

A GEOMETRICAL CHARACTERIZATION OF THE TWIN PARADOX AND ITS VARIANTS

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ABSTRACT. The aim of this paper is to provide a conceptual analysis of the twin paradox (TwP) within a first-order logic framework. We give a geometrical characterization of TwP and its variants, for example, one without differential aging (No-TwP). It is shown that TwP is not equivalent to the assumption of slowing down of moving clocks and No-TwP is not equivalent to the Newtonian assumption of the absoluteness of time. The connection of TwP and a symmetry axiom of Special Relativity is also studied.

1. INTRODUCTION

Our general aim is to turn spacetime theories into axiomatic theories of first-order logic and exhaustively investigate the relationship between the axioms and the predictions of the theories. We work in the first-order framework of [1], [2]. For reasons why to apply the axiomatic method to spacetime theories see, for example, [7], [16], [18]. For reasons why to stay within first-order logic when dealing with axiomatic foundations see, for example, [3], [2, §Why FOL?], [25], [22].

The twin paradox (TwP) is one of the most famous predictions of special relativity. It concerns a pair of twins. One of them stays at home while the other leaves and returns. The paradox is constituted by the fact that at the event of returning the traveler *twin* turns out to be *younger* than the stay-at-home one. In this paper we concentrate on the relation of TwP to the axioms and other consequences of special relativity. Since the axiom systems used here allows only inertial motions for observers, we formulate the inertial approximation of TwP, which is also called clock paradox in the literature. Logical investigation of the full version of TwP needs more complex mathematical apparatus, see [10], [21]. We also formulate variants of TwP where the stay-at-home twin turns out to be the younger one (Anti-TwP) and where no differential aging takes place (No-TwP).

Unfortunately, it is still not uncommon for people misinterpreting the word 'paradox' to look for contradictions in relativity theory, that is why we think it important to note here that its original meaning is "a statement that is seemingly contradictory and yet is actually true," that is, it has nothing to do with logical contradiction. Having the

nearly century long fruitless debate in view, perhaps it would be better to call the paradoxes of relativity theory simply effects, thus saying twin effect instead of twin paradox, but for the time being it appears to be a hopeless effort to have this usage generally accepted. Anyway, we would like to emphasize that it is absolutely pointless to try to find a logical contradiction in relativity theory, as its consistency is proved, see [2].

In Section 2 we introduce a very basic axiom system \mathbf{Kinem}_0 of kinematics in which no relativistic effect is assumed. \mathbf{Kinem}_0 is a subtheory of Newtonian kinematics and special relativity. In Section 3 we formulate and prove a geometrical characterization of TwP, Anti-TwP and No-TwP each within the models of \mathbf{Kinem}_0 , see Corollary 3.3 and Theorem 3.4. In Sections 4 and 5 we prove some surprising logical consequences of the characterization. In Theorem 4.1 we show that the absoluteness of time (in the Newtonian sense) is not equivalent to that there is no twin paradox (No-TwP). Similarly, in Theorem 5.3 we show that the slowing down of moving clocks is not equivalent to TwP. In Theorem 5.4 we show that a symmetry axiom of special relativity is strictly stronger than TwP.

We try to be as self-contained as possible. First occurrences of concepts used in this work are set in boldface to make them easier to find. We also use colored text and boxes to help the reader to find the axioms, notations, etc. Throughout this work, if-and-only-if is abbreviated to **iff**.

2. A FIRST-ORDER AXIOM SYSTEM OF KINEMATICS

Our basic concepts are explained as follows. Here we deal with kinematics, that is, we with the motion of *bodies*. We represent motion as the changing of spatial location in time. Thus we use reference frames for coordinatizing events (set of bodies). *Quantities* are used for marking time and space. The structure of quantities is assumed to be an *ordered field* in place of the field of real numbers. For simplicity, we associate reference frames with special bodies which we call *observers*. Observations are formulated by means of the *world-view relation*.

In an axiomatic approach to relativity, it is more natural taking relations of bodies (particles) as basic concepts instead of events. That is not uncommon in the literature, see, for example, Ax [3], Benda [4]. However, a large variety of basic concepts occur in the different axiomatizations of special relativity, see, for example, Goldblatt [6], Mundy [11], [12], Pambuccian [13], Rob [14], Suppes [19], Schutz [16], [15], [17].

Using ordered fields in place of the field of real numbers increases the flexibility of the theory and minimizes the amount of mathematical presuppositions. For further motivation in this direction see, for example, Ax [3]. Similar remarks apply to our other flexibility-oriented decisions, for example, to keep the dimension of spacetime as a variable.

Using observers in place of coordinate systems or reference frames is only a matter of didactic convenience and visualization. There are several reasons for using observers (or coordinate systems, or reference frames) instead of a single observer-independent spacetime structure. One is that it helps in weeding out unnecessary axioms from our theories. Nevertheless, we state and emphasize the equivalence between observer-oriented and observer-independent approaches to relativity theory, see, for example, [8, §4.5].

Keeping the foregoing in mind, let us now set up the first-order language of our axiom systems. First we fix a natural number $d \geq 2$ for the dimension of spacetime. We use a two-sorted language: \mathbf{B} is the sort of **bodies** and \mathbf{Q} is the sort of **quantities**. Our language contains the following non-logical symbols:

- unary relation symbol \mathbf{IOb} (for **observers**);
- binary function symbols $+$, \cdot and a binary relation symbol $<$ (for the field operations and the ordering on \mathbf{Q}); and
- a $2 + d$ -ary relation symbol \mathbf{W} (for **world-view relation**).

The variables of sort \mathbf{B} are denoted by m, k, a, b and c ; and those of sort \mathbf{Q} are denoted by p, q, r, x and y .

$\mathbf{IOb}(m)$ is translated as “ m is an observer.” We use the world-view relation \mathbf{W} to speak about coordinatization by translating $\mathbf{W}(m, b, x_1, \dots, x_d)$ as “observer m coordinatizes body b at spacetime location $\langle x_1, \dots, x_d \rangle$,” that is, at space location $\langle x_2, \dots, x_d \rangle$ at instant x_1 .

