Meaningless Divisions

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Abstract

In this article we revisit a number of disputes regarding significance logics—i.e., inferential frameworks capable of handling *meaningless*, although grammatical, sentences—that took place in a series of articles most of which appeared in the Australasian Journal of Philosophy between 1966 and 1978. These debates concern (i) the way in which *logical consequence* ought to be approached in the context of a significance logic, and (ii) the way in which the *logical vocabulary* has to be modified (either by restricting some notions, or by adding some vocabulary) to keep as much of Classical Logic as possible. Our aim is to show that the divisions arising from these disputes can be dissolved in the context of a novel and intuitive proposal that we put forward.

1 Introduction

From 1966 to 1978, a group of philosophers among which were Leonard Goddard, Richard Routley (later Richard Sylvan), Karel Lambert, Richard Haack, Ross Brady and Michael Bradley took part on a series of debates—incarnated in a number of articles most of which appeared in the Australasian Journal of Philosophy—concerning the legitimacy, the definition, and the appropriate resources needed to build *significance logics*. Systems of this kind are intended to be inferential frameworks that allows to handle *meaningless*, although grammatical, sentences. Items of this sort were taken by some of these philosophers to express neither a truth nor a falsehood, that is, they were taken as sentences completely lacking a truth-value—i.e., as a kind of truth-value gaps.

Motivation to admit sentences of this sort allegedly came from various corners of the philosophical landscape. Some thought that Russell's type theory rendered sentences violating its construction rules as meaningless, and so believed, too, those who were focused on category mistakes in the context of Ryle's philosophy of language. Others claimed that semantic paradoxes and the paradoxes of vagueness represented genuine cases of nonsignificance, independently of being counted as type or category mistakes by some plausible account. Additionally, there were those who thought that unverifiable sentences, classified as such by the work of the logical positivists, were deprived of meaning in this very same way.

Convinced of the fruitfulness of constructing a system allowing for reasoning with both meaningful and meaningless sentences—i.e. of building a significance logic—some of these scholars faced the task of designing proper technical means to formalize their ideas. In what pertains to the semantics of their target systems, this required not only making precise the way in which the usual logical connectives were supposed to work when meaningless sentences were around, but also choosing an appropriate notion of logical consequence for the resulting formalism. Both of these tasks were (knowingly or not) carried out against the backdrop of the work by Dmitri Bochvar and Sören Halldén on the so-called *logics of nonsense*.

The work done by this pair of philosophers embodies two of the main alternatives used to define *logical consequence* when working with many-valued non-classical systems. That is, siding either with truth-preservation from premises to conclusion (i.e., Bochvar's choice) or with non-falsity-preservation from premises to conclusion (i.e., Halldén's option). As is well-known, these two accounts need not coincide in non-classical environments—especially when truth-value gaps are admitted—and, for this very reason, they do not produce the same result for significance logics. Early in the discussion it was noticed that both alternatives had substantially undesirable consequences. Particularly, while Bochvar's alternative implied the lack of those tautologies present in Classical Logic, Halldén's option implied the invalidity of some inferences very dear to Classical Logic.

One way in which the participants of this discussion settled these differences—between those formulae they wished were logical truths and those which actually were logical truths in their systems, and between those inferences they wished were valid and those which were actually valid in their systems—was through the introduction of certain subtleties concerning the *logical vocabulary*. Indeed, when working with significance logics it became fairly common to distinguish an internal and an external vocabulary allowing, e.g., for two ways to negate a formula φ, as well as two ways to conjoin and disjoin formulae. For instance, while for some of these scholars saying "not φ" of a meaningless sentence resulted in an equally nonsignificant sentence, it so happened that saying "it is not true that φ" (thereby claiming that φ was non-truth-apt) resulted in a true sentence, nevertheless. Thus, the division between internal and external language allowed logicians to keep all the crown jewels. Although there were certain logical truths that were absent and certain valid inferences that were not properly ruled so, their thought was that inasmuch as the attention was restricted to formulae constructed with the appropriate vocabulary, these problems faded away.

In this article, we would like to show that these divisions (truth-preservation versus non-falsity-preservation, internal versus external vocabulary) are unnecessary. That in order to have a significance logic complying with the intuitive desiderata, enjoying all the advantages that we want from it and suffering none of the disadvantages we want to avoid, there is no need to opt for any of these dichotomous options. We will argue that there are alternative and more fruitful definitions of logical consequence that even in the context of significance logics render a system which has all the logical truths and the valid inferences of Classical Logic. We will claim, additionally, that we are not forced to choose between truth- or non-falsity-preservation, and that it is not mandatory to choose between expressing the key notions represented by the logical connectives in either an internal or an external language. The former, because both notions are going to be represented within our system in a qualified sense; the latter, because those inferences valid in our resulting system will be the same no matter which of these languages we employ.

For these purposes, this article is structured as follows. In Section 1, we discuss the reasons presented by the advocates of significance logics to endow these systems with some peculiar three-valued semantics. In Section 2, we review the main alternatives employed to understand logical consequence in the context of significance logics, paying special attention to each alternative's advantages and disadvantages. In Section 3, we revisit the strategy of appealing to various divisions and restrictions of the logical vocabulary in order to overcome the shortcomings of understanding validity either as truth-preservation or as non-falsity-preservation. Section 4 is where we build our own positive proposal. First, we prove that there is a perfectly cogent formal way of understanding logical consequence in significance logics that renders Classical Logic in all relevant respects, and which does not fall into the dichotomous alternatives discussed so far. Secondly, we show that adopting our system allows for the dissolution of the truth versus non-falsity divide, since both options are represented within our definition of logical consequence—in a sense that we will opportunely precise. Thirdly, we show that embracing this novel option makes the division of internal and external languages unnecessary, therefore dissolving the need for such a distinction. Finally, Section 5 outlines some concluding remarks.

2 Need for nonsense

The debate that took place in the articles to which we previously referred, concerning the legitimacy of working with significance logics, had two main opposing parties. On one side were those (Goddard, Routley, Brady) supporting the legitimacy of such an enterprise, believing there were enough reasons to develop a non-classical framework in which meaningless—and, thus, non-truth-valued—sentences could be hosted. On the other side were those (Lambert, Haack, Bradley) who did not share that view, either because they thought that meaningless sentences should remain outside of the dominions of logic, or because they thought that meaningless sentences could be somehow treated as truth-valued.

