Quantum Covers in Quantum Measure Theory

Citation for published version:
Surya, S & Wallden, P 2010, 'Quantum Covers in Quantum Measure Theory', *Foundations of Physics*, vol. 40, no. 6, pp. 585-606. https://doi.org/10.1007/s10701-010-9419-1

Digital Object Identifier (DOI):
10.1007/s10701-010-9419-1

Published In:
Foundations of Physics

Quantum Covers in Quantum Measure Theory

Sumati Surya & Petros Wallden
Raman Research Institute, Bangalore, India

October 15, 2008

Abstract

In standard measure theory the measure on the base set Ω is normalised to one, which encodes the statement that "Ω happens". Moreover, the rules imply that the measure of any subset A of Ω is strictly positive if and only if A cannot be covered by a collection of subsets of zero measure. In quantum measure theory on the other hand, simple examples suffice to demonstrate that this is no longer true. We propose an appropriate generalisation of a cover to quantum measure theory, the *quantum cover*, which in addition to being a cover of A, satisfies the property that if every one of its elements has zero quantum measure, then so does A. We show that a large class of inextendible antichains in the associated powerset lattice provide quantum covers for Ω, for a quantum measure that derives from a strongly positive decoherence functional. Quantum covers, moreover, give us a new perspective on the Peres-Kochen-Specker theorem and its role in the anhomomorphic logic approach to quantum interpretation.

1 Introduction

One of the goals of the decoherent histories approach to quantum theory is to formulate precisely how classicality emerges from quantum theory [3]. A partition Ω_c of the space of histories Ω is considered to be classical if the decoherence functional $D(A, B)$ between any two distinct elements $A, B \in \Omega_c$ is weakly zero. This allows one to confer to $D(A, A)$ the status of a classical probability, or measure, obeying the standard probability sum rules. To complete this picture, the decoherence functional on the full histories space $D(\Omega, \Omega)$ must be strictly positive, and thus normalisable to 1. From a realist's perspective, this gives us the coarsest possible sense in which we can say that "something happens".

The *quantum measure* is a natural generalisation of the classical measure and is defined for *any* subset $A \subseteq \Omega$ to be $|A| = D(A, A) \geq 0$,[1] with the same requirement $D(\Omega, \Omega) > 0$ [1]. This measure does not satisfy the Kolmogorov sum rule since it can involve a non-vanishing interference term

$$I_2(A, B) = |A \sqcup B| - |A| - |B| \tag{1}$$

for a pair of disjoint sets $A, B \subset \Omega$, where $\sqcup$ indicates disjoint union. Instead, as shown in [1], the quantum measure satisfies the quantum sum rule

$$I_3(A, B, C) = |A \sqcup B \sqcup C| - |A \sqcup B| - |A \sqcup C| - |B \sqcup C| + |A| + |B| + C| = 0. \tag{2}$$

Thus, while the classical measure satisfies the null test for the two slit experiment with $I_2(A, B) = 0$, the quantum measure in standard quantum theory satisfies the null test for the three slit, with $I_3(A, B, C) = 0$. The generalisation to the n-level interference term $I_n(A_1, \ldots, A_n)$ on n-tuples is obvious, and standard quantum theory satisfies a null test for all n-slit experiments, with $n > 2$. It was shown in [1] that if the nth-level interference vanishes on all n-tuples, then so do all higher level interferences. This implies a hierarchy of generalised quantum measures, with classical theory being level 1, and standard quantum theory being level 2. We will refer to as "standard" a quantum measure of level 2.

In classical measure theory the condition $|\Omega|_c > 0$[2] puts the following, somewhat trivial restriction on the choice of measure for the subsets of Ω. Namely, if Ω admits a covering of zero measure sets, then $|\Omega|_c = 0$, and hence such *zero covers* are disallowed. On the other hand, zero covers are allowed in quantum measure theory. As an example, consider the 3 slit experiment and restrict to three paths A, B and C, one through each slit, all of which arrive at the same non-dark spot on the screen.

[1]This inequality is often referred to as the condition of "positivity" which we will henceforth assume for a quantum measure.

[2] We will use $|.|_c$ to denote the classical measure.

It is possible for $|A \sqcup B| = 0$ and $|B \sqcup C| = 0$, but $|A \sqcup C| \neq 0$, so that $|\Omega| > 0$. Thus, while the pair of subsets $A \sqcup B$ and $B \sqcup C$ suffices to cover Ω, this does not imply that $|\Omega| = 0$. In this sense, the pair $\{A \sqcup B, B \sqcup C\}$ doesn't "cover" Ω sufficiently, when using the quantum measure. We use this idea to define the *quantum cover* of Ω as:

Definition 1 *For a quantum measure* $|.|$, $\{O_i\}$ *is said to be a quantum cover of* Ω *if* $\Omega = \bigcup_i O_i$ *and* $|O_i| = 0$ *for all* i *implies that* $|\Omega| = 0$.

Does a non-trivial quantum cover of Ω always exist? We begin in section (2) by showing that it does. The quantum cover finds its rightful place in the powerset lattice $\mathfrak{B}$ associated with the powerset $2^{|\Omega|}$, where set inclusion provides the order relation, and the minimal element is the empty set and the maximal element is Ω. We prove existence by showing that any k-level inextendible antichain in $\mathfrak{B}$ satisfies the criterion for a quantum cover, for *any* quantum measure, when $k \geq 2$. Motivated by this, we examine more general classes of inextendible antichains in $\mathfrak{B}$ and show that they also satisfy this criterion, when the quantum measure is derived from a strongly positive decoherence functional [6][3]. This is summarised as

Lemma 1 *Let* $\mathcal{A}$ *be an intextendible antichain in the powerset lattice* $\mathfrak{B}$ *over a finite histories space* Ω*, which belongs to the class* $\mathfrak{C}$*. If each of the elements in* $\mathcal{A}$ *has zero quantum measure, then* $|\Omega| = 0$*, when* $|.|$ *is assumed to come from a strongly positive decoherence functional* [4]*. Thus* $\mathcal{A}$ *is a quantum cover of* Ω*.*

Inextendible antichains in $\mathfrak{B}$ do not possess a universal characterisation, apart from satisfying somewhat weak inequalities [7], and hence it is unclear how to proceed most generally. However, the class $\mathfrak{C}$ examined so far is general enough for us to conjecture a deeper connection between inextendible antichains and quantum coverings:

[3] The assumption of strong positivity is natural – all known physical systems satisfy it.

[4] And possibly, more general quantum measures, which are only required to satisfy conditions (11) and (12) (see Section 2).

Conjecture 1 *For a quantum measure that is obtained from a strongly positive decoherence functional*[5]*, every inextendible antichain in* $\mathfrak{B}$ *is a quantum cover of* Ω*.*

We end this section with a brief diversion to the preclusion-based anhomomorphic logic interpretation of quantum theory. In particular, we point out the existence of a special inextendible antichain in the powerset lattice which spurred our search for a quantum cover.

One of the motivations for constructing a quantum measure theory is to be able to use it in a manner analogous to the classical measure, i.e., to say something more about physical reality than the standard Copenhagen interpretation allows. For example, one would like to be able to interpret a set of histories of zero measure to unambiguously imply that they do not occur. However, contrary to classical intuition, such sets can contain subsets of non-vanishing measure. An example of this is the two slit experiment, for which pairs of destructively interfering histories have zero measure, although each individual history has non-zero measure.

