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"Inference versus consequence" revisited: inference, consequence, conditional, implication

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Abstract *Inference versus consequence*, an invited lecture at the LOGICA 1997 conference at Castle Liblice, was part of a series of articles for which I did research during a Stockholm sabbatical in the autumn of 1995. The article seems to have been fairly effective in getting its point across and addresses a topic highly germane to the Uppsala workshop. Owing to its appearance in the *LOGICA Yearbook 1997*, Filosofia Publishers, Prague, 1998, it has been rather inaccessible. Accordingly it is republished here with only bibliographical changes and an afterword.

Keywords Inference · Consequence · Validity · Judgement · Proposition · Type theory

The following passage, hereinafter "the passage", could have been taken from a modern textbook.[1] It is prototypical of current logical orthodoxy:

> The inference
> (*) $A_1, \ldots, A_k$. Therefore: C
> is valid if and only if
> whenever all the premises $A_1, \ldots, A_k$ are true, the conclusion C is true also.
> When (*) is valid, we also say that C is a logical consequence
> of $A_1, \ldots, A_k$.
> We write $A_1, \ldots, A_k \models C$.

It is my contention that the passage does not properly capture the nature of inference, since it does not distinguish between valid inference and logical consequence. The

[1] Could have been so taken and almost was; cf. Tennant (1978, p. 2). In order to avoid misunderstanding let me note that I hold Tennant's book in high regard.

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view that the validity of inference is reducible to logical consequence has been made famous in our century by Tarski, and also by Wittgenstein in the *Tractatus* and by Quine, who both reduced valid inference to the logical truth of a suitable implication.[2] All three were anticipated by Bolzano.[3]

Bolzano considered *Urteile* (judgements) of the form
A is true

where A is a *Satz an sich* (proposition in the modern sense).[4] Such a judgement is correct (*richtig*) when the proposition A, that serves as the judgemental content, really is true.[5] A correct judgement is an *Erkenntnis,* that is, a piece of knowledge.[6] Similarly, for Bolzano, the general form I of inference

$$\frac{\mathrm{J}_1, \ldots, \mathrm{J}_\mathrm{k}}{\mathrm{J},}$$

where $\mathrm{J}_1, \ldots, \mathrm{J}_\mathrm{k}$ are judgements, becomes I′:

$$\frac{\mathrm{A}_1 \text{ is true}, \ldots, \mathrm{A}_\mathrm{k} \text{ is true}}{\mathrm{C} \text{ is true},}$$

where $\mathrm{A}_1, \ldots, \mathrm{A}_\mathrm{k}$, and C are propositions. The inference I' is valid when C is a logical consequence of $\mathrm{A}_1, \ldots, \mathrm{A}_\mathrm{k}$.[7] This is the notion of logical consequence that is explained in the passage: whenever all the antecedent propositions are true, the consequent proposition C is true also.[8]

One should note, however, that propositions and judgements are conflated in the passage. The relata in logical consequence are propositions, whereas an inference effects a passage from known judgements to a novel judgement that becomes known in virtue of the inference in question. Frege wrote:

Ein Schluss … ist eine Urteilsfällung, die auf grund schon früher gefällter Urteile nach logischen Gesetzen vollzogen wird. Jede der Prämissen ist ein bestimmter

[2] Tarski (1936); Wittgenstein, *Tractatus* 5.11, 5.132; Quine (1951, p. 7).

[3] Bolzano (1837).

[4] A proposition in the old sense is a *judgement*, usually of the [subject/copula/predicate] form **S is P** and its linguistic correlate is a complete declarative sentence, for instance, *Snow is white*. A proposition in the modern sense is not itself a judgement, but serves as the content of a judgement of the modern form **A is true**. Its linguistic correlate is a that-clause, for instance, *that snow is white*. The term 'proposition' without further qualification will be taken in the modern sense of a *Satz an sich* that was introduced by Bolzano (WL, §19).

[5] WL (§34).

[6] WL (§36).

[7] Bolzano's term was *Ableitbarkeit*, WL (§155(2)). The literal translation 'derivability' would prove too confusing against the background of current practice which uses the two metamathematical turnstiles $|=$ and $|-$. The semantic double turnstile is the analogue of Bolzano's *Ableitbarkeit*, whereas the (modern, non-Fregean) single turnstile expresses syntactic derivability according to certain derivation rules.