Body terms are just the variables of sort \mathbf{B} . Quantity terms are the variables of sort \mathbf{Q} and what can be built up from quantity terms by using the field operations. $\mathbf{IOb}(m)$, $\mathbf{W}(m, b, x_1, \dots, x_d)$, $m = b$, $x_1 = x_2$ and $x_1 < x_2$ are the so-called atomic formulas of our first-order language, where m, b, x_1, \dots, x_d can be arbitrary terms of the required sorts. The **formulas** of our first-order language are built up from these atomic formulas by using the logical connectives *not* (\neg), *and* (\wedge), *or* (\vee), *implies* (\implies), *if-and-only-if* (\iff) and the quantifiers *exists* x ($\exists x$) and *for all* x ($\forall x$) for every variable x . To abbreviate formulas of first-order logic we often omit parentheses according to the following convention. Quantifiers bind as long as they can, and \wedge binds stronger than \implies . For example, $\forall x \varphi \wedge \psi \implies \exists y \delta \wedge \eta$ means $\forall x((\varphi \wedge \psi) \implies \exists y(\delta \wedge \eta))$.

We use first-order set theory as a meta theory to be able to speak about model theoretical terms like models, validity, etc. The **models** of this language are of the form

$$\mathfrak{M} = \langle \mathbf{B}, \mathbf{Q}; \mathbf{IOb}_{\mathfrak{M}}, +_{\mathfrak{M}}, \cdot_{\mathfrak{M}}, <_{\mathfrak{M}}, \mathbf{W}_{\mathfrak{M}} \rangle, \quad (1)$$

where \mathbf{B} and \mathbf{Q} are nonempty sets and $\mathbf{IOb}_{\mathfrak{M}}$ is a unary relation on \mathbf{B} , $+_{\mathfrak{M}}$ and $\cdot_{\mathfrak{M}}$ are binary functions and $<_{\mathfrak{M}}$ is a binary relation on \mathbf{Q} , and $\mathbf{W}_{\mathfrak{M}}$ is a relation on $\mathbf{B} \times \mathbf{B} \times \mathbf{Q} \times \dots \times \mathbf{Q}$. Formulas are interpreted in \mathfrak{M} in the usual way.

We formulate each axiom at two levels. First we give an intuitive formulation, then a precise formalization using our logical notation (which can easily be translated into first-order formulas by inserting the first-order definitions into the formalizations). We seek to formulate easily understandable axioms in first-order logic.

We use the notation $\mathbf{Q}^n := \mathbf{Q} \times \dots \times \mathbf{Q}$ (n -times) for the set of all n -tuples of elements of \mathbf{Q} . If $p \in \mathbf{Q}^n$, then we assume that $p = \langle p_1, \dots, p_n \rangle$, that is, $p_i \in \mathbf{Q}$ denotes the i -th component of the n -tuple p . We write $W(m, b, p)$ in place of $W(m, b, p_1, \dots, p_d)$, and we write $\forall p$ in place of $\forall p_1, \dots, \forall p_d$, etc. To abbreviate formulas, we also use bounded quantifiers in the following way: $\forall x \in \varphi \psi$ and $\exists x \in \varphi \psi$ are abbreviated to $\forall x \in \varphi \psi$ and $\exists x \in \varphi \psi$, respectively. For example, we write

$$\forall m \in \mathbf{IOb} \exists b \in \mathbf{B} \exists p \in \mathbf{Q}^d \quad W(m, b, p) \quad (2)$$

instead of

$$\forall m \exists b \exists p \quad \mathbf{IOb}(m) \implies \mathbf{B}(b) \wedge \mathbf{Q}(p_1) \wedge \dots \wedge \mathbf{Q}(p_d) \wedge W(m, b, p) \quad (3)$$

to formulate that every observer observes a body somewhere.

To be able to add, multiply and compare measurements of observers, we provide an algebraic structure for the set of quantities with the help of the following axiom which can be formulated within first-order logic.

AxEOF: The **quantity part** $\langle \mathbf{Q}; +, \cdot, \langle \rangle$ is a Euclidean ordered field (i.e., a linearly ordered field in which positive elements have square roots).

For the first-order definition of linearly ordered field see, for example, [5]. We use the usual field operations $0, 1, -, /, \sqrt{\quad}$ definable within first-order logic. We also use the vector-space structure of \mathbf{Q}^n , that is, if $p, q \in \mathbf{Q}^n$ and $\lambda \in \mathbf{Q}$, then $p + q, -p, \lambda \cdot p \in \mathbf{Q}^n$; the **length** of $p \in \mathbf{Q}^n$ is defined as

$$|p| := \sqrt{p_1^2 + \dots + p_n^2} \quad (4)$$

for any $n \geq 1$, and $\mathbf{o} := \langle 0, \dots, 0 \rangle$ denotes the **origin**.

We need some definitions and notations to formulate our other axioms. \mathbf{Q}^d is called the **coordinate system** and its elements are referred to as **coordinate points**. We use the notations

$$p_\sigma := \langle p_2, \dots, p_d \rangle \quad \text{and} \quad p_\tau := p_1 \quad (5)$$

for the **space component** and for the **time component** of $p \in \mathbf{Q}^d$, respectively.

Our first axiom on observers simply states that each observer thinks that he is stationary in the origin of the space part of his coordinate system.