The space of subsets of the histories space Ω or the power set $2^{|\Omega|}$ forms a unital Boolean algebra $\mathfrak{A}$, with addition $A+B$ defined as symmetric difference, multiplication AB as set intersection, and with Ω being the unit element. Classical logic involves a homomorphism or *coevent* Φ_c from $\mathfrak{A}$ to $\mathbb{Z}_2$, the set of truth values, with

$$\Phi_c(A+B)=\Phi_c(A)+\Phi_c(B), \quad \Phi_c(AB)=\Phi_c(A)\Phi_c(B), \quad \Phi_c(\Omega)=1. \tag{3}$$

An actualisation or reality is a *primitive* coevent, Φ_c^p, i.e., one whose support $\mathrm{supp}(\phi_c^{\mathrm{p}})$ is a single fine grained history. Any primitive coevent ϕ_c^p for a classical system is also "preclusive": for all sets $P \in \mathfrak{A}$ of zero measure $|P|_c=0$,

$$\Phi_c^p(P) = 0, \tag{4}$$

$$\mathrm{supp}(\Phi_c^p) \not\subseteq P. \tag{5}$$

In other words, sets of zero measure cannot be realised.

[5]Or more generally one which satisfies (11) and (12).

The anhomomorphic logic proposal [4] generalises classical logic to coevents $\Phi : \mathfrak{A} \rightarrow \mathbb{Z}_2$ which are *not* homomorphisms (hence the term "an-homomorphic"), but are *preclusive* in that they satisfy Eqn (4). In addition, Φ could at best retain a part, but not all, of the Boolean structure of $\mathfrak{A}$. In the *multiplicative scheme* one retains

$$\Phi(AB) = \Phi(A)\Phi(B) \tag{6}$$

so that a preclusive coevent also satisfies the "modus ponens" of classical logic

$$\Phi(A) = 1 \Rightarrow \Phi(B) = 1 \ \forall \ B \supset A. \tag{7}$$

This means that not only is $\Phi(\Omega) = 1$, but Eqn (5) is also satisfied. Thus, a zero measure set always maps to the zero element in $\mathbb{Z}_2$, i.e., is "false". Moreover, every subset of this set also maps to the zero element. Thus, even though a set P of zero quantum measure can contain a subset $Q \subset P$ of non-zero quantum measure, neither P nor Q can be realised. From the example of the double slit, it means that for two destructively interfering paths A, B, not only is $A+B$ false, but so are A and B, individually, even though their quantum measures are strictly non-zero. Again, in analogy with the classical system, quantum reality is represented by a *primitive preclusive coevent*(PPC), i.e., preclusive coevents with the smallest possible support. The supports of PPCs are typically not fine grained histories and hence quantum reality can manifest itself at best as a collection of fine grained histories. As shown in [5] this scheme passes the stringent test of the Kochen-Specker theorem, and hence has emerged as a promising candidate for a realist interpretation of quantum theory.

Our proposal for a quantum cover comes from the observation that the set of supports of the set of PPC's forms an antichain $\mathcal{A}$ in $\mathfrak{B}$. The up set $\uparrow A \equiv \{b \supseteq a | a \in \mathcal{A}\}$ contains no zero measure sets, and hence all zero measure sets must lie either to the past of $\mathcal{A}$, or be incomparable to it. In particular, the set of maximal elements $\mathcal{M}'$ of $\mathfrak{B}' \equiv \mathfrak{B} \backslash \uparrow A$ is an antichain in $\mathfrak{B}$ containing only zero measure sets: any m that is not of zero measure either lies in $\uparrow \mathcal{A}$ or is contained in a zero measure set. $\mathcal{M}'$ is thus made up of the largest possible zero measure sets.

The maximal elements of $\mathfrak{B}\backslash(\uparrow A\backslash\mathcal{A})$ forms an antichain $\mathcal{A}'$ in $\mathfrak{B}$, with $\mathcal{A}' = \mathcal{A} \sqcup \mathcal{M}$ and $\mathcal{M} \subseteq \mathcal{M}'$. Any element of $\mathcal{M}'$ which is not in $\mathcal{M}$ must lie in the down set $\downarrow A$ (defined dually to $\uparrow A$). Moreover, since every zero measure set either lies in $\mathcal{M}'$ or is contained in an element of $\mathcal{M}'$, this means that every zero measure set lies in $\downarrow A'$. If a set is not precluded, then it either lies in $\uparrow A$ or is contained in an element of $\mathcal{M}'$ and hence contained in an element of $\mathcal{M}$ or $\mathcal{A}$.

Thus, $\mathcal{A}'$ is an inextendible antichain in $\mathfrak{B}$, with a portion of the elements reserved for what doesn't happen, and the remaining for what can; it is crucial that that the set of possible realisations (i.e., the supports of PPC's) are *not* of zero measure, i.e. that $\mathcal{A}'$ is not a zero cover. A natural question that arises from this analysis is whether, for a physically realisable quantum system $\mathfrak{B}$ admits an inextendible antichain containing only elements of zero measure. We make considerable progress in proving otherwise.

2 Constructing Quantum Covers

Let $A_1, A_2 \ldots A_n$ represent the fine grained histories in Ω. The associated powerset lattice of Ω is obtained by taking the empty set as the bottom element and Ω as the top element, with the order corresponding to set inclusion. Thus, $X \prec Y$ in $\mathfrak{B}$ iff $X \subset Y$ in Ω. Cardinality of a set defines a "level" in $\mathfrak{B}$ – for example, $(A_1 \sqcup A_2)$ is a level 2 element in $\mathfrak{B}$.

As mentioned in the introduction, a k-level quantum measure is defined as the smallest k for which the $k+1$ way interference

$$\begin{aligned} I_{k+1}(A_1, \ldots A_{k+1}) &= |A_1 \sqcup \ldots \sqcup A_{k+1}| \\ &\quad - \sum_{k-\text{subsets}} |k - \text{subset}| + \sum_{(k-1)-\text{subsets}} |(k-1) - \text{subset}| \ldots \\ &\quad + (-1)^{k+2}[|A_1| + |A_2| \ldots + |A_{k+1}|], \end{aligned} \tag{8}$$

is zero, where by a j-subset we mean a j-element subset of $\{A_1, \ldots A_{k+1}\}$, $j = 1, \ldots k+1$, and $\sum_{j-\text{subset}}$ is a sum over all possible j-subsets. If so, then all higher interference terms also vanish [1, 2], and one obtains, for $n > k$ the identity

$$|A_1 \sqcup \ldots \sqcup A_n| = \sum_{(n-1)-\text{subsets}} |(n-1)-\text{subsets}| - \sum_{(n-2)-\text{subsets}} |(n-2)-\text{subsets}| \ldots + (-1)^n [|A_1| + |A_2| \ldots + |A_n|]. \tag{9}$$

Using this identity recursively, one can show inductively that[6] for a 2-level quantum measure

$$|A_1 \sqcup \ldots \sqcup A_n| = (2-n)\sum_{i=1}^{n} |A_i| + \frac{1}{2}\sum_{i,j=1}^{n} |A_i \sqcup A_j|. \tag{10}$$

In order to avoid confusion with the use of the word "level" in $\mathfrak{B}$, we fix the quantum measure to be 2-level once and for all[7]. We will make crucial use of the following identities

$$|A \sqcup B| = 0 \Rightarrow |A| = |B| \tag{11}$$

and

$$|A| = 0 \Rightarrow |A \sqcup B| = |B|. \tag{12}$$

which follow from the strong positivity of the decoherence functional, as shown in the Appendix.