[8] As a representation of Bolzano this is substantially but not literally correct: Bolzano imposed certain compatibility conditions on the antecedents in *Ableitbarkeiten* that need not detain us further in the present context.

als wahr anerkannter Gedanke, und im Schlussurteil wird gleichfalls ein bestimmter Gedanke als wahr anerkannt.[9]

An *Erkenntnis*—what is known—is a judgement and may be of the form that a proposition is true.[10] Such a piece of knowledge gets known, or is obtained, in an act of judgement. Similarly, in an inference-act, the conclusion-judgement gets known on the basis of previously known premiss-judgements: the inference is an act of mediate judgement.

Thus we have two Bolzanian reductions, namely (i) that of the correctness of the judgement to that of the truth of the propositional content and (ii) that of the validity of an inference between judgements to a corresponding logical consequence among suitable propositions. From an epistemological point of view, we get the problem that the reduced notions may obtain *blindly*. This happy term was coined by Brentano for the case when an assertion without ground happens to agree with an evidenceable judgement.[11] An example would be when I hazard a guess as to the size of the fortune of a former Dutch premier and by fluke happen to hit bull's eye, even though my knowledge of the financial situation of Dutch statesmen is nil. On the Bolzano reduction, this unsubstantiated claim would be an *Erkenntnis,* in spite of its being completely unwarranted. In the same way, an act of inference between judgements whose contents happened to be true and happened to stand in the relation of logical consequence would be valid, even though no epistemic warrant had been offered.

Blind correct judgement—be it mediate or not—is not to my taste, whence I am concerned to find other explications of judgemental correctness and inferential validity that do not admit of such blindness. By the side of Bolzano, Frege is virtually the only other modern logician that is of any help in the philosophical study of the notion of inference. In my opinion his much decried view that inference starts from true, nay, known, premisses contains an important insight:

> Aus falschen Praemissen kann überhaupt nichts gesclossen werden. Ein blosser Gedanke, der nicht als wahr anerkannt ist, kann überhaupt nicht Praemisse sein. … Blosse Hypothesen können nicht als Praemissen gebraucht werden.[12]

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[9] Frege (1906, p. 387). (My) English translation:

An inference … is an act of judgement that is drawn according to logical laws from judgements previously made. Each premiss is a certain proposition which has been recognised as true, and also in the conclusion-judgement a certain proposition is recognised as true.

[10] Following Martin-Löf (1996, p. 26), I explain a judgement in terms of the knowledge required for having the right to make it. Alternatively the explanation might run in terms of what one has to do (namely, acquire the knowledge in question) in order to have the right to make the judgement in question.

[11] *Wahrheit und Evidenz*, Felix Meiner, Hamburg, 1974^{II} (1930, p. 135).

[12] Letter to Jourdain, Frege (1976, p. 118). (My) English translation:

Nothing at all can be inferred from false premisses. A mere thought, that has not been recognised as true, cannot be a premiss. … Mere hypotheses cannot be premisses.

Properly understood, this Fregean insight does not contradict Gentzen's views—when *they* are properly understood—concerning the use of assumptions within so called natural deduction derivations.[13] In general these derivations depend on open assumptions: accordingly the endfomula of a derivation-tree will express a proposition that is not true outright, but only dependently true, that is, true, *given* the truth of the propositions expressed by the assumption-formulae. Thus, the form of judgement used by Gentzen in his system of natural deduction is not

A is true,

but

C is true, provided that $A_1, \ldots, A_k$ are true.

Hence an inference effects an act of passage between known judgements of the latter dependent form, whence there is no contradiction with Frege. In Gentzen's sequential version of natural deduction, on the other hand, the form of the conclusion-judgement that is demonstrated is better thought of as being

S holds,

where the sequent S expresses a consequence.

However, in order to find further genuinely relevant views one has to turn to the Scholastics. Towards the end of the 13th century tracts entitled *De Consequentiis* begin to appear, by such authors as William of Ockham, Walter Burleigh, Richard Billingham, Ralph Strode, John Buridan, Marsilius of Inghen, Paul of Venice, ... A *consequence* is a hypothetical proposition (in the old sense) which can be recognised through the use of certain indicator words:

Indicator	**Example**	**Modern analogue**
Si (if)	If A, then B	conditional
Sequitur (follows)	From A follows B	consequence
Quia (because)	B because A	causal grounding
Igitur (therefore)	A. Therefore B	inference

These were all variants of one and the same notion. Thus, where today we would formulate four different theories with various and sometimes conflicting principles, the scholastics sought for principles that covered all four (modern) notions. An example of such a principle is, of course, *modus ponens*, which from the premisses A and the consequence of A and B draws the conclusion B.