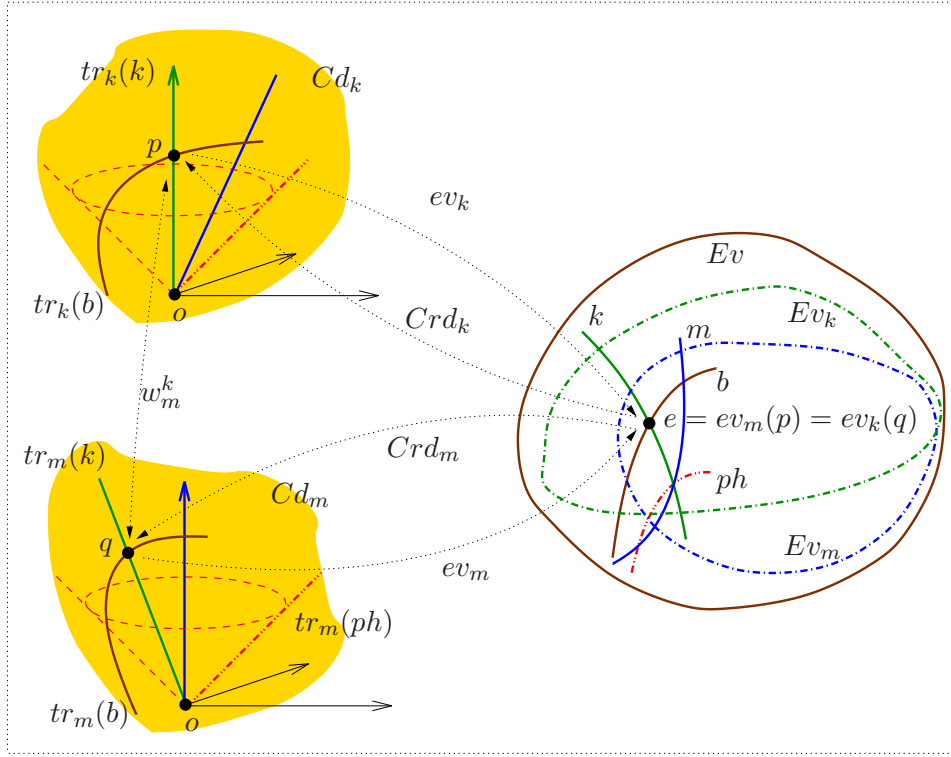


FIGURE 1. Illustration of the basic definitions

AxSelf: An observer observes himself at a coordinate point iff the space component of this point is the origin:

$$\forall m \in \text{IOb} \quad \forall p \in \mathbb{Q}^d \quad W(m, m, p) \iff p_\sigma = o. \quad (6)$$

The **event** (the set of bodies) observed by observer m at coordinate point p is denoted by $ev_m(p)$, that is,

$$ev_m(p) := \{ b \in \mathbb{B} : W(m, b, p) \}, \quad (7)$$

and the **event-function** of m is the function that maps coordinate point p to event $ev_m(p)$. Let Ev_m denote the set of nonempty events coordinatized by observer m , that is,

$$Ev_m := \{ ev_m(p) : ev_m(p) \neq \emptyset \}, \quad (8)$$

and Ev denote the set of all observed events, that is,

$$Ev := \{ e \in Ev_m : m \in \text{IOb} \}. \quad (9)$$

Our next axiom states that the events observed by the observers are the same.

AxEv: Every observer coordinatizes the same events:

$$\forall m, k \in \text{IOb} \quad \forall p \in \mathbb{Q}^d \quad \exists q \in \mathbb{Q}^d \quad ev_m(p) = ev_k(q). \quad (10)$$

We define the **coordinate-function** of observer m , in symbols Crd_m , as the inverse of the event-function, that is,

$$Crd_m := ev_m^{-1}, \quad (11)$$

where $R^{-1} := \{\langle y, x \rangle : \langle x, y \rangle \in R\}$ is the first-order definition of the **inverse** of binary relation R . We note that by this definition, coordinate-functions are only binary relations.

CONVENTION 2.1. Whenever we write $Crd_m(e)$, we mean that there is a unique $q \in \mathbb{Q}^d$ such that $ev_m(q) = e$, and this unique q is denoted by $Crd_m(e)$. That is, if we talk about the value $Crd_m(e)$, we postulate that it exists and is unique (by the present convention).

The **time of event** e according to observer m is defined as

$$\text{time}_m(e) := Crd_m(e)_\tau, \quad (12)$$

and the **elapsed time** between events e_1 and e_2 measured by observer m is defined as

$$\text{time}_m(e_1, e_2) := |\text{time}_m(e_1) - \text{time}_m(e_2)|; \quad (13)$$

$\text{time}_m(e_1, e_2)$ is called the *proper time* measured by m between e_1 and e_2 if $m \in e_1 \cap e_2$. We note that whenever we write time_m , we assume that the events in its argument have unique coordinates by Convention 2.1.

The **coordinate-domain** of observer m , in symbols Cd_m , is the set of coordinate points where m observes something, that is,

$$Cd_m := \{p \in \mathbb{Q}^d : ev_m(p) \neq \emptyset\}. \quad (14)$$

The **world-view transformation** between the coordinate-domains of observers k and m is defined as

$$w_m^k := \{\langle q, p \rangle \in Cd_k \times Cd_m : ev_k(q) = ev_m(p)\}. \quad (15)$$

We note that by this definition, world-view transformations are only binary relations.

CONVENTION 2.2. Whenever we write $w_m^k(q)$, we mean there is a unique $p \in \mathbb{Q}^d$ such that $\langle q, p \rangle \in w_m^k$, and this p is denoted by $w_m^k(q)$.

Let $\mathbf{1}_t := \langle 1, 0, \dots, 0 \rangle$. The **time-unit vector** of k according to m is defined as

$$\mathbf{1}_m^k := w_m^k(\mathbf{1}_t) - w_m^k(o). \quad (16)$$

The **world-line** of body b according to observer m is defined as the set of coordinate points where b was observed by m , that is,

$$wl_m(b) := \{p \in \mathbb{Q}^d : b \in ev_m(p)\}. \quad (17)$$

AxLinTime: The world-lines of observers are lines and time is elapsing uniformly on them:

$$\begin{aligned} \forall m, k \in \text{IOb} \quad & wl_m(k) = \{wl_m(o) + \lambda \cdot \mathbf{1}_m^k : \lambda \in \mathbb{Q}\} \wedge \\ \forall p, q \in wl_m(k) \quad & \text{time}_k(ev_m(p), ev_m(q)) \cdot |\mathbf{1}_m^k| = |p - q|. \end{aligned} \quad (18)$$

Let us collect the axioms introduced so far in an axiom system:

$$\boxed{\text{Kinem}_0 := \{ \text{AxEOF}, \text{AxSelf}, \text{AxLinTime}, \text{AxEv} \}} \quad (19)$$

Let us note that Kinem_0 is a very weak axiom system of kinematics.