2.1 Proof of existence

We show that the simplest example of an inextendible antichain $\mathcal{A}$ in $\mathfrak{B}$, namely the set of all level k elements, with $0 < k < n$ provides a quantum cover for Ω, for any choice of quantum measure.

Let each of the elements in $\mathcal{A}$ be of zero measure. If $k = 1$, then since each $|A_i| = 0$, by Eqn (12) $|\Omega| = |A_1 \sqcup A_2 \ldots \sqcup A_n| = 0$. Let $k \geq 2$. Using Eqn (10) for a level k element $(A_{i_1}, A_{i_2} \ldots A_{i_k})$ of $\mathfrak{B}$,

$$|A_{i_1} \sqcup A_{i_2} \ldots \sqcup A_{i_k}| = (2-k)(|A_{i_1}| + \ldots |A_{i_k}|) + (|A_{i_1} \sqcup A_{i_2}| + \ldots + |A_{i_{k-1}} \sqcup A_{i_k}|) = 0. \tag{13}$$

[6]Proof in the Appendix

[7]A similar type of identity may be obtained for higher level quantum measures.

Adding up all such $\binom{n}{k}$ terms, we obtain

$$\binom{n-1}{k-1}(2-k)\sum_{i=1}^{n}|A_i| + \binom{n-2}{k-2}\sum_{i,j=1,i<j}^{n}|A_i \sqcup A_j| \;=\; 0 \tag{14}$$

$$\Rightarrow \sum_{i,j=1,i<j}^{n}|A_i \sqcup A_j| = \frac{(n-1)(k-2)}{(k-1)}\sum_{i=1}^{n}|A_i|. \tag{15}$$

Substituting in the expression Eqn (10) for $|\Omega| = |A_1 \sqcup A_2 \ldots \sqcup A_n|$

$$|\Omega| = \left((2-n) + \frac{(n-1)(k-2)}{(k-1)}\right)\sum_{i=1}^{n}|A_i| \leq 0 \Rightarrow |\Omega| = 0, \tag{16}$$

since the coefficient simplifies to $\frac{(k-n)}{(k-1)} < 0$ for $2 \leq k < n$. This proves

Lemma 2 *The level $k \geq 2$-antichain in $\mathfrak{B}$ is a quantum cover of Ω.*

Note that the above Lemma doesn't require strong positivity. However, to prove that the $k = 1$ antichain is a quantum cover, we did need strong positivity. We now generalise this result to a large class of inextendible antichains obtained by systematically "whittling" away at the level k antichain.

2.2 Generalisation

The powerset lattice has a great deal of structure: every k-level element has exactly k-links to level $k-1$ and exactly $n-k$ links to level $k+1$. Thus, any antichain obtained by removing $m < k+1$ elements in the k-level has a unique completion to the k-level antichain if $n \geq 2k$ and similarly if $m < n-k+1$ for $n \leq 2k$.

We define the past and future shadows of an element $a \in \mathfrak{B}$ on the k-level antichain to be

$$\begin{aligned} S_k(a) &\equiv \{b \subset a|\text{card(b)} = \text{k}\} \qquad\qquad \text{card(a)} > \text{k} \\ S_k(a) &\equiv \{b \supset a|\text{card(b)} = \text{k}\} \qquad\qquad \text{card(a)} < \text{k}, \end{aligned} \tag{17}$$

where card(a) denotes the cardinality of the set a. The strategy we employ is to whittle away at the k-level antichain by (a) picking a set of m mutually unrelated elements $\Lambda'(\neq k) \equiv \{a_s\}$ from levels other than k (b) adding in elements in the k-level antichain $\Lambda(k)$ which do not lie in the shadows of the a_s (c) finding an inextendible extension of the resulting antichain by adding more elements to $\Lambda'(\neq k)$ to get $\Lambda(\neq k)$, without changing $\Lambda(k)$, or equivalently, without changing the set of elements on the k-level antichain which lie inside the shadows of the elements of $\Lambda'(\neq k)$. The resulting antichain is then $\mathcal{A} = \Lambda(k) \sqcup \Lambda(\neq k)$.

To illustrate, let us start with a single element, $\Lambda(k+1) = \{a_1\}$ in level $k+1$. Wlog, let $a_1 = (A_1 \ldots A_{k+1})$. Then $\Lambda(k) = \{(A_{\alpha_1}, \ldots A_{\alpha_{k-r}}, A_{\beta_1}, \ldots, A_{\beta_r})\}$ is the set of $\binom{n}{k} - (k+1)$ elements of the k-level antichain which do not lie in the shadow of a_1, where $\alpha_i \in [1, \ldots k+1]$, $\beta_j \in [k+2, \ldots n]$, and $r \in [1, \ldots r_0]$ with $r_0 = \min(k, n-k-1)$. $\mathcal{A}$ is clearly inextendible, since every k level element either lies in $\mathcal{A}$ or is contained in an element of $\mathcal{A}$, and every $k+1$ level element is either in $\mathcal{A}$ or contains an element of $\mathcal{A}$.

For the $k+1$ elements in $S_k(a_1)$, using Eqn (11)

$$\begin{aligned} |A_{\alpha_1} \sqcup A_{\alpha_2} \ldots \sqcup A_{\alpha_k}| &= |A_{\alpha_{k+1}}| \\ \Rightarrow (2-k) \sum_{i=1}^{k} |A_{\alpha_i}| + \frac{1}{2} \sum_{i,j=1}^{k} |A_{\alpha_i} \sqcup A_{\alpha_j}| &= |A_{\alpha_{k+1}}| \end{aligned} \tag{18}$$

For the remaining $\binom{n}{k} - (k+1)$ elements in $\Lambda(k)$,

$$\begin{aligned} (2-k)\Big(\sum_{i=1}^{k-r} |A_{\alpha_i}| + \sum_{j=1}^{r} |A_{\beta_j}|\Big) + \frac{1}{2} \sum_{i,j=1}^{k-r} |A_{\alpha_i} \sqcup A_{\alpha_j}| + & \\ \sum_{i1}^{k-r} \sum_{j=1}^{r} |A_{\alpha_i} \sqcup A_{\beta_j}| + \frac{1}{2} \sum_{i,j=1}^{r} |A_{\beta_i} \sqcup A_{\beta_j}| &= 0, \end{aligned} \tag{19}$$

where $\alpha_i \in [1, \ldots k+1]$ and $\beta_j \in [1, \ldots, r]$. Adding all the $\binom{n}{k}$ equations (18) and (19)

$$(2-k)\binom{n-1}{k-1} \sum_{i=1}^{n} |A_i| + \binom{n-2}{k-2} \frac{1}{2} \sum_{i,j=1}^{n} |A_i \sqcup A_j| = \sum_{i=1}^{k+1} |A_i| \tag{20}$$

Solving for the term $\frac{1}{2}\sum_{i,j=1}^{n}|A_i \sqcup A_j|$ and inserting into the rhs of Eqn (10) we find

$$\begin{aligned} |\Omega| &= \left[(2-n) + \frac{(k-2)(n-1)}{(k-1)} + \frac{1}{\binom{n-2}{k-2}}\right]\sum_{i}^{k+1}|A_{\alpha_i}| \\ &\quad + \left[(2-n) + \frac{(k-2)(n-1)}{(k-1)}\right]\sum_{i=k+2}^{n}|A_i| \leq 0, \end{aligned} \tag{21}$$

which means that $|\Omega| = 0$.

