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Partly Free Semantics for Some Anderson-Like Ontological Proofs

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Abstract Anderson-like ontological proofs, studied in this paper, employ *contingent* identity, *free* principles of quantification of the 1st order variables and *classical* principles of quantification of the 2nd order variables. All these theories are strongly complete wrt. classes of modal structures containing families of *world-varying objectual* domains of the 1st order and *constant conceptual* domains of the 2nd order. In such structures, terms of the 1st order receive only *rigid* extensions, which are elements of the union of all 1st order domains. Terms of the 2nd order receive extensions and intensions. Given a family of *preselected world-varying objectual* domains of the 2nd order, *non-rigid* extensions of the 2nd order terms belong always to a preselected domain connected with a given world. *Rigid* intensions of the 2nd order terms are chosen from among members of a conceptual domain of the 2nd order, which is the set of all functions from the set of worlds to the union of all 2nd order preselected domains such that values of these functions at a given world belong to a preselected domain connected with this world.

Keywords Ontological proof · God · Essence · Necessary existence · Positive properties · 2nd order free modal logic · Absolute and relative identity · Strong completeness

1 Introduction

In general, the term: *ontological proof*—used often interchangeably with the term: *ontological argument*—means a proof, for the conclusion that **God (necessarily) exists**, from premisses which are independent of an observation of the world.

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Anderson-like ontological proofs are understood in this paper as theories formulated in the 2nd order modal language with unary predicate of *positiveness* over 2nd order terms. The following have been adopted from Anderson's theory (see Anderson 1990): the definitions of a *god-like being* (*God* is any being that has necessarily all and only positive properties), *essence* (A property A is an *essence* of an object x if and only if A entails all and only the properties that x has necessarily) and *necessary existence* (An object x has the property of *necessary existence* if and only if its essence is necessarily exemplified), and the three axioms: (i) if a property is positive, then its complement is not positive; (ii) the property of a god-like being is positive; (iii) any property entailed by a positive property is positive.[1]

However, these three Anderson's axioms are not sufficient to prove the statement: **God necessarily exists** or **God exists**. They must still be supplemented with some non-modal and modal principles of the zero-, first- and second-order. But, in regard to the systems of axioms for Anderson-like ontological proofs, the aim should not only be to prove the statement: **Necessarily there exists a god-like being**. It should be possible to assert also other ontological statements. And it is certainly acceptable that some axioms and inference rules are forced by semantics with respect to them the strong completeness of particular Anderson-like ontological proofs is required to proving.

An interesting quality of Anderson-like ontological proofs considered in this work is that they employ *free* principles of quantification of the 1st order variables, in contrast to the *classical* treatment adopted by the present author in Szatkowski (2005, 2007), and *classical* principles of quantification of the 2nd order variables. The theories incorporate *relative* (in particular, *contingent*) identity, similar to them in Szatkowski (2007) but different from Anderson-like ontological proofs considered in Szatkowski (2005) incorporating *absolute* identity.

The rejection of the 1st or 2nd order *universal specification*: $\forall x\phi(x) \rightarrow \phi(x/t)$ or $\forall\alpha\phi(\alpha) \rightarrow \phi(\alpha/\tau)$ is the key criterion differentiating 2nd order free logic from the 2nd order classical logic (cf. Fitting 2002; Fitting and Mendelson 1998; Garson 1984, 1991; Lambert 1997, 2003). Because there exists no commitment to impose/reject one of these principles in the presence/absence of the other, it is clear that one may distinguish 2nd order logic in which exactly one of the two universal specifications is rejected. What is important is that the choices between classical or free quantifications in the modal setting interact also with decisions about the presence/absence of *Barcan formulas*: $\forall\xi\mathsf{L}\boldsymbol{\phi} \rightarrow \mathsf{L}\forall\xi\boldsymbol{\phi}$ and *converse Barcan formulas*: $\mathsf{L}\forall\xi\boldsymbol{\phi} \rightarrow \forall\xi\mathsf{L}\boldsymbol{\phi}$, where ξ is a variable of both sorts. In particular, Barcan formulas of the 1st and/or 2nd order are provable in all systems containing all instances of the axiom schemas $\mathsf{L}(\boldsymbol{\phi} \rightarrow \boldsymbol{\psi}) \rightarrow (\mathsf{L}\boldsymbol{\phi} \rightarrow \mathsf{L}\boldsymbol{\psi})$ and $\mathsf{ML}\boldsymbol{\phi} \rightarrow \boldsymbol{\phi}$, the *necessitation rule*: If $\boldsymbol{\phi}$ `is a thesis`, $\mathsf{L}\boldsymbol{\phi}$ `is a thesis`, and the classical principles of quantification of the 1st and/or 2nd order. Converse Barcan formulas of the 1st and/or 2nd order are already provable in all systems which are obtained from these above by omitting the axiom schema $\mathsf{ML}\boldsymbol{\phi} \rightarrow \boldsymbol{\phi}$. It is possible to consider classically quantified modal systems of the 1st and 2nd order with converse Barcan formulas and without Barcan formulas

[1] The authorship of the axioms (i) and (iii) belongs to K. Gödel, and the authorship of the axiom (ii) belongs to Dana Scott (see Adams 1995; Gödel 1995).

of corresponding orders.[2] Seen from a syntactical point of view, it is also possible to consider free logic of the 1st and 2nd order with Barcan formulas of corresponding orders, but the main problem still lies in the question of how to treat such systems semantically. Complications as well as motivations for free quantification of the 1st and/or 2nd order variables in the modal setting may just be revealed by using semantic instruments.

There is not, and there could not be, any general catch grasp of the notions: *absolute* identity or *relative* identity of the 1st order variables in a modal context; they are doctrine- or option-orientated (see, for example, Griffin 1977; Noonan 1980). In our case, given the symbol $\overset{1}{\approx}$ introduced by the definition: $(x \overset{1}{\approx} y) \overset{\text{df}}{=} \forall\alpha(\alpha(x) \leftrightarrow \alpha(y))$, where α is a 2nd order variable, the absolute identity is additionally determined by the axiom schema (**I1**). $\mathsf{L}(x \overset{1}{\approx} y) \rightarrow (\boldsymbol{\phi}(z/x) \leftrightarrow \boldsymbol{\phi}(z/y))$, where x and y are free for z in $\boldsymbol{\phi}(z)$, and the axioms: (**I2**). $(x \overset{1}{\approx} y) \rightarrow \mathsf{L}(x \overset{1}{\approx} y)$ and (**I3**). $(x \overset{1}{\not\approx} y) \rightarrow \mathsf{L}(x \overset{1}{\not\approx} y)$; or, according to another option, (**I1**) together with (**I2***). $\mathsf{L}\big((x \overset{1}{\approx} y) \rightarrow \mathsf{L}(x \overset{1}{\approx} y)\big)$ and (**I3***). $\mathsf{L}\big((x \overset{1}{\not\approx} y) \rightarrow \mathsf{L}(x \overset{1}{\not\approx} y)\big)$. In order to have at least a general idea of what relative identity is, we reserve this denomination for every weakness of the absolute identity theory in which (**I2**)- or, (**I2***)-cannot be derived. Weaknesses of the absolute identity theories in which (**I2**) and (**I3**)- or, (**I2***)- and (**I3***)-cannot be theses, are called *contingent identity* theories (for example, see Hughes and Cresswell (1968)). But all our Anderson-like ontological proofs with contingent identity make one important exception, namely, (**CI2**). $\mathsf{G}(x) \rightarrow \big((x \overset{1}{\approx} y) \rightarrow \mathsf{L}(x \overset{1}{\approx} y)\big)$ and (**CI3**). $\mathsf{G}(x) \rightarrow \big((x \overset{1}{\not\approx} y) \rightarrow \mathsf{L}(x \overset{1}{\not\approx} y)\big)$ are their axioms; or, according to anther option, (**CI2***). $\mathsf{L}\big(\mathsf{G}(x) \rightarrow \big((x \overset{1}{\approx} y) \rightarrow \mathsf{L}(x \overset{1}{\approx} y)\big)\big)$ and (**CI3***). $\mathsf{L}\big(\mathsf{G}(x) \rightarrow \big((x \overset{1}{\not\approx} y) \rightarrow \mathsf{L}(x \overset{1}{\not\approx} y)\big)\big)$ are their axioms. For reading $\mathsf{G}(x)$: *x is a god-like being* or simply x is *God*, the axiom (**CI2**) ((**CI3**)) says that an individual which is identical to (different from) God is such necessarily. The axioms (**CI2***) and (**CI3***) take the commitments of the axioms (**CI2**) and (**CI3**) to be necessary.

The optional definitions of the 2nd order *equality* $\overset{2}{\approx}$: $A \overset{2}{\approx} B \overset{\text{df}}{=} \forall x\big(A(x) \leftrightarrow B(x)\big)$ or $A \overset{2}{\approx} B \overset{\text{df}}{=} \mathsf{L}\forall x(A(x) \leftrightarrow B(x))$, affect different axioms legitimating the treatment of the properties of a god-like being and (in some systems) necessary existence as terms of the 2nd sort. Juxtapositions of these optional definitions of the 2nd order equality with the relative identity of the 1st order terms are also far from arbitrary. Provided the *modal operator free* definition of the 2nd order equality is companionable with accompanied the $\star$-less axioms (**CI2**), (**CI3**), then the *marked with modal operator* definition of the 2nd order equality is companionable with accompanied the $\star$-marked axioms (**CI2***), (**CI3***). Note, in this context, the treatments of so called singletons I_x, where $\mathsf{I}_x(y) \overset{\text{df}}{=} (x \overset{1}{\approx} y)$, as terms of the 2nd sort are also option-oriented.

[2] See, for example, Hughes and Cresswell (1968) and Cresswell (1995), where 1st order modal logic adopting all converse Barcan formulas and refusing Barcan formulas are presented.

Admittedly, the first step in the systematic study of any logic should be to establish its completeness or incompleteness with respect to some good class of model structures. And it should be said here that Kripke frames are strongly insufficient to characterize all modal propositional logic, and consequently—all their 1st and 2nd order extensions. Moreover, many well-known complete propositional modal systems have incomplete 1st order predicate extensions. It is an open question whether something similar holds between complete 1st order modal systems and their 2nd order extensions; in particular—it is valid for us—what consequences adding ontological axioms has. Some technical problems concerning the choices of a method to prove completeness of quantified modal logic additionally arise; cf. Garson (1991, 1984), where J. W. Garson describes the difficulties in finding one general method of providing such a proof.

Model structures contain families of *world-varying objectual* domains of the 1st order—for comparison, any model structures as introduced in Szatkowski (2005) contain constant objectual domains of the 1st order— and *constant conceptual* domains of the 2nd order. Terms of the 1st order receive only *rigid* (i.e. *world-independent*) extensions, which are elements of the union of all 1st order domains. Members of these 1st order domains are called *existing objects of the world* in question. Terms of the 2nd order receive *non-rigid* (i.e. *world-dependent*) extensions and *rigid* intensions. It is necessary to introduce families of *preselected world-varying objectual* domains of the 2nd order; the preselected world-varying objectual domains of the 2nd order are simply subsets of the power sets of corresponding 1st order domains. Members of such world-indexed subfamilies are called *existing properties of the world* in question. Extensions of the 2nd order terms are allowed to vary from one possible world to another but they are always required to belong to a preselected domain connected with a given world. Intensions of the 2nd order terms are chosen from among members of a conceptual domain of the 2nd order, which is the set of all functions from the set of worlds to the union of all 2nd order preselected domains such that values of these functions at a given world belong to a preselected domain connected with this world. Members of this conceptual domain are called *conceptual properties*.

The question is: How to treat "satisfiability of formulas" in such model structures? More precisely, what to do with the satisfiability of formulas in the worlds in which the extensions of their free variables of the 1st and 2nd order don't exist? There are two differing ways of dealing with this problem. One is to allow *undefined values*—a formula $\boldsymbol{\phi}$ is neither satisfied nor unsatisfied at a world w under an assignment $\mathfrak{a}$, if there exists a 1st order free variable x in $\boldsymbol{\phi}$ such that $\mathfrak{a}(x)$ does not belong to the 1st order domain indexed by the world w or if there exists a 2nd order free variable α in $\boldsymbol{\phi}$ such that $(\mathfrak{a}(\alpha))(w)$ does not belong to the 2nd order preselected domain indexed by the world w. The difficulty with this proposal is that the inference rules of generalization of both orders do not preserve their validity. For example, the formulas $\mathsf{L}(\forall\alpha\alpha(x) \rightarrow \beta(x))$ and $\mathsf{L}(\forall x\alpha(x) \rightarrow \alpha(y))$ are valid, but $\forall x\mathsf{L}(\forall\alpha\alpha(x) \rightarrow \beta(x))$ and $\forall\alpha\mathsf{L}(\forall x\alpha(x) \rightarrow \alpha(y))$ are not.[3] The other alternative, which we follow, says

[3] Hughes and Cresswell (1968) have given completeness results for some classical modal logics of the 1st order without classical or relative identity and without Barcan formulas wrt. classes of modal structures containing families of *nested* world-varying objectual domains of the 1st order, i.e. domains of accessible

that even though there exists a 1st order free variable x in $\boldsymbol{\phi}$ such that $\mathfrak{a}(x)$ does not belong to the 1st order domain indexed by the world w or there exists a 2nd order free variable α in $\boldsymbol{\phi}$ such that $(\mathfrak{a}(\alpha))(w)$ does not belong to the 2nd order preselected domain indexed by the world w, then they are members of other domains and therefore meaningful—consequently, the formula $\boldsymbol{\phi}$ is required to be in the world w as either satisfied or unsatisfied.

The philosophical motivation for world-varying objectual domains of the 1st order and preselected world-varying objectual domains of the 2nd order lies in the idea that the objects and the properties in one world may fail to exist in another.[4] As a result, the universal specification of the 1st order, Barcan formulas and converse Barcan formulas of the 1st order are invalid in such structures. Similarly, if 2nd order terms were treated only as rigid extensions in the union of all 2nd order preselected domains, then the universal specification of the 2nd order, Barcan formulas and converse Barcan formulas of the 2nd order wouldn't be valid. The question is whether the universal specifications of the 1st and 2nd order are sufficient on the conditions of the constancy of 1st and 2nd order domains, respectively. It is extremely hard, if not impossible, to construct Kripke-type semantics with constant objectual 1st and/or 2nd order domains in which the universal specification of the 1st and/or 2nd order is invalid. On the other hand, it is not difficult to verify semantically that for every model with world-varying objectual domains of the 1st and/or 2nd order, Barcan formulas of the 1st and/or 2nd order are valid in this model if and only if it satisfies the anti-monotonicity condition: if world v is accessible from world w then the 1st and/or 2nd order domain of v is a subset of the 1st and/or 2nd order domain of w. And also, for every model with world-varying objectual domains of the 1st and/or 2nd order, converse Barcan formulas of the 1st and/or 2nd order are valid in this model if and only if it satisfies the monotonicity condition: if world v is accessible from world w then the 1st and/or 2nd order domain of w is a subset of the 1st and/or 2nd domain of v. Both equivalences together say that any formula is valid in all world-varying objectual domain model structures for Barcan formulas of the 1st and/or 2nd order and their converses if and only if it is valid in all model structures with constant objectual domains of the 1st and/or 2nd order.

So, the more natural model structures for modal theories of the 2nd order without the universal specification, Barcan and converse Barcan formulas of the 1st order—on the one hand, and with the universal specification, Barcan and Barcan formulas of the 2nd order—on the other hand, should be model structures with world-varying objectual domains of the 1st order and one constant objectual domain of the 2nd order, in which terms of the 1st and 2nd order obtain rigid extensions, respectively. However, there are reasons for rejecting such a choice. Firstly, we don't know how to characterize a constant objectual domain of the 2nd order, which should have some

Footnote 3 continued
worlds from a given one are supersets of the domain of the world in question. The requirement of nested domains guarantees that the inference rule of generalization of the 1st order already preserves the validity, whole undefined values of formulas in worlds are allowed.

[4] In Szatkowski (2005) we proved strong completeness theorems for different Anderson-like variants of Gödel's theory with respect to classes of model structures containing constant objectual 1st order domains.

coherence to world-varying objectual domains of the 1st order. Secondly, if the sum of all 2nd order world-varying preselected objectual domains (without introducing the conceptual domain of the 2nd order) was treated as a constant objectual domain of the 2nd order in which terms of the 2nd order would obtain rigid extensions, then formulas of the form $\mathsf{L}\boldsymbol{\phi} \leftrightarrow \boldsymbol{\phi}$ would be verified for all formulas $\boldsymbol{\phi}$ not containing quantifiers of the 1st order. In order to avoid these difficulties we introduce preselected world-relative objectual 2nd order domains and a constant conceptual 2nd order domain.[5]

The aim of this paper is to prove strong completeness theorems for all distinguished theories with respect to corresponding classes of modal structures, which is certainly of philosophical and theoretical import. The strategy of establishing strong completeness is borrowed from Thomason (1970), which is essentially different from the method used by us in Szatkowski (2005, 2007). Of course, the statement: **God necessarily exists** is valid in all corresponding classes of modal structures.

Finally, a few words about related works. We limit ourselves only to works, in which ontological proofs considered were understand by the authors as modifications of Anderson's ontological proof, and additionally—they were based on free modal logic of the 2nd order. According to our knowledge of the literature, only Hájek's paper Hájek (2002a,b) satisfies such a criterion. Starting with the 2nd order modal logic, which is obtained by marrying the 2nd order classical logic—enlarged by absolute identity of the 1st order—with the propositional modal logic **S5** (see, for example, the notation in Hughes and Cresswell (1968)), Hájek firstly extends the language and the basis of this logic by adding an unary predicate E applied to 1st order variables and the axiom $\mathsf{L}\exists x \mathsf{E}(x)$, respectively; secondly, following Fitting (see, Fitting (2004) and Fitting and Mendelson (1998)), he introduces the relativized quantifiers of the 1st order $\forall^{\mathsf{E}}$ and $\exists^{\mathsf{E}}$ as follows: $\forall^{\mathsf{E}} x\boldsymbol{\phi}$ for $\forall x(\mathsf{E}(x) \rightarrow \boldsymbol{\phi})$ and $\exists^{\mathsf{E}} x\boldsymbol{\phi}$ for $\exists x(\mathsf{E}(x) \wedge \boldsymbol{\phi})$; and thirdly, he modifies Anderson's theory by replacing all 1st order unrelativized quantifiers by relativized quantifiers and then adds such a modified Anderson's theory to the whole, or he adds some versions of such a modified Anderson's theory which is obtained by applying at least one of the following three alterations: reducing the system of axioms, replacing some axioms by new axioms, or introducing a new definition of God-being. One can, of course, think of all these theories as having the free modal logic of the 2nd order as a basis, if relativized quantifiers of the 1st order are considered.

2 Anderson-Like Theories, Viewed Syntactically

The formal language $\mathcal{L}$ of *Anderson-like* theories is equipped with a 2nd order unary predicate **P** (**P**(A) is read: A is a *positive property* (or simply, A is *positive*), an existence determinator E, a necessity symbol L, two sorts of variables: $x, y, z, \ldots$ 1st order), $\alpha, \beta, \gamma, \ldots$ (2nd order), Boolean operator: $-$ (complementation), logical symbols: $\wedge, \neg$ (conjunction, negation) and $\forall$ (universal quantifier) for both sorts of variables. The only terms of 1st sort are variables of 1st sort and terms of 2nd sort are formed from variables of 2nd sort by applying complementation of any finite (possibly

[5] We recommend Fitting (2002), Fitting and Mendelson (1998) and Garson (1984, 1991) for a detailed discussion of the complications that arise in proving semantics for 1st order modal logics.

zero) numbers of times. Thus, the set of terms of the 2nd sort and the set of formulas are given by the grammars, respectively:

$$A \stackrel{\mathrm{df}}{=} \alpha \,\big|\, -A$$

$$\boldsymbol{\phi} \stackrel{\mathrm{df}}{=} A(x) \,\big|\, \mathsf{E}(x) \,\big|\, \mathbf{P}(A) \,\big|\, \boldsymbol{\phi} \wedge \boldsymbol{\psi} \,\big|\, \neg\boldsymbol{\phi} \,\big|\, \mathsf{L}\boldsymbol{\phi} \,\big|\, \forall x \boldsymbol{\phi} \,\big|\, \forall \alpha \boldsymbol{\phi}.$$

The remaining propositional connectives: $\vee$, $\rightarrow$, $\leftrightarrow$ as well as the strict implication $\prec$, the existential quantifier $\exists$, the possibility operator M, the identity $\stackrel{1}{\approx}$ and the inequality $\stackrel{1}{\not\approx}$ for terms of 1st sort are introduced as follows:

$$\boldsymbol{\phi} \vee \boldsymbol{\psi} \stackrel{\mathrm{df}}{=} \neg(\neg\boldsymbol{\phi} \wedge \neg\boldsymbol{\psi}), \quad \boldsymbol{\phi} \rightarrow \boldsymbol{\psi} \stackrel{\mathrm{df}}{=} \neg\boldsymbol{\phi} \vee \boldsymbol{\psi}, \quad \boldsymbol{\phi} \leftrightarrow \boldsymbol{\psi} \stackrel{\mathrm{df}}{=} (\boldsymbol{\phi} \rightarrow \boldsymbol{\psi}) \wedge (\boldsymbol{\psi} \rightarrow \boldsymbol{\phi}),$$

$$\boldsymbol{\phi} \prec \boldsymbol{\psi} \stackrel{\mathrm{df}}{=} \mathsf{L}(\boldsymbol{\phi} \rightarrow \boldsymbol{\psi}), \quad \exists\xi\boldsymbol{\phi} \stackrel{\mathrm{df}}{=} \neg\forall\xi\neg\boldsymbol{\phi} \text{ where } \xi \text{ is a variable of any sort,}$$

$$\mathsf{M}\boldsymbol{\phi} \stackrel{\mathrm{df}}{=} \neg\mathsf{L}\neg\boldsymbol{\phi}, \quad (x \stackrel{1}{\approx} y) \stackrel{\mathrm{df}}{=} \forall\alpha(\alpha(x) \leftrightarrow \alpha(y)) \text{ and } (x \stackrel{1}{\not\approx} y) \stackrel{\mathrm{df}}{=} \neg(x \stackrel{1}{\approx} y).$$

Some comments on the symbol E seem to be useful here. Why do we commit to E the status of the existence determinator and not of the existence predicate? This last terminology is already standard in literature. In the first place, if predicates of the 1st order are understood to be properties of individuals and predicates of the 2nd order—to be properties of properties, then E isn't a property of individuals, or of properties, or anything. For this reason, $\mathbf{P}(\mathsf{E})$ is not a formula of our language. In the second place, even those who give to E the name: *existence predicate*, don't always handle it as a predicate (cf. Lambert (1981), pp. 159–160). At this point it may be instructive to see how the formulas of the kinds: $\mathsf{E}(x)$ and $A(x)$ are usually valued in model structures. And so, informally, a formula $\mathsf{E}(x)$ is satisfied under an assignment $\mathfrak{a}$ at a world w iff the referent $\mathfrak{a}(x)$ of x is a member of the 1st order domain of w. This means, the symbol E *determines* (what justifies our name: *existence determinator*) referents of 1st order variables to be members of appropriate domains. Now a formula $A(x)$ is satisfied under an assignment $\mathfrak{a}$ (or, under an interpretation $\mathfrak{I}$, if 1st order logic is in work) at a world w iff $\mathfrak{a}(x)$ belongs to the referent of A with respect to the assignment $\mathfrak{a}$ (or, with respect to the interpretation $\mathfrak{I}$) at the world w. Clearly the semantic statuses of both kinds of formulas are different.

Further definitions of $\mathsf{G}(x)$, α Ess x and $\mathsf{NE}(x)$ adopted in Anderson-like theories are borrowed from Anderson (1990):

$$\mathsf{G}(x) \stackrel{\mathrm{df}}{=} \forall\alpha(\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)) \tag{2.1}$$

$\mathsf{G}(x)$ is read: *x is a god-like being* or simply *x is God*

$$A \;\mathsf{Ess}\; x \stackrel{\mathrm{df}}{=} \forall\beta\big(\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y(A(y) \rightarrow \beta(y))\big) \tag{2.2}$$

A Ess x is read: *a property A is an essence of entity x*, where A is a term of the 2nd sort

$$\mathsf{NE}(x) \stackrel{\mathrm{df}}{=} \forall\alpha(\alpha \;\mathsf{Ess}\; x \rightarrow \mathsf{L}\exists y\alpha(y)) \tag{2.3}$$

$\mathsf{NE}(x)$ is read: *x necessarily exists.*

Any Anderson-like theory will be determined by two groups of axioms and axiom schemas: *obligatory* axioms and axiom schemas and *optional* axioms and axiom schemas.