The former claimed the need for nonsense was justified in a number of different theories in philosophy of mathematics, logic, philosophy of language, and philosophy of science that had an implicit or explicit criterion of significance rendering intuitively grammatical but meaningless indicative sentences as rightfully existent. Russell's type theory was one of the places were most of these scholars found motivation to entertain nonsignificant sentences, as was also Ryle's philosophy of language—especially his considerations on category mistakes. An acknowledgment of these sources of inspiration, together with an assessment of the different ways in which they treated nonsense can be seen, for example, in the following quote from an article by Goddard:

> Thus, expressions such as 'The rainbow is happy', 'Mary kissed my thoughts' and 'Saturdays love Mary' are absurd for Ryle. They are absurd because individuals belong not only to different natural kinds but also to different logical kinds, and the predicates which are significant over the individuals of one logical kind (or category) may not be significant over the individuals of another logical kind. (...) Within Russell's theory, however, each of these expressions has to be taken as significant, i.e., as true-or-false. (...) Since, therefore, all individuals are of the same logical kind (type, or category), it follows that a predicate which is significant over any one individual is significant over all (...) There are, besides, other, more puzzling features of type-theory, since there are statements, such as 'The class of men is not a man', which must be true if the theory is true (...) but which nevertheless turn out to be meaningless in terms of the theory. [18, pp. 139-140]

An additional source of inspiration for considering nonsignificance as an authentic semantic category was the philosophy of the logical positivists—as claimed, e.g., in [5]. Especially, the idea that empirical verifiability could be seen as a criterion of significance according to which empirically unverifiable sentences were, properly speaking, cases of nonsense masquerading as sense. This applies, according to the aforementioned school of thought, to rather theological sentences like 'God is in Heaven', but also to metaphysical sentences such as 'The Absolute is eternal'.

In all fairness, that meaningless was a legitimate semantic category was defended long before arguments along these lines have been written. During the first half of the twentieth century, Dmitri Bochvar first, and Sören Halldén later argued that pathological phenomena like Russell's Paradox, the Liar Paradox, and the Sorites Paradox were constituted by properly speaking meaningless sentences, motivating them to develop their own *logics of nonsense* in [2] and [25], respectively.

In this respect, there was enough agreement among the philosophers favoring the significance logics project that the inclusion of meaninglessness, i.e., neither true nor false, sentences required an appropriate adjustment of the way in which the semantics for the usual logical connectives work. In other words, once nonsignificance was allowed, it remained to be seen how significant and nonsignificant sentences interacted with each other when coupled through the commonly employed logical connections—negation, conjunction, disjunction, and so on. Consensus was reached, in this regard, that a reasonable desiderata for this task was what Goddard and Routley came to refer to as the *Principle of Component Homogeneity*: the idea that whenever a compound sentence has a meaningless component, the compound itself is nonsignificant (see [21, pp. 260-261] and [19]).

The previous philosophical inspirations for allowing meaningless sentences seemed to agree with this. For example, these scholars tried to sustain the claim that the logical positivists advocated for something of this sort in ways like the following, taken from Goddard and Routley's book:

> It seems in any case apparent that the positivists were committed to the classical significance connectives by their commitment to the criterion of verification as a criterion of significance (...) Consider, first, simple indicative sentences without quantifiers, e.g. (1) The Absolute is green (2) Snow is green. If (1) is not empirically verifiable, then (3) The Absolute is not green is not empirically verifiable. For if it were, we could assess (3) by observation, experiment, etc., as true or false. In this case, however, (1) is false or true, as the case may be, so contradicting the assumption (...) Again, consider the conjunction (1) and (2). If (1) is not empirically verifiable, neither is (1) and (2), while if (1) and (2) is not, either (1) is not or (2) is not. Hence, in general, we may say: $P\&Q$ is empirically verifiable if, and only if, P is empirically verifiable and Q is empirically verifiable. That is, by [adopting empirical verifiability as a criterion of significance], $P\&Q$ is significant if, and only if, P is significant and Q is significant. And similarly for disjunction. [21, p. 263]

Furthermore, these authors claim, if Russell's theory of types is properly understood, then sentences involving type-equivocations such as 'The class of all humans is a human' should be as nonsignificant as 'The class of all humans is not a human'. According to them, Russell appears to adhere to the idea according to which conditional statements of the form $\varphi \supset \psi$ are significant if and only if both φ and ψ are significant—as can be seen in his [36, p. 45].[1]

Thus, accepting the Principle of Component Homogeneity had clear and crucial effect in formulating proper semantics for negation, conjunction, and disjunction in the context of significance logics. The idea was for these connectives to retain their usual truth- and falsity-conditions if and only if all their components were meaningful, i.e., either true or false. Thus, if truth and falsity are conceived as two distinct truth-values—**t** and **f**, respectively—while meaninglessness or nonsignificance is taken as a third semantic category—represented by the value **n**, below—these ideas render the following operations, represented by the so-called weak Kleene truth-tables of [26], that is:[2]

[1] Notice that if the usual definitions for conjunction and disjunction in terms of negation and the conditional are employed, it is straightforward to observe the point made by Goddard, Routley and others that Russell himself embraced the Principle of Component Homogeneity.

[2] The fact that **n**, as an algebraic element, behaves according to the Principle of Component Homogeneity describes it as *infectious*—using the terminology of, e.g., [32], [13] and [40]. By these we mean, in a few words, that if a formula φ is assigned **n**, any formula in which φ occurs is so assigned, too.

	$\neg$
$\mathbf{t}$	$\mathbf{f}$
$\mathbf{n}$	$\mathbf{n}$
$\mathbf{f}$	$\mathbf{t}$

$\wedge$	$\mathbf{t}$	$\mathbf{n}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{n}$	$\mathbf{f}$
$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{n}$	$\mathbf{f}$

$\vee$	$\mathbf{t}$	$\mathbf{n}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{n}$	$\mathbf{t}$
$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{n}$	$\mathbf{f}$

Figure 1: The weak Kleene truth-tables

However, even if this very important clarifications are in place, it takes little effort to observe that there is still some work to be done if the aim is to build an actual significance *logic*. As Brady and Routley have put it:

> Hence, to cope with the matter, a non-standard value, non-significance, should be introduced as part of the semantics of formal systems, and these systems should be constructed so as to be able to assess the validity of arguments involving significance or non-significance. [5, p. 223]

What is still missing, then, is an appropriate account of validity—i.e., a systematic way of telling good inferences from bad inferences in a context where reasoning with potentially meaningless sentences is to be carried out. Making this call is, in fact, what led to the first division among those working on significance logics. To discussing it we now turn.

3 Lines in the sand

3.1 The dispute over logical consequence

While discussing significance logics, the focus has been mainly placed on two well-known alternatives to define logical consequence in the context of many-valued systems, which we will detail shortly. Incidentally—although this was only partially acknowledged in the debate—these options embody the alternatives embraced by early works on these issues, due to Bochvar [2] and Halldén [25].