Alternatively, we can choose $\Lambda(k-1) = \{a_1\}$ to be an element in the $k-1$-level, i.e. $|A_1 \sqcup A_2 \ldots \sqcup A_{k-1}| = 0$. $S_k(a_1)$ is now the set of k-level elements that contain a_1 as a subset, i.e. all those of the form $(A_1, A_2 \ldots, A_{k-1}, A_i)$, with $i \in [k, \ldots n]$. Thus

$$|A_1 \sqcup A_2 \ldots \sqcup A_{k-1} \sqcup A_i| = |A_i| \, \forall \; i \in [k, \ldots, n]. \tag{22}$$

The remaining elements in $\Lambda(k)$ have measure zero, so that again, adding up all the measures of the k-level elements we get

$$(2-k)\binom{n-1}{k-1}\sum_{i=1}^{n}|A_i| + \binom{n-2}{k-2}\frac{1}{2}\sum_{i,j=1}^{n}|A_i \sqcup A_j| = \sum_{i=k}^{n}|A_i| \tag{23}$$

which implies

$$\begin{aligned} |\Omega| &= +\left[(2-n) + \frac{(k-2)(n-1)}{(k-1)}\right]\sum_{i=1}^{k-1}|A_i| \\ &\quad \left[(2-n) + \frac{(k-2)(n-1)}{(k-1)} + \frac{1}{\binom{n-2}{k-2}}\right]\sum_{i=k}^{n}|A_i| \leq 0, \end{aligned} \tag{24}$$

or $|\Omega| = 0$ as before.

The strategy for this line of reasoning should by now be clear. Instead of the k-level equations, one could "project" onto any level $2 < l < n$. Summing over the measure of all the level l elements gives the now recognisable term

$$(2-l)\binom{n-1}{l-1}\sum_{i=1}^{n}|A_i| + \binom{n-2}{l-2}\frac{1}{2}\sum_{i,j=1}^{n}|A_i \sqcup A_j|, \tag{25}$$

which we try to evaluate using the details of the inextendible antichain $\mathcal{A}$: the measure of each l-level element a_l, can be determined by knowing how it relates to an element of $\mathcal{A}$. If $a_l \geq a \in \mathcal{A}$, then strong positivity implies that $|a_l| = |a_l \backslash a|$ as in Eqn (18) and if $a_l \leq a \in \mathcal{A}$, then $|a_l| = |a \backslash a_l|$, as in Eqn (22). This should help us simplify the expression for $|\Omega|$ sufficiently to prove the result. This procedure clearly depends crucially on the details of $\mathcal{A}$.

We now focus on particular classes of generalisations for which $\Lambda(k)$ is "sufficiently" populated in the following sense. It is useful to define an *index set* Γ_s associated with any l-level element $a_s \in \mathfrak{B}$ as the set of labels $\{\alpha_j^{(s)}\}$, $j = 1, \ldots l$, where $a_s = A_{\alpha_1} \sqcup A_{\alpha_2} \ldots A_{\alpha_l}$. We will reserve the symbol Γ for the complete set of labels $\{1, \ldots, n\}$. Define $\widetilde{\Gamma} \equiv \Gamma \backslash \bigcup_s \Gamma_s$, where $a_s \in \Lambda(\neq k)$ and $p \equiv |\widetilde{\Gamma}|$. The requirement that $\Lambda(k)$ be sufficiently populated is the requirement that there be a lower bound on p. We will consider the three cases separately, $\Lambda(> k)$, $\Lambda(< k)$ and the mixed case $\Lambda(> k) \sqcup \Lambda(< k)$.

2.2.1 $\mathcal{A} = \Lambda(k) \sqcup \Lambda(> k)$

Let $\Lambda(> k) \equiv \{a_s\}$, $s \in [1, \ldots, m]$ and $p \geq 1$. This means that there is at least one fine grained history A_1 which is contained only in some of the elements of $\Lambda(k)$ but not in any element of $\Lambda(> k)$. The set of all k-level elements containing A_1 has zero measure, i.e. $|A_{\alpha_1} \sqcup \ldots A_{\alpha_{k-1}} \sqcup A_1| = 0$, $\forall\ \alpha_i \neq 1$. This implies that

$$|A_{\alpha_1} \sqcup \ldots A_{\alpha_k} \sqcup A_1| = |A_{\alpha_1}| = \ldots = |A_{\alpha_k}| \tag{26}$$

which means that all $|A_{\alpha_i}|$'s are equal for all $\alpha_i \neq 1$. Call this $|A_\alpha|$. Moreover,

$$|A_{\alpha_1} \sqcup \ldots A_{\alpha_{k+1}} \sqcup A_1| = |A_{\alpha_1} \sqcup A_{\alpha_2}| = \ldots = |A_{\alpha_k} \sqcup A_{\alpha_{k+1}}| \tag{27}$$

from which we deduce that all $|A_{\alpha_i} \sqcup A_{\alpha_j}|$'s are equal for $\alpha_i, \alpha_j \neq 1$. Call this $|A_\alpha \sqcup A_{\alpha'}|$. Using (10), the k-level zero measure sets give

$$(2-k)|A_1| + (2-k)(k-1)|A_\alpha| + \sum_{i=1}^{k-1} |A_1 \sqcup A_{\alpha_i}| + \frac{(k-1)(k-2)}{2} |A_\alpha \sqcup A_{\alpha'}| = 0. \tag{28}$$

Summing over all $\binom{n-1}{k-1}$ of these

$$(2-k)|A_1|+(2-k)(k-1)|A_\alpha|+\frac{k-1}{n-1}\sum_{i=2}^{n}|A_1\sqcup A_i|+\frac{(k-1)(k-2)}{2}|A_\alpha\sqcup A_{\alpha'}|=0. \tag{29}$$

Moreover, for any $a_s \in \Lambda(>k)$, $\{\alpha_i\}=\Gamma_s$

$$|A_{\alpha_1}\sqcup\ldots\sqcup A_{\alpha_s}|=0 \Rightarrow \frac{1}{2}|A_\alpha\sqcup A_{\alpha'}|=\frac{s-2}{s-1}|A_\alpha|. \tag{30}$$

Inserting into (10) we get

$$|\Omega|=(k-n)\bigg(\frac{1}{k-1}|A_1|+\frac{n-1}{s-1}|A_\alpha|\bigg)\leq 0. \tag{31}$$

2.2.2 $\mathcal{A}=\Lambda(k)\sqcup\Lambda(<k)$

Let s_0 be the lowest level in $\Lambda(<k)$, and let $r=k-s_0+1$. Let $p\geq r$, so that at least r fine grained histories $A_1,\ldots A_r$ are not contained in any of the elements of $\Lambda(<k)$. Define $P\equiv\{1,\ldots r\}\subset\Gamma$. For any set $Q\subset\Gamma\backslash P$, with $q\equiv|Q|=s_0-1$, $P\sqcup Q\not\supset\Gamma_s$ for any s, by construction. Hence, it is an index set of an element in $\Lambda(k)$

$$|A_1\sqcup\ldots A_r\sqcup A_{\alpha_1}\sqcup\ldots A_{\alpha_q}|=0,\ \alpha_i\notin P. \tag{32}$$