(I) *Obligatory* axioms and axiom schemas of Anderson-like theories are following (we assume that $\boldsymbol{\phi}$, $\boldsymbol{\psi}$ are formulas and ξ is a variable of any sort):

All what is needed for classical propositional logic, (2.4)

$$\forall\xi(\boldsymbol{\phi} \rightarrow \boldsymbol{\psi}) \rightarrow (\forall\xi\boldsymbol{\phi} \rightarrow \forall\xi\boldsymbol{\psi}), \tag{2.5}$$

$\forall\alpha\boldsymbol{\phi} \rightarrow \boldsymbol{\phi}(\alpha/A)$ where A is a term of the 2nd sort, (2.6)

$\boldsymbol{\phi} \leftrightarrow \forall\xi\boldsymbol{\phi}$ if ξ is not free in $\boldsymbol{\phi}$, (2.7)

$$\forall x\boldsymbol{\phi} \wedge \mathsf{E}(y) \rightarrow \boldsymbol{\phi}(x/y), \tag{2.8}$$

$$\forall x\mathsf{E}(x), \tag{2.9}$$

$$A(x) \rightarrow \mathsf{E}(x), \tag{2.10}$$

$$\mathsf{L}(\boldsymbol{\phi} \rightarrow \boldsymbol{\psi}) \rightarrow (\mathsf{L}\boldsymbol{\phi} \rightarrow \mathsf{L}\boldsymbol{\psi}), \tag{2.11}$$

$$\mathsf{L}\boldsymbol{\phi} \rightarrow \mathsf{M}\boldsymbol{\phi}, \tag{2.12}$$

$\forall\alpha\mathsf{L}\boldsymbol{\phi} \rightarrow \mathsf{L}\forall\alpha\boldsymbol{\phi}$ (Barcan formula of 2nd sort), (2.13)

$\mathsf{L}\exists\alpha\boldsymbol{\phi} \rightarrow \exists\alpha\mathsf{L}\boldsymbol{\phi}$ (principle of exportation of 2nd sort), (2.14)

$(x \overset{1}{\approx} y) \rightarrow (\boldsymbol{\phi}(z/x) \rightarrow \boldsymbol{\phi}(z/y))$ where z does not occur within the scope of a modal operator, and x, y are free for z in $\boldsymbol{\phi}$, (2.15)

$\forall x\big(A(x) \leftrightarrow B(x)\big) \rightarrow \big(\boldsymbol{\phi}(\alpha/A) \rightarrow \boldsymbol{\phi}(\alpha/B)\big)$ where A and B are free for α in $\boldsymbol{\phi}$ and α does not occur within the scope of a modal operator, (2.16)

$\mathsf{G}(x) \rightarrow \big(\mathsf{L}(x \overset{1}{\approx} y) \rightarrow (\boldsymbol{\phi}(z/x) \leftrightarrow \boldsymbol{\phi}(z/y))\big)$ where x and y are free for z in $\boldsymbol{\phi}$, (2.17)

$$\mathsf{G}(x) \rightarrow \big(\exists\alpha(\alpha(x) \wedge \neg\alpha(y)) \rightarrow \mathsf{L}\exists\alpha(\alpha(x) \wedge \neg\alpha(y))\big), \tag{2.18}$$

$$\mathbf{P}(\alpha) \rightarrow \neg\mathbf{P}(-\alpha), \tag{2.19}$$

$$\mathbf{P}(\alpha) \wedge \mathsf{L}\forall x(\alpha(x) \rightarrow \beta(x)) \rightarrow \mathbf{P}(\beta), \tag{2.20}$$

$$\mathsf{LG}(x) \rightarrow \mathsf{G}(x), \tag{2.21}$$

$$\mathsf{L}\exists x\mathsf{G}(x) \rightarrow \exists x\mathsf{LG}(x) \tag{2.22}$$

Every Anderson-like theory must be equipped also with an axiom saying that the property of a god-like being is positive and therefore it must be legitimate to treat this property as a term of the 2nd sort. Thus, the following two axioms are obligatory for all Anderson-like theories:

$$\exists\alpha(\alpha \overset{2}{\approx} \mathsf{G}), \tag{2.23}$$

$\exists\beta\big(\mathbf{P}(\beta) \wedge (\beta \overset{2}{\approx} \mathsf{G})\big)$ or shortly: $\mathbf{P}(\mathsf{G})$. (2.24) where, of course, the symbol $\overset{2}{\approx}$ stands for the relation of identity of objects of the 2nd sort i.e. *properties*.

However, the relation $\overset{2}{\approx}$ can be introduced in Anderson-like theories in two different ways by the following *optional definitions*:

$$A \overset{2}{\approx} B \overset{\text{df}}{=} \forall x(A(x) \leftrightarrow B(x)), \tag{2.25}$$

$$A \overset{2}{\approx} B \overset{\text{df}}{=} \mathsf{L}\forall x(A(x) \leftrightarrow B(x)) \tag{2.26}$$

and it is clear that the translation of (2.23) and (2.24) to the original language depends on which optional definition of $\overset{2}{\approx}$ has been applied.

Moreover, the choice of definition of $\overset{2}{\approx}$ affects other obligatory axioms of Anderson-like theories. Those axioms take the form:

$$(-\alpha)(x) \to \neg\alpha(x), \tag{2.27}$$

$$\forall x(\neg\alpha(x) \to (-\alpha)(x)) \text{ and} \tag{2.28}$$

$$\mathsf{G}(x) \to \big((x \overset{1}{\approx} y) \to \mathsf{L}(x \overset{1}{\approx} y)\big) \tag{2.29}$$

or

$$\mathsf{L}((-\alpha)(x) \to \neg\alpha(x)), \tag{2.30}$$

$$\mathsf{L}\forall x(\neg\alpha(x) \leftrightarrow (-\alpha)(x)) \text{ and} \tag{2.31}$$

$$\mathsf{G}(x) \to \mathsf{L}\big((x \overset{1}{\approx} y) \to \mathsf{L}(x \overset{1}{\approx} y)\big) \tag{2.32}$$

depending on which one of (2.25), (2.26) has been adopted.

What is important to note is that neither of two optional definitions of $\overset{2}{\approx}$ provides what one might have expected of an identity relation. Indeed, the formula: $(\alpha \overset{2}{\approx} \beta) \wedge \mathsf{E}(x) \to (\alpha(x) \to \beta(x))$ is unprovable on the basis of the definition (2.26), however, it can be proved if (2.25) is applied. On the other hand, the formula: $(\alpha \overset{2}{\approx} \beta) \to (\mathbf{P}(\alpha) \to \mathbf{P}(\beta))$ is unprovable on the basis of (2.25) but it can be proved if (2.26) is applied.

Some comments about particular axioms or axiom schemas are also desirable here. And so, the axiom (2.10)—called the thesis of *serious actualism*—can be seen surprising. Anticipating semantical considerations, this axiom says that if an atomic formula $A(x)$ is true at a world w then any extension of x is an element of the 1st order domain of w. What is important, atomic formulas $A(x)$ in the axiom (2.10) can not be replaced by arbitrary formulas $\phi(x)$; for example, substituting $\neg A(x)$ for $A(x)$ in (2.10) yields a non-valid formula $\neg A(x) \to \mathsf{E}(x)$ in our semantics. In view of the thesishood in Anderson-like theories the usefulness of the axiom $A(x) \to \mathsf{E}(x)$ lies in them that it interferes in our proofs of the ontologically meaningful theorems $\exists x\mathsf{G}(x) \to \mathsf{L}\exists x\mathsf{G}(x)$ (If there is a god-like being, then necessarily there is a god-like being), $\forall x\mathsf{NE}(x)$ ($\mathsf{NE}(x)$ holds for every object x) and $\mathbf{P}(\mathsf{NE})$ (The property of necessary existence is a positive property); see, T6, T10 and T16 in "Appendix". As a result of possesing the axiom (2.10) we are able to prove in any Anderson-like theory the formula: $\mathsf{M}\exists x\mathsf{G}(x) \rightarrow \mathsf{L}\exists x\mathsf{G}(x)$ (Possible existence of a god-like being implies necessary existence of a god-like being)—what is in the sprit of the ancient Anselm's principle. The formulas $\exists x\mathsf{G}(x) \rightarrow \mathsf{L}\exists x\mathsf{G}(x)$ and $\mathsf{M}\exists x\mathsf{G}(x) \rightarrow \mathsf{L}\exists x\mathsf{G}(x)$ were already proved by Anderson (1990) as an important part of his ontological proof. Further, as a result of possesing the axioms (2.21) and (2.22) we are able to prove in any Anderson-like theory the formula: $\mathsf{L}\exists x\mathsf{G}(x) \leftrightarrow \exists x\mathsf{G}(x)$ (If necessarily there is a god-like being, then indeed there is a god-like being); see, T9 in "Appendix". For clarity, if our Anderson-like theories would be based on the propositional modal logic **S5**, then the axioms (2.21) and (2.22) would be superfluous. Finally, it may be also instructive to make a small comment on the axiom (2.18). Perhaps, in the reader's opinion, the natural tendency would be use the axiom $\mathsf{G}(x) \rightarrow \big((x \overset{1}{\not\approx} y) \rightarrow \mathsf{L}(x \overset{1}{\not\approx} y)\big)$ instead of $\mathsf{G}(x) \rightarrow \big(\exists\alpha(\alpha(x) \wedge \neg\alpha(y)) \rightarrow \mathsf{L}\exists\alpha(\alpha(x) \wedge \neg\alpha(y))\big)$. For the justification of our choice, we answer that the formula $\mathsf{G}(x) \rightarrow \big(\exists\alpha(\alpha(x) \wedge \neg\alpha(y)) \rightarrow \mathsf{L}\exists\alpha(\alpha(x) \wedge \neg\alpha(y))\big)$ is deductively stronger than the formula $\mathsf{G}(x) \rightarrow \big((x \overset{1}{\not\approx} y) \rightarrow \mathsf{L}(x \overset{1}{\not\approx} y)\big)$, i.e. the first formula is deducible from the second one in the context of other axioms of Anderson-like theories, and the converse deduction does not hold.

Throughout this paper, we will consider different Anderson-like theories, which will be denoted by appropriate acronyms. The first symbol of each acronym will be $\mathbf{V}^A$. Any Anderson-like theory employing the definition (2.26) will be given an acronym ending with the symbol $\star$ and thus, theories employing (2.25) can be easily recognized by their $\star$-less acronyms.

(II) *Optional axioms* of Anderson-like theories are chosen according to the following criteria:

(i) treatment of the property of *necessary existence*,
(ii) treatment of so called *permanence*,
(iii) treatment of properties abstracted from expressions of the form $\mathsf{I}_x(y)$ defined by:

$$\mathsf{I}_x(y) \overset{\text{df}}{=} (x \overset{1}{\approx} y)\,, \tag{2.33}$$

(iv) characterization of modal operators.

As to (i), if we intend to treat the property of necessary existence as a term of the 2nd sort we should adopt an optional axiom:

$$\exists\alpha(\alpha \overset{2}{\approx} \mathsf{NE}) \tag{2.34}$$

and augment the acronym of theory with the symbol **n**.

As to (ii), we simply add the axiom schema

$$\forall x\mathsf{LE}(x) \tag{2.35}$$

and augment the acronym of theory with the symbol **p**.

As to (iii), if we intend to treat expressions of the form $\mathsf{I}_x(y)$ as terms of 2nd sort we should adopt an optional axiom:

$$\exists\alpha(\alpha \overset{2}{\approx} \mathsf{I}_x) \tag{2.36}$$

and augment the acronym of theory with the symbol **s**.

Once again we wish to emphasize that the form of optional axioms (2.34) and (2.36) depends on which definition (2.25 or 2.26) of the relation $\overset{2}{\approx}$ has been chosen.

As to (iv), we choose one of the following:

(5) $\mathsf{ML}\boldsymbol{\phi} \to \mathsf{L}\boldsymbol{\phi}$
(b) $\mathsf{ML}\boldsymbol{\phi} \to \boldsymbol{\phi}$
(c) $\mathsf{ML}\boldsymbol{\phi} \to \mathsf{L}\boldsymbol{\phi}$, $\mathsf{ML}\boldsymbol{\phi} \to \boldsymbol{\phi}$ and $\mathbf{P}(\alpha) \to \mathsf{L}\mathbf{P}(\alpha)$
(d) $\mathsf{ML}\boldsymbol{\phi} \to \mathsf{L}\boldsymbol{\phi}$, $\mathsf{L}\boldsymbol{\phi} \to \mathsf{LL}\boldsymbol{\phi}$ and $\mathbf{P}(\alpha) \to \mathsf{L}\mathbf{P}(\alpha)$

and augment the acronym of theory by symbol **5, b, c** or **d** indicating the choice that has been made.[6]

Each Anderson-like theory has the inference rule: *modus ponens*, *necessitation* and *generalization*, respectively:

$$\text{R1: } \frac{\boldsymbol{\phi}, \boldsymbol{\phi} \to \boldsymbol{\psi}}{\boldsymbol{\psi}} \quad \text{R2: } \frac{\boldsymbol{\phi}}{\mathsf{L}\boldsymbol{\phi}} \quad \text{R3: } \frac{\boldsymbol{\phi}}{\forall\xi\boldsymbol{\phi}}$$

and the following borrowed from Thomason (1970):

$$\text{R4}_0\text{: } \frac{\boldsymbol{\phi} \to \mathsf{L}\chi}{\boldsymbol{\phi} \to \mathsf{L}\forall x\chi}$$

where x is not free in $\boldsymbol{\phi}$,

$$\text{R4}_n\text{: } \frac{\boldsymbol{\phi} \to .\boldsymbol{\psi}_1 \prec . \cdots \prec .\boldsymbol{\psi}_n \prec \mathsf{L}\chi}{\boldsymbol{\phi} \to .\boldsymbol{\psi}_1 \prec . \cdots \prec .\boldsymbol{\psi}_n \prec \mathsf{L}\forall x\chi}$$

where x is not free in $\boldsymbol{\phi}$, $\boldsymbol{\psi}_1, \ldots,$ or $\boldsymbol{\psi}_n$, $n > 0$.

$$\text{R5}_0\text{: } \frac{\boldsymbol{\phi} \to (x \overset{1}{\not\approx} y)}{\neg\boldsymbol{\phi}}$$

where x is not free in $\boldsymbol{\phi}$,

$$\text{R5}_n\text{: } \frac{\boldsymbol{\phi} \to .\boldsymbol{\psi}_1 \prec . \cdots \prec .\boldsymbol{\psi}_n \prec (x \overset{1}{\not\approx} y)}{\boldsymbol{\phi} \to .\boldsymbol{\psi}_1 \prec . \cdots \prec .\boldsymbol{\psi}_{n-1} \prec \mathsf{L}\neg\boldsymbol{\psi}_n}$$

where x is not free in $\boldsymbol{\phi}$, $\boldsymbol{\psi}_1, \ldots,$ or $\boldsymbol{\psi}_n$, $n > 0$.

[6] Anderson's ontological proof is based on the propositional modal logic **S5**. But, its variants proposed by Hájek (1996, 2002a,b) and us in Szatkowski (2005) were already grounded on weaker than **S5** propositional modal logics.

By $\vdash_{\mathbf{Th}}$ we denote the *inference relation* determined by axioms and rules of the Anderson-like theory **Th**. Thus, for a set of formulas X and a formula $\boldsymbol{\phi}$ we write: $X \vdash_{\mathbf{Th}} \boldsymbol{\phi}$ to mean that there exists a **Th**-*derivation* of $\boldsymbol{\phi}$ from X. Such a derivation is a finite sequence of formulas (*derivation steps*) each of which has to be justified in an appropriate manner. Each step of derivation is therefore required to be an axiom of **Th** or an element of X or a result of applying an inference rule to preceding step (or steps). Moreover, applying inference rules is subject to the following important restriction:

rules other than $R1$ are applicable only to steps which are obtained without using elements of X.

The following remark is relevant here. We have defined a **Th**-*derivation of* $\boldsymbol{\phi}$ *from* X in a different way from this in Szatkowski (2005), where we have rejected the inference rules of necessitation and generalizations and we have worked with modus ponens as the only inference rule and with so called clothed axioms. If necessitation and generalizations had been introduced, then their use must be restricted to theorems of a **Th** theory. But then the axiomatic basis obtained could easily be proved to be equivalent to the one we have used. However, we have chosen the more "modern and elegant" treatment. It still seems to be technically difficult, if it is really possible, to apply this treatment to theories considered here. These theories are determined except necessitation and generalizations yet by other complex inference rules, which makes such a mutual translation difficult.

An easy proof of the following elementary properties of the inference relation $\vdash_{\mathbf{Th}}$ is left to the reader.

Proposition 2.1 *For any Anderson-like theory* ***Th****:*

(i) $X \cup \{\boldsymbol{\phi}\} \vdash_{\mathbf{Th}} \boldsymbol{\psi}$ *iff* $X \vdash_{\mathbf{Th}} \boldsymbol{\phi} \rightarrow \boldsymbol{\psi}$,
(ii) *if* $X \vdash_{\mathbf{Th}} \boldsymbol{\phi}$ *then* $\{\mathsf{L}\boldsymbol{\psi} : \boldsymbol{\psi} \in X\} \vdash_{\mathbf{Th}} \mathsf{L}\boldsymbol{\phi}$,
(iii) *if* $X \cup \{\boldsymbol{\phi}(\xi/\zeta)\} \vdash_{\mathbf{Th}} \boldsymbol{\psi}$ *and* ζ *is a variable not occurring in* $\boldsymbol{\psi}$ *or in any member of* X, *then* $X \cup \{\exists\xi\boldsymbol{\phi}\} \vdash_{\mathbf{Th}} \boldsymbol{\psi}$.

3 Anderson-Like Theories, Viewed Semantically

By a *model structure* we mean a quintuple of the form $\mathfrak{W} = \langle W, R, \mathscr{D}_1, \mathscr{D}_2, \mathbb{G}\rangle$ where $W \neq \emptyset$ is the set of *possible worlds*, $R \subseteq W^2$ is the *relation of accessibility*, $\mathscr{D}_1$ is the family $(D_w)_{w\in W}$ of 1st sort domains—members of $\bigcup_{w\in W} D_w$ are called *existing objects*, $\mathscr{D}_2$ is the family $(\mathcal{D}_w)_{w\in W}$ of 2nd sort domains $\mathcal{D}_w \subseteq 2^{D_w}$, $w \in W$—members of $\bigcup_{w\in W} \mathcal{D}_w$ are called *existing properties*. Apart from existing properties we also consider so called *conceptual properties* of the structure, by which we mean functions $f \in W \mapsto \bigcup_{w\in W} \mathcal{D}_w$ such that $f(w) \in \mathcal{D}_w$ for every $w \in W$. The set of all conceptual properties of the structure $\mathfrak{W}$ will be denoted by $C_{\mathfrak{W}}$. The additional conditions necessary for modal structures are:

$$\varnothing \neq \mathbb{G} \subseteq \bigcap_{w \in W} D_w \text{ and } \mathbb{G} \in \bigcap_{w \in W} \mathcal{D}_w, \tag{3.1}$$

$$\forall w \in W \forall a, b \in \mathbb{G} \forall X \in \mathcal{D}_w \big(a \in X \text{ iff } b \in X\big), \tag{3.2}$$

$$\forall w \in W \big(X \in \mathcal{D}_w \Longrightarrow D_w - X \in \mathcal{D}_w\big), \tag{3.3}$$

$$R \text{ is } serial, \text{i.e. } \forall \in W \exists v \in W\, wRv. \tag{3.4}$$

By an *assignment* in a model structure $\mathfrak{W}$ we mean a function $\mathfrak{a}$ which maps variables of the 1st sort to members of $\bigcup_{w \in W} D_w$ and variables of the 2nd sort to conceptual properties of the structure (i.e. members of $C_{\mathfrak{W}}$). An assignment $\mathfrak{a}$ is extended to all terms A by putting: $(\mathfrak{a}(-A))(w) \overset{\text{df}}{=} D_w - (\mathfrak{a}(A))(w)$, for every $w \in W$ and every term A of the 2nd sort. If $\mathfrak{a}$ is an assignment, then the symbol $\mathfrak{a}_\xi^o$ denotes the assignment defined by:

$$\mathfrak{a}_\xi^o(\zeta) \overset{\text{df}}{=} \begin{cases} o & \text{if } \zeta = \xi, \\ \mathfrak{a}(\zeta) & \text{if } \zeta \neq \xi. \end{cases}$$

Of course, o is tacitly assumed to be an entity suitable for the variable ξ depending on its sort and both $\mathfrak{a}$ and $\mathfrak{a}_\xi^o$ are assumed to be assignments in the same structure. We say that assignments $\mathfrak{a}$, $\mathfrak{b}$ *agree apart from* ξ (symbolically: $\mathfrak{a} \equiv_\xi^? \mathfrak{b}$) if for some o, $\mathfrak{a}_\xi^o = \mathfrak{b}$. Note that $\equiv_\xi^?$ is an equivalence relation on the set of all assignments of a model structure. The equivalence class of $\mathfrak{a}$ with respect to $\equiv_\xi^?$ will be further denoted by $\{\mathfrak{a}_\xi^?\}$. And for every $w \in W$, $\{\mathfrak{a}_{\xi,w}^?\}$ will be the subclass of $\{\mathfrak{a}_\xi^?\}$ defined as follows:

(i) for every 1st sort variable x, $\{\mathfrak{a}_{x,w}^?\} = \big\{\mathfrak{b} \mid \mathfrak{b} \in \{\mathfrak{a}_x^?\} \text{ and } \mathfrak{b}(x) \in D_w\big\}$,
(ii) for every 2nd sort variable α, $\{\mathfrak{a}_{\alpha,w}^?\} = \{\mathfrak{a}_\alpha^?\}$.

It is worth nothing that $\{\mathfrak{a}_{\xi,w}^?\}$ is an equivalence subclass of $\{\mathfrak{a}_\xi^?\}$ iff $\mathfrak{a} \in \{\mathfrak{a}_{\xi,w}^?\}$.

A pair of the form $\langle \mathfrak{W}, \mathfrak{a} \rangle$ will be called *model* and the symbol $\models$ will be used for the *satisfiability relation*—the expression $\mathfrak{W}, \mathfrak{a}, w \models \boldsymbol{\phi}$, where $w \in W$ reads: *the formula* $\boldsymbol{\phi}$ *is satisfied in the world w of model* $\langle \mathfrak{W}, \mathfrak{a} \rangle$. If no misunderstanding is likely as to the particular structure $\mathfrak{W}$ in which an assignment $\mathfrak{a}$ has been chosen, we simplify the notation by writing: $\mathfrak{a}, w \models \boldsymbol{\phi}$ instead of $\mathfrak{W}, \mathfrak{a}, w \models \boldsymbol{\phi}$. Given a model $\langle \mathfrak{W}, \mathfrak{a} \rangle$, the satisfiability relation $\models$ is defined as usual, for any possible world $w \in W$ by the following conditions, where x is a variable of the 1st sort, A is a term of the 2nd sort, ξ is a variable of arbitrary sort and $\boldsymbol{\phi}$, $\boldsymbol{\psi}$ are a formulas:

(i) $\mathfrak{a}, w \models \mathsf{E}(x)$ iff $\mathfrak{a}(x) \in D_w$,
(ii) $\mathfrak{a}, w \models A(x)$ iff $\mathfrak{a}(x) \in (\mathfrak{a}(A))(w)$,
(iii) $\mathfrak{a}, w \models \boldsymbol{\phi} \wedge \boldsymbol{\psi}$ iff $\mathfrak{a}, w \models \boldsymbol{\phi}$ and $\mathfrak{a}, w \models \boldsymbol{\psi}$,
(iv) $\mathfrak{a}, w \models \neg\boldsymbol{\phi}$ iff not $\mathfrak{a}, w \models \boldsymbol{\phi}$ (symbolically: $\mathfrak{a}, w \not\models \boldsymbol{\phi}$),
(v) $\mathfrak{a}, w \models \forall\xi\boldsymbol{\phi}$ iff $\mathfrak{b}, w \models \boldsymbol{\phi}$ for every $\mathfrak{b} \in \{\mathfrak{a}_{\xi,w}^?\}$,
(vi) $\mathfrak{a}, w \models \mathsf{L}\boldsymbol{\phi}$ iff $\mathfrak{a}, v \models \boldsymbol{\phi}$ for every $v \in W$ such that wRv,
(vii) $\mathfrak{a}, w \models \mathbf{P}(A)$ iff $\mathbb{G} \subseteq (\mathfrak{a}(A))(v)$ for every $v \in W$ such that wRv.