For what it is worth, most (though not all) of the discussion was carried out under the implicit assumption that defining validity amounts to choosing which truth-values are *designated*, i.e., which truth values should be preserved from premises to conclusion. This is in line with the analysis of logical systems as induced by logical *matrices*, in the sense of pairs $\langle \mathbf{A}, D\rangle$ of an algebra $\mathbf{A}$ and a set of designated values D, i.e., a distinguished subset of the universe of the algebra. In this vein, the elements of the algebra in question are usually referred to as 'truth-values', while its operations are commonly represented in the form of the so-called 'truth-tables'—just like at the end of §2, above. Thus, given a propositional language $\mathcal{L}$ and a corresponding formula algebra $\mathbf{FOR}(\mathcal{L})$ defined as usual, a valuation v is an homomorphism from $\mathbf{FOR}(\mathcal{L})$ to $\mathbf{A}$, where both algebras are of the same type, and $FOR(\mathcal{L})$ is the universe of $\mathbf{FOR}(\mathcal{L})$.

Finally, a logical matrix $\mathbf{M} = \langle \mathbf{A}, D\rangle$ induces a consequence relation $\vDash_{\mathbf{M}}$ as follows, where $\Gamma \subseteq FOR(\mathcal{L})$ and $\varphi \in FOR(\mathcal{L})$:

$$\Gamma \vDash_{\mathbf{M}} \varphi \Longleftrightarrow \text{ for every valuation } v\text{: if } v[\Gamma] \subseteq D, \text{ then } v(\varphi) \in D$$

During the past century, the discussion of which is the preferable significance logic was mostly centred on determining which logical matrix should be chosen to work with. However, other options not appealing to logical matrices were discussed, for example by Goddard and Routley, which we will present and discuss in the next section.[3]

The first account of logical consequence for significance logics that we will review is, definitely, the one that appears to have persuaded most of the people involved in the discussion. It is no other than the most widespread conception of logical consequence as concerns the contemporary study of Classical (propositional) Logic—CL, for short. With this we refer to the idea that whenever the premises are true, the conclusion must also be true. In other words, that *truth-preservation* is the right choice. Technically speaking, this will amount to $\mathbf{t}$ being the only designated value, because $\mathbf{f}$ represents falsity and $\mathbf{n}$ represents meaninglessness, a sort of truth-value gap.

Interestingly, as central and as independently motivated this point of view might seem, in the course of the works on significance logics that we are analyzing, most of the reasons presented in its favor were rather indirect. Both Brady and Routley [5, p. 219] and Routley and Goddard [21, p. 273] argued that just $\mathbf{t}$ should be designated. Their argument relies on the assumption that we ought to be committed to and that we ought to assert those formulae which are theorems of our logic (that is, those sentences which receive a designated value in every valuation). Thus, the latter pair says:

> It might perhaps seem obvious that $\mathbf{t}$ and only $\mathbf{t}$ can be adopted as a designated value. For clearly $\mathbf{f}$ cannot be taken as designated without undermining the whole purpose of the logic since this would commit us to the adoption of theses which take the value $\mathbf{f}$ for some or all assignments of values to the components [21, p. 273]

Although the quote goes on to extend these considerations to also apply to the case where the truth-value $\mathbf{n}$ is designated, we will postpone the analysis of this option for now in order to dissect what has been regarded as the key disadvantage of the truth-preservation alternative. This is no other than the fact (whose realization is unanimously credited to Presley in the course of these articles) that if the conception is retained that:

> a schema is valid if and only if it comes out true for all interpretations of its letters (...) some schemata that are valid in the classical

[3] Additionally, it shall be noted that in the early stages of the discussion revolving around significance logics the conversation was inclined towards the linguistic notion of *implication* rather than the meta-linguistic notion of *entailment*. This being said, many of the considerations made regarding implications were later exported to the analysis of entailment and its features—probably because the former was intended to formalize expressions of the form 'φ, therefore ψ'.

> propositional calculus will be invalid in the three-valued calculus. [31, pp. 225-234]

Whereas, in the present case, not only *some* schemata that are valid in Classical Logic will come out as invalid, but given the Principle of Component Homogeneity, *all* such schemata will come out as invalid. Thus, a significance logic built along the previous lines whose notion of validity is that of truth-preservation will have no tautologies—and, therefore, it will *not share any of the tautologies characteristic of Classical Logic.*

In order to make this clear, let us look at a significance logic thus defined. We will start with a propositional language $\mathcal{L}$ counting with the usual connectives $\neg, \wedge, \vee$, and we will consider the algebra $\mathbf{WK}$ whose universe is formed by the set of the three discussed truth-values $\{\mathbf{t}, \mathbf{n}, \mathbf{f}\}$ and whose operations are just as described in the weak Kleene truth-tables. Thus, above a valuation from $\mathbf{FOR}(\mathcal{L})$ to $\mathbf{WK}$ is a weak Kleene valuation (WK-valuation, hereafter), and a valuation from $\mathbf{FOR}(\mathcal{L})$ to its $\{\mathbf{t}, \mathbf{f}\}$-subalgebra (that is, the two-element Boolean algebra) is a Boolean valuation.

With these items we can build the matrix $\langle \mathbf{WK}, \{\mathbf{t}\}\rangle$, having solely $\mathbf{t}$ as its designated value. The logic induced by this matrix is a significance logic commonly referred to as weak Kleene logic, and usually denoted by $\mathsf{K}_3^{\mathsf{w}}$—see, e.g., [24]. Historically speaking, it appears accurate to say that Bochvar was the first to work with this system, as it represents the so-called 'internal' fragment of his significance logic developed in [2]. With regard to $\mathsf{K}_3^{\mathsf{w}}$, it is straightforward to prove the following as done, for instance, in [2, p. 94] and [21, pp. 273-274].

Observation 1. *The logic $\mathsf{K}_3^{\mathsf{w}}$ has no tautologies, i.e., there is no φ such that $\vDash_{\mathsf{K}_3^{\mathsf{w}}} \varphi$.*

The second account of logical consequence for significance logics that we will review in this section represents, in itself, a way to solve the inconveniences of the first account. It was mentioned a couple of times, by Goddard, Haack and Goddard and Routley, but it is really the early work of Halldén [25] which incarnated this alternative for the first time. It consists in nothing more than the idea that whenever the premises are not false, the conclusion must also be not false. Put differently, it amounts to *non-falsity-preservation.* Technically, speaking this requires taking not only $\mathbf{t}$ as a designated value, but also $\mathbf{n}$ as such.

It is important to highlight that this option was mainly motivated by indirect reasons. Thus, it is discussed by Haack in [22, p. 76] as a solution to the conundrum presented by Presley, and indeed as an option which allowed to regain all the tautologies of Classical Logic. In a similar vein, Goddard says that "if a formula expresses a logical law, if, and only if, it does not come out false for any values of the variables (but may be either true or non-significant) then all the usual laws of [Classical Logic] are 'laws' of the three-valued system so constructed" [19, p. 240]. Halldén himself only devised a logical matrix for a logic of this sort, because he was puzzled by intuitively incorrect arguments going from non-false premises to false premises.