Since $n\geq k+1$

$$|A_1\sqcup\ldots A_r\sqcup A_{\alpha_1}\sqcup\ldots A_{\alpha_{q+1}}|=|A_{\alpha_1}|=\ldots=|A_{\alpha_q+1}| \tag{33}$$

which means that all $|A_{\alpha_i}|$'s are equal for $\alpha_i\notin P$. Call this $|A_\alpha|$. If, in addition, $n\geq k+2$ then

$$|A_1\sqcup\ldots A_r\sqcup A_{\alpha_1}\sqcup\ldots A_{\alpha_{q+2}}|=|A_{\alpha_1}\sqcup A_{\alpha_2}|=\ldots=|A_{\alpha_{q+1}}\sqcup A_{\alpha_{q+2}}|, \tag{34}$$

which means that all $|A_{\alpha_i}\sqcup A_{\alpha_j}|$'s are equal for $\alpha_i,\alpha_j\notin P$. We call this $|A_\alpha\sqcup A'_\alpha|$ as before. Using (10), (32) reduces to

$$\begin{aligned}(2-k)\sum_{i=1}^{r}|A_i|+(2-k)q|A_\alpha|+\frac{1}{2}\sum_{i,j=1}^{r}|A_i\sqcup A_j|+\\ \sum_{i=1}^{r}\sum_{j=1}^{q}|A_i\sqcup A_{\alpha_j}|+\frac{q(q-1)}{2}|A_\alpha\sqcup A_{\alpha'}| &= 0.\end{aligned} \tag{35}$$

Summing over all $\binom{n-r}{q}$ of these, gives us

$$\binom{n-r}{q}\Bigg((2-k)\sum_{i=1}^{r}|A_i| + (2-k)q|A_\alpha| + \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}|A_i \sqcup A_j| + \frac{q(q-1)}{2}|A_\alpha \sqcup A_{\alpha'}|\Bigg) + \binom{n-r-1}{q-1}\sum_{i=r+1}^{n}\sum_{i=1}^{r}|A_i \sqcup A_j| \;=\; 0. \quad (36)$$

Moreover, for any $a_s \in \Lambda(<k)$, $\{\alpha_i\} \in \Gamma_s$,

$$|A_{\alpha_1} \sqcup \ldots \sqcup A_{\alpha_s}| = 0 \Rightarrow \frac{1}{2}|A_\alpha \sqcup A_{\alpha'}| = \frac{s-2}{s-1}|A_\alpha|. \quad (37)$$

Inserting into (10) we get

$$\begin{aligned} |\Omega| &= \frac{k-n}{q}\Bigg((2-r)\sum_{i=1}^{r}|A_i| + \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}|A_i \sqcup A_j|\Bigg) + \frac{(k-n)(n-r)}{s-1}|A_\alpha| \\ &= (k-n)\Bigg(\frac{1}{q}|A_1 \sqcup \ldots \sqcup A_r| + \frac{(n-r)}{s-1}|A_\alpha|\Bigg) \\ &\leq 0. \end{aligned} \quad (38)$$

For $k = n-1$, we cannot use the simplification (34). Noting that $\Omega = A_1 \sqcup \ldots A_r \sqcup A_{r+1} \sqcup \ldots A_n$, we see that $\Lambda(<k)$ consists of the single element $a_s = A_{r+1} \sqcup \ldots A_n$ in level $s_0 = n-r$. Replacing

$$\binom{n-r}{q}\frac{q(q-1)}{2}|A_\alpha \sqcup A_{\alpha'}| \rightarrow \binom{n-r-2}{q-2}\frac{1}{2}\sum_{i,j=r+1}^{n}|A_i \sqcup A_j| \quad (39)$$

in Eqn (36) and replacing (37), with

$$\frac{1}{2}\sum_{i=r+1}^{n}\sum_{j=r+1}^{n}|A_i \sqcup A_j| = (n-r)(n-r-2)|A_\alpha|, \quad (40)$$

we recover Eqn (38) with $s = s_0$, thus proving our result for all k.

2.2.3 $\mathcal{A} = \Lambda(k) \sqcup \Lambda(<k) \sqcup \Lambda(>k)$

Again, define r and q, P and Q via the lowest level s_0 in $\Lambda(<k)$. Then $P \sqcup Q \not\supset \Gamma_s$ for any Γ_s coming from $\Lambda(<k)$ and also, since $P \cap \Gamma_s = \emptyset$ for all s, $P \sqcup Q \not\subset \Gamma_s$

for any Γ_s coming from $\Lambda(> k)$. Hence the construction goes through as in Section 2.2.2].

In particular, if we redefine s_0 to be the lowest level in $\Lambda(< k) \sqcup \Lambda(k)$, then for the case $\Lambda(< k) = \emptyset$, $s_0 = k$ which means that $p \geq 1$ as in Section (2.2.1). The results Sections (2.2.1),(2.2.2) and (2.2.3) can then be summarised into

Lemma 3 *Let $\mathcal{A} = \Lambda(k) \sqcup \Lambda(> k) \sqcup \Lambda(< k)$ be an inextendible antichain in the powerset lattice $\mathfrak{B}$. Let s_0 be the lowest level in $\Lambda(< k) \sqcup \Lambda(k)$, and let there exist at least $k - s_0 + 1$ fine grained histories not contained in any of the elements of $\Lambda(> k) \sqcup \Lambda(< k)$. Then, if every element of $\mathcal{A}$ has zero measure, so does Ω. Hence $\mathcal{A}$ is a quantum cover of Ω.*

Lemma (3), while quite general, clearly does not cover all cases; while the conditions on p are sufficient, they are not necessary for proving $|\Omega| = 0$. Relaxing the conditions on p means that we can no longer use the equality of the measures $|A_{\alpha_i}|$ and $|A_{\alpha_i} \sqcup A_{\alpha_j}|$'s in a general way, independent of the details of $\mathcal{A}$. A more general technique than what we have used may exist, but eludes us at the present. Instead, we deal with specific examples and show in every case that the inextendible antichains we construct are quantum covers[8]. While in no way exhaustive, the following is a list of such examples.

1. $\mathcal{A}_1 \in \Big\{\Lambda(k) \sqcup \Lambda(> k)\Big\}$, $p = 0$, $n > 3$.

   Let $k = n - 2$ and $p = 0$. Consider the maximal antichain

$$\begin{aligned} \mathcal{A}_1 &= \{(n-1)\text{-level } : \{\bar{A}_1\}, \{\bar{A}_2\}; \\ &\quad \text{all } (n-2)\text{-level elements} \not\subset S_{n-2}(\bar{A}_1) \bigcup S_{n-2}(\bar{A}_2)\}, \end{aligned} \tag{41}$$

   where $\{\bar{A}_\alpha\}$ denotes the set of all but the fine grained history A_α, $\alpha = 1, 2$. $|\{\bar{A}_\alpha\}| = 0$ implies that $|\Omega| = |A_1| = |A_2|$. Using the fact that for any

[8]Indeed, in the tens of examples we have examined, we have not found a single counterexample.

$a \in S_{n-2}(\bar{A}_\alpha)$, $|a| = |\bar{A}_\alpha \backslash a|$, and adding up all the $(n-2)$-level elements as before gives

$$\frac{(4-n)(n-1)(n-2)}{2}\sum_{i=1}^{n}|A_i| + \frac{(n-2)(n-3)}{4}\sum_{i,j=1}^{n}|A_i \sqcup A_j| = 2\sum_{i=3}^{n}|A_i| + |A_1|. \tag{42}$$

Thus,

$$|\Omega| = \frac{-2n+8}{(n-2)(n-3)}\sum_{i=3}^{n}|A_i| + \frac{-2n+5}{(n-2)(n-3)}|A_1| \leq 0 \tag{43}$$

2. $\mathcal{A}_2 \in \Big\{\Lambda(k) \sqcup \Lambda(< k)\Big\}, \quad p = 1 < k - s_0 + 1$, n odd.