Today one would say that

- a conditional is a proposition that may be true;
- a consequence is a relation between propositions that may hold[14];

[13] Gentzen (1934–1935); Gentzen (1936).

[14] *Tenere* is the term that the scholastics applied to a *consequentia.*

- causal grounding is a relation (between states of affairs) that may obtain;
- an inference is an act of passage from judgement(s) to judgement that may be valid.

The task I set myself is to elucidate relationship between the second and fourth notions among these four alternatives.

One can discern two views concerning *consequentia* and their validity (holding) in the medieval logical tradition:[15]

(i) the containment theory which was adumbrated by Peter Abelard and advocated by "English" logicians at Padua from 1400 onwards;
(ii) the incompatibility theory, which is of Stoic origin and was advocated by Parisian logicians around 1400.

Aristotle held that in a valid syllogism, when the premisses are true, necessarily the conclusion must be true.

The Stoics refined this into:

[A. **Therefore** B] is *valid*
if and only if
A is true and B is false are *incompatible*.

Using elementary modal logic and Boolean combinations,

[A. **Therefore** B] is valid **iff**
$\neg\Diamond$ (A is true **and** B is false) **iff**
$\Box\neg$ (A is true **and** B is false) **iff**
$\Box$ (**if** A is true, **then** B is true).

When the necessity $\Box$ is read as "holds in every variant", or "in all terms", ordinary (Bolzano) logical consequence is the result. Thus on the *Incompatibility Theory*, inferential validity is reduced to the logical holding, that is, holding in all alternatives, of the consequence from A to B:

The inference [A. **Therefore** B] is valid
when the consequence $A| = B$ holds formally (*in omnibus terminis*).

Essentially, this is the theory that we found in Bolzano, Tarski, and Quine: the theory from the passage is an intellectual descendant of the medieval incompatibility theory thus construed. I am not satisfied with this reduction, though, since the above difficulties concerning blindly valid inference remain unresolved. Logic is an epistemological tool for obtaining new knowledge from known premisses. The incompatibility theory does not fully acknowledge this epistemic aspect of logic: the (logical) holding of a consequence, as well as propositional truth, will (in general) be "evidence transcendent".[16] In modern terms the incompatibility theory pertains not so much to the validity of inferences as to the (logical) holding of consequences.

Inference, like judgement, is primarily an act: one *draws* an inference and *makes* a judgement.[17] We have the diagram:

[15] The distinction was drawn by Martin (1986, pp. 564–572), and used by Boh (1993).

[16] The felicitous term 'evidence transcendent' derives from the realism/anti-realism debate: cf. Wright (1987, p. 2).

[17] Cf. the quote from Frege (1906) offered at footnote. 9.

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_____act_____ object.
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The object, however, is not the only objective correlate of the act. Coupled to the exercised act, the subject(ive) process, there is also the objective signified act, that is, the trace, or track, of the subjective act:[18]

```
         act    ————————> object
       /      \
subjective    objective
process       trace.
```

When applied to a concrete example, for instance, the preparation of a Sauce Béarnaise, this abstract scheme becomes concrete as:

```
                        (act)
                  Preparation of
                  Sauce Béarnaise       ————————>     the sauce (object)
                  /               \
Cooking (process)                 (trace)
                                   /      \
                  set of dirty pans,    blue-print, recipe
                  spats of butter,
                  twigs of tarragon
```

As we see the act-trace can be taken in two senses:

(i) as the actual (concrete) trace of the exercised act, and
(ii) as the blue-print of the signified act.[19]