3. THE GEOMETRICAL CHARACTERIZATION

To formulate TwP, first we have to formulate the situations in which it can occur. We say that observer m observes observers a, b and c in a **twin paradox situation** at events e, e_a and e_c iff $a \in e_a \cap e$, $b \in e_a \cap e_c$, $c \in e \cap e_c$, $b \notin e$ and $\text{time}_m(e_a) < \text{time}_m(e) < \text{time}_m(e_c)$ or $\text{time}_m(e_a) > \text{time}_m(e) > \text{time}_m(e_c)$, see Figure 2. This situation is denoted by $\text{TwP}_m(\widehat{ac}, b)(e_a, e, e_c)$.

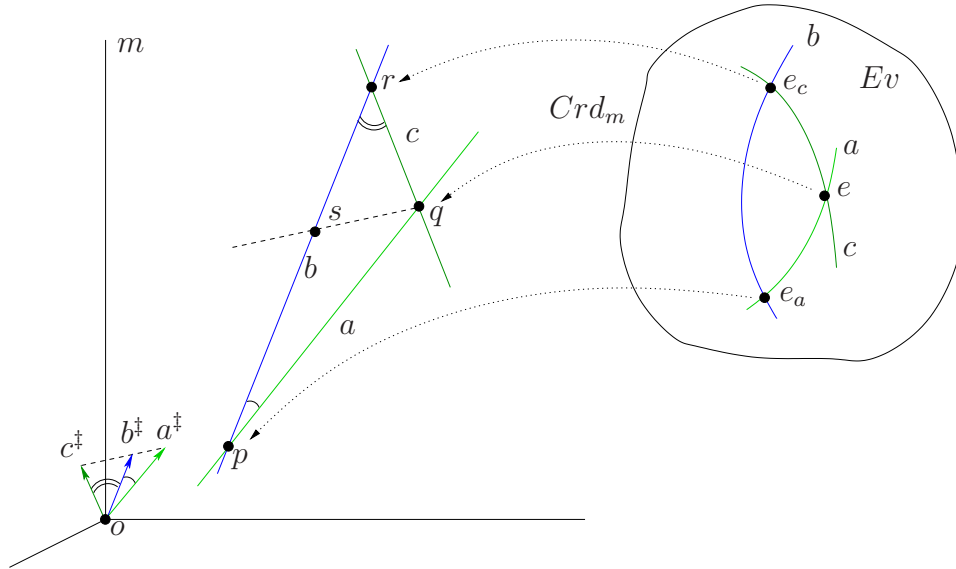


FIGURE 2. Illustration of relation $\text{TwP}_m(\widehat{ac}, b)(e_a, e, e_c)$ and the proof of Proposition 3.2

Let $a, b, c \in \text{IOb}$ and $e_a, e, e_b \in \text{Ev}$. Let $\text{time}(\widehat{ac} < b)(e_a, e, e_b)$ be the abbreviation of $\text{time}_a(e_a, e) + \text{time}_c(e, e_b) < \text{time}_b(e_a, e_b)$. The definitions of $\text{time}(\widehat{ac} = b)(e_a, e, e_b)$ and $\text{time}(\widehat{ac} > b)(e_a, e, e_b)$ are analogous.

Now we are able to formulate the twin paradox in our notations.

TwP: Every observer observes the twin paradox in every twin paradox situation:

$$\forall m, c, a, b \in \text{IOb} \quad \forall e, e_a, e_c \in \text{Ev}_m \quad \text{TwP}_m(\widehat{ac}, b)(e_a, e, e_c) \implies \text{time}(\widehat{ac} < b)(e_a, e, e_c). \quad (20)$$

We define **noTwP** and **antiTwP** by replacing $<$ by $=$ and $>$ in the formula TwP, respectively.

Remark 3.1. For convenience, we quantify over events too. That does not mean that we abandon our first order language. It is just a new abbreviation that simplifies the formalization of our axioms. Instead of events we could speak about observers and spacetime locations. For example, instead of $\forall e \in Ev_m \phi$ we could write $\forall p \in Cd_m \phi[e \rightsquigarrow ev_m(p)]$, where none of $p_1 \dots p_d$ occurs free in ϕ , and $\phi[e \rightsquigarrow ev_m(p)]$ is the formula achieved from ϕ by substituting $ev_m(p)$ for e in all occurrences. Similarly, we can replace $e \in Ev_m$ by $\exists p \in Cd_m e = ev_m(p)$ and $\forall e \in Ev$ by $\forall m \in IOb \forall e \in Ev_m$.

We say that $q \in \mathbb{Q}^d$ is (strictly) **between** $p \in \mathbb{Q}^d$ and $r \in \mathbb{Q}^d$ iff there is $\lambda \in \mathbb{Q}$ such that $q = \lambda p + (1 - \lambda)r$ and $0 < \lambda < 1$. This situation is denoted by **Bw**(p, q, r).

Let $p, q, r \in \mathbb{Q}^d$ and $\mu \in \mathbb{Q}$ such that **Bw**($p, \mu q, r$). In this case we use notations **Conv**(p, q, r) and **Conc**(p, q, r) if $0 < \mu < 1$ and $1 < \mu$, respectively.