The set of all formulas satisfied in a world w of a model $\langle \mathfrak{W}, \mathfrak{a} \rangle$ will be denoted by $\mathsf{Sat}(\mathfrak{W}, \mathfrak{a}, w)$ or simply by $\mathsf{Sat}(\mathfrak{a}, w)$, if the model structure in question is clear from the context.

As customary, we say that a formula $\boldsymbol{\phi}$ *is true* in a model structure $\mathfrak{W}$ (symbolically: $\mathfrak{W} \models \boldsymbol{\phi}$) iff $\mathfrak{a}, w \models \boldsymbol{\phi}$, for every assignment $\mathfrak{a}$ in $\mathfrak{W}$ and every world $w \in W$. The set of all true formulas will be denoted by $\mathsf{Th}(\mathfrak{W})$. We also put $\mathsf{Th}(\mathbb{K}) \stackrel{\text{df}}{=} \bigcap \{\mathsf{Th}(\mathfrak{W}) : \mathfrak{W} \in \mathbb{K}\}$, for an arbitrary class of structures $\mathbb{K}$. If X is a set of formulas, then we write $\mathfrak{W} \models X$, $\mathbb{K} \models X$ if $X \subseteq \mathsf{Th}(\mathfrak{W})$, $X \subseteq \mathsf{Th}(\mathbb{K})$ respectively. We write $X \models_{\mathbb{K}} \boldsymbol{\phi}$ to express that for every assignment $\mathfrak{a}$ in a structure $\mathfrak{W} \in \mathbb{K}$ and for every $w \in W$, if $X \subseteq \mathsf{Sat}(\mathfrak{W}, \mathfrak{a}, w)$ then $\boldsymbol{\phi} \in \mathsf{Sat}(\mathfrak{W}, \mathfrak{a}, w)$.

The following fact is sometimes called *substitution lemma*. Its proof—a routine induction on the degree of complexity of $\boldsymbol{\phi}$—will be omitted.

Proposition 3.1 *If A is a term of the same sort as a variable ξ then $\mathfrak{a}, w \models \boldsymbol{\phi}(\xi/A)$ iff $\mathfrak{a}_{\xi}^{\mathfrak{a}(A)}, w \models \boldsymbol{\phi}$.*

We will need a certain subset $W^{acc} \subseteq W$. Members of W^{acc} are called *accessible worlds* and W^{acc} is defined as the R-image of W. We also define *inaccessible worlds* putting $W^{inacc} \stackrel{\text{df}}{=} W - W^{acc}$. We will define a class of so called *special structures* in which inaccessible worlds will be treated in a special way—they will be provided with a separate family $\mathscr{E}_2$ of the 2nd sort domains. Thus, by a *special structure* we shall mean a sextuple of the form $\mathfrak{W} = \langle W, R, \mathscr{D}_1, \mathscr{D}_2, \mathscr{E}_2, \mathbb{G}\rangle$ where $\mathfrak{W} = \langle W, R, \mathscr{D}_1, \mathscr{D}_2, \mathbb{G}\rangle$ is an ordinary model structure and $\mathscr{E}_2 = (\mathcal{E}_w)_{w\in W}$, where $\emptyset \neq \mathcal{E}_w \subseteq \mathcal{D}_w$ for every $w \in W^{inacc}$. By conceptual properties of a special structure we shall mean those functions $f \in W \mapsto \bigcup_{w\in W}(\mathcal{D}_w \cup \mathcal{E}_w)$ such that for every $w \in W$: $f(w) \in \mathcal{D}_w$ if $w \in W^{acc}$, and $f(w) \in \mathcal{E}_w$ if $w \in W^{inacc}$. The above restriction on the set of conceptual properties of a special model structure forces a revision of treatment of terms of the 2nd sort. Indeed, if $\mathfrak{a}$ is an assignment in a special model structure $\mathfrak{W}$ and $w \in W^{inacc}$ then we can no longer put: $(\mathfrak{a}(-A))(w) \stackrel{\text{df}}{=} D_w - (\mathfrak{a}(A))(w)$ because the value $(\mathfrak{a}(-A))(w)$ has to belong to $\mathcal{E}_w$ which has not been assumed to be closed under complementation. Thus, for $w \in W^{inacc}$, we allow $(\mathfrak{a}(-A))(w)$ to be an arbitrary element of $\mathcal{E}_w$ and in effect, *in inaccessible worlds of special model structures, the complementation—operator is deprived of its usual sense*.

Now, we will define certain classes of model structures which will play the role of semantical counterparts of Anderson-like theories. To each class will be affixed the same acronym as to its corresponding Anderson-like theory, however, the symbols: $\mathbf{V}^A$, **5**, **b**, **c**, **d**, **n**, **p**, **s** and $\star$ will be interpreted in a different manner according to the following simple rules:

- (**V**) The first symbol of an acronym i.e. $\mathbf{V}^A$ stands for the class of all model structures which subsequently undergo restrictions forced by successive symbols of acronym ;
- (**5**) The symbol **5** in an acronym indicates that structures in the class are *Euclidean* i.e. they obey the condition: if wRv and wRv_1 then vRv_1, for every $w, v, v_1 \in W$;
- (**b**) The symbol **b** indicates that structures in the class are *symmetric* i.e. if wRv then vRw, for every $w, v \in W$;
- (**c**) The symbol **c** indicates that structures in the class are *Euclidean* and *symmetric*;
- (**d**) The symbol **d** indicates that structures in the class are *Euclidean* and *transitive* i.e. if wRv and vRv_1 then wRv_1, for every $w, v, v_1 \in W$;

(**n**) The symbol **n** indicates that structures in the class obey the following condition: $\forall w \in W \left(D_w \in \mathcal{D}_w\right)$;
(**p**) The symbol **p** indicates that structures in the class obey the *monotonicity condition* : $\forall w, v \in W \left(wRv \Longrightarrow D_w \subseteq D_v\right)$;
(**s**) The symbol **s** indicates that structures in the class obey the condition: $\forall w \in W \forall a \in \bigcup_{w\in W} D_w \left(\bigcap\{X \mid X \in \mathcal{D}_w \text{ and } a \in X\} \in \mathcal{D}_w\right)$;
($\star$) If an acronym ends with $\star$ then all structures in the class are required to be *special*.

Now, it can be showed that the statement: **God (necessarily) exists** is true in all model structures of any kind.

Lemma 3.2 *For every assignment* $\mathfrak{a}$ *in a model structure of any kind (ordinary or special) and for every world* $w \in W$ *(no matter whether accessible or not) the following holds:* $\mathfrak{a}, w \models \mathsf{L}\exists x\mathsf{G}(x)$.

Proof We start with a computation:

$$\mathfrak{a}, w \models \mathsf{G}(x) \text{ iff } \mathfrak{a}, w \models \forall\alpha(\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x))$$
$$\text{iff } \forall\mathfrak{b} \in \{\mathfrak{a}^?_{\alpha,w}\}\big(\mathfrak{b}, w \models \mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)\big)$$
$$\text{iff } \forall\mathfrak{b} \in \{\mathfrak{a}^?_{\alpha,w}\}\big(\mathfrak{b}, w \models \mathbf{P}(\alpha) \iff \mathfrak{b}, w \models \mathsf{L}\alpha(x)\big)$$
$$\text{iff } \forall\mathfrak{b} \in \{\mathfrak{a}^?_{\alpha,w}\}\big(\mathfrak{b}, w \models \mathbf{P}(\alpha) \iff \forall v \in R(w)(\mathfrak{b}, v \models \alpha(x))\big)$$
$$\text{iff } \forall\mathfrak{b} \in \{\mathfrak{a}^?_{\alpha,w}\}\big(\forall v \in R(w)(\mathbb{G} \subseteq (\mathfrak{b}(\alpha))(v)) \iff \forall v \in R(w)(\mathfrak{b}(x) \in (\mathfrak{b}(\alpha))(v))\big)$$
$$\text{iff } \forall\mathfrak{b} \in \{\mathfrak{a}^?_{\alpha,w}\}\big(\forall v \in R(w)(\mathbb{G} \subseteq (\mathfrak{b}(\alpha))(v)) \iff \forall v \in R(w)(\mathfrak{a}(x) \in (\mathfrak{b}(\alpha))(v))\big).$$

Now, since R is serial (see 3.4) then $R(w) \neq \emptyset$ and therefore, (3.1) yields the implication: $\mathfrak{a}, w \models \mathsf{G}(x) \Longrightarrow \mathfrak{a}(x) \in \mathbb{G}$. The converse implication is obvious. Thus, $\mathfrak{a}, w \models \mathsf{G}(x)$ iff $\mathfrak{a}(x) \in \mathbb{G}$. And since, $\mathfrak{a}, w \models \mathsf{L}\exists x\mathsf{G}(x)$ iff $\forall v \in R(w)\big(\mathfrak{a}, v \models \exists x\mathsf{G}(x)\big)$ iff $\forall v \in R(w)\exists\mathfrak{b} \in \{\mathfrak{a}^?_{x,v}\}\big(\mathfrak{b}, v \models \mathsf{G}(x)\big)$ iff $\forall v \in R(w)\exists\mathfrak{b} \in \{\mathfrak{a}^?_{x,v}\}\big(\mathfrak{b}(x) \in \mathbb{G}\big)$. Then, taking $\mathfrak{b} \in \{\mathfrak{a}^?_{x,v}\}$ such that $\mathfrak{b}(x) \in \mathbb{G}$ we immediately obtain $\mathfrak{a}, w \models \mathsf{L}\exists x\mathsf{G}(x)$. □

4 A Preliminary Machinery to Strong Completeness

Before we begin proofs of strong completeness theorems in the strict sense, we must present some preparatory technical results. Everything presented here is borrowed from Thomason (1970). We will simply adopt this semantic machinery proving the completeness theorem for the 1st order free modal logic **S4** to our Anderson-like theories.

We shall assume that all formulas have been arranged in some denumerable sequence: $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_i, \ldots$. We shall also suppose that some particular enumerations are fixed so that we may speak of the 1st, 2nd, ..., ith, ... variable of the 1st or 2nd sort, respectively.

Given a set X of formulas of $\mathcal{L}$, we say that X is **Th**-*consistent* if there exists no formula $\boldsymbol{\phi}$ of $\mathcal{L}$ such that both $X \vdash_{\mathbf{Th}} \boldsymbol{\phi}$ and $X \vdash_{\mathbf{Th}} \neg\boldsymbol{\phi}$; **Th**-*inconsistent*, otherwise.

X is *maximally* **Th**-consistent if it is **Th**-consistent and for any formula $\boldsymbol{\phi}$ of $\mathcal{L}$ that does not belong to X, $X \cup \{\boldsymbol{\phi}\}$ is **Th**-inconsistent.

Lemma 4.1 *Let X be a* ***Th****-consistent set of formulas of $\mathcal{L}$ and $\mathsf{M}(\boldsymbol{\phi}_1 \wedge \cdots \wedge \boldsymbol{\phi}_n) \in X$, $n \geq 1$. Then, $\{\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n\}$ is also* ***Th****-consistent.*

Proof By an easy verification. □

Lemma 4.2 *Let X be a maximally* ***Th****-consistent set of formulas of $\mathcal{L}$. Then: $X \vdash_{\mathbf{Th}} \boldsymbol{\phi}$ iff $\boldsymbol{\phi}$ belongs to X.*

Proof By an easy verification. □

Lemma 4.3 *Let X be a maximal* ***Th****-consistent set of formulas of $\mathcal{L}$ and $\mathsf{M}(\boldsymbol{\phi}_1 \wedge \cdots \wedge \boldsymbol{\phi}_n) \in X$, $n \geq 1$. Then, for any formula $\boldsymbol{\psi}$ of $\mathcal{L}$: $\mathsf{M}(\boldsymbol{\phi}_1 \wedge \cdots \wedge \boldsymbol{\phi}_n \wedge \boldsymbol{\psi}) \in X$ or $\mathsf{M}(\boldsymbol{\phi}_1 \wedge \cdots \wedge \boldsymbol{\phi}_n \wedge \neg\boldsymbol{\psi}) \in X$.*

Proof Let the assumptions of the lemma be satisfied and let $\mathsf{M}(\boldsymbol{\phi}_1 \wedge \cdots \wedge \boldsymbol{\phi}_n \wedge \boldsymbol{\psi}) \notin X$ and $\mathsf{M}(\boldsymbol{\phi}_1 \wedge \cdots \wedge \boldsymbol{\phi}_n \wedge \neg\boldsymbol{\psi}) \notin X$. Then, by the maximality of X, $\neg\mathsf{M}(\boldsymbol{\phi}_1 \wedge \cdots \wedge \boldsymbol{\phi}_n \wedge \boldsymbol{\psi}) \in X$ and $\neg\mathsf{M}(\boldsymbol{\phi}_1 \wedge \cdots \wedge \boldsymbol{\phi}_n \wedge \neg\boldsymbol{\psi}) \in X$. And hence, $\boldsymbol{\phi}_1 \wedge \cdots \wedge \boldsymbol{\phi}_n \prec \boldsymbol{\psi} \in X$ and $\boldsymbol{\phi}_1 \wedge \cdots \wedge \boldsymbol{\phi}_n \prec \neg\boldsymbol{\psi} \in X$. Consequently, $\boldsymbol{\psi} \vee \neg\boldsymbol{\psi} \prec \neg(\boldsymbol{\phi}_1 \wedge \cdots \wedge \boldsymbol{\phi}_n) \in X$, and further, $\mathsf{L}\neg(\boldsymbol{\phi}_1 \wedge \cdots \wedge \boldsymbol{\phi}_n) \in X$, i.e. $\neg\mathsf{M}(\boldsymbol{\phi}_1 \wedge \cdots \wedge \boldsymbol{\phi}_n) \in X$—a contradiction. □

Two sequences of functions: $f_0, f_1, \ldots$ and $h_0, h_1, \ldots$ are defined as follows:

$$\text{(i)}\quad f_0(\exists\xi\boldsymbol{\phi}, \zeta) = \begin{cases} \mathsf{M}\exists\xi\boldsymbol{\phi} \to \mathsf{M}(\mathsf{E}(\zeta) \wedge \boldsymbol{\phi}(\xi/\zeta)) & \text{if } \xi \text{ is a 1st order variable,} \\ \mathsf{M}\exists\xi\boldsymbol{\phi} \to \mathsf{M}\boldsymbol{\phi}(\xi/\zeta) & \text{if } \xi \text{ is a 2nd order variable;} \end{cases}$$

$$\text{(ii)}\quad f_1(\boldsymbol{\psi}_1, \exists\xi\boldsymbol{\phi}, \zeta) = \mathsf{M}\boldsymbol{\psi}_1 \to \mathsf{M}\big(\boldsymbol{\psi}_1 \wedge f_0(\exists\xi\boldsymbol{\phi}, \zeta)\big);$$

$$\text{(iii)}\quad f_{i+1}(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{i+1}, \exists\xi\boldsymbol{\phi}, \zeta) = \mathsf{M}\boldsymbol{\psi}_{i+1} \to \mathsf{M}\big(\boldsymbol{\psi}_{i+1} \wedge f_i(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_i, \exists\xi\boldsymbol{\phi}, \zeta)\big)$$

and

$$\text{(iv)}\quad h_1(\boldsymbol{\psi}_1, x, y) = \mathsf{M}\boldsymbol{\psi}_1 \to \mathsf{M}\big(\boldsymbol{\psi}_1 \wedge (x \overset{1}{\approx} y)\big);$$

$$\text{(v)}\quad h_{i+1}(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{i+1}, x, y) = \mathsf{M}\boldsymbol{\psi}_{i+1} \to \mathsf{M}\big(\boldsymbol{\psi}_{i+1} \wedge h_i(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_i, x, y)\big).$$

Lemma 4.4 *For all $i > 0$, if $X \vdash_{\mathbf{Th}} \neg f_i(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_i, \exists\chi\boldsymbol{\phi}, \zeta)$ where ζ does not occur free in $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_i, \exists\xi\boldsymbol{\phi}$, or any member of X, then X is* ***Th****-inconsistent.*

Proof By induction on i, $i > 0$.

Case 1: $i = 1$.

Suppose that $X \vdash_{\mathbf{Th}} \neg f_1(\boldsymbol{\psi}_1, \exists\xi\boldsymbol{\phi}, \zeta)$ and ζ does not occur free in $\boldsymbol{\psi}_1$, $\exists\xi\boldsymbol{\phi}$, or any member of X. Hence, it follows that: (i) $X \vdash_{\mathbf{Th}} \mathsf{M}\boldsymbol{\psi}_1$ and $X \vdash_{\mathbf{Th}} \boldsymbol{\psi}_1 \prec \big(\mathsf{M}\exists\xi\boldsymbol{\phi} \wedge \mathsf{L}(\mathsf{E}(\zeta) \to \neg\boldsymbol{\phi}(\xi/\zeta))\big)$, if ξ and ζ are 1st sort variables; or (ii) $X \vdash_{\mathbf{Th}} \mathsf{M}\boldsymbol{\psi}_1$ and $X \vdash_{\mathbf{Th}} \boldsymbol{\psi}_1 \prec \big(\mathsf{M}\exists\xi\boldsymbol{\phi} \wedge \mathsf{L}\neg\boldsymbol{\phi}(\xi/\zeta)\big)$, if ξ and ζ are 2nd sort variables. In the case (i), we obtain $X \vdash_{\mathbf{Th}} \boldsymbol{\psi}_1 \prec \mathsf{M}\exists\xi\boldsymbol{\phi}$ and $X \vdash_{\mathbf{Th}} \boldsymbol{\psi}_1 \prec \mathsf{L}(\mathsf{E}(\zeta) \to \neg\boldsymbol{\phi}(\xi/\zeta))$. Which, by $R4_1$, yields $X \vdash_{\mathbf{Th}} \boldsymbol{\psi}_1 \prec \mathsf{L}\forall\zeta(\mathsf{E}(\zeta) \to \neg\boldsymbol{\phi}(\xi/\zeta))$. Applying now (2.5), (2.11), (2.9), R2 and R1 to the last, we get $X \vdash_{\mathbf{Th}} \boldsymbol{\psi}_1 \prec \mathsf{L}\forall\xi\neg\boldsymbol{\phi}$, and consequently, $X \vdash_{\mathbf{Th}} \mathsf{M}\exists\xi\boldsymbol{\phi} \prec \neg\boldsymbol{\psi}_1$. Therefore, $X \vdash_{\mathbf{Th}} \mathsf{M}\exists\xi\boldsymbol{\phi} \vee \neg\mathsf{M}\exists\xi\boldsymbol{\phi} \prec \neg\boldsymbol{\psi}_1$, which implies $X \vdash_{\mathbf{Th}} \mathsf{L}\neg\boldsymbol{\psi}_1$, i.e. $X \vdash_{\mathbf{Th}} \neg\mathsf{M}\boldsymbol{\psi}_1$. And we conclude that X is **Th**-inconsistent. In the case (ii), the argument is similar.

Case 2: $i > 1$.

Suppose that Lemma holds for $i = l$. Additionally, assume that $X \vdash_{\mathbf{Th}} \neg f_{l+1}(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_l, \boldsymbol{\psi}_{l+1} \exists\xi\boldsymbol{\phi}, \zeta)$ and ζ does not occur free in $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{l+1}$, $\exists\xi\boldsymbol{\phi}$, or any member of X. Thus, $X \vdash_{\mathbf{Th}} \neg\big(\mathsf{M}\boldsymbol{\psi}_{l+1} \rightarrow \mathsf{M}\left(\boldsymbol{\psi}_{l+1} \wedge f_j(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_l, \exists\xi\boldsymbol{\phi}, \zeta)\right)\big)$. It follows that $X \vdash_{\mathbf{Th}} \mathsf{M}\boldsymbol{\psi}_{l+1}$ and $X \vdash_{\mathbf{Th}} \mathsf{M}\boldsymbol{\psi}_{l+1} \prec \neg f_l(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_l, \exists\xi\boldsymbol{\phi}, \zeta)$. Consequently, $X \vdash_{\mathbf{Th}} \neg f_l(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_l, \exists\xi\boldsymbol{\phi}, \zeta)$. And from the latter result, on the strength of the induction hypothesis, we obtain that X is **Th**-inconsistent.

In this way we have finished the proof of the lemma. □

Lemma 4.5 *For all $i > 0$, if $X \vdash_{\mathbf{Th}} \neg h_i(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_i, x, y)$ where y is different from x and y does not occur free in $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_i$ or any member of X, then X is **Th**-inconsistent.*

Proof Like that of Lemma 4.4. □

Let X be a set of formulas of $\mathcal{L}$ and **Th** be of one of Anderson-like theories in $\mathcal{L}$. We shall say that X is **Th**-*saturated* in $\mathcal{L}$ if it meets the following conditions:

(i) X is maximally **Th**-consistent;
(ii) For every formula $\boldsymbol{\phi}$ and variable ξ of $\mathcal{L}$, $\forall\xi\boldsymbol{\phi} \in X$ if $\boldsymbol{\phi}(\xi/\zeta) \in X$ for all variables ζ of $\mathcal{L}$;
(iii) For every 1st sort variable y there is a 1st sort variable x such that $(x \overset{1}{\approx} y) \in X$;
(iv) For every formula $\boldsymbol{\phi}$ and variable ξ of $\mathcal{L}$ there is a variable ζ of $\mathcal{L}$ such that $f_0(\exists\xi\boldsymbol{\phi}, \zeta) \in X$;
(v) For all $n > 0$, for all sets $\{\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n, \exists\xi\boldsymbol{\phi}\}$ of formulas of $\mathcal{L}$ there is a variable ζ of $\mathcal{L}$ such that $f_n(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n, \exists\xi\boldsymbol{\phi}, \zeta) \in X$;
(vi) For all $n > 0$, for every 1st sort variable y and all sets $\{\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n\}$ of formulas of $\mathcal{L}$ there is a 1st sort variable x such that $h_n(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n, (x \overset{1}{\approx} y)) \in X$.