   The maximal antichain is defined as

$$\begin{aligned} \mathcal{A}_2 \quad = \quad & \Big\{\frac{n+1}{2} - \text{level} : \{A_1, A_2, \cdots, A_{\frac{n+1}{2}}\}, \{A_1, A_{\frac{n+3}{2}}, \cdots A_n\}; \\ & 2 - \text{level} : \{A_i, A_j\} \quad i \in [2, \frac{n+1}{2}] \text{ and } j \in [\frac{n+3}{2}, n]\Big\}. \end{aligned} \tag{44}$$

   Here, $k = \frac{n+1}{2}$, $s_0 = 2$, so that $r = \frac{n-1}{2}$ which implies that $p = 1 < r$. However, $\mathcal{A}_2$ can also be viewed as an antichain in the set $\Big\{\Lambda(k) \sqcup \Lambda(> k)\Big\}$ with $k = 2$ and $p = 0$.

   The 2-level elements in $\mathcal{A}_2$ imply that the $|A_i|$ are all equal for $i \neq 1$. Using $|A_2 \sqcup \cdots \sqcup A_{\frac{n+1}{2}}| = |A_1| = |A_{\frac{n+3}{2}} \sqcup \cdots \sqcup A_n|$, we see that

$$\sum_{i_1, i_2 \in [2, \frac{n+1}{2}]} |A_{i_1} \sqcup A_{i_2}| = \sum_{j_1, j_2 \in [\frac{n+3}{2}, n]} |A_{j_1} \sqcup A_{j_2}| = |A_1| + \frac{(n-1)(n-5)}{4}|A_i| \tag{45}$$

   Moreover, $|A_1 \sqcup A_i| = |A_1 \sqcup A_j|$ for all $i \in [2, \frac{n+1}{2}]$ and $j \in [\frac{n+3}{2}, n]$. Finally, using the $(\frac{n+1}{2})$-level zero sets and Eqn 10,

$$|A_1 \sqcup A_i| = \frac{n-5}{n-1}|A_1| + |A_i|, \tag{46}$$

   which gives

$$|\Omega| = -|A_1| - \frac{(n-1)^2}{2}|A_i| \leq 0. \tag{47}$$

3. $\mathcal{A}_3 \in \Big\{\Lambda(k) \sqcup \Lambda(< k)\Big\}, \quad p = 1 < k - s_0 + 1,\ m, l \equiv \frac{n-1}{m}$ positive integers.

This generalizes the previous example. Consider the maximal antichain

$$\begin{aligned}\mathcal{A}_3 &= \Big\{(l+1) - \text{level} : \{A_1, A_2, \cdots, A_{l+1}\}, \{A_1, A_{l+2}, \cdots A_{2l+1}\} \cdots, \\ &\quad \{A_1, A_{(m-1)l+2}, \cdots A_{ml+1=n}\}; \\ &\quad 2 - \text{level} : \text{all}\{A_i, A_j\} \not\subset S_2(\text{(l+1)-level elements}), \Big\} \qquad (48)\end{aligned}$$

The previous example is the special case, $m = 2$. Note that the $(l+1)$-level elements only overlap pairwise at A_1. This antichain belongs to the $\Lambda(k) \sqcup \Lambda(< k)$ case, $p = 1 < r$ with $k = \frac{n-1}{m} + 1$, and $s_0 = 2$ so that $r = \frac{n-1}{m}$ and $p = 1$. However, it can also be thought of as an element of $\Lambda(k) \sqcup \Lambda(> k)$ with $k = 2$ and $p = 0$.

Retracing the arguments in the previous example, we see that all the $|A_i|$'s are equal for $i \neq j$. Moreover,

$$\begin{aligned}\sum_{i_1, i_2 \in [(k-1)l+2, kl+1]} |A_{i_1} \sqcup A_{i_2}| &= \sum_{j_1, j_2 \in [(k'-1)l+2, k'l+1]} |A_{j_1} \sqcup A_{j_2}| \quad \forall k, k' \\ &= |A_1| + \frac{(n-1)(n-2m-1)}{m^2} |A_i| \qquad (49)\end{aligned}$$

and

$$|A_1 \sqcup A_i| = \left(1 - \frac{2m}{n-1}\right) |A_1| + |A_i|. \qquad (50)$$

This implies that

$$|\Omega| = (1-m)|A_1| + (n-1)(l+1-n)|A_i| \leq 0. \qquad (51)$$

4. $\mathcal{A}_4 \in \Big\{\Lambda(k) \sqcup \Lambda(< k) \sqcup \Lambda(> k)\Big\}, \quad p = 1 < k - s_0 + 1.$

We consider the maximal antichain:

$$\begin{aligned}\mathcal{A}_4 &= \Big\{(n-2) - \text{level} : \{\bar{A}_1, \bar{A}_2\}, \{\bar{A}_2, \bar{A}_3\}; \\ &\quad l - \text{level} : \{A_1, A_2, A_{\alpha_1}, \cdots, A_{\alpha_{l-2}}\}, \{A_2, A_3, A_{\alpha_1} \cdots, A_{\alpha_{l-2}}\}, \\ &\quad \{A_2, \cdots, A_{\alpha_{l-1}}\},\ \alpha_i \in [4, \cdots, n]; \\ &\quad 2 - \text{level} : \{A_1, A_3\}\Big\} \qquad (52)\end{aligned}$$

This antichain belongs to the mixed case with $p = 1 < r$ for $k = l$, and since $s_0 - 2$, $r = l - 1$. Again, it can be viewed as an element of $\Lambda(k) \sqcup \Lambda(< k)$ with $k = n - 2$ and $p = 0$.

From the l-level elements we conclude that the $|A_i|$'s are all equal for $i \neq 2$. Moreover, $|A_{\alpha_i} \sqcup A_{\alpha_j}| = |A_1 \sqcup A_{\alpha_k}| = |A_3 \sqcup A_{\alpha_k}|$ for all $\alpha_{i,j,k} \in [4, \cdots, n]$, and $|A_2 \sqcup A_i|$'s are all equal for all $i \neq 2$. $|A_1 \sqcup A_2 \sqcup A_3| = |A_2|$ gives us the useful equality $|A_1| + |A_2| = |A_1 \sqcup A_2|$, while $|A_1 \sqcup A_3 \sqcup A_i| = |A_i|$ gives $|A_1| + |A_i| = |A_1 \sqcup A_i|$. Using this in the expression for a l-level zero set yields the equation $(l - 1)|A_1| + |A_2| = 0$ which implies that $|A_1| = |A_2| = 0$, and hence $|\Omega| = 0$. Note that while we used the l-level and the 2-level elements we didn't need the $(n - 2)$-level elements to prove this result.

We thus obtain our main result, Lemma (1), where $\mathfrak{C}$ includes all classes and specific examples of antichains considered so far. Tens of other examples have also been examined, in every case, verifying Conjecture 1.

We have in this work used a particular characterisation of the classical measure derived from zero sets to define our quantum cover. In this we were motivated by the preclusion-based approaches to quantum measure theory. Instead, one might want to consider the "complementary" property satisfied by a classical measure. Namely, that given a cover $\{O_i\}$ of a classical measure space Ω, $\sum_i |O_i|_c \geq |\Omega|_c$. Indeed, as we now show, this property is satisfied by the k-level quantum cover.