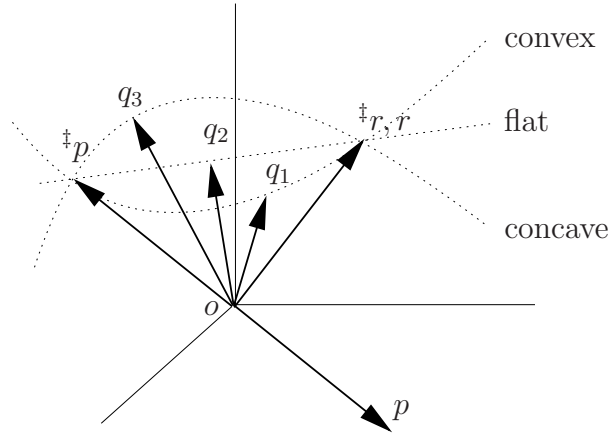


FIGURE 3. Illustration of relations **Conv**(p, q_1, r), **Bw**(p, q_2, r) and **Conc**(p, q_3, r)

For convenience we introduce the following notation:

$$\dagger p := \begin{cases} p & \text{if } p_t \geq 0, \\ -p & \text{if } p_t < 0. \end{cases} \quad (21)$$

Proposition 3.2. Assume **Kinem**₀. Let m, a, b , and c be observers and e, e_a and e_c events such that $\text{Tw}P_m(\widehat{ac}, b)(e_a, e, e_c)$. Then

$$\text{time}(\widehat{ac} < b)(e_a, e, e_c) \iff \text{Conv}(\dagger 1_m^a, \dagger 1_m^b, \dagger 1_m^c) \quad (22)$$

$$\text{time}(\widehat{ac} = b)(e_a, e, e_c) \iff \text{Bw}(\dagger 1_m^a, \dagger 1_m^b, \dagger 1_m^c) \quad (23)$$

$$\text{time}(\widehat{ac} > b)(e_a, e, e_c) \iff \text{Conc}(\dagger 1_m^a, \dagger 1_m^b, \dagger 1_m^c) \quad (24)$$

proof. Let m, a, b , and c be observers and e, e_a and e_c events such that $\text{Tw}P_m(\widehat{ac}, b)(e_a, e, e_c)$. Let us abbreviate the time-unit vectors

$\dagger 1_m^k$ to k^\dagger throughout this proof. Let $p = \text{Crd}_m(e_a)$, $q = \text{Crd}_m(e)$ and $r = \text{Crd}_m(e_c)$. We have that $p \neq r$ since $p_\tau < r_\tau$ or $r_\tau < p_\tau$. Therefore, by **AxLinTime**, the triangle pqr is nondegenerate since $p, r \in \text{wl}_m(k)$ but $q \notin \text{wl}_m(b)$. Let us first show that b measures the same length of time between e_a and e_c as a and c do together if $\text{Bw}(a^\dagger, b^\dagger, c^\dagger)$ holds. Let s be the intersection of line pr and the line parallel to $a^\dagger c^\dagger$ through q , see Figure 2. Then the triangles $oa^\dagger b^\dagger$ and pqs are similar; and the triangles $ob^\dagger c^\dagger$ and rsq are also similar. Thus

$$\frac{|p - q|}{|a^\dagger|} = \frac{|p - s|}{|b^\dagger|} \quad \text{and} \quad \frac{|q - r|}{|c^\dagger|} = \frac{|s - r|}{|b^\dagger|} \quad (25)$$

hold. Thus by **AxLinTime**, we have that

$$\begin{aligned} \left| \text{time}_a(e_a, e) \right| + \left| \text{time}_c(e, e_c) \right| &= \frac{|p - q|}{|a^\dagger|} + \frac{|q - r|}{|c^\dagger|} \\ &= \frac{|p - s| + |s - r|}{|b^\dagger|} = \frac{|r - p|}{|b^\dagger|} = \left| \text{time}_c(e_a, e_c) \right|. \end{aligned} \quad (26)$$

Hence $\text{time}(\widehat{ac} = b)(e_a, e, e_c)$ holds if $\text{Bw}(a^\dagger, b^\dagger, c^\dagger)$. By **AxLinTime**, b measures more (less) time between e_a and e_c iff his time-unit vector is shorter (longer). Thus we get that $\text{time}(\widehat{ac} < b)(e_a, e, e_c)$ holds if $\text{Conv}(a^\dagger, b^\dagger, c^\dagger)$, and $\text{time}(\widehat{ac} > b)(e_a, e, e_c)$ holds if $\text{Conc}(a^\dagger, b^\dagger, c^\dagger)$. The converse implications also hold since one of the relations **Conv**, **Bw** and **Conc** holds for a^\dagger , b^\dagger and c^\dagger , and only one of the relations $\text{time}(\widehat{ac} < b)$, $\text{time}(\widehat{ac} = b)$ and $\text{time}(\widehat{ac} > b)$ can hold for events e_a , e and e_c . This completes the proof of Proposition 3.2. ■

A set $H \subseteq \mathbb{Q}^d$ is called **convex** iff $\text{Conv}(p, q, r)$ for all $p, q, r \in H$ if there is $\mu \in Q$ such that $\text{Bw}(p, \mu q, r)$. We call H **flat** or **concave** if $\text{Conv}(p, q, r)$ is replaced by $\text{Bw}(q, r, p)$ or $\text{Conc}(r, p, q)$, respectively. Let us define the **Minkowski sphere** here as

$$MS_m^\dagger := \{ \dagger 1_m^k : k \in \text{IOb} \}. \quad (27)$$

Now we can state the following corollary of Proposition 3.2.

Corollary 3.3. Assume Kinem_0 . Then

$$\forall m \in \text{IOb} \quad MS_m^\dagger \text{ is convex} \quad \implies \quad \text{TwP}, \quad (28)$$

$$\forall m \in \text{IOb} \quad MS_m^\dagger \text{ is flat} \quad \implies \quad \text{noTwP}, \quad (29)$$

$$\forall m \in \text{IOb} \quad MS_m^\dagger \text{ is concave} \quad \implies \quad \text{antiTwP}. \quad (30)$$

The implications of the above corollary cannot be reversed because there may be observers who are not part of any twin paradox situation. We can solve this problem by using the following axiom to shift observers in order to create twin paradox situations.