Let X be a **Th**-consistent set of formulas of $\mathcal{L}$. Let $\mathcal{L}'$ be a language obtained from $\mathcal{L}$ by adding an infinite number of new 1st order variables $X' = \{x'_1, x'_2, \ldots\}$ and an infinite number of new 2nd order variables $Y' = \{\alpha'_1, \alpha'_2, \ldots\}$. Moreover, suppose that the set of nonnegative integers was partitioned into denumerably many denumerable sets $S_0, S_1, S_2, \ldots$. We define the infinite sequence of Thomason's *sets* (in short, t-*sets*) $X_0, X_1, X_2, \ldots$ of formulas of $\mathcal{L}'$ in this way that $X_0 = X$ and if X_i was already introduced, then according to the following cases:

(0) $i \in S_0$. Let $\exists\xi\boldsymbol{\phi}$ be the alphabetically first formula of $\mathcal{L}'$ of the kind $\exists\zeta\delta$ such that: (i) for all $x' \in X'$, $\big((\exists\xi\boldsymbol{\phi} \rightarrow \boldsymbol{\phi}(\xi/x')) \wedge \mathsf{E}(x')\big) \notin X_i$, if ξ is a 1st sort variable, or (ii) for all $\alpha' \in Y'$, $\big(\exists\xi\boldsymbol{\phi} \rightarrow \boldsymbol{\phi}(\xi/\alpha')\big) \notin X_i$, if ξ is a 2nd sort variable. Then, in the case (i) we put $X_{i+1} = X_i \cup \{(\exists\xi\boldsymbol{\phi} \rightarrow \boldsymbol{\phi}(\xi/x')) \wedge \mathsf{E}(x')\}$ where x' is the first member of X' not occurring in any member of X_i or $\exists\xi\boldsymbol{\phi}$, in the case (ii) we put $X_{i+1} = X_i \cup \big\{(\exists\xi\boldsymbol{\phi} \rightarrow \boldsymbol{\phi}(\xi/\alpha')\big\}$ where α' is the first member of Y' not occurring in any member of X_i or $\exists\xi\boldsymbol{\phi}$;
(1) $i \in S_1$. Let $\exists\xi\boldsymbol{\phi}$ be the alphabetically first formula of $\mathcal{L}'$ of the kind $\exists\zeta\delta$ such that for all ς of X' (or, Y'), $f_0(\exists\xi\boldsymbol{\phi}, \varsigma) \notin X_i$. Then, we put $X_{i+1} = X_i \cup \{f_0(\exists\xi\boldsymbol{\phi}, \tau)\}$

where τ is the first member of X' (or, Y') not occurring in any member of X_i or $\exists\xi\boldsymbol{\phi}$;

(2) $i \in S_2$. Let y be the alphabetically first 1st sort variable of $\mathcal{L}'$ such that for all $x' \in X'$, $(x' \overset{1}{\approx} y) \notin X_i$. Then, we put $X_{i+1} = X_i \cup \{z' \overset{1}{\approx} y\}$ where z' is the first member of X' not occurring in any member of X_i and z' is different from y;

(3) $i \in S_{2n+1}$, where $n > 0$. Let $\boldsymbol{\psi}_1 \vee \cdots \vee \boldsymbol{\psi}_n \vee \exists\xi\boldsymbol{\phi}$ be the alphabetically first formula of $\mathcal{L}'$ of the kind $\delta_1 \vee \cdots \vee \delta_n \vee \exists\zeta\delta$ such that for all ς of X' (or, Y'), $f_i(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n, \exists\xi\boldsymbol{\phi}, \varsigma) \notin X_i$. Then, we put $X_{i+1} = X_i \cup \{f_n(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_i, \exists\xi\boldsymbol{\phi}, \tau)\}$ where τ is the first member of X' (or, Y') not occurring in any member of X_i or in $\boldsymbol{\psi}_1 \vee \cdots \boldsymbol{\psi}_n \vee \exists\xi\boldsymbol{\phi}$;

(4) $i \in S_{2n+2}$, where $n > 0$. Let $\boldsymbol{\psi}_1 \vee \cdots \vee \boldsymbol{\psi}_n \vee (x \overset{1}{\approx} y)$ be the alphabetically first formula of $\mathcal{L}'$ of the kind $\delta_1 \vee \cdots \vee \delta_n \vee (a \overset{1}{\approx} b)$ such that for all $x' \in X'$, $h_i(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n, x', y) \notin X_i$. Then, we put $X_{i+1} = X_i \cup \{h_n(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n, z', y)\}$ where z' is the first member of X' not occurring in any member of X_i or $\boldsymbol{\psi}_1 \vee \ldots \boldsymbol{\psi}_n$, and z' is different from y.

Lemma 4.6 *Let $X_0, X_1, X_2, \ldots$ be a sequence of t-sets of formulas of $\mathcal{L}'$. Then, the union $X_\infty = \bigcup_{i\geq 0} X_i$ is **Th**-consistent set of formulas of $\mathcal{L}'$.*

Proof We show by induction that for all $i \geq 0$, X_i is a **Th**-consistent set of formulas of $\mathcal{L}'$.

Case (0) If X_{i+1} were **Th**-inconsistent, then we would have: (i) $X_i \cup \{(\exists\xi\boldsymbol{\phi} \rightarrow \boldsymbol{\phi}(\xi/x')) \wedge \mathsf{E}(x')\} \vdash_{\mathbf{Th}} \chi \wedge \neg\chi$ where x' is a 1st sort variable not occurring in any member of X_i, $\exists\xi\boldsymbol{\phi}$ or χ, or (ii) $X_i \cup \{\exists\xi\boldsymbol{\phi} \rightarrow \boldsymbol{\phi}(\xi/\alpha')\} \vdash_{\mathbf{Th}} \chi \wedge \neg\chi$ where α' is a 2nd sort variable not occurring in any member of X_i, $\exists\xi\boldsymbol{\phi}$ or χ. Hence, using Proposition 2.1(iii), we would obtain $X_i \cup \{\exists x'\big((\exists\xi\boldsymbol{\phi} \rightarrow \boldsymbol{\phi}) \wedge \mathsf{E}(x')\big)\} \vdash_{\mathbf{Th}} \chi \wedge \neg\chi$ or $X_i \cup \{\exists\alpha'(\exists\xi\boldsymbol{\phi} \rightarrow \boldsymbol{\phi})\} \vdash_{\mathbf{Th}} \chi \wedge \neg\chi$. From this, by `T2` ("Appendix"), we would get $X_i \vdash_{\mathbf{Th}} \chi \wedge \neg\chi$, i.e, that X_i is **Th**-inconsistent—a contradiction.

Case (1) If X_{i+1} were **Th**-inconsistent, then we would have $X_i \cup \{f_0(\exists\xi\boldsymbol{\phi}, \tau)\} \vdash_{\mathbf{Th}} \chi \wedge \neg\chi$ where τ is the member of X' (or, Y') not occurring in any member of X_i or $\exists\xi\boldsymbol{\phi}$. From this it follows that $X_i \vdash_{\mathbf{Th}} \neg f_0(\exists\xi\boldsymbol{\phi}, \tau)$. If τ were a member of X', we would have $X_i \vdash_{\mathbf{Th}} \mathsf{M}\exists\xi\boldsymbol{\phi}$ and $X_i \vdash_{\mathbf{Th}} \mathsf{E}(\tau) \prec \neg\boldsymbol{\phi}(\xi/\tau)$. From the latter, by virtue of $R4_1$, we would obtain $X_i \vdash_{\mathbf{Th}} \mathsf{L}\forall\xi(\mathsf{E}(\xi) \rightarrow \neg\boldsymbol{\phi})$, and further, by (2.5), (2.11), (2.9), R2 and R1, $X_i \vdash_{\mathbf{Th}} \mathsf{L}\forall\xi\neg\boldsymbol{\phi}$, i.e. $X_i \vdash_{\mathbf{Th}} \neg\mathsf{M}\exists\xi\boldsymbol{\phi}$—a contradiction. If τ were a member of Y', we would have $X_i \vdash_{\mathbf{Th}} \mathsf{M}\exists\xi\boldsymbol{\phi}$ and $X_i \vdash_{\mathbf{Th}} \neg\mathsf{M}\boldsymbol{\phi}(\xi/\tau)$. Hence, we would get $X_i \vdash_{\mathbf{Th}} \forall\tau\mathsf{L}\neg\boldsymbol{\phi}$, and therefore, by (2.13), $X_i \vdash_{\mathbf{Th}} \mathsf{L}\forall\tau\neg\boldsymbol{\phi}$. Finally, we would obtain $X_i \vdash_{\mathbf{Th}} \neg\mathsf{M}\exists\xi\boldsymbol{\phi}$—a contradiction.

Case (2) If X_{i+1} were **Th**-inconsistent, then we would have $X_i \cup \{z' \overset{1}{\approx} y\} \vdash_{\mathbf{Th}} \chi \wedge \neg\chi$ where z' is a 1st sort variable not occurring in any member of X_i and z' is different from y. Hence, we would obtain $X_i \vdash_{\mathbf{Th}} \neg(z' \overset{1}{\approx} y)$. But then, after applying $R5_0$, X_i would be **Th**-inconsistent—a contradiction.

Case (3) If X_{i+1} were **Th**-inconsistent, then we would have $X_i \cup \{f_n(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_i, \exists\xi\boldsymbol{\phi}, \tau)\} \vdash_{\mathbf{Th}} \chi \wedge \neg\chi$ where τ is the member of X' (or, Y') not occurring in any member of X_i or in $\boldsymbol{\psi}_1 \vee \cdots \boldsymbol{\psi}_n \vee \exists\xi\boldsymbol{\phi}$. From this it follows that $X_i \vdash_{\mathbf{Th}} \neg f_i(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_i, \exists\xi\boldsymbol{\phi}, \tau)$ holds. But then, in view of Lemma 4.4, we would obtain that X_i is **Th**-inconsistent—a contradiction.

Case (4) If X_{i+1} were **Th**-inconsistent, then we would have $X_i \cup \{h_n(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n, z', y)\} \vdash_{\mathbf{Th}} \chi \wedge \neg\chi$ where z' is a 1st sort variable not occurring in any member of X_i or $\boldsymbol{\psi}_1 \vee \cdots \boldsymbol{\psi}_n$, and z' is different from y. From this, we would get $X_i \vdash_{\mathbf{Th}} \neg h_n(\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n, z', y)$, and by Lemma 4.5, X_i would be **Th**-inconsistent—a contradiction. □

Let X be a **Th**-consistent set of formulas of $\mathcal{L}$. Let $X_0, X_1, X_2, \ldots$ be a sequence of t-sets of formulas of $\mathcal{L}'$. By the *normal* Thomason's **Th**-*extension* (in short, *normal* t-**Th**-*extension*) of X in $\mathcal{L}'$ we shall understand the extension of $X_\infty = \bigcup_{i\geq 0} X_i$ to a maximal **Th**-consistent set of formulas of $\mathcal{L}'$.

Lemma 4.7 *Let X be a **Th**-consistent set of formulas of $\mathcal{L}$. Then, the normal t-**Th**-extension of X in $\mathcal{L}'$ is a **Th**-saturated set in $\mathcal{L}'$.*

Proof Let Δ be a normal t-**Th**-extension of X in $\mathcal{L}'$. Then, Lemma 4.6 and the definition of the sequence of t-sets guarantee that Δ fulfills the clause (i) of the definition of **Th**-saturated sets. To prove that Δ fulfills the condition (ii) of this latter definition, let us assume that for every variable ζ of $\mathcal{L}'$, $\boldsymbol{\phi}(\xi/\zeta) \in \Delta$, and $\forall\xi\boldsymbol{\phi} \notin \Delta$. Hence, $\exists\xi\neg\boldsymbol{\phi} \in \Delta$. Thus, by the clause (0) of the definition of the sequence of t-sets: (i) there exists a 1st sort variable x of $\mathcal{L}'$ such that $\big((\exists\xi\neg\boldsymbol{\phi} \rightarrow \neg\boldsymbol{\phi}(\xi/x) \wedge \mathsf{E}(x)\big) \in \Delta$, if ξ is a 1st sort variable, or (ii) there exists a 2nd sort variable α of $\mathcal{L}'$ such that $\exists\xi\neg\boldsymbol{\phi} \rightarrow \neg\boldsymbol{\phi}(\xi/\alpha) \in \Delta$, if ξ is a 2nd sort variable. In both cases, there exists a variable ζ of $\mathcal{L}'$ such that $\exists\xi\neg\boldsymbol{\phi} \rightarrow \neg\boldsymbol{\phi}(\xi/\zeta) \in \Delta$. Consequently, $\neg\boldsymbol{\phi}(\xi/\zeta) \in \Delta$ for some variable ζ of $\mathcal{L}'$—a contradiction. Evidently, the clauses (1)–(4) of the definition of the sequence of t-sets guarantee that Δ fulfills the clauses (iii)–(vi) of the definition of **Th**-saturated sets, respectively. □

Let X be a **Th**-saturated set in $\mathcal{L}$ and $\mathsf{M}\boldsymbol{\psi} \in X$. Moreover, suppose that the set of nonnegative integers was partitioned into denumerably many denumerable sets $S_0, S_1, S_2, \ldots$. By the *special* Thomason's **Th**-*extension* (in short, *special* t-**Th**-*extension*) of $\boldsymbol{\psi}$ in $\mathcal{L}$ we shall understand the union $X_\infty = \bigcup_{i\geq 0} X_i$, where $X_0 = \{\boldsymbol{\psi}_0\} = \{\boldsymbol{\psi}\}$ and if $X_i = \{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_i\}$ then X_{i+1} is given according to the following cases:

(0) $i \in S_0$. Let χ be the alphabetically first formula of $\mathcal{L}$ such that $\chi \notin X_i$ and $\neg\chi \notin X_i$. Then, we put $X_{i+1} = X_i \cup \{\boldsymbol{\psi}_{j+1}\}$, where $\boldsymbol{\psi}_{i+1}$ is χ if $\mathsf{M}\big(\boldsymbol{\psi}_0 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge \chi\big) \in X$, and $\boldsymbol{\psi}_{i+1}$ is $\neg\chi$ otherwise.
$\big($According to Lemma 4.3, $\boldsymbol{\psi}_{i+1}$ is defined and $\mathsf{M}\big(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i, \boldsymbol{\psi}_{i+1}\big) \in X\big)$;

(1) $i \in S_1$. If there is no formula of the kind $\exists\varsigma\delta$ such that $\exists\varsigma\delta \in X_i$, we put $\boldsymbol{\psi}_{i+1}$ to be $\boldsymbol{\psi}_i$, i.e. $X_{i+1} = X_i$. If there is a formula of the kind $\exists\varsigma\delta$ such that $\exists\varsigma\delta \in X_i$, then we choose the alphabetically first formula $\exists\tau\chi \in X_i$ and according to the sort of τ: (i) if τ a 1st sort variable, we choose the first 1st sort variable u of $\mathcal{L}$ such

that $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i) \rightarrow \mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge \mathsf{E}(u) \wedge \chi(\tau/u)) \in X$ holds, and put $X_{i+1} = X_i \cup \{\boldsymbol{\psi}_{i+1}\}$, where $\boldsymbol{\psi}_{i+1}$ is $\mathsf{E}(u) \wedge \chi(\tau/u)$; (ii) if τ a 2nd sort variable, we choose the first 2nd sort variable γ of $\mathcal{L}$ such that $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i) \rightarrow \mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge \chi(\tau/\gamma)) \in X$ holds, and put $X_{i+1} = X_i \cup \{\boldsymbol{\psi}_{i+1}\}$, where $\boldsymbol{\psi}_{i+1}$ is $\chi(\tau/\gamma)$.
(Suppose that for some $k \leq i$, $\boldsymbol{\psi}_k$ is $\exists\tau\chi$. Let u be the alphabetically first 1st sort variable u of $\mathcal{L}$ such that not occurring in any formula of $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_i\}$. Because $\vdash_{\mathbf{Th}} \mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_{k-1} \wedge \exists\tau\chi \wedge \boldsymbol{\psi}_{k+1} \wedge \cdots \wedge \boldsymbol{\psi}_i) \rightarrow \mathsf{M}\exists u(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_{k-1} \wedge \chi(\tau/u) \wedge \boldsymbol{\psi}_{k+1} \wedge \cdots \wedge \boldsymbol{\psi}_i)$ and $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_{k-1} \wedge \exists\tau\chi \wedge \boldsymbol{\psi}_{k+1} \wedge \cdots \wedge \boldsymbol{\psi}_i) \in X$, then $\mathsf{M}\exists u(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_{k-1} \wedge \chi(\tau/u) \wedge \boldsymbol{\psi}_{k+1} \wedge \cdots \wedge \boldsymbol{\psi}_i) \in X$. Hence, by the **Th**-saturation of X in $\mathcal{L}$, $\mathsf{M}\exists u(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_{k-1} \wedge \chi(\tau/u) \wedge \boldsymbol{\psi}_{k+1} \wedge \cdots \wedge \boldsymbol{\psi}_i) \rightarrow \mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_{k-1} \wedge \mathsf{E}(z) \wedge \chi(\tau/z) \wedge \boldsymbol{\psi}_{k+1} \wedge \cdots \wedge \boldsymbol{\psi}_i) \in X$ for some 1st sort variable z of $\mathcal{L}$. Finally, in view of T3 ("Appendix"), $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge \mathsf{E}(z) \wedge \chi(\tau/z)) \in X$, i.e. $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_{i+1}) \in X$. The reasoning is similar, if τ is a first 2nd sort variable.)

(2) $i \in S_2$. Let y be the alphabetically first 1st sort variable of $\mathcal{L}$ such that for all 1st sort variables u of $\mathcal{L}$, $(u \overset{1}{\approx} y) \notin X_i$. Let z be the alphabetically first 1st sort variable of $\mathcal{L}$ such that $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i) \rightarrow \mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge (z \overset{1}{\approx} y)) \in X$. Then, we put $X_{i+1} = X_i \cup \{\boldsymbol{\psi}_{i+1}\}$, where $\boldsymbol{\psi}_{i+1}$ is $(z \overset{1}{\approx} y)$.
(Because X is **Th**-saturated in $\mathcal{L}$, therefore there exists a 1st sort variable z of $\mathcal{L}$ such that $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i) \rightarrow \mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge (z \overset{1}{\approx} y)) \in X$. But, by induction hypothesis, $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i) \in X$. Thus, $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge (z \overset{1}{\approx} y)) \in X$, i.e. $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_{i+1}) \in X$.)

(3) $i \in S_3$. If there is no formula of the kind $\mathsf{M}\exists\varsigma\delta$ such that $\mathsf{M}\exists\varsigma\delta \in X_i$, we put $\boldsymbol{\psi}_{i+1}$ to be $\boldsymbol{\psi}_i$, i.e. $X_{i+1} = X_i$. If there is a formula of the kind $\mathsf{M}\exists\varsigma\delta$ such that $\mathsf{M}\exists\varsigma\delta \in X_i$, then then we choose the alphabetically first formula $\mathsf{M}\exists\tau\chi \in X_i$ and the first variable τ' of $\mathcal{L}$ such that $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i) \rightarrow \mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge f_0(\exists\tau\chi, \tau')) \in X$ holds, and put $X_{i+1} = X_i \cup \{\boldsymbol{\psi}_{i+1}\}$, where $\boldsymbol{\psi}_{i+1}$ is $f_0(\exists\tau\chi, \tau')$.
(Suppose that for some $k \leq i$, $\boldsymbol{\psi}_k$ is $\mathsf{M}\exists\tau\chi$. Let u be the alphabetically first 1st sort variable of $\mathcal{L}$ not occurring in any formula of $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_i\}$. Because $(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_{k-1} \wedge \mathsf{M}\exists\tau\chi \wedge \boldsymbol{\psi}_{k+1} \wedge \cdots \wedge \boldsymbol{\psi}_i) \in X$ and $\mathsf{M}\exists x\chi \rightarrow \mathsf{M}(\mathsf{E}(u) \wedge \boldsymbol{\phi}(\tau/u)) \in X$, then $(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_{k-1} \wedge \mathsf{M}(\mathsf{E}(u) \wedge \boldsymbol{\phi}(\tau/u)) \wedge \boldsymbol{\psi}_{k+1} \wedge \cdots \wedge \boldsymbol{\psi}_i) \in X$. Therefore, in view of T3 ("Appendix"), $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge \mathsf{M}(\mathsf{E}(u) \wedge \boldsymbol{\phi}(\tau/u))) \in X$, and consequently, $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge f_0(\exists\tau\chi, u)) \in X$ i.e. $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_{i+1}) \in X$. The reasoning is similar, if τ is a first 2nd sort variable.)

(4) $i \in S_{2n+2}$, where $n > 0$. Let $\chi_1 \vee \cdots \vee \chi_n \vee (x \overset{1}{\approx} y)$ be the alphabetically first formula of $\mathcal{L}$ of the kind $\delta_1 \vee \cdots \vee \delta_n \vee (a \overset{1}{\approx} b)$ such that for all 1st sort variables u of $\mathcal{L}$, $h_n(\chi_1, \ldots, \chi_n, u, y) \notin X_i$. Let z be the first 1st sort variable of $\mathcal{L}$ such that $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i) \rightarrow \mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge h_n(\chi_1, \ldots, \chi_n, z, y)) \in X$. Then, we put $X_{i+1} = X_i \cup \{\boldsymbol{\psi}_{i+1}\}$ where $\boldsymbol{\psi}_{i+1}$ is $h_n(\chi_1, \ldots, \chi_n, z, y)$.

(Because X is **Th**-saturated in $\mathcal{L}$, therefore there exists a 1st sort variable z of $\mathcal{L}$ such that $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i) \rightarrow \mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge h_n(\chi_1, \ldots, \chi_n, z, y)) \in X$. But, by induction hypothesis, $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i) \in X$. Thus, $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge h_n(\chi_1, \ldots, \chi_n, z, y)) \in X$, i.e. $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_{i+1}) \in X$.)

(5) $i \in S_{2n+3}$, where $n > 0$. Let $\chi_1 \vee \cdots \vee \chi_i \vee \exists\xi\chi$ be the alphabetically first formula of $\mathcal{L}$ of the kind $\delta_1 \vee \cdots \vee \delta_i \vee \exists\zeta\delta$ such that for all variables ς of $\mathcal{L}$ $f_n(\chi_1, \ldots, \chi_n, \exists\xi\chi, \varsigma) \notin X_i$. Then, we put $X_{i+1} = X_i \cup \{\boldsymbol{\psi}_{i+1}\}$ where $\boldsymbol{\psi}_{i+1}$ is $f_n(\chi_1, \ldots, \chi_n, \exists\xi\chi, \tau)\}$ and τ is the alphabetically first variable such that $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i) \rightarrow \mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge f_n(\chi_1, \ldots, \chi_n, \exists\xi\chi, \tau)) \in X$.

(Because X is **Th**-saturated in $\mathcal{L}$, therefore there exists variable τ of $\mathcal{L}$ such that $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i) \rightarrow \mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge f_n(\chi_1, \ldots, \chi_n, \exists\xi\chi, \tau)) \in X$. But, by induction hypothesis, $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i) \in X$. Thus, $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i \wedge f_n(\chi_1, \ldots, \chi_n, \exists\xi\chi, \tau)) \in X$, i.e. $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_{i+1}) \in X$.)

Lemma 4.8 *Let X be a **Th**-saturated set in $\mathcal{L}$ and $\mathsf{M}\boldsymbol{\psi} \in X$. Then, the special t-**Th**-extension X_∞ of $\{\boldsymbol{\psi}\}$ in $\mathcal{L}$ is **Th**-saturated in $\mathcal{L}$. Moreover, $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in X\} \subseteq X_\infty$.*

Proof Let the assumptions of the lemma be satisfied. To prove that X_∞ is a **Th**-consistent set of formulas of $\mathcal{L}$, it suffices to see that $\mathsf{M}(\chi_1 \wedge \cdots \wedge \chi_n) \in X$ for every finite subset $\{\chi_1, \ldots, \chi_n\}$ of X_∞. Hence, on the strength of Lemma 4.1, all finite subsets $\{\chi_1, \ldots, \chi_n\}$ of X_∞ are **Th**-consistent, which yields the **Th**-consistency of X_∞. The clause (0) of the definition of special t-**Th**-extensions of formulas guarantees that X_∞ is maximally **Th**-consistent. On the other hand, clauses (1)–(5) of this definition guarantee that X_∞ fulfills clauses (ii)–(vi) of the definition of **Th**-saturated sets. Consequently, X_∞ is **Th**-saturated in $\mathcal{L}$. To conclude the proof of this lemma, it must be still established that $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in X\} \subseteq X_\infty$. For otherwise, suppose that there exists a formula $\boldsymbol{\phi}$ of $\mathcal{L}$ such that $\mathsf{L}\boldsymbol{\phi} \in X$ and $\boldsymbol{\phi} \notin X_\infty$. But then, owing to the maximal **Th**-consistency of X_∞, $\neg\boldsymbol{\phi} \in X_\infty$. Thus, by construction of sets X_i, $i \geq 0$, there exists a set X_i such that $X_i = \{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_i\}$, $\boldsymbol{\psi}_i$ is $\neg\boldsymbol{\phi}$ and $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_i) \in X$. Consequently, $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_{i-1}) \rightarrow \mathsf{M}\neg\boldsymbol{\phi} \in X$, and because $\mathsf{M}(\boldsymbol{\psi}_0 \wedge \boldsymbol{\psi}_1 \wedge \cdots \wedge \boldsymbol{\psi}_{i-1}) \in X$, it follows that $\mathsf{M}\neg\boldsymbol{\phi} \in X$, i.e. $\neg\mathsf{L}\boldsymbol{\phi} \in X$—a contradiction. □

5 Strong Completeness

Utilizing the above machinery, it is now possible to prove that for every set of formulas X and every formula $\boldsymbol{\phi}$: if $\boldsymbol{\phi}$ is true in the class of all model structures corresponding to a given Anderson-like theory in which X is true, then $\boldsymbol{\phi}$ is provable from X on the basis of that theory. We leave to the reader proofs of the converse implications, because we have no space for them here.