Be that as it may, it is true that a significance logic built along these lines will help retain the classical tautologies. To make this precise enough, let us consider again a formal system so defined. We will start with the propositional language $\mathcal{L}$ and we will consider the matrix $\langle \mathbf{WK}, \{\mathbf{t}, \mathbf{n}\}\rangle$, having both $\mathbf{t}$ and $\mathbf{n}$ as its designated values, letting $\mathbf{WK}$ be the weak Kleene algebra, as before. The logic induced by this matrix is a significance logic commonly referred to as Paraconsistent weak Kleene logic, and usually denoted by PWK—see, e.g., [3]. Historically speaking, Halldén was the first to work with this system, as it represents the so-called 'internal' fragment of his significance logic developed in [25]. With regard to PWK, it is straightforward to prove the following as done, e.g., in [25, pp. 48-49] and [21, p. 300].

Observation 2. *The logic* PWK *has the same tautologies that* CL*, i.e.,* $\vDash_{\mathsf{PWK}} \varphi$ *iff* $\vDash_{\mathsf{CL}} \varphi$*.*

This advantage notwithstanding, some scholars pointed out a few problems with this option.[4] On the one hand, Goddard and Routley, and more vociferously Brady and Routley, underlined the highly unintuitive choice of designating the meaninglessness value $\mathbf{n}$, given their previously commented adherence to the idea that we must commit ourselves, and that we ought to assert the sentences which receive a designated value in every valuation—a view referred to as assertion-designation harmony in [7]. This conception appears to straightforwardly lead to the acceptability of committing ourselves, or of asserting logical nonsense—see, e.g., [5, p. 219]. Halldén himself appears to have anticipated such allegations, when he says that:

> a formula is to be taken as asserting something only about those values of which it can meaningfully assert something. The formula is true if the property or relation it asserts applies to all those values of which it can be meaningfully asserted. [25, p. 47]

This response is taken by Ferguson [14, p. 67] as a pointer to the fact that Halldén's concern is mainly with validity, and not with truth, assertibility or commitment. This explains why he is not preoccupied with rather 'local' properties, such as the assertibility of individual possibly nonsignificant sentences one at a time, but with 'global' properties like validity instead—looking how to define logical consequence in general. Whether or not this would be a satisfactory answer to someone sharing the worries of Brady, Goddard and Routley on designating $\mathbf{n}$, is something we will, nevertheless, not discuss here.

There is a remaining problem associated to this second alternative of taking logical consequence to be defined by non-falsity-preservation. Namely, that the *Modus Ponens* rule is rendered as an invalid inference schema. To see this,

[4] As pointed out by an anonymous reviewer, there might not even be an advantage for PWK over $\mathsf{K}^{\mathsf{w}}_{3}$. The reason is that, while the latter does not coincide with CL in what pertains to tautologies or theorems, it does nevertheless coincide with CL in what pertains to anti-theorems—namely, formulae that entail everything. The situation of the former is symmetrical, and results of replacing theorems for anti-theorems in the previous sentence. Thus, both systems lack certain classical inference rules.

observe that if an implication $\varphi \supset \psi$ is defined as a material conditional in the vein of $\neg\varphi \vee \psi$, then if the value of φ is $\mathbf{n}$ and the value of ψ is $\mathbf{f}$, the inference going from φ and $\varphi \supset \psi$ to ψ does not preserve designation in PWK. However, this is one of the most recognized valid inference rules, and at the same time one of the least contested.[5] Furthermore, as suggested by an anonymous reviewer, the non-standard features of PWK are not limited to this, as another inference schema dear to Classical Logic is invalid to it—namely, the principle going from a conjunction $\varphi \wedge \psi$ to either of the conjuncts φ or ψ, sometimes referred to as *Conjunction Simplification*.

Therefore, both alternatives (truth-preservation and non-falsity-preservation) seem to have their own deficiencies. On the one side, we have the lack of classical tautologies, on the other, the lack of classically valid inferences like *Modus Ponens*. Notice moreover that trying to get the best of both approaches by focusing on the intersection of $\mathsf{K}_3^{\mathsf{w}}$ and PWK, that is, by asking both for truth- and non-falsity-preservation is no good.[6] The reason is that the resulting logic will count as valid only those inferences which are valid according to both conceptions, as it will count as tautologies those that are tautologies according to both conceptions. Thus, it will not actually enjoy the advantages of both $\mathsf{K}_3^{\mathsf{w}}$ and PWK, but instead it will suffer from the disadvantages of both.

It is important to highlight, though, that the discussion around significance logics reached an insightful conclusion concerning this trouble. Indeed, many of the philosophers involved independently argued that these difficulties could be solved by introducing some modifications pertaining to the logical vocabulary. Saliently, Bochvar, Halldén, Goddard and Routley proposed linguistic workarounds to the previously mentioned shortcomings, in the form of more logical connectives (intended to express notions that are undefinable in terms of the basic language) or of distinguished propositional variables, claiming that those interested in significance logics should actually be looking at restricted forms of the target inferences and logical truths—where either only these new connectives, or only these distinguished propositional variables were involved. In the next section, we discuss these alternatives in full detail.

3.2 The dispute over the logical vocabulary

Goddard and Routley summarize in a rather succinct manner the sources of the linguistic modifications that were discussed in the works we are commenting

[5] In fact, as pointed out by an anonymous reviewer, this does not depend on the implication in question being material implication. Rather, it shall be noted—as done in [3, Theorem 49]—that the logic PWK is not protoalgebraic, meaning that there is no set of formulae $\Delta(x, y)$ such that $\vDash_{\mathsf{PWK}} \Delta(x, x)$ and $x, \Delta(x, y) \vDash_{\mathsf{PWK}} y$. This means, ultimately, that there is no term-definable implication-like connective for which *Modus Ponens* holds in PWK.

[6] In this vein, the resulting system will stand with regard to $\mathsf{K}_3^{\mathsf{w}}$ and PWK in the same relation that the system $\mathsf{RM}_{\mathsf{fde}}$ stands with regard to strong Kleene logic K_3 and Priest's Logic of Paradox LP. Namely, just like $\mathsf{RM}_{\mathsf{fde}} = \mathsf{K}_3 \cap \mathsf{LP}$, the aforementioned system will represent the intersection of $\mathsf{K}_3^{\mathsf{w}}$ and PWK. To the best of our knowledge, such a system was mentioned in the relevant literature only once as the first-degree entailment fragment of the logic EM discussed by Francesco Paoli in [28], whence it can be denoted as $\mathsf{EM}_{\mathsf{fde}}$. A further reference about this logic is [1].

on here. In [21, p. 274], they point out the fact that the technical problems we reviewed in the previous section (the absence of classical tautologies, or the failure of Modus Ponens) were due to two assumptions of linguistic provenance that were taken for granted and that need not be so, if the aim is to construct a well-functioning and satisfactory significance logic.