AxShift: If an observer observes another observer with a certain time-unit vector, he also observes still another observer with

the same time-unit vector in every coordinate point of his coordinate domain:

$$\forall m, k \in \text{IOb} \quad \forall p \in Cd_m \quad \exists h \in \text{IOb} \quad h \in ev_m(p) \wedge 1_m^k = 1_m^h. \quad (31)$$

Now we can reverse the above implications.

Theorem 3.4. Assume Kinem_0 and AxShift . Then

$$\text{TwP} \iff \forall m \in \text{IOb} \quad MS_m^\ddagger \text{ is convex}, \quad (32)$$

$$\text{noTwP} \iff \forall m \in \text{IOb} \quad MS_m^\ddagger \text{ is flat}, \quad (33)$$

$$\text{antiTwP} \iff \forall m \in \text{IOb} \quad MS_m^\ddagger \text{ is concave}. \quad (34)$$

proof. By Corollary 3.3, we have to prove the “ \implies ” part only. For that, let us take three points from MS_m^\ddagger : a' , b' and c' , such that there is $\mu \in Q$ for which $\text{Bw}(\ddagger a', \mu b', \ddagger c')$. By AxShift there are observers a , b and c in a twin paradox situation such that $1_m^a = a'$, $1_m^b = b'$ and $1_m^c = c'$. Thus from Proposition 3.2 we get that MS_m^\ddagger has the desired property. ■

In the sections below we will use the following concept. Let Σ and Γ be sets of formulas, and let φ and ψ be formulas of our language. Then Σ **logically implies** φ , in symbols $\Sigma \models \varphi$, iff φ is true in every model of Σ . To simplify our notations, we use the plus sign between formulas and sets of formulas in the following way: $\Sigma + \Gamma := \Sigma \cup \Gamma$, $\varphi + \psi := \{\varphi, \psi\}$ and $\Sigma + \varphi := \Sigma \cup \{\varphi\}$.

Remark 3.5. Convexity as used here is not far from convexity as understood in geometry or in the case of functions. For example, in the models of $\text{Kinem}_0 + \text{AxThExp}^+$ or $\text{SecRel}_0 + \text{AxThExp}$, that we are going to introduce below, the Minkowski Sphere MS_m^\ddagger is convex in our sense if and only if the set of points above it, that is $\{p \in Q^d : \exists q \in MS_m^\ddagger \quad p_\tau \geq q_\tau\}$, is convex in the geometrical sense.

Remark 3.6. Let us note that theorem $\Sigma \models \varphi$ is the stronger, the fewer axioms Σ contains, and similarly, $\Sigma \not\models \varphi$ is the stronger, the more axioms Σ contains.

Remark 3.7. All the theorems would remain valid if we replaced \models by \vdash , that is, by the deductibility relation of first-order logic.

4. CONSEQUENCES ON KINEMATICS

First let us investigate the connection of No-TwP and the Newtonian assumption on the absoluteness of time.

AbsTime: Any observer measures the same elapsed time between any two events:

$$\forall m, k \in \text{IOb} \quad \forall e_1, e_2 \in Ev \quad \text{time}_m(e_1, e_2) = \text{time}_k(e_1, e_2). \quad (35)$$

To strengthen our axiom system, we introduce an axiom that ensures the existence of several observers.

AxThExp⁺: Observers can move in any direction at any finite speed:

$$\begin{aligned} \forall m \in \mathbf{IOb} \ \forall p, q \in Q^d \quad p_\tau \neq q_\tau \\ \implies \exists k \in \mathbf{IOb} \quad k \in ev_m(p) \cap ev_k(q). \end{aligned} \quad (36)$$

By the following Theorem, **NoTwP** does not logically imply **AbsTime**. Which is an astonishing fact since it means that we would not been able to conclude that the time is absolute in the Newtonian sense even if there had been no twin paradox in our world.

Theorem 4.1.

$$\mathbf{AxEOF} + \mathbf{AbsTime} \models \mathbf{noTwP}, \text{ but} \quad (37)$$

$$\mathbf{Kinem}_0 + \mathbf{AxThExp}^+ + \mathbf{noTwP} \not\models \mathbf{AbsTime}. \quad (38)$$

proof. Item (37) is easy.

To prove (38), we construct a model of **Kinem₀**, **AxThExp⁺** and **noTwP**, in which **AbsTime** does not hold. Let $\langle \mathbf{Q}; +, \cdot, < \rangle$ be any Euclidean ordered field. Let $B := \mathbf{Q}^d \times \mathbf{Q}^d$. Let $\mathbf{IOb} := \{ \langle p, q \rangle \in B : p_\tau \neq q_\tau \}$. Let

$$MS_{\langle 1, 0 \rangle}^\ddagger := \{ x \in \mathbf{Q}^d : x_\tau - x_2 = 1 \wedge x_\tau > 0 \}. \quad (39)$$

Let $W(\langle 1, 0 \rangle, \langle p, q \rangle, r)$ hold iff r is in line pq . Now the world-view relation is given for observer $\langle 1, 0 \rangle$. For any other observer $\langle p, q \rangle$, let $w_{\langle 1, 0 \rangle}^{\langle p, q \rangle}$ be an affine transformation that takes o to p while its linear part takes 1_t to $MS_{\langle 1, 0 \rangle}^\ddagger \cap \{ \lambda \cdot (p - q) : \lambda \in \mathbf{Q} \}$ and fixes the other basis vectors. From these world-view transformations, it is easy to define the world-view relations of other observers. So the model is given. It is also easy to see that **Kinem₀** and **AxThExp⁺** are true in this model. Since $MS_{\langle 1, 0 \rangle}^\ddagger$ is flat and the world-view transformations are affine ones, it is clear that MS_m^\ddagger is flat for all $m \in \mathbf{IOb}$. Hence **noTwP** is also true in this model by Corollary 3.3. It is easy to see that **AbsTime** implies that $(1_m^k)_\tau = \pm 1$ for all $m, k \in \mathbf{IOb}$. Hence **AbsTime** is not true in this model; and that is what we wanted to prove. ■

5. CONSEQUENCES ON SPECIAL RELATIVITY THEORY

Now we are going to investigate the consequences on special relativity of the characterization. To do so, let us extend our language by a new unary relation **Ph** on **B** for **photons** and formulate an axiom about the constancy of the speed of photons. For convenience, this speed is chosen to be 1.