Theorem 5.1 (Strong completeness) *Let X be a set of formulas. Then:*

(i) $X \models_{\mathbf{V}^A\mathbf{5}} \boldsymbol{\phi}$ *implies* $X \vdash_{\mathbf{V}^A\mathbf{5}} \boldsymbol{\phi}$,
(ii) $X \models_{\mathbf{V}^A\boldsymbol{b}} \boldsymbol{\phi}$ *implies* $X \vdash_{\mathbf{V}^A\boldsymbol{b}} \boldsymbol{\phi}$,

(iii) $X \models_{\mathbf{V}^A\mathbf{c}} \boldsymbol{\phi}$ *implies* $X \vdash_{\mathbf{V}^A\mathbf{c}} \boldsymbol{\phi}$,
(iv) $X \models_{\mathbf{V}^A\mathbf{5s}} \boldsymbol{\phi}$ *implies* $X \vdash_{\mathbf{V}^A\mathbf{5s}} \boldsymbol{\phi}$,
(v) $X \models_{\mathbf{V}^A\mathbf{bs}} \boldsymbol{\phi}$ *implies* $X \vdash_{\mathbf{V}^A\mathbf{bs}} \boldsymbol{\phi}$,
(vi) $X \models_{\mathbf{V}^A\mathbf{cs}} \boldsymbol{\phi}$ *implies* $X \vdash_{\mathbf{V}^A\mathbf{cs}} \boldsymbol{\phi}$,
(vii) $X \models_{\mathbf{V}^A\mathbf{5n}} \boldsymbol{\phi}$ *implies* $X \vdash_{\mathbf{V}^A\mathbf{5n}} \boldsymbol{\phi}$,
(viii) $X \models_{\mathbf{V}^A\mathbf{bn}} \boldsymbol{\phi}$ *implies* $X \vdash_{\mathbf{V}^A\mathbf{bn}} \boldsymbol{\phi}$,
(ix) $X \models_{\mathbf{V}^A\mathbf{cn}} \boldsymbol{\phi}$ *implies* $X \vdash_{\mathbf{V}^A\mathbf{cn}} \boldsymbol{\phi}$,
(x) $X \models_{\mathbf{V}^A\mathbf{5ns}} \boldsymbol{\phi}$ *implies* $X \vdash_{\mathbf{V}^A\mathbf{5ns}} \boldsymbol{\phi}$,
(xi) $X \models_{\mathbf{V}^A\mathbf{bns}} \boldsymbol{\phi}$ *implies* $X \vdash_{\mathbf{V}^A\mathbf{bns}} \boldsymbol{\phi}$,
(xii) $X \models_{\mathbf{V}^A\mathbf{cns}} \boldsymbol{\phi}$ *implies* $X \vdash_{\mathbf{V}^A\mathbf{cns}} \boldsymbol{\phi}$.

Proof

Proof of (i):
We consider only the non-trivial case, when $X \nvdash_{\mathbf{V}^A\mathbf{5}} \boldsymbol{\phi}$. Hence, $X \cup \{\neg\boldsymbol{\phi}\}$ is $\mathbf{V}^A\mathbf{5}$-consistent. Our aim is to find a model $\langle \mathfrak{W}, \mathfrak{a}\rangle$, where $\mathfrak{W} \in \mathbf{V}^A\mathbf{5}$, such that for every $\boldsymbol{\psi} \in X \cup \{\neg\boldsymbol{\phi}\}$ and for each $w \in W$, $\mathfrak{W}, \mathfrak{a}, w \models \boldsymbol{\psi}$. The proof is organized in three parts:

A. Construction of the frame $\langle \boldsymbol{W}, \boldsymbol{R}\rangle$,
B. Introduction of the 1st and 2nd sort domains,
C. Proof of the *Truth Lemma*.

Step A:

We define $\boldsymbol{W}$ to be the family consisting of the normal t-$\mathbf{V}^A\mathbf{5}$-extension $\boldsymbol{w_1}$ in $\mathcal{L}'$ of the set $X \cup \{\neg\boldsymbol{\phi}\}$, the special t-$\mathbf{V}^A\mathbf{5}$-extensions in $\mathcal{L}'$ of formulas $\boldsymbol{\psi}$ such that $\mathsf{M}\boldsymbol{\psi} \in \boldsymbol{w_1}$, the special t-$\mathbf{V}^A\mathbf{5}$-extensions in $\mathcal{L}'$ of formulas $\boldsymbol{\psi}$ such that $\mathsf{M}\boldsymbol{\psi}$ is a member of a special t-$\mathbf{V}^A\mathbf{5}$-extension in $\mathcal{L}'$ of formulas $\boldsymbol{\psi}$ such that $\mathsf{M}\boldsymbol{\psi} \in \boldsymbol{w_1}$, etc.

The members of $\boldsymbol{W}$ will be ordered in four following steps:[7]

Step 1 We assign a *rank* to each $\boldsymbol{w} \in \boldsymbol{W}$ (*rank* ($\boldsymbol{w}$), for short). And so, we declare *rank* ($\boldsymbol{w_1}$) = 1; and *rank* ($\boldsymbol{v}$) = *rank* ($\boldsymbol{w}$) + 1 if $\boldsymbol{v}$ is a special t-$\mathbf{V}^A\mathbf{5}$-extension in $\mathcal{L}'$ of a formula $\boldsymbol{\psi}$ such that $\mathsf{M}\boldsymbol{\psi} \in \boldsymbol{w}$ and $\boldsymbol{v}$ has not yet got a *rank* ($\boldsymbol{v}$) $\leq$ *rank* ($\boldsymbol{w}$).

Step 2 For every $\boldsymbol{w} \in \boldsymbol{W}$, we order the various special t-$\mathbf{V}^A\mathbf{5}$-extensions in $\mathcal{L}'$ of formulas $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ such that $\{\mathsf{M}\boldsymbol{\phi}, \mathsf{M}\boldsymbol{\psi}\} \subseteq \boldsymbol{w}$. So, suppose $\boldsymbol{w'}$ and $\boldsymbol{w''}$ are distinct special t-$\mathbf{V}^A\mathbf{5}$-extensions in $\mathcal{L}'$ of formulas $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$, respectively, such that $\{\mathsf{M}\boldsymbol{\phi}, \mathsf{M}\boldsymbol{\psi}\} \subseteq \boldsymbol{w}$. Then $\boldsymbol{w'}$ is to *precede* or *follow* $\boldsymbol{w''}$ according to whether $\boldsymbol{\phi}$ precedes or follows $\boldsymbol{\psi}$.

Step 3 We partition $\boldsymbol{W}$ into cells $\boldsymbol{W^1}, \boldsymbol{W^2}, \ldots, \boldsymbol{W^r}, \ldots$, consisting for each r, $r \geq 1$, of the members of $\boldsymbol{W}$ of rank r, and next we order the members of each cell. If $\boldsymbol{W^r}$ has exactly one member, we declare it *the first member of* $\boldsymbol{W^r}$. Otherwise, we employ the following inductive procedure:

[7] We order the members of $\boldsymbol{W}$ in the same way as in Szatkowski (2005). For the reader's convenience we make this paper self-contained however.

Case 1 $r = 2$. Then the members of $\boldsymbol{W^2} = \{\boldsymbol{w} \mid \boldsymbol{w}\, is\ a\ special$ $t\text{-}\mathbf{V}^A\mathbf{5}\text{-}extension\ in\ \mathcal{L}'\ of\ a\ formula\ \boldsymbol{\psi}\ such\ that\ \mathsf{M}\boldsymbol{\psi} \in \boldsymbol{w_1}\}$ already come in an order of their own (see, **Step 2**);

Case 2 $r > 2$. Given any two members of $\boldsymbol{W^r}$, one—call it $\boldsymbol{w'}$, is sure to be for some j' and k', the k'th special t-$\mathbf{V}^A\mathbf{5}$-extension in $\mathcal{L}'$ of some formula $\boldsymbol{\psi}'$ such that $\mathsf{M}\boldsymbol{\psi}'$ belongs to the j'th member of $\boldsymbol{W^{r-1}}$, and the other—call it $\boldsymbol{w''}$—is sure to be for some j'' and k'', the k''th special t-$\mathbf{V}^A\mathbf{5}$-extension in $\mathcal{L}'$ of some formula $\boldsymbol{\psi}''$ such that $\mathsf{M}\boldsymbol{\psi}''$ belongs to the j''th member of $\boldsymbol{W^{r-1}}$. Then $\boldsymbol{w'}$ *will precede* $\boldsymbol{w''}$ in $\boldsymbol{W^r}$ if $j' + k' < j'' + k''$ or , when $j' + k' = j'' + k''$ and $j' < j''$; otherwise, $\boldsymbol{w'}$ *will follow* $\boldsymbol{w''}$ in $\boldsymbol{W^r}$.

Step 4 We now order the members of $\boldsymbol{W}$ in a single run:

(i) $\boldsymbol{w_1}$, the one member of $\boldsymbol{W^1}$, is to *precede* all other members of $\boldsymbol{W}$;

(ii) $\boldsymbol{w'}$ being the j'th member of $\boldsymbol{W^{r'}}$ $(r' > 1)$, and $\boldsymbol{w''}$ the j''th member of $\boldsymbol{W^{r''}}$ $(r'' > 1)$, $\boldsymbol{w'}$ is to *precede* $\boldsymbol{w''}$ if $j' + r' < j'' + r''$ or, when $j' + r' = j'' + r''$ and $r' < r''$; otherwise, $\boldsymbol{w'}$ *follows* $\boldsymbol{w''}$.

Now, let us suppose for induction that the set $\boldsymbol{w_n}$, $n > 1$, is already defined. Thus, there exist parameters $j \geq 1$ and $r \geq 2$ such that $\boldsymbol{w_n}$ is the jth member of the $\boldsymbol{W^r}$. For each i, $2 \leq i < r + j$, we next put

$$\boldsymbol{V^i} = \boldsymbol{W^i} - \{\boldsymbol{v} \mid \boldsymbol{v} \in \boldsymbol{W^i} \text{ and } \boldsymbol{v} \text{ precedes or equals } \boldsymbol{w_n}\},\ and$$

$$\boldsymbol{V} = \{\boldsymbol{v} \mid \boldsymbol{v} \text{ is the first member of some } \boldsymbol{V^i},\ 2 \leq i < r + j\}.$$

In the case of $\boldsymbol{V} = \emptyset$, $\boldsymbol{w_n}$ is the last member of $\boldsymbol{W}$. Supposing then that $\boldsymbol{V} \neq \emptyset$, we define $\boldsymbol{w_{n+1}}$ to be the first member of $\boldsymbol{V}$. It is easily shown, when $\boldsymbol{w_n}$ is not the last member of $\boldsymbol{W}$, that there not exist a member of $\boldsymbol{W}$ which follows $\boldsymbol{w_n}$ and precedes $\boldsymbol{w_{n+1}}$.

We define now the accessibility relation $\boldsymbol{R}$ on $\boldsymbol{W}$:

$$\text{For every } \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{W},\ \boldsymbol{wRv} \text{ if and only } \{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}\} \subseteq \boldsymbol{v}. \tag{R}$$

And we can then prove that

$$\text{For every formula } \boldsymbol{\phi} \text{ and all } \boldsymbol{w} \in \boldsymbol{W},\ \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w} \text{ if and only if } \boldsymbol{\phi} \in \boldsymbol{v} \text{ for each } \boldsymbol{v} \in \boldsymbol{W} \text{ such that } \boldsymbol{wRv}. \tag{$\bullet$}$$

Let $\boldsymbol{\phi}$ be any formula and $\boldsymbol{w}$ any member of $\boldsymbol{W}$. We leave it to the reader to verify that for every axiom $\boldsymbol{\phi}$ of $\mathbf{V}^A\mathbf{5}$, $\vdash_{\mathbf{V}^A\mathbf{5}} \mathsf{M}\boldsymbol{\phi}$. Hence, trivially, $\boldsymbol{w}$ has members of the sort $\mathsf{M}\boldsymbol{\psi}$. And therefore, if $\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}$, then by Lemma 4.8 and (**R**), $\boldsymbol{\phi} \in \boldsymbol{v}$ for each $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$. Suppose, on the other hand, that $\boldsymbol{\phi} \in \boldsymbol{v}$ for each $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$, and let $\mathsf{L}\boldsymbol{\phi} \notin \boldsymbol{w}$. Because $\boldsymbol{w}$ is $\mathbf{V}^A\mathbf{5}$-saturated, then with respect to Lemma 4.2, $\mathsf{M}\neg\boldsymbol{\phi} \in \boldsymbol{w}$. Hence, by the construction of members of $\boldsymbol{W}$, there exists $\boldsymbol{v} \in \boldsymbol{W}$ such that $\neg\boldsymbol{\phi} \in \boldsymbol{v}$, which contradicts the assumption that $\boldsymbol{\phi} \in \boldsymbol{v}$ and $\boldsymbol{v}$ is $\mathbf{V}^A\mathbf{5}$-consistent.

To prove that the relation $\boldsymbol{R}$ is serial, let us again note that each $\boldsymbol{w} \in \boldsymbol{W}$ has members of the sort $\mathsf{M}\boldsymbol{\psi}$. Consequently, by our construction of members of $\boldsymbol{W}$, for each $\boldsymbol{w} \in \boldsymbol{W}$ there exists $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$.

Finally, we must also require that $\boldsymbol{R}$ is Euclidean. Let us assume to the contrary that for some members $\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{v_1} \in \boldsymbol{W} : \boldsymbol{wRv}, \boldsymbol{wRv_1}$ and not $\boldsymbol{vRv_1}$. Hence, by the definition (**R**), $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}\} \subseteq \boldsymbol{v}$, $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}\} \subseteq \boldsymbol{v_1}$, and there exists a formula $\boldsymbol{\phi}$ such that $\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{v}$ and $\boldsymbol{\phi} \notin \boldsymbol{v_1}$. Hence, $\mathsf{L}\boldsymbol{\phi} \notin \boldsymbol{w}$ and, by the maximality of $\boldsymbol{w}$, $\neg\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}$. Thus $\mathsf{M}\neg\boldsymbol{\phi} \in \boldsymbol{w}$, and since $\mathsf{M}\neg\boldsymbol{\phi} \to \mathsf{LM}\neg\boldsymbol{\phi} \in \boldsymbol{w}$, then in view of Proposition 2.1 and Lemma 4.2, $\mathsf{LM}\neg\boldsymbol{\phi} \in \boldsymbol{w}$. The last implies $\mathsf{M}\neg\boldsymbol{\phi} \in \boldsymbol{v}$, which is equivalent to $\neg\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{v}$—a contradiction. So, $\boldsymbol{R}$ is Euclidean.

Step B:

We first prove the following two facts:

$$\text{If } \mathsf{G}(x) \in \boldsymbol{v} \text{ for some } \boldsymbol{v} \in \boldsymbol{W},\ \text{ then } \mathsf{G}(x) \in \boldsymbol{w} \text{ for each } \boldsymbol{w} \in \boldsymbol{W}. \qquad (\bullet\bullet)$$

Suppose that $\mathsf{G}(x) \in \boldsymbol{v}$ for some $\boldsymbol{v} \in \boldsymbol{W}$. And, for the non-trivial case, suppose that the family $\boldsymbol{W}$ is at least two-element.

Notice first that $\vdash_{\mathbf{V}^A\mathbf{5}} \exists y\mathsf{G}(y)$; see, the step 4 in the proof of T9 ("Appendix"). Hence, with respect to Lemma 4.2, $\exists y\mathsf{G}(y) \in \boldsymbol{w}$ for all $\boldsymbol{w} \in \boldsymbol{W}$. And because each $\boldsymbol{w} \in \boldsymbol{W}$ is $\mathbf{V}^A\mathbf{5}$-saturated, therefore for every $\boldsymbol{w} \in \boldsymbol{W}$ there exists 1st sort variable x_w such that $\mathsf{G}(x_w) \in \boldsymbol{w}$.

In order to show that $\mathsf{G}(x_{w_1}) \in \boldsymbol{w}$ for any $\boldsymbol{w} \in \boldsymbol{W}$, $\boldsymbol{w} \neq \boldsymbol{w_1}$, let $\boldsymbol{w}$ be the any choice element of $\boldsymbol{W}$. We may suppose that $\boldsymbol{w}$ is member of $\boldsymbol{W^r}$. So, by the **Step 3** of the definition of the order on $\boldsymbol{W}$, there exists a finite sequence $\boldsymbol{w_1}, \boldsymbol{w_2}, \ldots, \boldsymbol{w_r}$, $r \geq 2$ such that $\boldsymbol{w_r} = \boldsymbol{w}$ and for each i, $1 < i \leq r$, $\boldsymbol{w_i}$ is a special t-$\mathbf{V}^A\mathbf{5}$-extension of some formula $\boldsymbol{\psi}_{i-1}$ such that $\mathsf{M}\boldsymbol{\psi}_{i-1} \in \boldsymbol{w_{i-1}}$. *For* (i): $r = 2$. Since $\vdash_{\mathbf{V}^A\mathbf{5}} \mathsf{G}(x_{w_1}) \to \mathsf{LG}(x_{w_1})$, therefore by Lemma 4.2 and Proposition 2.1(ii), $\mathsf{LG}(x_{w_1}) \in \boldsymbol{w_1}$. And, by applying to the latter the definition (**R**), we obtain that $\mathsf{G}(x_{w_1}) \in \boldsymbol{w}$. *For* (ii): $r > 2$. Suppose now that $\mathsf{G}(x_{w_1}) \in \boldsymbol{w_p}$, $p < r$. Similar to (i), it can be shown that $\mathsf{G}(x_{w_p}) \in \boldsymbol{w_{p+1}}$. Since $\vdash_{\mathbf{V}^A\mathbf{5}} \mathsf{G}(x_{w_1}) \wedge \mathsf{G}(x_{w_p}) \to (x_{w_1} \overset{1}{\approx} x_{w_p})$, therefore by Lemma 4.2, the axiom 2.15 and Proposition 2.1(ii), $(x_{w_1} \overset{1}{\approx} x_{w_p}) \in \boldsymbol{w_{p+1}}$. It can be also proved that $(x_{w_p} \overset{1}{\approx} x_{w_{p+1}}) \in \boldsymbol{w_{p+1}}$. And hence, it follows that $(x_{w_1} \overset{1}{\approx} x_{w_{p+1}}) \in \boldsymbol{w_{p+1}}$. And since, $\vdash_{\mathbf{V}^A\mathbf{5}} \mathsf{G}(w_{p+1}) \wedge (x_{w_1} \overset{1}{\approx} x_{w_{p+1}}) \to \mathsf{G}(x_{w_1})$, then after applying Lemma 4.2, $\mathsf{G}(x_{w_1}) \in \boldsymbol{w_{p+1}}$. Therefore, $\mathsf{G}(x_{w_1}) \in \boldsymbol{w}$.

Furthermore, by the same argument, we obtain $(x_{w_1} \overset{1}{\approx} x) \in \boldsymbol{w}$, and consequently, $\mathsf{G}(x) \in \boldsymbol{w}$, which finishes the proof of $(\bullet\bullet)$.

$$\text{If } \mathsf{G}(x) \in \boldsymbol{v} \text{ for some } \boldsymbol{v} \in \boldsymbol{W},\ \text{ then } \mathsf{E}(x) \in \boldsymbol{w} \text{ for each } \boldsymbol{w} \in \boldsymbol{W}. \qquad (\bullet\bullet\bullet)$$

Suppose that $\mathsf{G}(x) \in \boldsymbol{v}$ for some $\boldsymbol{v} \in \boldsymbol{W}$. Therefore, by $(\bullet\bullet)$, $\mathsf{G}(x) \in \boldsymbol{w}$ for every $\boldsymbol{w} \in \boldsymbol{W}$. Next, on the strength of T9 ("Appendix") and Lemma 4.2 we obtain that $\exists x(\mathsf{G}(x) \wedge \mathsf{E}(x)) \in \boldsymbol{w}$ for each $\boldsymbol{w} \in \boldsymbol{W}$. Hence, for every $\boldsymbol{w} \in \boldsymbol{W}$ there exists 1st sort variable x_w such that $(\mathsf{G}(x_w) \wedge \mathsf{E}(x_w)) \in \boldsymbol{w}$. Consequently, $\mathsf{G}(x_w) \in \boldsymbol{w}$

and $\mathsf{E}(x_w) \in \boldsymbol{w}$ for each $\boldsymbol{w} \in \boldsymbol{W}$. And in the same way as before we obtain that, for every $\boldsymbol{w} \in \boldsymbol{W}$, $(x \overset{1}{\approx} x_w) \in \boldsymbol{w}$, which, by (2.29) and Lemma 4.2, implies that $\mathsf{L}(x \overset{1}{\approx} x_w) \in \boldsymbol{w}$ for every $\boldsymbol{w} \in \boldsymbol{W}$. But hence, by applying (2.17) and Lemma 4.2, we obtain that $\mathsf{E}(x) \in \boldsymbol{w}$ for each $\boldsymbol{w} \in \boldsymbol{W}$, which finishes the proof of $(\bullet\bullet\bullet)$.

Given some (any chosen) member $\boldsymbol{w}$ of $\boldsymbol{W}$, we define

$$\mathbb{G} = \{x \mid \mathsf{G}(x) \in \boldsymbol{w}\},$$

By $(\bullet\bullet)$, it can easy be seen that this definition is correct.

Now with each $\boldsymbol{w} \in \boldsymbol{W}$ we associate the 1st order domain

$$\boldsymbol{D_w} = \{x \mid x \text{ is a 1st order variable of } \mathcal{L}' \text{ and } \mathsf{E}(x) \in \boldsymbol{w}\},$$

and we put

$$\mathscr{D}_\mathbf{1} = (\boldsymbol{D_w})_{\boldsymbol{w}\in\boldsymbol{W}}.$$

Clearly, by $(\bullet\bullet\bullet)$, $\mathbb{G} \subseteq \bigcap_{w\in W} \boldsymbol{D_w}$.

Next, with each 2nd order term A and $\boldsymbol{w} \in \boldsymbol{W}$ we associate the set

$$F(A, \boldsymbol{w}) = \{a \mid A(a) \in \boldsymbol{w}\}$$

(According to 2.10, Proposition 2.1(i) and Lemma 4.2, $\mathsf{E}(a) \in \boldsymbol{w}$ if $A(a) \in \boldsymbol{w}$. Hence, $F(A, \boldsymbol{w}) \in 2^{\boldsymbol{D_w}}$ for every $\boldsymbol{w} \in \boldsymbol{W}$),
and we put

$$\text{for every } \boldsymbol{w} \in \boldsymbol{W},\ \boldsymbol{\mathcal{D}_w} \text{ to be the family of all sets } F(A, \boldsymbol{w}) \in 2^{\boldsymbol{D_w}},$$
$$\mathscr{D}_\mathbf{2} = (\boldsymbol{\mathcal{D}_w})_{\boldsymbol{w}\in\boldsymbol{W}}$$
$$\boldsymbol{C_{\mathfrak{W}}} = \{\boldsymbol{f} \in \boldsymbol{W} \mapsto \bigcup_{w\in W} \boldsymbol{\mathcal{D}_w} \mid \boldsymbol{f}(\boldsymbol{w}) \in \boldsymbol{\mathcal{D}_w} \text{ for every } \boldsymbol{w} \in \boldsymbol{W}\}.$$

To prove that $\mathbb{G} \in \bigcap_{w\in W} \boldsymbol{\mathcal{D}_w}$ let us notice that, by $(\bullet\bullet)$, $F(\mathsf{G}, \boldsymbol{w}) = F(\mathsf{G}, \boldsymbol{w_1})$ for every $\boldsymbol{w}, \boldsymbol{w_1} \in \boldsymbol{W}$. Hence, trivially, $\mathbb{G} = F(\mathsf{G}, \boldsymbol{w}) \in \bigcap_{w\in W} \boldsymbol{\mathcal{D}_w}$.

To prove that for each $\boldsymbol{w} \in \boldsymbol{W}$, every $x, y \in \mathbb{G}$ and each $X \in \boldsymbol{\mathcal{D}_w} : x \in X$ iff $y \in X$, let us suppose that $x, y \in \boldsymbol{G}$ and $X = F(A, \boldsymbol{w}) \in \boldsymbol{\mathcal{D}_w}$ for some 2nd order term A and $\boldsymbol{w} \in \boldsymbol{W}$. It follows from (2.17) and (2.29) that $\vdash_{\mathbf{V}^{A}\mathbf{5}} \mathsf{G}(x) \wedge \mathsf{G}(y) \to \big((x \overset{1}{\approx} y) \to (A(x) \leftrightarrow A(y))\big)$. And because $\vdash_{\mathbf{V}^{A}\mathbf{5}} \mathsf{G}(x) \wedge \mathsf{G}(y) \to (x \overset{1}{\approx} y)$, then $\vdash_{\mathbf{V}^{A}\mathbf{5}} \mathsf{G}(x) \wedge \mathsf{G}(y) \to (A(x) \leftrightarrow A(y))$. Hence, by $(\bullet\bullet)$ and Lemma 4.2, $(A(x) \leftrightarrow A(y)) \in \boldsymbol{w}$, i.e., $x \in F(A, \boldsymbol{w})$ iff $y \in F(A, \boldsymbol{w})$.