This battery of distinctions can now be applied to the act of demonstration (judgement):

```
                 Act of proof
                 Act of judgement                 Theorem
                 Demonstration        ——————>     Judgement made
      /                               \
process of                            (trace)
getting to                             /      \
know                 set of coffee-cups,   written proof
(exercised act)      chalk-marks,          in mathematical text
                     scrap-papers          (signified act)
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The object (product) of an act of judgement (demonstration) is the judgement made (theorem proved). Also an act of inference, though, has a theorem (judgement) as its product. An inference-figure is not so much the product as the trace of an act of inference. An inference, be it immediate or not, is a mediate act of judgement. Inferences are discursive (acts of) judgement. Immediate, or intuitive, acts of judgement, on the other hand, have axioms as products, that is, known judgements that rest upon no other knowledge. Following Martin-Löf, a judgement is actually true when it is

[18] I am indebted to Per Martin-Löf for drawing my attention to this notion of an act-trace. He spoke about it in an as yet unpublished lecture in Paris, April 1992.

[19] Concerning 'The distinction *actus exercitus/actus significatus* in medieval semantics', see Nuchelmans (1988, pp. 57–90).

known (evident) and potentially true when it can be made evident (is evidenceable, justifiable, warrantable, demonstrable, knowable, etc.).[20] This notion of potential truth of a judgement corresponds to the "objective correctness" of a statement or assertion that is familiar from the anti-realist literature.[21]

With these distinctions at our disposal we can now deal with the other proposal for inferential validity, namely the *Containment Theory*:

> An inference is valid when the conclusion is "contained" (in some suitable sense) in the premisses.

Already Aristotle used an idea of this kind when he wished to ground the validity of a syllogism in the existence of a chain of linking terms.

It is often said that a valid inference is a truth-preserving one. What kind of truth has to be preserved? True propositions? Actually true judgements? Objectively correct judgements? Preservation of propositional truth can hardly be what is at issue here: that gives us not the validity of an inference, but the holding of a consequence. Preservation of actual truth for judgements is also ruled out as an explication of inferential validity. On such an account the completely general inference I above would be valid when the premisses $J_1, \ldots, J_k$ are *unknown*.

Preservation of objective correctness, that is, potential truth for judgements, is the only viable option. The question remains *how* such truth is going to be preserved from the premisses to the conclusion of a valid inference. Scholastic logic proves helpful also here. Robert Kilwardby (ca. 1215–1279) writes:

> Consequence is twofold, namely essential or natural, as when a consequent is naturally understood in its antecedent, and accidental consequence.[22]

This, I take it, is an early formulation of the reduction of valid inference to analytic containment: when the premisses of the inference are understood and known, and the conclusion is understood, that is, one knows the definitions of the essences of the terms that occur in the conclusion, nothing more is called for in order to come to know the conclusion. It is analytically contained in the premisses. We have then an instance of an inference per se nota, whose evidence is not founded upon anything but the knowledge of the terms out of which the judgements of the inference has been put together: the inference accordingly rests upon *evidentia ex terminis*.

In his attack upon the notion of analyticity, Quine remarked that

> meaning is what essence becomes when it is divorced from the object of reference and wedded to the word.[23]

20 Martin-Löf (1998).

21 Dummett (1976, pp. 119–120).

22 Quoted from Bochenski (1970, §30.07, p. 190). Latin text in Kneale and Kneale (1961, p. 275).

23 Quine (1963, p. 22).

This linguistic turn transforms the evidence conferred through the understanding of natures (essences) into "self-evidence in virtue of meaning".[24] Not every inference, though, will be conceptually self-evident from meaning. Only an immediate inference, that is, an inference that is not supportable further by other inferences has this character. Examples are the standard introduction and elimination rules for the intuitionistic logical constants.[25]

Consider the completely general inference-figure I once more:

$$\frac{J_1, \ldots, J_k}{J.}$$