The first of these assumptions is that *all the logical connectives behave according to the Principle of Component Homogeneity*, discussed above. The reasoning behind the claim that this induces the aforementioned problems goes as follows. If different connectives, for example different sort of negations, conjunctions and disjunctions were available, then it could be possible to employ such connectives to produce appropriate technical surrogates of the tautologies and inferences of Classical Logic which will be rendered as true or valid, in the supplemented system. More concretely then, even if the classical tautologies are not true (and the classical inferences are not valid) when formulated with the help of $\neg, \wedge$ and $\vee$, perhaps they are when formulated with the help of other connectives not constrained by the aforementioned principle.

The second of these assumptions is that valuations for all propositional variables range over the set containing the values $\mathbf{t}, \mathbf{f}$ and $\mathbf{n}$, that is to say, *all propositional variables available could, in principle, be either true, false, or meaningless.* In this case, the reasoning behind the claim that this induces the aforementioned problems goes along the following lines. If distinguished propositional variables were available—whose valuations did not range over the elements $\mathbf{t}, \mathbf{f}$ and $\mathbf{n}$, but just over the first two—then it could be possible to define validity only over formulae built using those distinguished propositional variables, in order to retain all the tautologies and valid inferences of Classical Logic. In other words, even if the classical tautologies are not true (and the classical inferences are not valid) when formulated over the full collection of propositional variables, perhaps they are when formulated over the rather restricted set of distinguished propositional variables, which are always meaningful—for they cannot receive the value $\mathbf{n}$ in any valuation.

In a nutshell, the core of these reflections is a coin that has two sides, one constituted by a negative claim, the other by a positive claim. The negative claim is that, as long as meaningless sentences can be around, the tautologies and inferential principles of Classical Logic are *invalid* in significance logics. The positive claim is that the tautologies and inferential principles of Classical Logic are *valid* in significance logics, when restricted to meaningful sentences, or when expressed with the help of suitable connectives (i.e., connectives not obeying the Principle of Component Homogeneity).

Linguistic considerations along these lines were present in the literature revolving around significance logics from the very beginning. In fact, Bochvar himself acknowledged the existence of two variants of logical connectives used to express assertion, denial, negation, conjunction, disjunction, and so on: the *internal* and the *external* connectives. The former operated following the Principle of Component Homogeneity, whereas the latter did not. According to everyone involved in the conversation, paradigmatic examples of logical operations of this last kind are, e.g., the predication of truth, falsity, meaninglessness and meaningfulness to any sentence whatsoever. This is all in line with Bochvar's reflection on the matter, e.g., in [2, p. 89] but also notably with others. For example, in this regard Goddard says:

> it is always significant to say of an absurd expression that it is absurd; and if we restrict the application of the logic to indicatives, we may say that an expression is absurd if and only if it is neither true nor false. To say, then, of an absurd expression that it is not true and not false is to make a significant, i.e., a two-valued, statement. But if 'not true and not false' is a significant description, so are its component parts 'not true' and 'not false'. And again, if 'not true' is a significant description of an absurdity, so is 'true'; and similarly, so is 'false'. So it is always significant, though false, to say of an absurdity that it is true or that it is false [18, p. 147]

It should be noticed, though, that contrary to what is nowadays common practice the truth and the falsity predicates were back then not properly understood as predicates that apply to quotation names for the target sentences, but as truth and falsity *operators* that apply to the proposition expressed by the target sentences. This being said, Bochvar claims the internal form of the assertion of φ is represented by the formula φ itself, whereas the external form of its assertion is represented by the expression 'φ is true', formalized by the proposition $\mathbf{T}\varphi$. Accordingly, the internal form of the conjunction and disjunction of sentences φ and ψ are represented by the formulae $\varphi \wedge \psi$ and $\varphi \vee \psi$, respectively. Meanwhile, their external form is expressed by the phrases 'φ is true and ψ is true', and φ is true or ψ is true', formalized by the propositions $\varphi \sqcap \psi$ and $\varphi \sqcup \psi$. These last two are, according to Bochvar, definable as $\mathbf{T}\varphi \wedge \mathbf{T}\psi$ and $\mathbf{T}\varphi \vee \mathbf{T}\psi$—in line with his remark that the "external forms (...) represent nothing other than the corresponding internal forms, in which φ and ψ have been replaced by their external assertions" [2, p. 90].[7]

	$\mathbf{T}$
t	**t**
n	**f**
f	**f**

	$\mathbf{F}$
t	**f**
n	**f**
f	**t**

$\sqcap$	**t**	**n**	**f**
t	**t**	**f**	**f**
n	**f**	**f**	**f**
f	**f**	**f**	**f**

$\sqcup$	**t**	**n**	**f**
t	**t**	**t**	**t**
n	**t**	**f**	**f**
f	**t**	**f**	**f**

	$\downarrow$
t	**t**
n	**f**
f	**t**

	$\uparrow$
t	**f**
n	**t**
f	**f**

	$\sim$
t	**f**
n	**t**
f	**t**

Figure 2: Some external connectives

There is yet another couple of logical operations definable with the help of these external notions, that Bochvar and the remaining scholars pay special attention to. On the one hand, there is the predication of meaninglessness to any sentence whatsoever which, following our previous clarifications, was treated by

[7] As pointed out by an anonymous reviewer, it should be highlighted that the external connectives were also introduced in Segerberg's study of Halldén's external logic, in [38].

those involved in the discussion as an operator—instead of a proper predicate. This sort of operation, represented by the expression 'φ is meaningless' and formalized as $\downarrow\varphi$, can in fact be defined as $\neg(\mathbf{T}\varphi \vee \mathbf{F}\varphi)$. It is straightforward to see, although he did not focus on it, that the related notion of meaningfulness is of equal if not more interest—as witnessed by the discussion below. This notion, formalizable as $\uparrow\varphi$, can be defined either as $\neg\downarrow\varphi$ or, more conspicuously, as $\mathbf{T}\varphi \vee \mathbf{F}\varphi$.

Bochvar's own idea was that an external expression of the first kind, claiming the meaningless nature of a sentence φ did not have any proper internal form counterpart [2, p. 90]. Halldén, on the other hand, shared the opinion that it was necessary to expand the language with connectives not obeying the Principle of Component Homogeneity, but he believed that expressions such as 'φ is meaningful' were *ambiguous*, i.e., that they had several meanings attached to them [25, p. 36]. More particularly, concerning the concept of meaningfulness he thought that it had a tight relation with the idea of a sentence being true or false, although he objected that these notions had a recognizably different range of application. He argues along the following lines:

> In certain cases we can meaningfully assert ['p is meaningful'], but not ['p is true or p is false], namely in those cases in which p is meaningless. The concept of meaningfulness is such that it can be meaningfully asserted of entities which are meaningless. But [the concept of being true or false] cannot be meaningfully asserted of such entities. Then, the concepts (...) are not identical [25, p. 38]

Although probably not all the scholars involved in the discussion would agree with these remarks, from the point of view of Halldén this helped him advance the claim that it is necessary to have different linguistic ways of expressing both ideas, letting the former be formalized by a meaningful operator working like $\uparrow$ above.