AxPh₀ : For every observer, there is a photon through two coordinate points p and q iff the slope of $p - q$ is 1:

$$\begin{aligned} \forall m \in \text{IOb} \quad \forall p, q \in \mathbb{Q}^d \quad |p_\sigma - q_\sigma| &= |p_\tau - q_\tau| \\ \iff \text{Ph} \cap \text{ev}_m(p) \cap \text{ev}_m(q) &\neq \emptyset. \end{aligned} \quad (40)$$

Let us introduce a weakened axiom system of special relativity:

$$\boxed{\text{SpecRel}_0^d := \{ \text{AxEOF}, \text{AxSelf}, \text{AxPh}_0, \text{AxEv} \}} \quad (41)$$

We note that if $d \geq 3$, SpecRel_0^d is strong enough to prove the most important predictions of special relativity such as that moving clocks get out of synchronism, see, for example, [1]. However, SpecRel_0^d is also weak enough not to prove every prediction of special relativity. For example, it does not prove TwP or the slowing down of relatively moving clocks. Thus it is possible to compare these predictions within the models of SpecRel_0^d . To investigate the logical connection between them, let us formulate the slowing down effect on moving clocks within our first-order logic framework.

SlowTime : Relatively moving observers' clocks slow down:

$$\forall m, k \in \text{IOb} \quad w_{l_m}(k) \neq w_{l_m}(m) \implies |(1_m^k)_\tau| > 1. \quad (42)$$

To prove a theorem about the logical connection between **SlowTime** and **TwP**, we need some definitions and theorems. A map $\tilde{\varphi} : \mathbb{Q}^d \rightarrow \mathbb{Q}^d$ is called **field-automorphism-induced map** iff there is an automorphism φ of the field $\langle \mathbb{Q}, \cdot, + \rangle$ such that $\tilde{\varphi}(p) = \langle \varphi(p_1), \dots, \varphi(p_d) \rangle$ for every $p \in \mathbb{Q}^d$.

Theorem 5.1. Let $d \geq 3$. Let $m, k \in \text{IOb}$. Then

- (1) If **AxPh₀** and **AxEv** are assumed, w_m^k is a Poincaré transformation composed by a dilation D and a field-automorphism-induced map $\tilde{\varphi}$.
- (2) If **AxPh₀**, **AxEv** and **AxSymDist** (defined below) are assumed, w_m^k is a Poincaré transformation.

On the proof. It is not hard to see that **AxPh₀** and **AxEv** imply that w_m^k is a bijection from \mathbb{Q}^d to \mathbb{Q}^d that preserves lines of slope 1. Hence Item (1) is a consequence of the Alexandrov-Zeeman theorem generalized for fields, see, for example, [23], [24].

Now let us see why Item (2) is true. By Item (1), it is easy to see that there is a line l such that both l and its w_m^k image are orthogonal to the time-axis. Thus by **AxSymDist**, w_m^k restricted to l is distance preserving. Consequently, both dilation D and field-automorphism-induced map $\tilde{\varphi}$ in Item (1) has to be the identity map. Hence w_m^k is a Poincaré transformation. ■

Lemma 5.2. Let $d \geq 3$. Assume **AxEOF**, **AxPh₀**, **AxEv** and **AxLinTime**. Let $m, a, b, c \in \text{IOb}$ and let $e_a, e, e_b \in \text{Ev}$. Then

$$\text{Tw}P_m(\widehat{ac}, b)(e_a, e, e_c) \iff \text{Tw}P_b(\widehat{ac}, b)(e_a, e, e_c). \quad (43)$$

proof. By (1) of Theorem 5.1, **AxEOF**, **AxPh₀** and **AxEv** imply that w_m^b is a composition of a Poincaré transformation, a dilation and a field-automorphism-induced map. By **AxLinTime**, the field-automorphism is trivial. Hence $\text{time}_m(e)$ is between $\text{time}_m(e_a)$ and $\text{time}_m(e_c)$ iff $\text{time}_b(e)$ is between $\text{time}_b(e_a)$ and $\text{time}_b(e_c)$. This completes the proof since the other parts of the definition of **TwP** do not depend on observers m and b . ■

We have to weaken **AxThExp⁺** since **SpecRel₀^d** implies the impossibility of faster than light motion of observers if $d \geq 3$, see, for example, [1].

AxThExp: Observers can move in any direction at any speed less than 1, that is, less than the speed of light:

$$\begin{aligned} \forall m \in \text{IOb} \quad \forall p, q \in Q^d \quad |p_\sigma - q_\sigma| < |p_\tau - q_\tau| \\ \implies \exists k \in \text{IOb} \quad k \in \text{ev}_m(p) \cap \text{ev}_k(q). \end{aligned} \quad (44)$$

The following theorem shows that the slowing down of moving clocks (**SlowTime**) is not logically equivalent to **TwP**.

Theorem 5.3. Let $d \geq 3$. Then

$$\text{SpecRel}_0^d + \text{AxLinTime} + \text{SlowTime} \models \text{TwP}, \text{ but} \quad (45)$$

$$\text{SpecRel}_0^d + \text{AxLinTime} + \text{AxThExp} + \text{TwP} \not\models \text{SlowTime}. \quad (46)$$

proof. Item (45) is clear by Lemma 5.2.