What does it mean for I to be valid?[26] We consider how an inference according to I is used. In such use one takes it for granted that the premisses $J_1, \ldots, J_k$ are known and goes on to obtain knowledge of J. Thus, under the epistemic assumption that the judgements $J_1, \ldots, J_k$ are all known, one has to make the judgement J known.[27] Given the knownness of $J_1, \ldots, J_k$, the knowability of J is secured through a chain of immediately evident axioms and inferences that begins in the premisses and ends in the conclusion. In order to have the right to infer according to I, one must posses the chain in question. When such a chain can be found, the inference-schema (as signified act) is potentially valid. For the exercised act this is not enough: then one needs the actual validity. One must actually possess the chain of immediate evidences, be they

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24 *Evidence* is here taken in the sense of *the property of being evident* and not in the sense *support for the truth of a proposition.*

25 See Martin-Löf's (1996) treatment. The justification of the elimination rules in terms of the introduction rules does not constitute a *derivation* of the former from the latter. To know the meaning of an intuitionistic propositional connective C is to know how canonical, that is, introductory, *proof-objects* for propositions of C-form may be put together (and when two such introductory proofs are equal). That knowledge is enough to make plain the validity of the (immediate) elimination *inferences.* Note further that the introduction-/elimination-rule distinction operates on two different levels. On the one hand, on the level of propositions, it concerns how propositional proof-objects may be put together; for instance when a is a proof-object for A and when b is a proof-object for b, then &I(A, B, a, b) is a proof-object for A&B. On the other hand, at the epistemic level of judgements and inferences, it concerns for instance the inference rules

A is true, B is true. Therefore: A&B is true

and

A&B is true. Therefore: B is true,

or, when we use the fully explicit for of judgement including the proof-objects:

$$\frac{c{:}A\&B}{q(c){:}B}$$

26 Martin-Löf's (1987) notion "validity of a proof" is different from the validity of an inference. The former notion results from applying the notion of *rightness* to proofs: a valid (right, real, true, conclusive, …) proof is one in which each axiom really is true and each inference really is valid.

27 Note the difference between alethic assumptions that propositions are true and epistemic assumptions that judgements are known (knowable). The former are used in natural-deduction consequences between propositions. The latter are used when making evident the validity of inferences.

axiomatic or inferential, and actually carry out each of the immediate component steps thereof.[28]

1 Afterword

1.1 Implication, conditional and consequence

Inference versus consequence stressed the distinction between inference from judgement to judgement and (logical) consequence among propositions, while resisting the customary reduction of inferential validity to the holding of consequence, be it logical or not, that is, the blind preservation of truth from antecedent propositions to consequent proposition, possibly under all variations. Other articles of mine considered also the implicational proposition, and drew a further distinction between open and closed consequences, which are connected to the two different styles of natural deduction derivation that are familiar from the works of Genzten.[29]

The *vernacular* conditional is naturally expressed by means of an *if-then* construction. This mode of expression, however, prevents the conditional from taking that-clauses as arguments: complete declaratives are called for, for instance, as in:

If grass is green, then snow is white,

whereas nonsense result from using that-clauses:

If that grass is green, then that snow is white.

Here we have to draw upon truth in order to restore grammaticality:

If that grass is green is true, then that snow is white is true.

Accordingly, when A and B are propositions, the conditional is regimented as:

if A is true, then B is true

or as:

	given presupposed provided	
B is true	on condition dependent on	that A is true
	under hypothesis under assumption	

The conditional is a form of judgement: just as we get a judgment

A is true

by applying the form

28 The picture outlined in the present paper is presented in more detail in my articles (1996, 1998, 2000).

29 See also Sundholm (1997, 2006).

… is true

to the proposition A, a judgement

B is true, on condition that A is true,

results from applying the *dependent* form of judgement

… is true, on condition that A is true

to the proposition B. As a suitable notation we may here use

B true (A true),

mirroring the notation of Martin-Löf's type theory for dependent objects in contexts

$$c:C \quad (x_1: A_1, \ldots, x_k: A_k),$$

that is, c is a proof of C, on condition (assumption) that $x_1, \ldots, x_k$, respectively, are proofs of $A_1, \ldots, A_k$, which corresponds to the general dependent truth

C is true, on condition that A_1 is true…A_k is true.[30]