To prove that for every $\boldsymbol{w} \in \boldsymbol{W}$, $-X \in \boldsymbol{\mathcal{D}_w}$ if $X \in \boldsymbol{\mathcal{D}_w}$, let us assume that $F(A, \boldsymbol{w}) \in \boldsymbol{\mathcal{D}_w}$. Then, for every $a \in F(A, \boldsymbol{w})$, $\neg A(a) \notin \boldsymbol{w}$. It follows that for every $x \in \boldsymbol{D_w} - F(A, \boldsymbol{w})$, $-A(x) \in \boldsymbol{w}$. Thus, $F(-A, \boldsymbol{w}) \in \boldsymbol{\mathcal{D}_w}$. But, $F(-A, \boldsymbol{w}) = -F(A, \boldsymbol{w})$, therefore $-F(A, \boldsymbol{w}) \in \boldsymbol{\mathcal{D}_w}$.

Step C:

The assignment $\mathfrak{a}$ in the canonical $\mathbf{V}^A\mathbf{5}$-model structure such that for any 1st order variable x, $\mathfrak{a}(x) = x$, and for any 2nd sort term A and each $\boldsymbol{w} \in \boldsymbol{W}$, $\mathfrak{a}(A)(\boldsymbol{w}) = F(A, \boldsymbol{w})$, will be called a *canonical assignment*.

One can show that

(**TL**) Given the canonical $\mathbf{V}^A\mathbf{5}$-model structure $\mathfrak{W} = \langle \boldsymbol{W}, \boldsymbol{R}, \mathscr{D}_1, \mathscr{D}_2, \mathbb{G} \rangle$ and the canonical assignment $\mathfrak{a}$ in it; for any formula $\boldsymbol{\phi}$ and each $\boldsymbol{w} \in \boldsymbol{W}$, $\mathfrak{a}, \boldsymbol{w} \models \boldsymbol{\phi}$ if and only if $\boldsymbol{\phi} \in \boldsymbol{w}$.

The proof of (**TL**) proceeds by simultaneous induction on the complexity of $\boldsymbol{\phi}$.

$\boldsymbol{\phi}$ is of the form $\mathsf{E}(x)$:
Then, $\mathfrak{a}, \boldsymbol{w} \models \mathsf{E}(x)$ iff $\mathfrak{a}(x) \in \boldsymbol{D}_w$ iff $x \in \boldsymbol{D}_w$, by the definition of $\boldsymbol{D}_w$, this last iff $\mathsf{E}(x) \in \boldsymbol{w}$.

$\boldsymbol{\phi}$ is of the form $A(x)$:
Then, $\mathfrak{a}, \boldsymbol{w} \models A(x)$ iff $\mathfrak{a}(x) \in \mathfrak{a}(A)(\boldsymbol{w})$ iff $x \in F(A, \boldsymbol{w})$, by the definition of $F(A, \boldsymbol{w})$), this last iff $A(x) \in \boldsymbol{w}$.

$\boldsymbol{\phi}$ is of the form $\boldsymbol{\psi} \wedge \chi$:
Then, $\mathfrak{a}, \boldsymbol{w} \models \boldsymbol{\psi} \wedge \chi$ iff $\mathfrak{a}, \boldsymbol{w} \models \boldsymbol{\psi}$ and $\mathfrak{a}, \boldsymbol{w} \models \chi$, by the inductive hypothesis, this last iff $\boldsymbol{\psi} \in \boldsymbol{w}$ and $\chi \in \boldsymbol{w}$, so on the strength of Lemma 4.2 and (2.4), this last iff $\boldsymbol{\psi} \wedge \chi \in \boldsymbol{w}$.

$\boldsymbol{\phi}$ is of the form $\neg\boldsymbol{\psi}$:
Then, $\mathfrak{a}, \boldsymbol{w} \models \neg\boldsymbol{\psi}$ iff $\mathfrak{a}, \boldsymbol{w} \not\models \boldsymbol{\psi}$, and by the inductive hypothesis, this last iff $\boldsymbol{\psi} \notin \boldsymbol{w}$, and owing to the maximality of $\boldsymbol{w}$, this last iff $\neg\boldsymbol{\psi} \in \boldsymbol{w}$.

$\boldsymbol{\phi}$ is of the form $\forall\xi\boldsymbol{\psi}$:
Then, $\mathfrak{a}, \boldsymbol{w} \models \forall\xi\boldsymbol{\phi}$ iff $\mathfrak{b}, \boldsymbol{w} \models \boldsymbol{\phi}$ for every $\mathfrak{b} \in \{\mathfrak{a}^?_{\xi,\boldsymbol{w}}\}$, and further on the strength of Proposition 3.1, this last iff $\mathfrak{b}, \boldsymbol{w} \models \boldsymbol{\phi}(\xi/\mathfrak{b}(\xi))$ for every $\mathfrak{b} \in \{\mathfrak{a}^?_{\xi,\boldsymbol{w}}\}$, and by the inductive hypothesis, iff $\boldsymbol{\phi}(\xi/\mathfrak{b}(\xi)) \in \boldsymbol{w}$ for every assignment $\mathfrak{b} \in \{\mathfrak{a}^?_{\xi,\boldsymbol{w}}\}$. This last, because $\boldsymbol{w}$ is $\mathbf{V}^A\mathbf{5}$-saturated, implies that $\forall\xi\boldsymbol{\psi} \in \boldsymbol{w}$. Suppose now that $\forall\xi\boldsymbol{\psi} \in \boldsymbol{w}$ and that ξ is a 1st order variable. Then, by applying (2.8) and Lemma 4.2, it follows that $(\mathsf{E}(\zeta) \to \boldsymbol{\psi}(\xi/\zeta)) \in \boldsymbol{w}$ for every 1st order variable ζ. This means that, for every assignment $\mathfrak{b} \in \{\mathfrak{a}^?_{\xi,\boldsymbol{w}}\}$, $\boldsymbol{\phi}(\xi/\mathfrak{b}(\xi)) \in \boldsymbol{w}$. In exactly the same way, using (2.6) instead of (2.8), we obtain that for every assignment $\mathfrak{b} \in \{\mathfrak{a}^?_{\xi,\boldsymbol{w}}\}$, $\boldsymbol{\phi}(\xi/\mathfrak{b}(\xi)) \in \boldsymbol{w}$ if $\forall\xi\boldsymbol{\psi} \in \boldsymbol{w}$ and ξ is a 2nd order variable.

$\boldsymbol{\phi}$ is of the form $\mathsf{L}\boldsymbol{\psi}$:
Then, $\mathfrak{a}, \boldsymbol{w} \models \mathsf{L}\boldsymbol{\psi}$ iff $\mathfrak{a}, \boldsymbol{v} \models \boldsymbol{\psi}$ for every $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$, therefore by the inductive hypothesis, this last iff $\boldsymbol{\psi} \in \boldsymbol{v}$ for every $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$, and further on the strength of the condition (•), this last iff $\mathsf{L}\boldsymbol{\psi} \in \boldsymbol{w}$.

$\boldsymbol{\phi}$ is of the form $\mathbf{P}(A)$:
Then, $\mathfrak{a}, \boldsymbol{w} \models \mathbf{P}(A)$ iff $\mathbb{G} \subseteq \mathfrak{a}(A)(\boldsymbol{v})$ for every $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$ iff $\mathbb{G} \subseteq F(A, \boldsymbol{v})$ for every $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$ iff $x \in F(A, \boldsymbol{v})$ for every $x \in \mathbb{G}$ and every $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$ iff $A(x) \in \boldsymbol{v}$ for every $x \in \mathbb{G}$ and every $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$ iff, on the strength of the condition (•), $\mathsf{L}A(x) \in \boldsymbol{w}$ for every $x \in \mathbb{G}$. We have already demonstrated that $\mathsf{G}(x) \in \boldsymbol{w}$ for every $x \in \mathbb{G}$. Hence, by definition (2.1), $\forall\alpha(\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)) \in \boldsymbol{w}$ for every $x \in \mathbb{G}$. But then, by (2.6) and Lemma 4.2,

$\mathbf{P}(A) \leftrightarrow \mathsf{L}A(\boldsymbol{g}) \in \boldsymbol{w}$ for every $x \in \mathbb{G}$. Therefore, $\mathsf{L}A(x) \in \boldsymbol{w}$ for every $x \in \mathbb{G}$ iff $\mathbf{P}(A) \in \boldsymbol{w}$.

This concludes our proof of (**TL**).

Reminding ourselves of the assumption $X \nvdash_{\mathbf{V}^A\mathbf{5}} \boldsymbol{\phi}$ we apply now the semantic instrument, which we have here introduced. So, let $\mathfrak{W} = \langle \boldsymbol{W}, \boldsymbol{R}, \mathcal{D}_1, \mathcal{D}_2, \mathbb{G} \rangle$ be the canonical model structure and $\mathfrak{a}$ be the canonical assignment in it. Because $X \cup \{\neg\boldsymbol{\phi}\} \subseteq \boldsymbol{w}_1$, then, for all $\boldsymbol{\psi} \in X \cup \{\neg\boldsymbol{\phi}\}$, $\mathfrak{W}, \mathfrak{a}, \boldsymbol{w}_1 \models \boldsymbol{\psi}$. Therefore, $X \nvDash_{\mathbf{V}^A\mathbf{5}} \boldsymbol{\phi}$, which completes the proof of (**i**).

Proof of (ii):
By the proof of (**ii**), we must only show that the relation $\boldsymbol{R}$ is symmetric. Let us assume to the contrary that for some members $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{W} : \boldsymbol{wRv}$ and not $\boldsymbol{vRw}$. So, by definition (**R**), $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}\} \subseteq \boldsymbol{v}$ and there exists a formula $\boldsymbol{\phi}$ such that $\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{v}$ and $\boldsymbol{\phi} \notin \boldsymbol{w}$. Since, $(\mathsf{ML}\boldsymbol{\phi} \rightarrow \boldsymbol{\phi}) \in \boldsymbol{w}$, then Proposition 2.1(ii) and Lemma 4.2, would still guarantee that $\mathsf{ML}\boldsymbol{\phi} \notin \boldsymbol{w}$, and hence, $\mathsf{LM}\neg\boldsymbol{\phi} \in \boldsymbol{w}$. But it is not possible, because then we would obtain $\mathsf{M}\neg\boldsymbol{\phi} \in \boldsymbol{v}$, and thus $\neg\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{v}$—a contradiction. So, $\boldsymbol{R}$ is symmetric.

Proof of (iii):
By dint of (**i**) and (**ii**).

Proofs of (iv)–(vi):
Relying on these results **(i)–(iii)**, it suffices only to show that $\bigcap\{X \mid X \in \mathcal{D}_{\boldsymbol{w}} \text{ and } x \in X\} \in \mathcal{D}_{\boldsymbol{w}}$ for every $x \in \bigcup_{w \in W} \boldsymbol{D}_{\boldsymbol{w}}$ and every $\boldsymbol{w} \in \boldsymbol{W}$. And so, according to (2.36) with the definition (2.25), for each 1st sort variable x, I_x is a term of the 2nd order. Thus, $F(\mathsf{I}_x, \boldsymbol{w}) \in \mathcal{D}_{\boldsymbol{w}}$ for every $x \in \bigcup_{w \in W} \boldsymbol{D}_{\boldsymbol{w}}$. It remains to show that $F(\mathsf{I}_x, \boldsymbol{w}) = \bigcap\{X \mid X \in \mathcal{D}_{\boldsymbol{w}} \text{ and } x \in X\}$ for every $x \in \bigcup_{w \in W} \boldsymbol{D}_{\boldsymbol{w}}$. First, we prove that $\bigcap\{X \mid X \in \mathcal{D}_{\boldsymbol{w}} \text{ and } x \in X\} \subseteq F(\mathsf{I}_x, \boldsymbol{w})$ for every $x \in \bigcup_{w \in W} \boldsymbol{D}_{\boldsymbol{w}}$ and every $\boldsymbol{w} \in \boldsymbol{W}$. Clearly, $\bigcap\{X \mid X \in \mathcal{D}_{\boldsymbol{w}} \text{ and } x \in X\} = \emptyset$ if $x \notin \boldsymbol{D}_{\boldsymbol{w}}$, and consequently, the required inclusion holds. If $x \in \boldsymbol{D}_{\boldsymbol{w}}$, then $x \in F(\mathsf{I}_x, \boldsymbol{w})$, and of course the required inclusion also holds. Now, we prove that $F(\mathsf{I}_x, \boldsymbol{w}) \subseteq \bigcap\{X \mid X \in \mathcal{D}_{\boldsymbol{w}} \text{ and } x \in X\}$ for every $x \in \bigcup_{w \in W} \boldsymbol{D}_{\boldsymbol{w}}$ and every $\boldsymbol{w} \in \boldsymbol{W}$. It should be clear that the required inclusion holds if $x \notin \boldsymbol{D}_{\boldsymbol{w}}$. Finally, suppose that for some $\boldsymbol{w} \in \boldsymbol{W}$ and for some $x \in \boldsymbol{D}_{\boldsymbol{w}} : a \in F(\mathsf{I}_x, \boldsymbol{w})$ and $a \notin \bigcap\{X \mid X \in \mathcal{D}_{\boldsymbol{w}} \text{ and } x \in X\}$. Since the following equivalences hold: $a \in F(\mathsf{I}_x, \boldsymbol{w})$ iff $a \in \{y \in \boldsymbol{D}_{\boldsymbol{w}} \mid \mathsf{I}_x(y) \in \boldsymbol{w}\}$ iff $a \in \boldsymbol{D}_{\boldsymbol{w}}$ and $\forall\alpha(\alpha(x) \leftrightarrow \alpha(a)) \in \boldsymbol{w}$, in the former case, we consequently have: $a \in \boldsymbol{D}_{\boldsymbol{w}}$ and $\forall\alpha(\alpha(x) \leftrightarrow \alpha(a)) \in \boldsymbol{w}$. In the latter one, the assumption: $a \notin \bigcap\{X \mid X \in \mathcal{D}_{\boldsymbol{w}} \text{ and } x \in X\}$ means that there exists $X \in \mathcal{D}_{\boldsymbol{w}}$ such that $a \notin X$, $x \in X$ and $X = F(A, \boldsymbol{w})$ for some 2nd sort term A. Therefore, $a \notin \{y \in \boldsymbol{D}_{\boldsymbol{w}} \mid A(y) \in \boldsymbol{w}\}$, i.e., $a \notin \boldsymbol{D}_{\boldsymbol{w}}$ or $A(a) \notin \boldsymbol{w}$, whence $a \notin \boldsymbol{D}_{\boldsymbol{w}}$ or $\forall\alpha(\alpha(x) \leftrightarrow \alpha(a)) \notin \boldsymbol{w}$—a contradiction, which means that the required inclusion holds.

Proofs of (vii)–(ix):
Banking on these results **(i)–(iii)**, it suffices only to prove that $\boldsymbol{D}_{\boldsymbol{w}} \in \mathcal{D}_{\boldsymbol{w}}$ for every $\boldsymbol{w} \in \boldsymbol{W}$. And so, by `T10` ("Appendix"), for every theory $\mathbf{Th} \in \{\mathbf{V}^A\mathbf{5n}, \mathbf{V}^A\mathbf{bn}, \mathbf{V}^A\mathbf{cn}\}$, $\vdash_{\mathbf{Th}} \forall x \mathsf{NE}(x)$. Hence, with respect to Lemma 4.2, $\forall x \mathsf{NE}(x) \in \boldsymbol{w}$ for all $\boldsymbol{w} \in \boldsymbol{W}$. And finally, $\boldsymbol{D}_{\boldsymbol{w}} = F(\mathsf{NE}, \boldsymbol{w}) \in \mathcal{D}_{\boldsymbol{w}}$ for every $\boldsymbol{w} \in \boldsymbol{W}$.

Proofs of (x)–(xii):
By dint of **(iv)–(vi)** and **(vii)–(ix)**, respectively.

In this way we have finished the proof of Theorem 5.1. □

Theorem 5.2 (Strong completeness) *Let X be a set of formulas. Then:*

(i) $X \models_{V^A\mathbf{5p}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{5p}} \boldsymbol{\phi}$,
(ii) $X \models_{V^A\mathbf{bp}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{bp}} \boldsymbol{\phi}$,
(iii) $X \models_{V^A\mathbf{cp}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{cp}} \boldsymbol{\phi}$,
(iv) $X \models_{V^A\mathbf{5ps}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{5ps}} \boldsymbol{\phi}$,
(v) $X \models_{V^A\mathbf{bps}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{bps}} \boldsymbol{\phi}$,
(vi) $X \models_{V^A\mathbf{cps}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{cps}} \boldsymbol{\phi}$,
(vii) $X \models_{V^A\mathbf{5np}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{5np}} \boldsymbol{\phi}$,
(viii) $X \models_{V^A\mathbf{bnp}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{bnp}} \boldsymbol{\phi}$,
(ix) $X \models_{V^A\mathbf{cnp}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{cnp}} \boldsymbol{\phi}$,
(x) $X \models_{V^A\mathbf{5nps}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{5nps}} \boldsymbol{\phi}$,
(xi) $X \models_{V^A\mathbf{bnps}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{bnps}} \boldsymbol{\phi}$,
(xii) $X \models_{V^A\mathbf{cnps}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{cnps}} \boldsymbol{\phi}$.

Proof

Proof of (i):
According to the proof of Theorem 5.1(i), we need only to show that for every $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{W}$, $\boldsymbol{D_w} \subseteq \boldsymbol{D_v}$ if $\boldsymbol{wRv}$. So, using (2.35), (2.8) and Lemma 4.2, we obtain that for every $\boldsymbol{w} \in \boldsymbol{W}$ and every 1st sort variable x of $\mathcal{L}'$, $(\mathsf{E}(x) \to \mathsf{LE}(x)) \in \boldsymbol{w}$. And supposing that $x \in \boldsymbol{D_w}$, by the definition of $\boldsymbol{D_w}$ we have that x is a 1st sort variable of $\mathcal{L}'$ and $\mathsf{E}(x) \in \boldsymbol{w}$, and hence, $\mathsf{LE}(x) \in \boldsymbol{w}$. Consequently, $\mathsf{E}(x) \in \boldsymbol{v}$, i.e., $x \in \boldsymbol{D_v}$, for every $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$. This means, $\boldsymbol{D_w} \subseteq \boldsymbol{D_v}$ for every $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$.

Proofs of (ii)–(xii)
Putting together the proofs of Theorem 5.1(ii)–(xii), respectively, and of Theorem 5.2 (i). □

Theorem 5.3 (Strong completeness) *Let X be a set of formulas. Then:*

(i) $X \models_{V^A\mathbf{5\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{5\star}} \boldsymbol{\phi}$,
(ii) $X \models_{V^A\mathbf{5s\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{5s\star}} \boldsymbol{\phi}$,
(iii) $X \models_{V^A\mathbf{d\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{d\star}} \boldsymbol{\phi}$,
(iv) $X \models_{V^A\mathbf{5n\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{5n\star}} \boldsymbol{\phi}$,
(v) $X \models_{V^A\mathbf{ds\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{ds\star}} \boldsymbol{\phi}$,
(vi) $X \models_{V^A\mathbf{5ns\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{5ns\star}} \boldsymbol{\phi}$,
(vii) $X \models_{V^A\mathbf{dn\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{dn\star}} \boldsymbol{\phi}$,
(viii) $X \models_{V^A\mathbf{dns\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{V^A\mathbf{dns\star}} \boldsymbol{\phi}$.

Proof

Proof of (i):
Like that of Theorem 5.1(i), but by using of several distinct or new points in **Step B**.

In particular, the proof of the condition (••) is now a bit mysterious.

$$\mathsf{G}(x) \in \boldsymbol{v} \text{ for some } \boldsymbol{v} \in \boldsymbol{W}, \text{ then } \mathsf{G}(x) \in \boldsymbol{w} \text{ for each } \boldsymbol{w} \in \boldsymbol{W}. \qquad (\bullet\bullet)$$

We first prove that if $\mathsf{G}(x) \in \boldsymbol{v}$ for some $\boldsymbol{v} \in \boldsymbol{W^{acc}}$, then $\mathsf{G}(x) \in \boldsymbol{w}$ for each $\boldsymbol{w} \in \boldsymbol{W^{acc}}$.

Suppose, for the non-trivial case, that the family $\boldsymbol{W}$ is at least two-element. We shall first prove that for any $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{W^{acc}}$, $\boldsymbol{wRv}$. So, let $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{W^{acc}}$. Employing **Step 3** of the definition of the order on $\boldsymbol{W}$, we may suppose that $\boldsymbol{w} \in \boldsymbol{W^r}$ and $\boldsymbol{v} \in \boldsymbol{W^{r'}}$ for some $r, r' \geq 2$. Therefore there exists a finite sequences $\boldsymbol{w_1}, \boldsymbol{w_2}, \ldots, \boldsymbol{w_r}$ and $\boldsymbol{w_1}, \boldsymbol{w'_2}, \ldots, \boldsymbol{w'_{r'}}$ such that $\boldsymbol{w_r} = \boldsymbol{w}$, $\boldsymbol{w'_{r'}} = \boldsymbol{v}$ and for each i, $1 < i \leq r$, $\boldsymbol{w_i}$ is a special t-$\mathbf{V}^A\mathbf{5}\star$-extension of some formula ψ_{i-1} such that $\mathsf{M}\psi_{i-1} \in \boldsymbol{w_{i-1}}$, and for each j, $1 < j \leq r'$, $\boldsymbol{w'_j}$ is a special t-$\mathbf{V}^A\mathbf{5}\star$-extension of some formula ψ'_{j-1} such that $\mathsf{M}\psi_{j-1} \in \boldsymbol{w'_{j-1}}$. The fact that $a\boldsymbol{R}b$, where $a, b \in \{\boldsymbol{w_2}, \boldsymbol{w'_2}\}$, follows directly from the Euclideanness of $\boldsymbol{R}$. Assuming next that $\boldsymbol{w_2Rw_p}$ and $\boldsymbol{w_pRw_2}$, where $r > 2$ and $r > p$, we obtain that $\boldsymbol{w_2Rw_{p+1}}$ and $\boldsymbol{w_{p+1}Rw_2}$. Consequently, $\boldsymbol{w_2Rw}$ and $\boldsymbol{wRw_2}$. Similarly, we can prove that $\boldsymbol{w'_2Rv}$ and $\boldsymbol{vRw'_2}$. Finally, from $\boldsymbol{w_2Rw}$ and $\boldsymbol{w_2Rw'_2}$ follows $\boldsymbol{w'_2Rw}$, and from $\boldsymbol{w'_2Rw}$ and $\boldsymbol{w'_2Rv}$ follows $\boldsymbol{wRv}$.

Suppose that $\mathsf{G}(x) \in \boldsymbol{v}$ for some $\boldsymbol{v} \in \boldsymbol{W^{acc}}$. Since $\vdash_{\mathbf{V}^A\mathbf{5}\star} \mathsf{G}(x) \to \mathsf{LG}(x)$, then after applying Lemma 4.2, we obtain that $\big(\mathsf{G}(x) \to \mathsf{LG}(x)\big) \in \boldsymbol{v}$. Hence, it follows that $\mathsf{LG}(x) \in \boldsymbol{v}$, and consequently, $\mathsf{G}(x) \in \boldsymbol{w}$ for every $\boldsymbol{w} \in \boldsymbol{W^{acc}}$.