As a matter of fact, not only Bochvar and Halldén thought that significance logics needed to be further equipped with operators expressing the concept of meaningfulness and meaninglessness, but also Goddard, Routley and Brady did. As discussed in the previous section, they believed that significance logics were motivated by considerations around Russell's type-theory and Ryle's philosophy of language. In accordance with these reflections, there was the need to express some of the ideas about these topics in a rather formal manner. This implies that enough formal resources, i.e., logical connectives, have to be available so as to formalize several principles of significance. Goddard and Routley point out that Russell falls short of this task, failing to provide a proper formalization of his ideas expressed in *Principia Mathematica*, because he has no means to express in the symbolism of his system the conditions that, e.g., if a predicate applied to an individual of certain type is significant, then it is also significant as applied to another different individual of the same type [21, p. 225]. For such tasks, connectives expressing the idea that propositions are meaningless or meaningful are required.

A further logical connective behaving in the way just described above was also discussed in the literature on significance logics, namely the so-called exclusion negation. This notion is usually expressed by phrases like 'φ is not true' and formalized by the proposition $\sim\varphi$. Bochvar identified this as the external variant of denial. Thus, while the internal form of the denial of φ is represented by the formula $\neg\varphi$, its external form is represented by the expression 'φ is not true'. Whence, as discussed by Brady and Routley in [5, p. 219], this sort of negation can be defined as $\neg\mathbf{T}\varphi$.

Discussion of such a connective arose when some, like Haack, Lambert, and Bradley opposed the view that there was a need for a three-valued account of nonsignificant sentences claiming that these can be fairly well accommodated in a two-valued contexts—e.g., in [22], [23], [27], and [4]. To this extent, meaningless sentences like Carnap's paradigmatic example 'This stone is thinking about Vienna' would be rendered as (necessarily) false, in virtue of the fact that their negative-style counterparts such as 'This stone is not thinking about Vienna' appeared to some as less queer [20, p. 14]. But this certainly violates the constraint according to which if a sentence is meaningless, so is its negation. Thus, if meaningless sentences were to be necessarily false, so would be their negations—which implies in itself a hugely revisionary account of the relation between sentences and their negations.

Routley's own way of explaining out this problem is clear enough. He says that:

> it might be thought that 'Tables do not talk' is significant, in fact true, while 'Tables talk' is non-significant. Here, however, the 'not' in 'Tables do not talk' is taken to mean 'cannot be significantly said to' and hence 'Tables talk' should be interpreted as 'Tables (can significantly be said to) talk'; in this case, both sentences are significant [35, p. 200]

Following Goddard [17, p. 147], these sort of considerations allow for an indirect definition of meaningfulness. For, on the one hand, meaningful sentences are those sentence for which negating them in the usual manner and negating them in the special way just described renders the same results. So, if negating a sentence delivers the same result in an external or an internal way, then the sentence is meaningful. Otherwise, if negating a sentence in the usual manner renders one thing (i.e., a meaningless sentence) and negating them in the external way renders another (i.e., a true sentence), then the sentence is meaningless.

This being said, let us now turn to a more formal account of our initial remarks to the extent that classical tautologies and inferences could be retained if linguistic measures were properly taken.

First, let $\mathcal{L}_{\mathsf{E}}$ be the expansion of the language $\mathcal{L}$ with the previously discussed connectives $\mathbf{T}, \mathbf{F}, \sqcap, \sqcup, \downarrow, \uparrow$ and $\sim$, and let $FOR(\mathcal{L}_{\mathsf{E}})$ be its set of well-formed formulae, defined as usual. Similarly, let $\mathcal{L}_{\uparrow}$ be the expansion of the language $\mathcal{L}$ with the previously discussed connective $\uparrow$, and let $FOR(\mathcal{L}_{\uparrow})$ be its set of well-formed formulae, defined as usual. In this respect, it is useful to have in mind a terminology due to Krister Segerberg. In [37, 38] it is said that a propositional variable p is *open* in a formula φ if and only if at least one occurrence of p in φ is not in the scope of a connective of the external language. Thus, a propositional variable p is *covered* in a formula φ if and only if it is not open in φ. It is easy to observe that if a formula has all of its propositional variables covered, then it is guaranteed to receive a classical truth-value—and thus can be considered to represent a meaningful proposition.

Secondly, let $\mathbf{WK}_{\mathsf{E}}$ be the extension of the weak Kleene algebra $\mathbf{WK}$ with the operations associated to the external connectives $\mathbf{T}, \mathbf{F}, \sqcap, \sqcup, \downarrow, \uparrow$ and $\sim$. Similarly, let $\mathbf{WK}_{\uparrow}$ be the extension of the weak Kleene algebra $\mathbf{WK}$ with the meaningfulness operator associated to the connective $\uparrow$. Given this, let Bochvar's external logic $\mathsf{\Sigma}$ be the system induced by the logical matrix $\langle \mathbf{WK}_{\mathsf{E}}, \{\mathbf{t}\}\rangle$, and let Halldén's external logic C be the system induced by the logical matrix $\langle \mathbf{WK}_{\uparrow}, \{\mathbf{t}, \mathbf{n}\}\rangle$.

Furthermore, let Var be the set of all propositional variables of the language, let $Var(\varphi)$ be the set of all propositional variables occurring in φ, and for a set of sentences Γ, let $Var(\Gamma) = \bigcup\{Var(\gamma) \mid \gamma \in \Gamma\}$ be defined as usual. Finally, let the ext-translation from $\mathbf{FOR}(\mathcal{L})$ to $FOR(\mathcal{L}_{\mathsf{E}})$ be recursively defined as follows:

$$\begin{aligned}
(p)^{\mathsf{ext}} &= p \\
(\neg\varphi)^{\mathsf{ext}} &= \sim(\varphi)^{\mathsf{ext}} \\
(\varphi \wedge \psi)^{\mathsf{ext}} &= (\varphi)^{\mathsf{ext}} \sqcap (\psi)^{\mathsf{ext}} \\
(\varphi \vee \psi)^{\mathsf{ext}} &= (\varphi)^{\mathsf{ext}} \sqcup (\psi)^{\mathsf{ext}}
\end{aligned}$$

Having these pieces in place helps appreciating the following observations that can be easily proved as done, for example, in [2, pp. 95-96] and [12, pp. 96-97], respectively. Notice that tautologies or theorems can be considered as limit cases of valid inferences of the form $\Gamma \vDash \varphi$, where $\Gamma = \emptyset$.[8] As highlighted by an anonymous reviewer, it should be pointed out that in the observations below the set $Var(\Gamma \cup \{\varphi\})$ is finite, because CL is a finitary logic.

Observation 3. *All the tautologies and valid inferences of* CL *are valid in* $\mathsf{\Sigma}$*, if the usual logical connectives of negation, conjunction and disjunction are replaced by their external forms, i.e.,* $(\Gamma)^{\mathsf{ext}} \vDash_{\mathsf{\Sigma}} (\varphi)^{\mathsf{ext}}$ *iff* $\Gamma \vDash_{\mathsf{CL}} \varphi$.