Gentzen's notion of a sequent (German *Sequenz)* $\Gamma \rightarrow$ C, where Γ stands for a list of propositions, indicates a relation of consequence between the antecedent proposition and the consequent proposition C. Accordingly we have here yet another extension of the form of judgement

proposition A is true

into

sequent $\Gamma \rightarrow$ *C holds*,

where *A is true* may now be seen as the special sequent case with an empty list of antecedents:

$\rightarrow$*A holds*.

The implication, finally, is a *proposition* (A $\supset$ B) that is made up from the connective $\supset$ and the constitutive propositions A and B.

The Bolzano reductions of inferential validity can be expressed using either of these three notions at the level of propositions. Commonly, the inference I is said to be valid if the matching sequent $A_1, \ldots, A_k \rightarrow$ C holds *logically*, "in all variations", but one could equally well use the *logical* truth of an iterated implication, and similarly for the dependent logical truth in all variations of the conditional B true (A_1 true, …, A_k true). One reason why it has proved difficult to tell these notions—implication, conditional, consequence—apart is that the matching judgements, as to truth, dependent truth, and holding, are *equiassertible*, that is, if one is assertible, then so are the other two. Furthermore, they, as well as the inference I of the conclusion C true, from the premises A_1true, …, A_k true, are all refuted by the same counter-example, namely a situation in which the antecedent propositions are true and the consequent is false.

30 See also the explanation of x + 5:N (x:N) in Sect. 1.3 below.

1.2 Constructive semantics and verification objects

My preferred constructive semantics, namely that of Per Martin-Löf, explains the notion of a proposition via the "Curry-Howard isomorphism".[31] To each proposition A belongs a type Proof(A) explained in terms of how *canonical* proof-objects for A may be put together from their parts (and what it is for two canonical proofs of A to be equal). In order to assert that the proposition A is true, it is not necessary to possess a canonical, or *direct*, proof of A; an indirect proof, such as those given by the elimination rules for the logical connectives, will also do, provided only that it admits evaluation to canonical form. Thus, for example, the crucial clause for *implication* says that, given a dependent proof-object b:Proof(B) (x:Proof(A)) one obtains a canonical proof-object for the proposition A ⊃ B by means of the ⊃-introduction rule:

$$\supset \mathrm{I}\,(\mathrm{A}, \mathrm{B}, (\mathrm{x})\mathrm{b}) : \mathrm{Proof}(\mathrm{A} \supset \mathrm{B}).$$

Here (x)b is a function obtained by *abstraction* from the dependent object b and, when a:Proof(A), it obeys the evaluation rule $(\mathrm{x})\mathrm{b}(\mathrm{a}) =_{\mathrm{df}} \mathrm{b}[\mathrm{a}/\mathrm{x}]$:Proof(B). Accordingly, we may justify the elimination rule ⊃ E (A, B, c, a):Proof(B), where c:Proof(A ⊃ B) and a:Proof(A), as follows. Since c is a proof-object it admits evaluation into canonical form:

$$\mathrm{c} \;= \supset \mathrm{I}\,(\mathrm{A}, \mathrm{B}, (\mathrm{x})\mathrm{b}){:}\mathrm{Proof}(\mathrm{A} \supset \mathrm{B}), \text{ for a suitable b. Hence,}$$

$$\supset \mathrm{E}\,(\mathrm{A}, \mathrm{B}, \mathrm{c}, \mathrm{a}) = \supset \mathrm{E}(\mathrm{A}, \mathrm{B}, \supset \mathrm{I}\,(\mathrm{A}, \mathrm{B}, (\mathrm{x})\mathrm{b}),\; \mathrm{a}) = (\mathrm{x})\mathrm{b}\,(\mathrm{a}) = \mathrm{b}\,[\mathrm{a}/\mathrm{x}]\,{:}\mathrm{Proof}\,(\mathrm{B}),$$

because b is a dependent proof of B, given x:Proof(A). This equation, as those schooled in the proof theory of Natural Deduction will recognize, is nothing but a linearization of Prawitz's ⊃-reduction.[32]

The assertion conditions for conditionals and consequences also ask for suitable *verification objects*. The conditional *B true* (*A true*) is verified by a *dependent* proof-object

b:Proof(B)(x:Proof(A)),

whereas the judgement *sequent A → B holds* is verified by a (higher-level) function, or mapping,

f:Proof(A) → Proof(B).

1.3 Different notions of function

These verification objects are all functions, but belong to *different* notions of function that are well known from the mathematical literature.[33] The verification objects

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31 See Martin-Löf (1984), Nordström et al. (1990), and Ranta (1994), for details and notations.

32 Prawitz (1971, 3.3.1.3, p. 252).

33 Rüthing (1984) offers a survey of different classical definitions of the notion of a function.

