\circ\alpha.$$

PROOF. The functors $\mathsf{F_p}$ and E are well-defined by Theorem 3.9 and Lemma D.5.

(a). Take any perfect-information preform $\Pi=(T,C,\otimes)$ with its t^o, p, and q. Importantly, Lemma C.2, Lemma D.2, and the perfect information of Π imply

$$q:T\smallsetminus\{t^o\}\to C\text{ is bijective.}\tag{75}$$

Further, by the definitions of F_{p0} and E_0,

$$\mathsf{E}_0\circ\mathsf{F}_{\mathsf{p0}}(\Pi) = \mathsf{E}_0(T,p) = (T, T\smallsetminus\{t^o\}, \{(t,t^\sharp,t^\sharp)|(t^\sharp,t)\in p\}).$$

By the definition of η_Π and the previous sentence,

$$\begin{aligned}\eta_\Pi &= [\ \Pi,\ \mathsf{E}_0\circ\mathsf{F}_{\mathsf{p0}}(\Pi),\ \mathsf{id}_T,\ q^{-1}\ ] \\ &= [\ (T,C,\otimes),\ (T,T\smallsetminus\{t^o\},\{(t,t^\sharp,t^\sharp)|(t^\sharp,t)\in p\}),\ \mathsf{id}_T,\ q^{-1}\ ].\end{aligned} \tag{76}$$

This paragraph shows that η_Π is a morphism in $\mathbf{NCP_p}$. By definition (17) and the equalities of (76), this is equivalent to showing

$$\Pi \text{ and } \mathsf{E}_0\circ\mathsf{F}_{\mathsf{p0}}(\Pi) \text{ are perfect-information preforms,} \tag{77a}$$

$$\mathsf{id}_T : T\to T,\ q^{-1}: C\to T\smallsetminus\{t^o\},\ \text{and} \tag{77b}$$

$$\{(t,q^{-1}(c),t^\sharp)|(t,c,t^\sharp)\in\otimes\}\ \subseteq\ \{(t,t^\sharp,t^\sharp)|(t^\sharp,t)\in p\}. \tag{77c}$$

(77a) holds since Π is a perfect-information preform by assumption and since $\mathsf{E}_0\circ\mathsf{F}_{\mathsf{p0}}(\Pi)$ is a perfect-information preform by Lemma D.3. The first half of (77b) is obvious. The second half of (77b) follows from (75). Finally, for (77c), I argue

$$\begin{aligned}&\{(t,q^{-1}(c),t^\sharp)\mid(t,c,t^\sharp)\in\otimes\}\\ &\quad\subseteq\{(t,t^\sharp,t^\sharp)\mid(\exists c)\ (t,c,t^\sharp)\in\otimes\}\\ &\quad=\{(t,t^\sharp,t^\sharp)\mid(t^\sharp,t)\in p\}.\end{aligned}$$

To see the set inclusion, take any $(t,c,t^\sharp)\in\otimes$. By the definition of q, $c=q(t^\sharp)$. Thus by (75), $q^{-1}(c)=t^\sharp$. The equality follows from the definition of p.

Lastly, this paragraph shows that η_Π is an isomorphism in $\mathbf{NCP_p}$. By the previous paragraph and Theorem 3.7, it suffices to show that the transformations id_T and q^{-1} are bijective. Obviously, id_T is bijective. Further, q^{-1} is bijective by (75).

(b). Consider any arrow $\alpha=[\Pi,\Pi',\tau,\delta]$ in $\mathbf{NCP_p}$. Suppose $\Pi=(T,C,\otimes)$ and derive its t^o and q. Similarly, suppose $\Pi'=(T',C',\otimes')$ and derive its t'^o and q'.

First, I argue

$$\begin{aligned}&(\mathsf{E}_1\circ\mathsf{F}_{\mathsf{p1}})(\alpha)\\ &\quad=(\mathsf{E}_1\circ\mathsf{F}_{\mathsf{p1}})([\Pi,\Pi',\tau,\delta])\\ &\quad=\mathsf{E}_1([\mathsf{F}_{\mathsf{p0}}(\Pi),\mathsf{F}_{\mathsf{p0}}(\Pi'),\tau])\\ &\quad=[\mathsf{E}_0\circ\mathsf{F}_{\mathsf{p0}}(\Pi),\mathsf{E}_0\circ\mathsf{F}_{\mathsf{p0}}(\Pi'),\tau,\tau|_{T\smallsetminus\{t^o\}}].\end{aligned} \tag{78}$$

The first equality holds by the definition of α. The second equality holds by the definition of F_{p1}. The third equality holds [1] by the definition of E_1, and [2] because the root node of $\mathsf{F}_{\mathsf{p0}}(\Pi)$ is t^o by Lemma C.1(b).