Suppose now that $\mathsf{G}(x) \in \boldsymbol{w_1}$. Then, as a direct consequence of the axiom (2.21) and Lemma 4.2 we obtain that $\mathsf{LG}(x) \in \boldsymbol{w_1}$. Therefore, $\mathsf{G}(x) \in \boldsymbol{v}$ for all $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{w_1Rv}$, and consequently, $\mathsf{G}(x) \in \boldsymbol{w}$ for all $\boldsymbol{w} \in \boldsymbol{W}$. On the other hand, observe that if $\mathsf{G}(x) \notin \boldsymbol{w_1}$, then by the maximal $\mathbf{V}^A\mathbf{5}\star$-consistency of $\boldsymbol{w_1}$, $\neg\mathsf{G}(x) \in \boldsymbol{w_1}$. Hence, with the help of the axiom (2.21) and Lemma 4.2, $\mathsf{M}\neg\mathsf{G}(x) \in \boldsymbol{w_1}$. But then, $\neg\mathsf{G}(x) \in \boldsymbol{v}$ for some $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{w_1Rv}$, and consequently, $\mathsf{G}(x) \notin \boldsymbol{w}$ for all $\boldsymbol{w} \in \boldsymbol{W}$, which completes the proof of (••).

$$\text{If } \mathsf{G}(x) \in \boldsymbol{v} \text{ for some } \boldsymbol{v} \in \boldsymbol{W}, \text{ then } \mathsf{E}(x) \in \boldsymbol{w} \text{ for each } \boldsymbol{w} \in \boldsymbol{W}. \qquad (\bullet\bullet\bullet)$$

Suppose that $\mathsf{G}(x) \in \boldsymbol{v}$ for some $\boldsymbol{v} \in \boldsymbol{W}$. Therefore, by (••), $\mathsf{G}(x) \in \boldsymbol{w}$ for every $\boldsymbol{w} \in \boldsymbol{W}$. Next, on the strength of T9 ("Appendix") and Lemma 4.2 we obtain that $\exists x(\mathsf{G}(x) \wedge \mathsf{E}(x)) \in \boldsymbol{w}$ for each $\boldsymbol{w} \in \boldsymbol{W}$. Hence, for every $\boldsymbol{w} \in \boldsymbol{W}$ there exists 1st sort variable x_w such that $(\mathsf{G}(x_w) \wedge \mathsf{E}(x_w)) \in \boldsymbol{w}$. Consequently, $\mathsf{G}(x_w) \in \boldsymbol{w}$ and $\mathsf{E}(x_w) \in \boldsymbol{w}$ for each $\boldsymbol{w} \in \boldsymbol{W}$. And since $\vdash_{\mathbf{V}^A\mathbf{5}\star} \mathsf{G}(x) \wedge \mathsf{G}(x_w) \to \mathsf{L}(x \overset{1}{\approx} x_w)$, then by Lemma 4.2, $\mathsf{L}(x \overset{1}{\approx} x_w) \in \boldsymbol{w}$ for every $\boldsymbol{w} \in \boldsymbol{W}$. But hence, by applying (2.17) and Lemma 4.2, we obtain that $\mathsf{E}(x) \in \boldsymbol{w}$ for each $\boldsymbol{w} \in \boldsymbol{W}$, which finishes the proof of (• • •).

The definitions of the distinguished set $\mathbb{G}$ and the family $\mathscr{D}_1 = (\boldsymbol{D_w})_{\boldsymbol{w}\in\boldsymbol{W}}$ of 1st order domains are the same as in the proof of Theorem 5.1(i). Naturally, $\mathbb{G} \subseteq \bigcap_{w\in W} \boldsymbol{D_w}$.

Again, with each 2nd term A and $\boldsymbol{w} \in \boldsymbol{W}$ we associate the set

$$F(A, \boldsymbol{w}) = \{a \mid A(a) \in \boldsymbol{w}\},$$

and we put

$$\mathcal{E}_{\boldsymbol{w_1}} \text{ to be the family of all sets } F(A, \boldsymbol{w_1}) \in 2^{\boldsymbol{D_{w_1}}},$$

$$\mathcal{D}_{\boldsymbol{w_1}} = 2^{\boldsymbol{D_{w_1}}},$$

for every $\boldsymbol{w} \in \boldsymbol{W^{acc}}$, $\mathcal{D}_{\boldsymbol{w}}$ to be the family of all sets $F(A, \boldsymbol{w}) \in 2^{\boldsymbol{D_w}}$,

$$\mathscr{D}_2 = (\mathcal{D}_{\boldsymbol{w}})_{\boldsymbol{w} \in \boldsymbol{W}}$$

$$\boldsymbol{C_{\mathfrak{W}}} = \left\{ \boldsymbol{f} \in \boldsymbol{W} \mapsto \bigcup_{w \in W} \boldsymbol{D_w} \mid \boldsymbol{f}(\boldsymbol{w_1}) \in \mathcal{E}_{\boldsymbol{w_1}} \text{ and } \boldsymbol{f}(\boldsymbol{w}) \in \mathcal{D}_{\boldsymbol{w}} \text{ for every } \boldsymbol{w} \in \boldsymbol{W^{acc}} \right\}.$$

Proof of (ii):
By the proof of **(i)**, we must here only show that $\bigcap\{X \mid X \in \mathcal{D}_{\boldsymbol{w}} \text{ and } x \in X\} \in \mathcal{D}_{\boldsymbol{w}}$ for every $x \in \bigcup_{w \in W} \boldsymbol{D_w}$ and every $\boldsymbol{w} \in \boldsymbol{W^{acc}}$. It follows by the same argument as in the proof of Theorem 5.1(iv), but using (2.36) with the definition (2.25) in lieu of (2.36) with the definition (2.26).

Proof of (iii):
Relying on **(i)**, it suffices here to show that $\boldsymbol{R}$ is transitive. So, suppose that $\boldsymbol{wRv}$ and $\boldsymbol{vRv_1}$. Therefore, by definition (**R**), $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}\} \subseteq \boldsymbol{v}$ and $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{v}\} \subseteq \boldsymbol{v_1}$. In order to show that $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}\} \subseteq \boldsymbol{v_1}$ let us assume that $\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}$. Because $\mathsf{L}\boldsymbol{\phi} \to \mathsf{LL}\boldsymbol{\phi} \in \boldsymbol{w}$, then by Lemma 4.2(i) and Proposition 2.1(ii), $\mathsf{LL}\boldsymbol{\phi} \in \boldsymbol{w}$. Hence, $\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{v}$ and $\boldsymbol{\phi} \in \boldsymbol{v_1}$. Thus $\boldsymbol{wRv_1}$, i.e. $\boldsymbol{R}$ is transitive.

Proof of (iv):
By the proof of **(i)**, we must here only show that $\boldsymbol{D_w} \in \mathcal{D}_{\boldsymbol{w}}$ for every $\boldsymbol{w} \in \boldsymbol{W^{acc}}$. It follows by the same argument as in the proof of Theorem 5.1(vii), but using $\vdash_{\mathbf{V}^A\mathbf{5n\star}}$ $\mathsf{L}\forall x \mathsf{NE}(x)$ in lieu of $\vdash_{\mathbf{V}^A\mathbf{5n}} \forall x \mathsf{NE}(x)$.

Proof of (v)–(viii):
Adapting the arguments of **(i)–(iv)** to suit the proofs of **(v)–(viii)** are a bit more work. □

Theorem 5.4 (Strong completeness) *Let X be a set of formulas. Then:*

(i) $X \models_{\boldsymbol{V^A5p\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{V^A5p\star}} \boldsymbol{\phi}$,
(ii) $X \models_{\boldsymbol{V^A5ps\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{V^A5ps\star}} \boldsymbol{\phi}$,
(iii) $X \models_{\boldsymbol{V^Adp\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{V^Adp\star}} \boldsymbol{\phi}$,
(iv) $X \models_{\boldsymbol{V^A5np\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{V^A5np\star}} \boldsymbol{\phi}$,
(v) $X \models_{\boldsymbol{V^Adps\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{V^Adps\star}} \boldsymbol{\phi}$,
(vi) $X \models_{\boldsymbol{V^A5nps\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{V^A5nps\star}} \boldsymbol{\phi}$,
(vii) $X \models_{\boldsymbol{V^Adnp\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{V^Adnp\star}} \boldsymbol{\phi}$,
(viii) $X \models_{\boldsymbol{V^Adnps\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{V^Adnps\star}} \boldsymbol{\phi}$.

Proof Putting together the proofs of Theorem 5.3(i)–(viii), respectively, and of Theorem 5.2 (i). □

Appendix

We shall prove here a number of interesting theorems.

T1: For every $\mathbf{Th} \in \{\mathbf{V}^A\mathbf{5}, \mathbf{V}^A\mathbf{b}, \mathbf{V}^A\mathbf{5}\star\}$, $\vdash_{\mathbf{Th}} \exists x \mathsf{E}(x)$

Proof

1.	$\forall x \neg \mathsf{E}(x) \rightarrow (\mathsf{E}(y) \rightarrow \neg \mathsf{E}(y))$	(2.4), (2.8), R1
2.	$\mathsf{E}(y) \rightarrow (\forall x \neg \mathsf{E}(x) \rightarrow \neg \mathsf{E}(y))$	(2.4), **1**, R1
3.	$\mathsf{E}(y) \rightarrow (\mathsf{E}(y) \rightarrow \exists x \mathsf{E}(x))$	(2.4), **2**, R1
4.	$\mathsf{E}(y) \rightarrow \exists x \mathsf{E}(x)$	(2.4), **3**, R1
5.	$\forall y(\mathsf{E}(y) \rightarrow \exists x \mathsf{E}(x))$	**4**, R3
6.	$\forall y \mathsf{E}(y) \rightarrow \exists x \mathsf{E}(x)$	(2.4), (2.5), (2.7), **5**, R1
7.	$\exists x \mathsf{E}(x)$	**6**, (2.9), R1

□

T2: For every $\mathbf{Th} \in \{\mathbf{V}^A\mathbf{5}, \mathbf{V}^A\mathbf{b}, \mathbf{V}^A\mathbf{5}\star\}$, $\vdash_{\mathbf{Th}} \exists x((\exists x\phi \rightarrow \phi) \wedge \mathsf{E}(x))$

Proof

1.	$\forall x(\mathsf{E}(x) \rightarrow (\exists x\phi \wedge \neg\phi)) \rightarrow \forall x(\exists x\phi \wedge \neg\phi)$	(2.5), (2.9), R1
2.	$\forall x(\exists x\phi \wedge \neg\phi) \rightarrow \exists x\phi$	(2.4), R3, (2.5), (2.7), R1
3.	$\neg\exists x\phi \rightarrow \exists x(\exists x\phi \rightarrow \phi)$	(2.4), **2**, R1
4.	$\forall x(\exists x\phi \wedge \neg\phi) \rightarrow \forall x \neg\phi$	(2.4), R3, (2.5), (2.7), R1
5.	$\exists x\phi \rightarrow \exists x(\exists x\phi \rightarrow \phi)$	(2.4), **4**, R1
6.	$\neg\exists x\phi \vee \exists x\phi \rightarrow \exists x(\exists x\phi \rightarrow \phi)$	(2.4), **3**, **5**, R1
7.	$\exists x(\exists x\phi \rightarrow \phi)$	(2.4), **6**, R1
8.	$\exists x(\exists x\phi \rightarrow \phi) \rightarrow \exists x(\mathsf{E}(x) \wedge \neg(\exists x\phi \wedge \neg\phi))$	(2.4), **1**, R1
9.	$\exists x(\exists x\phi \rightarrow \phi) \rightarrow \exists x((\exists x\phi \rightarrow \phi) \wedge \mathsf{E}(x))$	(2.4), **8**, R1
10.	$\exists x((\exists x\phi \rightarrow \phi) \wedge \mathsf{E}(x))$	**7**, **9**, R1

□

T3: For every $\mathbf{Th} \in \{\mathbf{V}^A\mathbf{5}, \mathbf{V}^A\mathbf{b}, \mathbf{V}^A\mathbf{5}\star\}$, $\vdash_{\mathbf{Th}} \mathsf{E}(y) \wedge \phi(x/y) \rightarrow \exists x\phi$

Proof Trivially, by (2.8). □

T4: For every $\mathbf{Th} \in \{\mathbf{V}^A\mathbf{5}, \mathbf{V}^A\mathbf{b}, \mathbf{V}^A\mathbf{5}\star\}$, $\vdash_{\mathbf{Th}} \exists\alpha(\exists\alpha\phi \rightarrow \phi)$

Proof

1.	$\forall\alpha(\exists\alpha\phi \wedge \neg\phi) \rightarrow \exists\alpha\phi$	(2.4), R3, (2.5), (2.7), R1
2.	$\neg\exists\alpha\phi \rightarrow \exists\alpha(\exists\alpha\phi \rightarrow \phi)$	(2.4), **1**, R1
3.	$\forall\alpha(\exists\alpha\phi \wedge \neg\phi) \rightarrow \forall\alpha\neg\phi$	(2.4), R3, (2.5), (2.7), R1
4.	$\exists\alpha\phi \rightarrow \exists\alpha(\exists\alpha\phi \rightarrow \phi)$	(2.4), **3**, R1
5.	$\neg\exists\alpha\phi \vee \exists\alpha\phi \rightarrow \exists\alpha(\exists\alpha\phi \rightarrow \phi)$	(2.4), **2**, **4**, R1
6.	$\exists\alpha(\exists\alpha\phi \rightarrow \phi)$	(2.4), **5**, R1

□

T5 : For every **Th** $\in \{\mathbf{V}^A\mathbf{5}, \mathbf{V}^A\mathbf{b}, \mathbf{V}^A\mathbf{5}\star\}$, $\vdash_{\mathbf{Th}} \mathsf{M}\exists y \mathsf{G}(y)$

Proof

1.	$\neg\mathsf{G}(x) \to (\mathsf{G}(x) \to \neg\mathsf{G}(x))$	(2.4)
2.	$\forall x \neg\mathsf{G}(x) \to \forall x(\mathsf{G}(x) \to \neg\mathsf{G}(x))$	**1**, R3, (2.5), R1
3.	$\forall x(\mathsf{G}(x) \to \neg\mathsf{G}(x)) \to \forall x(\mathsf{G}(x) \to -\mathsf{G}(x))$	(2.4), R3, (2.5), (2.28), R1
4.	$\forall x \neg\mathsf{G}(x) \to \forall x(\mathsf{G}(x) \to -\mathsf{G}(x))$	(2.4), R3, (2.5), **2**, **3**, R1
5.	$\mathsf{L}\forall x \neg\mathsf{G}(x) \to \mathsf{L}\forall x(\mathsf{G}(x) \to -\mathsf{G}(x))$	**4**, R2, (2.11), R1
6.	$\mathsf{L}\forall x \neg\mathsf{G}(x) \to \mathbf{P}(\mathsf{G}) \wedge \mathsf{L}\forall x(\mathsf{G}(x) \to -\mathsf{G}(x))$	(2.4), **5**, (2.24), R1
7.	$\mathsf{L}\forall x \neg\mathsf{G}(x) \to \mathbf{P}(-\mathsf{G})$	(2.4), (2.20), **6**, R1
8.	$\neg\mathbf{P}(-\mathsf{G}) \to \mathsf{M}\exists x \mathsf{G}(x)$	(2.4), **7**, the definition of M and of $\exists$, R1
9.	$\mathsf{M}\exists x \mathsf{G}(x)$	(2.4), **8**, (2.19), (2.24), R1

□

T6 : For every **Th** $\in \{\mathbf{V}^A\mathbf{5}, \mathbf{V}^A\mathbf{b}, \mathbf{V}^A\mathbf{5}\star\}$, $\vdash_{\mathbf{Th}} \exists x \mathsf{G}(x) \to \mathsf{L}\exists x \mathsf{G}(x)$

Proof

1.	$\mathsf{G}(x) \to \mathsf{LG}(x)$	(2.4), (2.1), (2.6), (2.24), R1
2.	$\exists x \mathsf{G}(x) \to \exists x \mathsf{LG}(x)$	**1**, (2.4), R3, (2.5), the definition of $\exists$, R1
3.	$\mathsf{E}(x) \wedge \mathsf{G}(x) \to \exists x \mathsf{G}(x)$	T3
4.	$\mathsf{G}(x) \to \mathsf{E}(x) \wedge \mathsf{G}(x)$	(2.4), (2.23), (2.10), R1
5.	$\mathsf{G}(x) \to \exists x \mathsf{G}(x)$	(2.4), **3**, **4**, R3
6.	$\mathsf{LG}(x) \to \mathsf{L}\exists x \mathsf{G}(x)$	**5**, R2, (2.11), R1
7.	$\exists x \mathsf{LG}(x) \to \mathsf{L}\exists x \mathsf{G}(x)$	**6**, R3, (2.5), the definition of $\exists$, (2.7), R1
8.	$\exists x \mathsf{G}(x) \to \mathsf{L}\exists x \mathsf{G}(x)$	(2.4), **2**, **7**, R1

□

T7 : For every **Th** $\in \{\mathbf{V}^A\mathbf{5}, \mathbf{V}^A\mathbf{5}\star\}$, $\vdash_{\mathbf{Th}} \mathsf{L}\exists x \mathsf{G}(x)$

Proof

1.	$\mathsf{M}\exists x \mathsf{G}(x) \to \mathsf{ML}\exists x \mathsf{G}(x)$	(2.4), T6, R2, (2.11), the definition of M, R1
2.	$\mathsf{ML}\exists x \mathsf{G}(x)$	**1**, T5, R1
3.	$\mathsf{L}\exists x \mathsf{G}(x)$	**2**, (**5**), R1

□

T8 : $\vdash_{\mathbf{V}^A\mathbf{b}} \mathsf{L}\exists x \mathsf{G}(x)$

Proof The steps **1–2** are the same as for T7

3.	$\exists x \mathsf{G}(x)$	**2**, (**b**), R1
4.	$\mathsf{L}\exists x \mathsf{G}(x)$	**3**, R2

□

T9 : For every $\mathbf{Th} \in \{\mathbf{V}^A\mathbf{5}, \mathbf{V}^A\mathbf{b}, \mathbf{V}^A\mathbf{5\star}\}$, $\vdash_{\mathbf{Th}} \mathsf{L}\exists x\mathsf{G}(x) \to \exists x\mathsf{G}(x)$

Proof

1.	$\mathsf{LG}(x) \to \mathsf{G}(x)$	(2.21)
2.	$\exists x\mathsf{LG}(x) \to \exists x\mathsf{G}(x)$	(2.4), **1**, R3, (2.5), the definition of $\exists$, R1
3.	$\mathsf{L}\exists x\mathsf{G}(x) \to \exists x\mathsf{G}(x)$	(2.4), (2.22), **2**, R1

□

T10 : For every $\mathbf{Th} \in \{\mathbf{V}^A\mathbf{5}, \mathbf{V}^A\mathbf{b}, \mathbf{V}^A\mathbf{5\star}\}$, $\vdash_{\mathbf{Th}} \exists x(\mathsf{G}(x) \wedge \mathsf{E}(x))$

Proof

1.	$\exists x\mathsf{G}(x)$	T9, T7(*or*, T8), R1
2.	$\exists x\mathsf{G}(x) \to ((\exists x\mathsf{G}(x) \to \mathsf{G}(x)) \to \mathsf{G}(x))$	(2.4)
3.	$(\exists x\mathsf{G}(x) \to \mathsf{G}(x)) \to \mathsf{G}(x)$	**1**, **2**, R1
4.	$(\exists x\mathsf{G}(x) \to \mathsf{G}(x)) \wedge \mathsf{E}(x) \to \mathsf{G}(x) \wedge \mathsf{E}(x)$	(2.4), **3**, R1
5.	$\exists x((\exists x\mathsf{G}(x) \to \mathsf{G}(x)) \wedge \mathsf{E}(x)) \to \exists x(\mathsf{G}(x) \wedge \mathsf{E}(x))$	(2.4), **4**, R3, (2.5), R1
6.	$\exists x(\mathsf{G}(x) \wedge \mathsf{E}(x))$	**5**, T2, R1

□

T11 : For every $\mathbf{Th} \in \{\mathbf{V}^A\mathbf{5}, \mathbf{V}^A\mathbf{b}, \mathbf{V}^A\mathbf{5\star}\}$, $\vdash_{\mathbf{Th}} \forall x\mathsf{NE}(x)$

Proof

1.	$\forall\alpha(\alpha(x) \to \exists y\alpha(y))$	(2.4), (2.10), T3, R1, R3
2.	$\forall x\mathsf{L}\forall\alpha(\alpha(x) \to \exists y\alpha(y))$	**1**, R2, R3
3.	$\forall x\forall\alpha\mathsf{L}(\alpha(x) \to \exists y\alpha(y))$	(2.6), R2, (2.11), R3, (2.5), **2**, R1
4.	$\forall x\forall\alpha(\mathsf{L}\alpha(x) \to \mathsf{L}\exists y\alpha(y))$	(2.11), R3, (2.5), **3**, R1
5.	$\forall\beta(\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y(\alpha(y) \to \beta(y))) \to (\mathsf{L}\alpha(x) \leftrightarrow \mathsf{L}\forall y(\alpha(y) \to \alpha(y)))$	(2.6)
6.	$\forall\beta(\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y(\alpha(y) \to \beta(y))) \to \mathsf{L}\alpha(x)$	(2.4), R3, R2, **5**, R1
7.	$\forall x\forall\alpha(\forall\beta(\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y(\alpha(y) \to \beta(y))) \to \mathsf{L}\exists y\alpha(y))$	(2.4), R3, (2.5), **6**, **4**, R1
8.	$\forall x\forall\alpha(\alpha\ \mathsf{Ess}\ x \to \mathsf{L}\exists y\alpha(y))$	**7**, (2.2)
9.	$\forall x\mathsf{NE}(x)$	**8**, (2.3)

□

T12 : For every $\mathbf{Th} \in \{\mathbf{V}^A\mathbf{5}, \mathbf{V}^A\mathbf{b}, \mathbf{V}^A\mathbf{5\star}\}$, $\vdash_{\mathbf{Th}} \mathsf{G}(x) \wedge \mathsf{G}(y) \to (x \overset{1}{\approx} y)$

Proof

1.	$\exists\alpha(\alpha(x) \wedge \neg\alpha(y)) \to (x \overset{1}{\not\approx} y)$	(2.4), R3, (2.5), the definition of $\overset{1}{\not\approx}$, R1
2.	$\mathsf{G}(x) \to (\exists\alpha(\alpha(x) \wedge \neg\alpha(y)) \to \mathsf{L}\exists\alpha(\alpha(x) \wedge \neg\alpha(y)))$	(2.18)
3.	$\mathsf{G}(x) \to (\exists\alpha(\alpha(x) \wedge \neg\alpha(y)) \to \exists\alpha\mathsf{L}(\alpha(x) \wedge \neg\alpha(y)))$	(2.4), **2**, (2.14), R1
4.	$\mathsf{G}(x) \to (\exists\alpha(\alpha(x) \wedge \neg\alpha(y)) \to \exists\alpha(\mathsf{L}\alpha(x) \wedge \mathsf{L}\neg\alpha(y)))$	(2.4), (2.5), (2.11), **3**, R1
5.	$\mathsf{G}(x) \to (\neg\exists\alpha(\mathsf{L}\alpha(x) \wedge \mathsf{L}\neg\alpha(y)) \to \neg\exists\alpha(\alpha(x) \wedge \neg\alpha(y)))$	(2.4), **4**, R1
6.	$\mathsf{G}(x) \to (\forall\alpha(\mathsf{L}\alpha(x) \to \mathsf{M}\alpha(y)) \to \forall\alpha(\alpha(x) \to \alpha(y)))$	(2.4), **5**, R1
7.	$\exists\alpha(\alpha(y) \wedge \neg\alpha(x)) \to (x \overset{1}{\not\approx} y)$	(2.4), R3, (2.5), the definition of $\overset{1}{\not\approx}$, R1