of conditionals are *dependent* objects of lowest level, that is, they are Euler(–Frege) "unsaturated" functions, which are given by analytical expressions in free variables, for instance, $x+5{:}N(x{:}N)$, that is, $x+5$ is a natural number given that (on condition that) x is a natural number, and application goes via *substitution,* for instance $(x+5)[2/x] = (2+5) = 7$. Similarly, the verification object of consequences are *independent* objects of higher level, that is, Riemann–Dedekind(–Church) general *mappings*, for instance $(x)x+5{:}N \to N$ is obtained by ("lambda"-) *abstraction*, and application goes via the primitive notion of application, $(x)x+5(2){:}N$, for which, in the present case, we may draw upon the properties of abstraction and substitution to obtain $(x)x+5(2) = (x+5)[2/x] = 2+5 = 7$.[34]

Finally, the proof-objects of implications are *courses-of-value* ("graphs"), that is, elements λ (A, B, (x)b)of Π-sets, and where application goes via an "application function" ap(x, y).[35] Frege's use of courses-of-value $\grave{\varepsilon}\varphi(\varepsilon)$ and the concomitant application function $x \wedge y$ such that $a \wedge \grave{\varepsilon}\varphi(\varepsilon) = \varphi(a)$, according to the *Grundgesetze* theorem that is a direct consequence of the fatal *Grundgesetz* V, is also of this kind. Similarly, the modern set-theoretic construal of functions as sets of ordered pairs that are unique in the second component needs to be supplemented with an application function, which itself *cannot be construed as a set of ordered pairs*. Here, for instance, Whitehead and Russell, Von Neumann, Bernays, Gödel, Quine, Shoenfield and Takeuti get it right, using, say, an elevated comma as application function $x'y$, for when x and y are sets.[36] For sets f which are function graphs, $f'a$ is then the second component b of the unique ordered pair $\langle a,b \rangle$ that belongs to the function graph f. However, on their own, graphs are just *sets* and cannot play the role of mappings.

1.4 Bolzano, Frege, and Gentzen

Bolzano's (1837) account of *Ableitbarkeit*, when taken in the sense of*logisch analytisch,* is a(n almost perfect) account of a consequence's holding logically. Today, after Gentzen, the consequent of a multiple conclusion consequence $A_1, \ldots, A_k \to B_1, \ldots, B_m$ is taken disjunctively, as $A_1 \& \cdots \& A_k \supset B_1 \vee \cdots \vee B_m$, whereas Bolzano preferred to read such consequences conjunctively as $A_1 \& \cdots \& A_k \supset B_1 \& \cdots \& B_m$. Furthermore, he also insisted that the antecedent propositions be compatible, a demand dropped by Gentzen.[37] Frege, on the other hand, did NOT consider (logical) *consequence*, but inference only, concerning which much criticism, for instance, by Dummett (1973, p. 309ff), has come his way. However, when we consider that Frege was not concerned with the alethic notion of consequence, but with the epistemic notion of inference, much of the criticism is beside the mark. In fact, it is only with Gentzen's (1936) *sequential account of Natural Deduction*—from his first consistency proof for arithmetic—that we get a theory that is able to cope both with inference and with consequence. Gentzen's account also makes it clear that holding of consequence, rather than

34 See Nordström et al. (1990, Chap. 3) and Ranta (1994, §8.2).

35 See Ranta (1994, p. 165).

36 See, for instance, Shoenfield (1967, p. 245).

37 See Siebel (1996) for an excellent study of of Bolzano's notion.

logical holding in all variations, is the central notion.[38] After all, his natural deduction sequents hold *arithmetically*, but certainly not in all variations. For instance, the rule of complete induction

$$\frac{\Gamma \rightarrow F(0) \qquad F(a), \Gamma \rightarrow F(a+1)}{\Gamma \rightarrow F(t)}$$

where a is an *eigen*-parameter, does not hold in all variations, but only arithmetically: when both premises hold arithmetically, then so does the conclusion.

1.5 Holding in "all variations"

The scholastics already knew that a consequence could hold in all terms—the Latin tag is *tenetur in omnibus terminis*. Such consequences were called formal. Terms, as they knew, can be taken in various suppositions, to wit *material* (syntactic), *simple* (conceptual), and *personal* (referential) supposition.[39] Modern theories of consequence to a surprising degree match these different kinds of variation. The theory of Carnap from *Logical Syntax* varies syntactic terms in *material supposition*, there called the *formal* mode of speech.[40] Bolzano's account varies *Vorstellungen an sich* (ideas-in-themselves) that are counterparts of words at the conceptual level, whence in *suppositio simplex*. Finally, Wittgenstein in the *Tractatus* varies components in the world, as does Tarski's model-theoretic account, whence the terms have referential use ("personal suppositions").[41]

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[38] Sundholm (2006), which was written in 1998, deals with the semantic interpretation of Gentzen's two styles of Natural Deduction formalisms in considerable detail.

[39] Bochenski (1970, §27), explains the various nations of supposition via well-chosen quotes.

[40] See Carnap (1937).

[41] See Tarski and Vaught (1957) for the model-theoretic account of logical consequence.

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