Second, Proposition 3.4(22c) for α states that

$$q'\circ\tau|_{T\smallsetminus\{t^o\}}=\delta\circ q$$

holds in **Set** (the proposition takes the codomain of $\tau|_{T\smallsetminus\{t^o\}}$ to be $T'\smallsetminus\{t'^o\}$). Also, Lemma D.2 and the perfect information of Π imply that q is bijective. Similarly, q'

is bijective. By the previous three sentences,

$$\tau|_{T\smallsetminus\{t^o\}}\circ q^{-1} = (q')^{-1}\circ\delta. \tag{79}$$

Finally, I argue

$$\begin{aligned}
&(\mathsf{E}_1\circ\mathsf{F}_{\mathsf{p}1})(\alpha)\circ\eta_{\varPi} \\
&\quad= [\mathsf{E}_0\circ\mathsf{F}_{\mathsf{p}0}(\varPi), \mathsf{E}_0\circ\mathsf{F}_{\mathsf{p}0}(\varPi'), \tau, \tau|_{T\smallsetminus\{t^o\}}]\circ[\varPi, \mathsf{E}_0\circ\mathsf{F}_{\mathsf{p}0}(\varPi), \mathsf{id}_T, q^{-1}] \\
&\quad= [\varPi, \mathsf{E}_0\circ\mathsf{F}_{\mathsf{p}0}(\varPi'), \tau, \tau|_{T\smallsetminus\{t^o\}}\circ q^{-1}] \\
&\quad= [\varPi, \mathsf{E}_0\circ\mathsf{F}_{\mathsf{p}0}(\varPi'), \tau, (q')^{-1}\circ\delta] \\
&\quad= [\varPi', \mathsf{E}_0\circ\mathsf{F}_{\mathsf{p}0}(\varPi'), \mathsf{id}_{T'}, (q')^{-1}]\circ[\varPi, \varPi', \tau, \delta] \\
&\quad= \eta_{\varPi'}\circ\alpha.
\end{aligned}$$

The first equality holds by (78) and the lemma's definition of $\eta_{\varPi}$. The second equality holds by the definition of $\circ$ in $\mathbf{NCP_p}$. The third equality holds by (79). The fourth equality holds by the definition of $\circ$ in $\mathbf{NCP_p}$. The fifth equality holds by the definitions of $\eta_{\varPi'}$ and α. ■

LEMMA D.7. *Define the functors* F_p *and* E *as in Theorem 3.13. Then* $\mathsf{F}_\mathsf{p}\circ\mathsf{E}$ *equals the identity functor for* **Tree**.

PROOF. The functors F_p and E are well-defined by Theorem 3.9 and Lemma D.5. Step 0 will show that $\mathsf{F}_{\mathsf{p}0}\circ\mathsf{E}_0$ maps each functioned tree to itself. Step 1 will show that $\mathsf{F}_{\mathsf{p}1}\circ\mathsf{E}_1$ maps each functioned-tree morphism to itself.

Step 0 Take any functioned tree (T,p) with its t^o. By the definition of E_0, $\mathsf{E}_0(T,p) = (T,C,\otimes)$ where

$$C = T\smallsetminus\{t^o\} \text{ and } \otimes = \{(t,t^\sharp,t^\sharp) \mid (t^\sharp,t)\in p\}.$$

Thus, by the definition of $\mathsf{F}_{\mathsf{p}0}$, $\mathsf{F}_{\mathsf{p}0}\circ\mathsf{E}_0(T,p) = (T,p^*)$, where

$$p^* = \{(t^\sharp,t) \mid (\exists c)\ (t,c,t^\sharp)\in\otimes\}.$$

Hence it suffices to show that $p^* = p$. I argue

$$\begin{aligned}
p^* &= \{(t^\sharp,t) \mid (\exists c)\ (t,c,t^\sharp)\in\otimes\} \\
&= \{(t^\sharp,t) \mid (\exists c)\ (t,c,t^\sharp)\in\{(t,t^\sharp,t^\sharp)|(t^\sharp,t)\in p\}\} \\
&= \{(t^\sharp,t) \mid (t^\sharp,t)\in p\} \\
&= p,
\end{aligned}$$

where the first equality is the definition of p^*, the second follows from the definition of $\otimes$, and the third holds by the equivalence of the sets' predicates.

Step 1 Take any functioned-tree morphism $[(T,p),(T',p'),\tau]$, and let (T,p) determine t^o. Then

$$\begin{aligned}
&\mathsf{F}_{\mathsf{p}1}\circ\mathsf{E}_1([(T,p),(T',p'),\tau]) \\
&\quad= \mathsf{F}_{\mathsf{p}1}([\mathsf{E}_0(T,p),\mathsf{E}_0(T',p'),\tau,\tau|_{T\smallsetminus\{t^o\}}])
\end{aligned}$$

$$= [\mathsf{F}_{\mathsf{p0}}\circ\mathsf{E}_0(T,p), \mathsf{F}_{\mathsf{p0}}\circ\mathsf{E}_0(T',p'), \tau]$$
$$= [(T,p),(T',p'),\tau],$$

where the first equality follows from the definition of E_1, the second equality follows from the definition of F_{p1}, and the third equality follows from two applications of Step 0. ■

Proof D.8. (for Theorem 3.13) (a) is established by Lemma D.6. (b) is established by Lemma D.7. ■

Proof D.9. (for Corollary 3.14) (a) follows from Theorem 3.13(a) because $(\mathsf{E}\circ\mathsf{H})\circ(\mathsf{G}\circ\mathsf{F}_\mathsf{p}) = \mathsf{E}\circ\mathsf{F}_\mathsf{p}$ by Theorem 2.8. (b) follows from Theorem 2.8 because $(\mathsf{G}\circ\mathsf{F}_\mathsf{p})\circ(\mathsf{E}\circ\mathsf{H}) = \mathsf{G}\circ\mathsf{H}$ by Theorem 3.13(b). ■

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