Eqn (25) gives the sum of the measures of all k-level elements and can be rearranged to

$$\frac{(n-2)!}{(k-2)!(n-k)!} \left(\frac{(2-k)(n-1)}{(k-1)} \sum_{i=1}^{n} |A_i| + \frac{1}{2} \sum_{i,j=1}^{n} |A_i \sqcup A_j| \right). \tag{53}$$

Using

$$\frac{(2-k)(n-1)}{(k-1)} = (2-n) + \frac{n-k}{k-1} \tag{54}$$

we conclude that

$$\binom{n-2}{k-2} \left(|\Omega| + \frac{n-k}{k-1} \sum_{i=1}^{n} |A_i| + \frac{1}{2} \sum_{i,j=1}^{n} |A_i \sqcup A_j| \right) \geq |\Omega|. \tag{55}$$

Whether this result holds for all inextendible antichains or not is still an open question.

3 The Peres-Kochen-Specker Theorem and Quantum Covers

In [5], Dowker and Ghazi-Tabatabai recast the work of Peres [8] on the Kochen-Specker(KS) theorem [9] in the framework of quantum measure theory and showed that it is consistent with the anhomomorphic logic interpretation of quantum theory. We show how their construction can help reinterpret the Peres-Kochen-Specker(PKS) result in the language of covers.

We very briefly review the PKS set up as decribed in [5] and refer the reader to the orginal papers for more detail. One starts off with the Peres Set(PS) of 33 rays in $\mathbb{R}^3$, which cluster into 16 orthogonal bases, some overlapping with each other. Using color labels for truth values, green is assigned for true and red for false. An outcome γ is a particular assignment of red or green to each of the 33 rays in PS, and the base set Ω is the set of all possible outcomes. In the classical realist picture, a particle cannot simultaneously be in two orthogonal spin states. Thus, a classical-realist path for the particle corresponds to a "consistent" coloring of the 33 rays in the PS; a simultaneous assignment of green to just one out of three rays in every one of the 16 bases, and such that no pair of mutually orthogonal rays in the PS are both green. The PKS proof against classical realism is the proof that there exists no "consistent" γ.

The PKS result is regarded as a definitive proof against realism, but is more accurately a proof against classical realism. As shown in [5], this subtle difference allows the anhomomorphic picture of "quantum realism" to be accomodated within the strictures of the theorem.

The Dowker-Ghazi-Tabatabai version of the PKS result states:

Lemma 4 (DGT): *Let* $|.|$ *be a measure on the space* Ω *of colourings of PS that is zero valued on the PKS sets. Then there is no preclusive classical coevent for this system.*

The PKS sets correspond to the basis-wise and pair-wise sets of inconsistent colorings: (a) if B is a basis in the PS, then the set $R_B \subset \Omega$ is the assignement of red to all three rays in B and (b) if P is a mutually orthogonal pair of rays in the PS, the set $G_P \subset \Omega$ is an assignment of green to both rays in P. The PKS collection is the set of all such subsets R_B and G_P of Ω. Classical realism then requires that each PKS set is of measure zero, but the converse is not true, as implied by the PKS theorem. Note that the classical coevent referred to in the DGT Lemma corresponds to a coevent with support on a single "classical" (or fine grained) element γ of Ω, which of course exists, even though a classical-realistic γ (corresponding to a consistent colouring) does not.

In general, the non-existence of preclusive classical covents in the anhomomorphic set up means that every $\gamma \in \Omega$ is contained in a set of zero measure. Thus, there exists a collection of sets of zero measure, which covers Ω, but since the (normalised) measure of Ω is 1, it is not a quantum cover. Conversely, if there exists a non-quantum covering of Ω of zero measure sets, then there can be no preclusive classical coevent. That the PKS collection is a non-quantum cover is obvious from the PKS theorem: the statement that there exists a $\gamma \in \Omega$ such that $\gamma \notin A$ for every A in the PKS collection is equivalent to the statement that there is a consistent coloring. Thus a quantum cover avatar of the PKS-DGT result is:

Lemma 5 *The PKS sets provide a non-quantum covering of* Ω*.*

While ruling out classical coevents does not come into conflict with the anhomomorphic logic interpretation, one must also check if too little information remains. Namely, is there only one (trivial) primitive preclusive coevent whose support is Ω? In the PKS set up, it was shown in [5] that this is not the case. More generally, we note that non-triviality is an obvious consequence of Lemma 2, for Ω of finite cardinality. Since $|\Omega| = 1$, all the elements in the $n-1$ level antichain cannot have zero measure. Hence there exists at least one element a in this level, such that $|a| \neq 0$, and since it is not itself contained in a zero measure set, $a \subseteq \mathrm{supp}(\Phi)$, where Φ is a primitive preclusive coevent. Thus, $\mathrm{supp}(\Phi) \subset \Omega$ is a strict inclusion, which means that the statement that "something happens", i.e. $|\Omega| = 1$, can be refined to one with more content. Therefore, in general,

Lemma 6 *For any quantum system with an Ω of finite cardinality, the set of primitive preclusive coevents is non-trivial.*

Finally, as further support for our Conjecture 1 within the PKS set up, we show that:

Lemma 7 *The PKS collection $\mathcal{P}$ does not form an inextendible antichain in the associated Boolean lattice $\mathfrak{B}$.*

Proof Consider the colorings of PS

$$
\begin{aligned}
\gamma &= \{\{u_1, u_2, u_3\} \rightarrow red,\ \{u_4, \ldots u_{33}\} \rightarrow green\} \\
\widetilde{\gamma} &= \{\{u_1, u_2, u_3\} \rightarrow green,\ \{u_4, \ldots u_{33}\} \rightarrow red\},
\end{aligned} \tag{56}
$$

where $B = \{u_1, u_2, u_3\}$ is the basis $(001, 010, 100)$ of [8, 5]. Let $B^c = \{u_4, \ldots u_{33}\}$ which is the remaining set of rays. Then, $\gamma \subset R_B$ and $\gamma \subset G_{P_s}$, for all mutually orthogonal pairs P_s in B^c. If B_i, $i = 1, \ldots 6$ are the 6 of the 16 bases contained wholly within B^c, and $\widetilde{P}_j$, $j = 1, 2, 3$, the 3 mutually orthogonal pairs in B, then $\widetilde{\gamma} \subset R_{B_i}$ for all i and $\widetilde{\gamma} \subset G_{\widetilde{P}_j}$ for all j. Neither γ nor $\widetilde{\gamma}$ are contained in any other sets in $\mathcal{P}$ besides. In particular, $\gamma \not\subset R_{B_i}$ for all i and $\gamma \not\subset G_{\widetilde{P}_j}$ for all j and $\widetilde{\gamma} \not\subset R_B$ and $\widetilde{\gamma} \not\subset G_{P_s}$ for all s. Thus the set $\gamma \sqcup \widetilde{\gamma} \nsubseteq A$ for any $A \in \mathcal{P}$. Since the PKS sets have cardinality > 2, $\gamma \sqcup \widetilde{\gamma}$ cannot contain any set in $\mathcal{P}$ either, and hence $\mathcal{P}$ is not an inextendible antichain in $\mathfrak{B}$. $\square$

Acknowledgements: We would like to thank Fay Dowker, Yousef Ghazi-Tabatabai and Rafael Sorkin for discussions on the PKS theorem. This work was supported in part by the Royal Society-British Council International Joint Project 2006/R2.