Observation 4. *All the tautologies and valid inferences of* CL *are valid in* C*, if it is assumed that all the propositional variables involved are meaningful, i.e., if* $Var(\Gamma \cup \{\varphi\}) = \{p_1, \ldots, p_n\}$*,* $\uparrow p_1, \ldots, \uparrow p_n, \Gamma \vDash_{\mathsf{C}} \varphi$ *iff* $\Gamma \vDash_{\mathsf{CL}} \varphi$.

Notice that this last result says that all the inferences that are valid in CL hold in the logic of nonsense C if and only if all the propositional variables appearing in the formulae involved in the inference in question are *assumed* to be meaningful. The emphasis in the assumption here is crucial to understand the result. To use Segerberg's terminology introduced above, the result does not say that all the propositional variables appearing in the formula involved in the

[8]Therefore, if a logic L_1 has the same valid inferences as the logic L_2, then it has also the same tautologies as L_2.

inference in question are as a matter of fact covered, and therefore meaningful. Instead, the result considers *what would follow if we were to assume* that all such propositional variables are meaningful—thus, treating this as a working hypothesis rather than an established fact.[9]

To close this section, let us now consider another alternative of cashing out these linguistic restrictions, due to Goddard and Routley. As we previously said, these authors attribute the failure of the truth- and non-falsity-preservation accounts in the context of significance logics to the implicit assumption that valuations for all propositional variables range over the **WK** algebra. Entertaining the idea of abandoning this feature is represented, in terms of inferential principles, by giving up *substitution-invariance.*

This property, usually taken as a defining feature of what a logic is, is actually explicitly criticized by Goddard and Routley. It can be succinctly described in the following way. Recall that an $\mathcal{L}$-substitution is an endomorphism σ of the formula-algebra $\mathbf{FOR}(\mathcal{L})$. Then, a logic L is *substitution-invariant* if and only if for all $\Gamma \subseteq FOR(\mathcal{L})$ and $\varphi \in FOR(\mathcal{L})$ and all substitution functions σ, if $\Gamma \vDash_{\mathsf{L}} \varphi$, then $\sigma[\Gamma] \vDash_{\mathsf{L}} \sigma(\varphi)$. Thus, regarding substitution-invariance, these authors claim that:

> It needs only a few examples to show that whatever credence is given to the rule of uniform substitution in ordinary logics should be withheld when uniform substitutions of nonsignificant sentences in theses is considered. Consider some examples: substituting in $[p \supset p]$ we obtain such nonsignificant sentences as 'If the Absolute is green then the Absolute is green'; 'If quadruplicity drinks procrastination, then it does'; and by substituting in $[\forall x Fx \supset Fa]$, we have 'If all numbers like soup, then the number six likes soup.' Hence, significance is not preserved under uniform substitution. [21, p. 275]

Before moving on with some further observations by Goddard and Routley, let us notice that in the previous quote the rejection of uniform substitution or of substitution-invariance as a desirable property of a significance logic is grounded in the fact that such a principle does not preserve significance.[10] By this it is meant that some inference whose premises and conclusion are significant can be turned into an inference whose premises and conclusion are not significant, by means of substitution-invariance. To speak properly, this failure

[9] We would like to thank an anonymous reviewer for urging us to clarify this.

[10] Notice that the discussion about substitution-invariance, as presented by some authors in the literature, does not rely on the logics in question being first-order. In fact, like the quote about highlights—even in propositional frameworks—substitution-invariance could lead to transitions from significant or meaningful instances of a schema like $p \supset p$ (such as "If it is sunny, then it is sunny") to otherwise nonsignificant instances (such as the referred "if the Absolute is green, then the Absolute is green"). Admittedly, this may or may not be a good reason to abandon substitution-invariance, but our aim is not to assess this matter. All we wanted to do is reconstruct the discussion as it historically happened, for we think new ideas and inspirations can arise from a close examination thereof. We would like to thank an anonymous review for urging us to clarify this issue.

is witnessed at some sort of meta level, where some property of an inference is not preserved while transitioning to another inference. Thus, significance-preservation appears to be cherished by these authors as a sort of desideratum governing transitions from inferences to inferences—in nowadays terminology, a 'metainferential' principle.

Interestingly, although discussed and motivated only at a meta level, Goddard and Routley's rejection of this principle might point towards a further option for the ground level—not entertained by them or by any other party in the literature on significance logics—that we briefly stop to analyze here. This alternative is nothing other that the conjunction of *truth-preservation* and *significance-preservation*, i.e., the idea according to which not only truth has to be preserved from premises to conclusion, but also significance has to. This option implies the rejection of non-falsity-preservation, but it does not condone the mere adoption of truth-preservation as an alternative. For, on the one hand, non-falsity-preservation allows to transition from true (hence, significant) premises to nonsignificant conclusions. But, on the other hand, truth-preservation accepts as appropriate cases where significance is not preserved, i.e., in the case of inferences with false (hence significant) premises and nonsignificant conclusions. Interestingly, inferences of this kind were considered as harmless both by the truth-preservation account and the non-falsity-preservation account. Therefore, asking for significance-preservation renders a notion of logical consequence for significance logics that is considerably stricter than truth-preservation, as one would expect. To wit, under truth-preservation the Principle of *Explosion* or *Ex Contradictione Quodlibet*—i.e., the inference $\varphi, \neg\varphi \vDash \psi$—is valid (since premises cannot possibly be true), but under significance-preservation this is invalid (since premises can be false, while the conclusion is nonsignificant). To close these exploratory remarks, let us highlight that adopting truth- and significance-preservation does not imply the rejection of substitution-invariance, as can be easily checked.[11]

Goddard and Routley, however, advocate for the abandonment of substitution-invariance and propose to formally model their choice along the following lines:

> [T]he semantical equivalent of restricted substitution (...) is restricted assignments of values to the sentential variables, and the particular restriction suggested amounts to specifying that certain variables should vary only over the set $\{\mathbf{t}, \mathbf{f}\}$ and not the full set $\{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$ [21, p. 275]

Let us, then, consider the simplest of Goddard and Routley's systems constructed following this recommendations, the logic $\mathsf{S_0}$. Before going into the details, we should bear in mind that the terminology and techniques used by the authors to construct their system do not perfectly align with the distinctions and standards implemented nowadays—for instance in Abstract Algebraic

[11] For more on significance-preservation in the context of three-valued logics, see [6, p. 2198]. For an application of significance- preservation to the definition of containment logics, see [30].

Logic. This, however, does not deprive this system of historical interest, which is why we chose to present it here, nevertheless.