8. $\mathsf{G}(y) \to (\exists\alpha(\alpha(y)\wedge\neg\alpha(x)) \to \mathsf{L}\exists\alpha(\alpha(y)\wedge\neg\alpha(x)))$ (2.18)
9. $\mathsf{G}(y) \to (\exists\alpha(\alpha(y)\wedge\neg\alpha(x)) \to \exists\alpha\mathsf{L}(\alpha(y)\wedge\neg\alpha(x)))$ (2.4), **8**, (2.14), R1
10. $\mathsf{G}(y) \to (\exists\alpha(\alpha(y)\wedge\neg\alpha(x)) \to \exists\alpha(\mathsf{L}\alpha(y)\wedge\mathsf{L}\neg\alpha(x)))$ (2.4), (2.5), (2.11), **9**, R1
11. $\mathsf{G}(y) \to (\neg\exists\alpha(\mathsf{L}\alpha(y)\wedge\mathsf{L}\neg\alpha(x)) \to \neg\exists\alpha(\alpha(y)\wedge\neg\alpha(x)))$ (2.4), **10**, R1
12. $\mathsf{G}(y) \to (\forall\alpha(\mathsf{L}\alpha(y) \to \mathsf{M}\alpha(x)) \to \forall\alpha(\alpha(y) \to \alpha(x)))$ (2.4), **11**, R1
13. $\mathsf{G}(x) \wedge \mathsf{G}(y) \to (\forall\alpha(\mathsf{L}\alpha(x) \leftrightarrow \mathsf{L}\alpha(y)) \to \forall\alpha(\mathsf{L}\alpha(x) \to \mathsf{M}\alpha(y))\wedge\forall\alpha(\mathsf{L}\alpha(y) \to \mathsf{M}\alpha(x)))$ (2.4), (2.12), R3, (2.5), R1
14. $\mathsf{G}(x) \wedge \mathsf{G}(y) \to (\forall\alpha(\mathsf{L}\alpha(x) \leftrightarrow \mathsf{L}\alpha(y)) \to \forall\alpha(\alpha(x) \leftrightarrow \alpha(y)))$ (2.4), **13**, **6**, **12**, R1
15. $\mathsf{G}(x)\wedge\mathsf{G}(y) \to \forall\alpha(\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x))\wedge\forall\alpha(\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(y))$ (2.4), (2.1), R1
16. $\mathsf{G}(x)\wedge\mathsf{G}(y) \to \forall\alpha(\mathsf{L}\alpha(x) \leftrightarrow \mathsf{L}\alpha(y))$ (2.4), (2.5), **15**, R1
17. $\mathsf{G}(x)\wedge\mathsf{G}(y) \to (x \overset{1}{\approx} y)$ (2.4), **16**, **14**, R1, the definition of $\overset{1}{\approx}$

□

T13: For every **Th** $\in \{\mathbf{V}^A\mathbf{5}, \mathbf{V}^A\mathbf{b}, \mathbf{V}^A\mathbf{5}\star\}$, $\vdash_{\mathbf{Th}} \mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\forall x(\mathsf{G}(y) \to \alpha(x))$

Proof

1. $(x \overset{1}{\approx} y) \to (\alpha(y) \to \alpha(x))$ (2.15)
2. $\mathsf{G}(x)\wedge\mathsf{G}(y) \to (\alpha(y) \to \alpha(x))$ (2.4), **T12**, **1**, R1
3. $\mathsf{G}(y)\wedge\alpha(y) \to (\mathsf{G}(x) \to \alpha(x))$ (2.4), **2**, R1
4. $\mathsf{G}(y)\wedge\alpha(y) \to \forall x(\mathsf{G}(x) \to \alpha(x))$ **3**, R3, (2.5), (2.7), R1
5. $\mathsf{G}(y) \to (\alpha(y) \to \forall x(\mathsf{G}(x) \to \alpha(x)))$ (2.4), **4**, R1
6. $\mathsf{L}\mathsf{G}(y) \to (\mathsf{L}\alpha(y) \to \mathsf{L}\forall x(\mathsf{G}(x) \to \alpha(x)))$ **5**, R2, (2.11), R1
7. $(\mathsf{L}\mathsf{G}(y) \to \mathsf{L}\alpha(y)) \to (\mathsf{L}\mathsf{G}(y) \to \mathsf{L}\forall x(\mathsf{G}(x) \to \alpha(x)))$ (2.4), **6**, R1
8. $(\mathsf{G}(y) \to \mathsf{L}\alpha(y)) \to (\mathsf{L}\mathsf{G}(y) \to \mathsf{L}\forall x(\mathsf{G}(x) \to \alpha(x)))$ (2.4), **7**, (2.21), R1
9. $\mathbf{P}(\alpha) \to (\mathsf{L}\mathsf{G}(y) \to \mathsf{L}\forall x(\mathsf{G}(x) \to \alpha(x)))$ (2.4), (2.1), (2.6), **8**, R1
10. $\mathbf{P}(\alpha) \to (\exists y\mathsf{L}\mathsf{G}(y) \to \mathsf{L}\forall x(\mathsf{G}(x) \to \alpha(x)))$ **9**, R3, (2.5), (2.7), R1
11. $\mathbf{P}(\alpha) \to (\mathsf{L}\exists y\mathsf{G}(y) \to \mathsf{L}\forall x(\mathsf{G}(x) \to \alpha(x)))$ (2.4), **10**, (2.22), R1
12. $\mathbf{P}(\alpha) \to \mathsf{L}\forall x(\mathsf{G}(x) \to \alpha(x))$ (2.4), **11**, **T7** (or, **T8**), R1
13. $\mathbf{P}(\mathsf{G})\wedge\mathsf{L}\forall x(\mathsf{G}(x) \to \alpha(x)) \to \mathbf{P}(\alpha)$ (2.20)
14. $\mathsf{L}\forall x(\mathsf{G}(x) \to \alpha(x)) \to \mathbf{P}(\alpha)$ (2.4), **13**, (2.24), R1
15. $\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\forall x(\mathsf{G}(x) \to \alpha(x))$ (2.4), **12**, **14**, R1

□

T14: For every **Th** $\in \{\mathbf{V}^A\mathbf{5}, \mathbf{V}^A\mathbf{b}, \mathbf{V}^A\mathbf{5}\star\}$, $\vdash_{\mathbf{Th}} \forall x\mathsf{L}(\mathsf{G}(x) \to \alpha(x)) \to \mathsf{L}\forall x(\mathsf{G}(x) \to \alpha(x))$

Proof

1. $\mathsf{G}(x) \to (\mathsf{L}\alpha(x) \to \mathbf{P}(\alpha))$ (2.4), (2.1), (2.6), R1
2. $(\mathsf{G}(x) \to \mathsf{L}\alpha(x)) \to (\mathsf{G}(x) \to \mathbf{P}(\alpha))$ (2.4), **1**, R1
3. $(\mathsf{L}\mathsf{G}(x) \to \mathsf{L}\alpha(x)) \to (\mathsf{G}(x) \to \mathbf{P}(\alpha))$ (2.4), (2.1), (2.6), (2.24), **2**, R1
4. $\mathsf{L}(\mathsf{G}(x) \to \alpha(x)) \to (\mathsf{G}(x) \to \mathbf{P}(\alpha))$ (2.4), (2.11), **3**, R1
5. $\forall x\mathsf{L}(\mathsf{G}(x) \to \alpha(x)) \to \forall x(\mathsf{G}(x) \to \mathbf{P}(\alpha))$ **4**, R3, (2.5), R1
6. $\forall x\mathsf{L}(\mathsf{G}(x) \to \alpha(x)) \to (\exists x\mathsf{G}(x) \to \mathbf{P}(\alpha))$ (2.4), (2.5), **5**, R1

7. $\forall x \mathsf{L}(\mathsf{G}(x) \to \alpha(x)) \to (\exists x \mathsf{L}\mathsf{G}(x) \to \mathbf{P}(\alpha))$ (2.4), (2.21), R3, (2.5), **6**, R1
8. $\forall x \mathsf{L}(\mathsf{G}(x) \to \alpha(x)) \to (\mathsf{L}\exists x \mathsf{G}(x) \to \mathbf{P}(\alpha))$ (2.4), (2.22), **7**, R1
9. $\forall x \mathsf{L}(\mathsf{G}(x) \to \alpha(x)) \to \mathbf{P}(\alpha)$ (2.4), **8**, **T7** (or, **T8**), R1
10. $\forall x \mathsf{L}(\mathsf{G}(x) \to \alpha(x)) \to \mathsf{L}\forall x(\mathsf{G}(x) \to \alpha(x))$ (2.4), **9**, **T13**, R1

□

T15: For every $\mathbf{Th} \in \{\mathbf{V}^A\mathbf{5}, \mathbf{V}^A\mathbf{b}\}$, $\vdash_{\mathbf{Th}} \mathsf{L}\forall x(\mathsf{G}(x) \to \alpha(x)) \to \forall x \mathsf{L}(\mathsf{G}(x) \to \alpha(x))$

Proof

1. $\mathsf{G}(x) \to \mathsf{E}(x)$ (2.10)
2. $(\mathsf{E}(x) \to (\mathsf{G}(x) \to \alpha(x))) \to (\mathsf{G}(x) \to (\mathsf{G}(x) \to \alpha(x)))$ (2.4), **1**, R1
3. $(\mathsf{E}(x) \to (\mathsf{G}(x) \to \alpha(x))) \to (\mathsf{G}(x) \to \alpha(x))$ (2.4), **2**, R1
4. $\forall x(\mathsf{G}(x) \to \alpha(x)) \to (\mathsf{G}(x) \to \alpha(x))$ (2.4), (2.8), **3**, R1
5. $\mathsf{L}\forall x(\mathsf{G}(x) \to \alpha(x)) \to \mathsf{L}(\mathsf{G}(x) \to \alpha(x))$ **4**, R2, (2.11), R1
6. $\mathsf{L}\forall x(\mathsf{G}(x) \to \alpha(x)) \to \forall x \mathsf{L}(\mathsf{G}(x) \to \alpha(x))$ **5**, R3, (2.5), (2.7), R1

□

T16: For every $\mathbf{Th} \in \{\mathbf{V}^A\mathbf{5}, \mathbf{V}^A\mathbf{b}\}$, $\vdash_{\mathbf{Th}} A \text{ Ess } x \wedge B \text{ Ess } x \to \mathsf{L}(A \overset{2}{\approx} B)$

Proof

1. $A \text{ Ess } x \to \forall \beta(\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y(A(y) \to \beta(y)))$ (2.2)
2. $\forall \beta(\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y(A(y) \to \beta(y))) \to (\mathsf{L}\forall y(A(y) \to A(y)) \to \mathsf{L}A(x))$ (2.5), (2.4), R1
3. $A \text{ Ess } x \to \mathsf{L}A(x)$ (2.4), **1**, **2**, R1
4. $B \text{ Ess } x \to \forall \alpha(\mathsf{L}\alpha(x) \leftrightarrow \mathsf{L}\forall y(B(y) \to \alpha(y)))$ (2.2)
5. $\forall \alpha(\mathsf{L}\alpha(x) \leftrightarrow \mathsf{L}\forall y(B(y) \to \alpha(y))) \to (\mathsf{L}\forall y(B(y) \to B(y)) \to \mathsf{L}B(x))$ (2.5), (2.4), R1
6. $B \text{ Ess } x \to \mathsf{L}B(x)$ (2.4), **4**, **5**, R1
7. $A \text{ Ess } x \wedge B \text{ Ess } x \to (\mathsf{L}B(x) \leftrightarrow \mathsf{L}\forall y(A(y) \to B(y))) \wedge \mathsf{L}B(x)$ (2.4), (2.2), (2.5), **6**, R1
8. $A \text{ Ess } x \wedge B \text{ Ess } x \to \mathsf{L}\forall y(A(y) \to B(y))$ (2.4), **7**, R1
9. $A \text{ Ess } x \wedge B \text{ Ess } x \to (\mathsf{L}A(x) \leftrightarrow \mathsf{L}\forall y(B(y) \to A(y))) \wedge \mathsf{L}A(x)$ (2.4), (2.2), (2.5), **3**, R1
10. $A \text{ Ess } x \wedge B \text{ Ess } x \to \mathsf{L}\forall y(B(y) \to A(y))$ (2.4), **9**, R1
11. $A \text{ Ess } x \wedge B \text{ Ess } x \to \mathsf{L}\forall y(A(y) \leftrightarrow B(y))$ (2.4), (2.11), **8**, **10**, R1
12. $A \text{ Ess } x \wedge B \text{ Ess } x \to \mathsf{L}(A \overset{2}{\approx} B)$ **11**, (2.25)

□

T17: $\vdash_{\mathbf{V}^A\mathbf{5}_\star} A \text{ Ess } x \wedge B \text{ Ess } x \to (A \overset{2}{\approx} B)$

Proof The steps **1–11** are the same as for **T16**

12. $A \text{ Ess } x \wedge B \text{ Ess } x \to (A \overset{2}{\approx} B)$ (2.4), (2.26), R1

□

T18: $\vdash_{\mathbf{V}^A\mathbf{d}\star} A\ \mathsf{Ess}\ x \wedge B\ \mathsf{Ess}\ x \to \mathsf{L}(A \overset{2}{\approx} B)$

Proof The steps **1–11** are the same as for **T16**

12.	$A\ \mathsf{Ess}\ x \wedge B\ \mathsf{Ess}\ x \to \mathsf{LL}\forall y(A(y) \leftrightarrow B(y))$	(2.4), **11**, **d**, R1
13.	$A\ \mathsf{Ess}\ x \wedge B\ \mathsf{Ess}\ x \to \mathsf{L}(A \overset{2}{\approx} B)$	**12**, (2.26)

□

T19: For every **Th** ∈ {$\mathbf{V}^A$**5n**, $\mathbf{V}^A$**bn**, $\mathbf{V}^A$**5n**⋆}, $\vdash_{\mathbf{Th}} \mathbf{P}(\mathsf{NE})$

Proof

1.	$\forall\alpha\mathsf{L}(\alpha(x) \to \exists y\alpha(y))$	(2.4), (2.10), T3, R1, R2, R3
2.	$\forall\alpha(\mathsf{L}\alpha(x) \to \mathsf{L}\exists y\alpha(y))$	(2.11), R3, (2.5), **1**, R1
3.	$\forall\alpha(\forall\beta(\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y(\alpha(y) \to \beta(y)))$ $\to (\mathsf{L}\alpha(x) \leftrightarrow \mathsf{L}\forall y(\alpha(y) \to \alpha(y))))$	(2.6), R3
4.	$\forall\alpha(\forall\beta(\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y(\alpha(y) \to \beta(y)))$ $\to \mathsf{L}\alpha(x))$	(2.4), R3, R2, (2.5), (2.11), **3**, R1
5.	$\forall\alpha(\forall\beta(\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y(\alpha(y) \to \beta(y))) \to \mathsf{L}\exists y\alpha(y))$	(2.4), R3, (2.5), **4**, **2**, R1
6.	$\forall\alpha(\mathsf{L}\exists y\alpha(y) \to ((\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)) \to \mathsf{L}\exists y\alpha(y)))$	(2.4), R3
7.	$\forall\alpha(\forall\beta(\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y(\alpha(y) \to \beta(y)))$ $\to ((\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)) \to \mathsf{L}\exists y\alpha(y)))$	(2.4), R3, (2.5), **5**, **6**, R1
8.	$\forall\alpha(\alpha\ \mathsf{Ess}\ x \to ((\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)) \to \mathsf{L}\exists y\alpha(y)))$	(2.4), R3, (2.5), (2.2), **7**, R1
9.	$\forall\alpha(\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)) \to \forall\alpha(\alpha\ \mathsf{Ess}\ x \to \mathsf{L}\exists y\alpha(y))$	(2.4), R3, (2.5), **8**, R1
10.	$(\mathsf{G}(x) \to \forall\alpha(\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)))$ $\to (\mathsf{G}(x) \to \forall\alpha(\alpha\ \mathsf{Ess}\ x \to \mathsf{L}\exists y\alpha(y)))$	(2.4), **9**, R1
11.	$\mathsf{G}(x) \to \forall\alpha(\alpha\ \mathsf{Ess}\ x \to \mathsf{L}\exists y\alpha(y))$	**10**, (2.1), R1
12.	$(\forall\alpha(\alpha\ \mathsf{Ess}\ x \to \mathsf{L}\exists y\alpha(y)) \to \mathsf{NE}(x)) \to (\mathsf{G}(x) \to \mathsf{NE}(x))$	(2.4), **11**, R1
13.	$\mathsf{L}\forall x(\mathsf{G}(x) \to \mathsf{NE}(x))$	**12**, (2.3), R3, R2
14.	$\mathbf{P}(\mathsf{G}) \wedge \mathsf{L}\forall x(\mathsf{G}(x) \to \mathsf{NE}(x)) \to \mathbf{P}(\mathsf{NE})$	(2.20), (2.34)
15.	$\mathbf{P}(\mathsf{NE})$	**14**, (2.24), **13**, R1

□

T20: For any **Th** ∈ {$\mathbf{V}^A$**5s**, $\mathbf{V}^A$**bs**, $\mathbf{V}^A$**5s**⋆}, $\vdash_{\mathbf{Th}} \alpha\ \mathsf{Ess}\ x \wedge \alpha\ \mathsf{Ess}\ y \to \mathsf{L}(\mathsf{E}(x) \vee \mathsf{E}(y) \to (x \overset{1}{\approx} y))$

Proof

1.	$\forall\beta(\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y(\alpha(y) \to \beta(y)))$ $\to (\mathsf{L}\mathsf{I}_x(x) \leftrightarrow \mathsf{L}\forall y(\alpha(y) \to \mathsf{I}_x(y)))$	(2.8), (2.36)
2.	$\mathsf{L}\mathsf{I}_x(x)$	(2.33), R2
3.	$\alpha\ \mathsf{Ess}\ x \to \mathsf{L}\forall y(\alpha(y) \to (x \overset{1}{\approx} y))$	(2.4), (2.2), **1**, **2**, R1
4.	$\alpha\ \mathsf{Ess}\ x \to \mathsf{L}(\mathsf{E}(y) \to (\alpha(y) \to (x \overset{1}{\approx} y)))$	(2.8), R2, (2.11), (2.4), **3**, R1
5.	$\forall\beta(\mathsf{L}\beta(y) \leftrightarrow \mathsf{L}\forall x(\alpha(x) \to \beta(x)))$ $\to (\mathsf{L}\alpha(y) \leftrightarrow \mathsf{L}\forall x(\alpha(x) \to \alpha(x)))$	(2.8)
6.	$\alpha\ \mathsf{Ess}\ y \to \mathsf{L}\alpha(y)$	(2.4), R3, R2, **5**, R1

7. $\alpha\ \mathsf{Ess}\ x \wedge \alpha\ \mathsf{Ess}\ y \to \mathsf{L}(\mathsf{E}(y) \to (x \overset{1}{\approx} y))$ (2.4), R2, (2.11), **4**, **6**, R1
8. $\forall\beta(\mathsf{L}\beta(y) \leftrightarrow \mathsf{L}\forall x(\alpha(x) \to \beta(x)))$
$\to (\mathsf{LI}_y(y) \leftrightarrow \mathsf{L}\forall x(\alpha(x) \to \mathsf{I}_y(x)))$ (2.8), (2.36)
9. $\mathsf{LI}_y(y)$ (2.33), R2
10. $\alpha\ \mathsf{Ess}\ y \to \mathsf{L}\forall x(\alpha(x) \to (y \overset{1}{\approx} x))$ (2.4), (2.2), **8**, **9**, R1
11. $\alpha\ \mathsf{Ess}\ y \to \mathsf{L}(\mathsf{E}(x) \to (\alpha(x) \to (y \overset{1}{\approx} x)))$ (2.8), R2, (2.11), (2.4), **10**, R1
12. $\forall\beta(\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y(\alpha(y) \to \beta(y)))$
$\to (\mathsf{L}\alpha(x) \leftrightarrow \mathsf{L}\forall y(\alpha(y) \to \alpha(y)))$ (2.8)
13. $\alpha\ \mathsf{Ess}\ y \to \mathsf{L}\alpha(x)$ (2.4), R3, R2, **12**, R1
14. $\alpha\ \mathsf{Ess}\ x \wedge \alpha\ \mathsf{Ess}\ y \to \mathsf{L}(\mathsf{E}(x) \to (y \overset{1}{\approx} x))$ (2.4), R2, (2.11), **11**, **13**, R1
15. $\alpha\ \mathsf{Ess}\ x \wedge \alpha\ \mathsf{Ess}\ y \to \mathsf{L}(\mathsf{E}(x) \vee \mathsf{E}(y) \to (x \overset{1}{\approx} y))$ (2.4), R2, (2.11), **7**, **14**, R1

□

`T21:` For any **Th** $\in \{\mathbf{V}^A\mathbf{5ps}, \mathbf{V}^A\mathbf{bps}, \mathbf{V}^A\mathbf{5ps}\star\}$, $\vdash_{\mathbf{Th}} \forall y(\alpha\ \mathsf{Ess}\ x \wedge \alpha\ \mathsf{Ess}\ y \to \mathsf{L}(x \overset{1}{\approx} y))$

Proof The steps **1–7** are the same as for `T20`

8. $\mathsf{LE}(y) \to (\alpha\ \mathsf{Ess}\ x \wedge \alpha\ \mathsf{Ess}\ y \to \mathsf{L}(x \overset{1}{\approx} y))$ (2.4), (2.11), **7**, R1
9. $\forall y\mathsf{LE}(y) \to \forall y(\alpha\ \mathsf{Ess}\ x \wedge \alpha\ \mathsf{Ess}\ y \to \mathsf{L}(x \overset{1}{\approx} y))$ (2.4), R3, (2.5), R1
10. $\forall y(\alpha\ \mathsf{Ess}\ x \wedge \alpha\ \mathsf{Ess}\ y \to \mathsf{L}(x \overset{1}{\approx} y))$ **9**, (2.35), R1

References

Adams, R. M. (1995). Introductory note to *1970. In S. Feferman, J. W. Dawson, Jr., W. Goldfarb, C. Parsons, & R. M. Solovay (Eds.), *Kurt Gödel collected works* (Vol. III, pp. 388–402). New York and Oxford: Oxford University Press.
Anderson, R. A. (1990). Some emendations of Gödel's ontological proof. *Faith and Philosophy, 7*, 291–303.
Cresswell, M. J. (1995). Incompleteness and the Barcan formula. *Journal of Philosophical Logic, 24*, 379–403.
Fitting, M. C. (2002). *Types, tableaus, and Gödel's god*. Dordrecht/Boston/London: Kluwer Academic Publishers.
Fitting, M. C. (2004). First-order intensional logic. *Annals of Pure and Applied Logic, 127*, 171–193.
Fitting, M. C., & Mendelson, R. (1998). *First-order modal logic*. Dordrecht/Boston/London: Kluwer Academic Publishers.
Garson, J. W. (1984). Quantification in modal logic. In D. Gabbay & F. Guenthner (Eds.), *Handbook of philosophical logic* (Vol. II, pp. 249–307). Dordrecht, Boston and Lancaster: D. Reidel Publishing Company.
Garson, J. W. (1991). Applications of free logic to quantified intensional logic. In K. Lambert (Ed.), *Philosophical applications of free logic* (pp. 111–142). New York and Oxford: Oxford University Press.
Griffin, N. (1977). *Relative identity*. Oxford: Clarendon Press.
Gödel, K. (1995). Ontological proof. In S. Feferman, J. W. Dawson, Jr., W. Goldfarb, C. Parsons, & R. M. Solovay (Eds.), *Kurt Gödel collected works* (Vol. III, pp. 403–404). New York and Oxford: Oxford University Press.
Hájek, P. (1996). Magari and others on Gödel's ontological proof. In A. Ursini & P. Agliano (Eds.), *Logic and algebra* (pp. 125–135). New York: Marcel Dekker Inc.

Hájek, P. (2002a). A new small emendation of Gödel's ontological proof. *Studia Logica, 71*, 149–164.
Hájek, P. (2002b). Der Mathematiker und die Frage der Existenz Gottes. In B. Buld et al. (Eds.), *Kurt Gödel—Wahrheit und Beweisbarkeit II* (pp. 325–335). Wien: bv et hpt.
Hughes, G. E., Cresswell, M. J. (1968). *An introduction to modal logic*. London: Methuen and Co LTD.
Lambert, K. (1981). On the philosophical foundations of free logic. *Inquiry, 24*, 147–203.
Lambert, K. (1997). *Free logics: Their fundations, character, and some applications thereof*. Sankt Augustin: Academia Verlag.
Lambert, K. (2003). *Free logic*. Cambridge: Cambridge University Press.
Noonan, H. W. (1980). *Objects and identity: An examination of the relative identity thesis and its consequences*. The Hauge: Martinus Nijhoff Publishers.
Szatkowski, M. (2005). Semantic analysis of some variants of Anderson-like ontological proofs. *Studia Logica, 79*, 317–355.
Szatkowski, M. (2007). Contingent modal semantics for some variants of Anderson-like ontological proofs. *Journal of Applied Non-Classical Logics, 17*, 91–114.
Thomason, R. H. (1970). Some completeness results for modal predicate calculi. In K. Lambert (Ed.), *Philosophical problems in logic* (pp. 56–76). Dordrecht: D. Reidel Publishing Company.