To prove Item (46), let us construct a model of **SpecRel₀^d**, **AxLinTime**, **AxThExp** and **TwP**, in which **SlowTime** does not hold. Let $\langle \mathbb{Q}; +, \cdot, < \rangle$ be any Euclidean ordered field. Let $B := \mathbb{Q}^d \times \mathbb{Q}^d$. Let $\text{IOb} := \{ \langle p, q \rangle \in B : |p_\sigma - q_\sigma| < |p_\tau - q_\tau| \}$. It is easy to see that there is a convex subset M of \mathbb{Q}^d such that $1_t \in M$ and $|p_\tau| < 1$ for some $p \in M$. Let $MS_{\langle 1, 0 \rangle}^\ddagger$ be such a convex subset of \mathbb{Q}^d . Let $W(\langle 1, 0 \rangle, \langle p, q \rangle, r)$ hold iff r is in line pq . Now the world-view relation is given for observer $\langle 1, 0 \rangle$. For any other observer $\langle p, q \rangle$, let $w_{\langle 1, 0 \rangle}^{\langle p, q \rangle}$ be such a composition of a Lorentz transformation, a dilation and a translation which takes o to p while its linear part takes 1_t to $MS_{\langle 1, 0 \rangle}^\ddagger \cap \{ \lambda \cdot (p - q) : \lambda \in \mathbb{Q} \}$ and fixes the other basis vectors. It is easy to see that there is such a transformation. From these world-view transformations, it is easy to define the world-view relations of the other observers. So the model is given. It is also easy to see that **SpecRel₀^d**, **AxLinTime** and **AxThExp⁺** are true in this model. Since $MS_{\langle 1, 0 \rangle}^\ddagger$ is convex and the world-view transformations are affine ones, it is clear that MS_m^\ddagger is convex for all $m \in \text{IOb}$. Hence

TwP is also true in this model by Corollary 3.3. It is easy to see that SlowTime is not true in this model since there is $p \in MS_{(1,0)}^\ddagger$ such that $|p_\tau| < 1$; and that is what we wanted to prove. \blacksquare

To see one more consequence of our characterization, we introduce a symmetry axiom called the symmetric distance axiom.

AxSymDist: If events e_1 and e_2 are simultaneous for both the observers m and k , then m and k agree as for the spatial distance between e_1 and e_2 :

$$\begin{aligned} \forall m, k \in \text{IOb} \quad \forall e_1, e_2 \in \text{Ev} \quad \text{time}_m(e_1, e_2) = \text{time}_k(e_1, e_2) = 0 \\ \implies \text{dist}_m(e_1, e_2) = \text{dist}_k(e_1, e_2), \end{aligned} \quad (47)$$

where $\text{dist}_m(e_1, e_2)$ is an abbreviation of $|\text{Crd}_m(e_1)_\sigma - \text{Crd}_m(e_2)_\sigma|$, and it is called the **spatial distance** between events e_1 and e_2 according to observer m .

Like the similar results of [20] and [21], the following theorem also answers Question 4.2.17 of Andr eka–Madar asz–N emeti [2]. It shows that TwP is not logically equivalent to the symmetric distance axiom of SpecRel.

Theorem 5.4. Let $d \geq 3$. Then

$$\text{SpecRel}_0^d + \text{AxSymDist} \models \text{TwP}, \text{ but} \quad (48)$$

$$\text{SpecRel}_0^d + \text{AxThExp} + \text{AxLinTime} + \text{TwP} \not\models \text{AxSymDist}. \quad (49)$$

proof. By (2) of Theorem 5.1, SpecRel_0^d and AxSymDist imply that w_m^k is a Poincar e transformation for all $m, k \in \text{IOb}$. Hence

$$MS_m^\ddagger \subseteq \{p \in \mathbb{Q}^d : p_\tau^2 - |p_\sigma|^2 = 1 \wedge p_\tau > 0\}. \quad (50)$$

Consequently, MS_m^\ddagger is convex. So by Corollary 3.3, TwP follows from SpecRel_0^d and AxSymDist .

Since SpecRel_0^d and AxSymDist imply SlowTime if $d \geq 3$, Item (48) follows from Theorem 5.3. \blacksquare

It is interesting that AxSymDist and SlowTime are equivalent in the models of SpecRel_0^d (and some auxiliary axiom) if the quantity part is the field of real numbers. However, the assumption that the quantity part is the field of real numbers cannot be formulated in any first-order language of spacetime theories. Thus this equivalence cannot be derived from first-order axioms in our language either.

Theorem 5.5. Assume SpecRel_0^d , AxThExp , AxLinTime , AxShift , and that \mathbb{Q} is the field of real numbers. Then

$$\text{SlowTime} \iff \text{AxSymDist}. \quad (51)$$

For proof of Theorem 5.5, see [21, §3].

This theorem is interesting because it shows that assuming only that the moving clocks slow down in some degree implies the exact ratio of the slowing down of moving clocks (since $\text{SpecRel}_0 + \text{AxSymDist}$ implies the Poincaré group, see Theorem 5.1).

Question 5.6. Does Theorem 5.5 retains its validity if the assumption that \mathbb{Q} is the field of real numbers is removed? If not, is it still possible to replace it by a first-order assumption, for example, by axiom schema IND used in [9], [10], [21]?

6. CONCLUDING REMARKS

We have seen that (the inertial approximation of) TwP can be characterized geometrically within a weak axiom system of kinematics. We have seen some consequences of this characterization, among others, that TwP is logically weaker than the assumption of slowing down of moving clocks or the AxSymDist axiom of special relativity. A future task is to explore the logical connections between other assumptions and predictions of relativity theories. For example, in [10] and [21], $\text{SpecRel}_0^d + \text{AxSymDist}$ is extended to axiom system AccRel that logically implies the accelerated version of TwP, but the natural question below raised by Theorem 5.4 has not been answered yet.

Question 6.1. Is it possible to weaken AxSymDist to TwP in AccRel without losing the accelerated version of TwP as a consequence? See Question 3.8 in [10].

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