Appendix

Proof of the Identity (10)

For convenience, we use the notation $[A_1](n) = \sum_{i=1}^{n} |A_i|$ and $[A_1A_2](n) = \sum_{i,j=1,i<j}^{n} |A_i \sqcup A_j|$, $[A_1A_2A_3](n) = \sum_{i,j,k=1,i<j<k}^{n} |A_i \sqcup A_j \sqcup A_k|$, etc. The fact that $I_n(A_1, A_2 \ldots A_n) = 0$ for $n > 2$ means that

$$\begin{aligned} |A_1 \sqcup \ldots \sqcup A_n| \quad = \\ & (-1)^0[A_1 \ldots A_{n-1}](n) + (-1)^1[A_1 \ldots A_{n-2}](n) + \ldots \\ & +(-1)^{n-3}[A_1A_2](n) + (-1)^{n-2}[A_1](n). \end{aligned} \tag{57}$$

For $n = 3$,

$$|A_1 \sqcup A_2 \sqcup A_3| = |A_1 \sqcup A_2| + |A_2 \sqcup A_3| + |A_1 \sqcup A_3| - |A_1| - |A_2| - |A_3|, \tag{58}$$

thus satisfying the identity (10). Now assume

$$|A_1 \sqcup A_2 \ldots \sqcup A_k| = (2-k)[A](k) + [AA'](k), \, \forall \, 1 < k \leq n-1. \tag{59}$$

Then,

$$|A_1 \sqcup \ldots \sqcup A_n| = Q \times [A_1](n) + P \times [A_1A_2](n) \tag{60}$$

from symmetry. Consider the contribution to P from the term $[A_1..A_k](n)$. Using (59) we see that each term $|A_i \sqcup A_j|$ for a given $i, j \in [1, \ldots n]$, $i \neq j$, appears $\binom{n-2}{k-2}$ times. Adding up the contributions from all k we get

$$P = \sum_{l=1}^{n-2} (-1)^{l-1} \binom{n-2}{l} = 1, \tag{61}$$

where we have used the identity

$$\sum_{l=0}^{n-2} (-1)^{l} \binom{n-2}{l} = 0. \tag{62}$$

Similarly, each term $|A_i|$ for a given $i \in [1, \dots n]$ appears in $[A_1..A_k](n)$ $\binom{n-1}{k-1}$ times. Thus,

$$
\begin{aligned}
Q &= (-1)^0(2-n+1)\begin{pmatrix} n-1 \\ n-2 \end{pmatrix} + (-1)^1(2-n+2)\begin{pmatrix} n-1 \\ n-3 \end{pmatrix} \\
&\quad \dots (-1)^{n-k-1}(2-k)\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} + \dots (-1)^{n-4}(2-3)\begin{pmatrix} n-1 \\ 2 \end{pmatrix} + (-1)^{n-2} \\
&= Q_1 + Q_2 + (-1)^{n-2}, \qquad (63)
\end{aligned}
$$

where

$$
\begin{aligned}
Q_1 &= -(2-n)\sum_{l=1}^{n-3}(-1)^l \begin{pmatrix} n-1 \\ l \end{pmatrix} \\
Q_2 &= -\sum_{l=1}^{n-3}(-1)^l \begin{pmatrix} n-1 \\ l \end{pmatrix} \times l. \qquad (64)
\end{aligned}
$$

Using (62) and

$$\sum_{l=0}^{n-1}(-1)^l \begin{pmatrix} n-1 \\ l \end{pmatrix} \times l = 0, \qquad (65)$$

Q simplifies to $(2-n)$, thus proving inductively, the identity (10). □.

Proof of the identities (11) and (12).

The identities (11) and (12) satisfied by the quantum measure can be obtained from a strongly positive decoherence functional, with $D(A,B)$, where $|A| = D(A,A)$ is the quantum measure. A decoherence functional is required to satisfy the following conditions

$$
\begin{aligned}
D(A \sqcup B, C) &= D(A,C) + D(B,C) \qquad \text{(Biadditivity)} \qquad (66) \\
D(A,B) &= D^*(B,A) \qquad \text{(Hermiticity)} \qquad (67) \\
D(A,A) &\geq 0 \qquad \text{(Positivity)} \qquad (68) \\
D(\Omega,\Omega) &= 1 \qquad \text{(Normalisability).} \qquad (69)
\end{aligned}
$$

While Eqn (68) is sufficient for satisfying the probability interpretation for a classical partition, standard unitary quantum mechanics satisfies the stronger condition of *strong positivity*, which allows a Hilbert space to be associated to Ω [6]. Namely, for any $A \subset \Omega$, and $A = \sum_i \gamma_i$ the matrix

$$M_{ij} = \sum_{i,j} D(\gamma_i, \gamma_j) \tag{70}$$

is positive, i.e. its eigenvalues are ≥ 0. For two disjoint sets A and B, this gives us the important inequality,

$$D(A,A)D(B,B) \geq |D(A,B)|^2 \Rightarrow D(A,A)D(B,B) \geq D_R(A,B)^2, \tag{71}$$

where $D_R(A,B) = \mathrm{Re}D(A,B)$. Now, biadditivity tells us that

$$D(A \sqcup B, A \sqcup B) = D(A,A) + D(B,B) + 2D_R(A,B). \tag{72}$$

Combining this with (71),

$$(\sqrt{D(A,A)} - \sqrt{D(B,B)})^2 \leq D(A \sqcup B, A \sqcup B) \leq (\sqrt{D(A,A)} + \sqrt{D(B,B)})^2. \tag{73}$$

From (71)

$$D(A,A) = 0 \Rightarrow |D(A,B)| = 0 \Rightarrow D(A \sqcup B, A \sqcup B) = D(B,B), \tag{74}$$

thus proving (12). Next, from (73)

$$D(A \sqcup B, A \sqcup B) = 0 \Rightarrow (\sqrt{D(A,A)} - \sqrt{D(B,B)})^2 = 0 \Rightarrow D(A,A) = D(B,B), \tag{75}$$

thus proving (11).

References

[1] R. D. Sorkin, Mod. Phys. Lett. A **9**, 3119 (1994); R. D. Sorkin, *Quantum Classical Correspondence: Proceedings of the 4th Drexel Symposium on Quantum Nonintegrability*, Philadelphia, September 8-11, 1994, pages 229-251, (International Press, Cambridge Mass. 1997) D.H. Feng and B-L Hu, editors. [arXiv:gr-qc/9507057].

[2] R. B. Salgado, Mod. Phys. Lett. A **17**, 711 (2002).

[3] M.Gell-Mann and J.B.Hartle, Phys.Rev. **D47**, 3345 (1993); J. J. Halliwell, Phys. Rev. D **60**, 105031 (1999).

[4] R. D. Sorkin, J. Phys. A **40**, 3207 (2007); R. D. Sorkin, J. Phys. Conf. Ser. **67**, 012018 (2007).

[5] F. Dowker and Y. Ghazi-Tabatabai, J. Phys. A **41**, 105301 (2008).

[6] X. Martin, D. O'Connor and R. D. Sorkin, Phys. Rev. D **71**, 024029 (2005).

[7] Konrad Engle, "Sperner Theory", Cambridge University Press, 1996.

[8] A. Peres, J.Phys. A: Math. Gen. **24** L175-LI78 (1991).

[9] S. Kochen and E. Specker, J. Math. Mech. **17** 5983 (1967).