This being said, let Var be the set of all propositional variables of the system, and let Var^* be a copy of it, where for every $p \in Var$, we have a $p^* \in Var^*$. The set of well-formed formulae built using variables either in Var or in $Var*$ is called $\mathbf{FOR}(\mathcal{L})$ as before, whereas the set of well-formed formulae built using only propositional variables in Var^* can be called $FOR(\mathcal{L}^*)$. Valuations for S_0 map every member of Var to a truth-value in $\{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$, but they map every member of Var^* to a truth-value in the restricted set $\{\mathbf{t}, \mathbf{f}\}$—in both cases, according to the weak Kleene truth-tables.

Thus, by an easy induction we can see that the formulae in $FOR(\mathcal{L}^*)$ (built using only the propositional variables in Var^*) are going to be two-valued and, hence, meaningful too. This induces a division between the unrestricted and possibly meaningless sentences, and the restricted and forcefully meaningful sentences. In this spirit, logical consequence for S_0 is defined only for sets of such restricted or distinguished formulae. Validity, as a technical notion, is not defined for formulae belonging to $\mathbf{FOR}(\mathcal{L})$, in general. This allows the said authors to prove the following fact in [21, p. 303].[12]

Observation 5. *All the tautologies and valid inferences of* CL *are valid in* S_0*, if it is assumed that all the propositional variables involved are meaningful, i.e.,* $\Gamma \vDash_{\mathsf{S}_0} \varphi$ *iff* $\Gamma \vDash_{\mathsf{CL}} \varphi$.

However—as advertised by Goddard and Routley—substitution-invariance is invalid in their system. Nevertheless, being it a constitutive feature of what a logical system is, it would be a nice thing to retain, if possible, along with all the tautologies and valid inferences of Classical Logic. Having a proper significance logic that enjoys all of these advantages at the same time has proven an elusive task, but in the next section we will show how it can be achieved. We not only do this, but at the same time show how embracing such a novel framework allows for the dissolution of the previously discussed divisions, which then prove to be unnecessary.

[12] Although these authors did not present their system using the terminology of Abstract Algebraic Logic, we could perhaps partially reconstruct their ideas in the following way. S_0 valuations are homomorphisms between $\mathbf{FOR}(\mathcal{L})$ and $\mathbf{WK}$ such that they are also homomorphisms between $FOR(\mathcal{L}^*)$ and the 2-element Boolean algebra. Furthermore, and perhaps here is where the biggest divergence with the contemporary terminology becomes apparent, these authors somehow took S_0 to be "defined" in some sense over $FOR(\mathcal{L}^*)$ but also perhaps in some sense over $\mathbf{FOR}(\mathcal{L})$. In this vein, the result of Observation 5 can be read as saying that whenever $\Gamma \cup \{\varphi\} \subseteq FOR(\mathcal{L}^*)$, $\Gamma \vDash_{\mathsf{S}_0} \varphi$ iff $\Gamma \vDash_{\mathsf{CL}} \varphi$. Given the fact that if any of the formulae involved is substituted by a formula with a propositional variable not in Var^*, then some inferences become invalid, these authors take it that substitution-invariance is not valid in their system. We would like to thank an anonymous reviewer for urging us to clarify this matter.

4 Building bridges

In this section we present a significance logic which enjoys all the desired features highlighted in the paragraphs above, without suffering any of the shortcomings of the previously discussed systems. Not only that, but we will also claim that adopting our favored logic allows to dissolve the need for the divisions that were subject to comment in earlier sections. To this extent, in this section we aim to establish three main results.

The first—and perhaps the main—result consists in showing that the set of inferences valid in the significance logic that we put forward here coincides with the set of those inferences that are valid in Classical Logic. Furthermore, since tautologies can be seen as inferences with empty set of premises, it is also the case that the set of tautologies of the significance logic that we favor coincides with the set of tautologies of Classical Logic. Whence, we claim, there are enough reasons to refer to the significance logic that we defended here as *a classical logic of nonsense*.

The second result consists in showing that within our significance logic the divide between truth-preservation and non-falsity-preservation can be dissolved. In other words, in what pertains to logical consequence, embracing our approach to non-significance makes it unnecessary to choose between truth-preservation and non-falsity-preservation, because—in some qualified sense—both are represented in our definition of logical entailment.

The third result consists in showing that within our approach the need for introducing any linguistic restrictions effectively disappears. The reasons for this are twofold. On the one hand, adopting our approach makes it unnecessary to choose between the internal and external connectives, since the inferences valid involving either of them will be the same—modulo translation. On the other hand, it also makes it unnecessary to divide the propositional variables between those that are meaningful and those which can be meaningless, because the set of valid inferences involving variables of each of these sorts will be the same—namely, those valid in Classical Logic.

Let us explain in simple terms, then, how we will understand validity moving forward. In a nutshell, for an inference to be valid in our framework it will be required that there is no valuation complying with the weak Kleene truth-tables which makes all the premises true (or assigns all the premises the value $\mathbf{t}$) while at the same time making the conclusion false (or assigning the conclusion the value $\mathbf{f}$). These sort of requirements were motivated by Cobreros, Egré, Ripley and van Rooij in the following way, where the truth-values $\mathbf{t}$, $\mathbf{f}$, and $\mathbf{n}$ are identified with 1, 0, and $\frac{1}{2}$, respectively:[13]

[13] Recall that these authors identify $\mathbf{t}$ with 1, $\mathbf{f}$ with 0, and $\mathbf{n}$ with $\frac{1}{2}$. Thus, 'having value greater than 0' should be understood in reference to the usual order of the rational numbers—i.e., of having value either 1 or $\frac{1}{2}$, that is to say, either $\mathbf{t}$ or $\mathbf{n}$. Notice that in the strong Kleene case this order coincides with the order induced by the operations $\wedge$ and $\vee$. As an anonymous reviewer points out, this is not true of the weak Kleene case, where these operations induce two different orders in none of which $\frac{1}{2}$ is intermediate between 1 and 0—for more on this see, e.g., [29].

> A "good" premise (a premise good enough to produce a sound argument) is one that takes value 1. A "good" conclusion, on the other hand (a conclusion that is not false enough to produce a counterexample) is one that takes value greater than 0.[10, p. 79]

Thus, an inference having a good premise (or set of good premises) and a bad conclusion is an *invalid* inference. Notice that, in this sense, all it takes for an inference to be invalid is to be invalid in Classical Logic, that is, to have true premises and a false conclusion—something we will come back to a few paragraphs below. Given validity and invalidity are mutually exclusive and jointly exhaustive, if an inference is not invalid, it is valid. Therefore, all the remaining cases where the premises are not true while at the same time the conclusion is false, are not counterexamples of the inference in question.

These authors, however, were busy defining a satisfying notion of logical consequence for frameworks employing another three-valued valuation schema—the *strong* Kleene truth-tables. The logic stemming out of applying the previous definition to said valuation schema is called ST in the context of several of their works.[14] Whence, given our proposed significance logic will implement their approach to entailment to the *weak* Kleene truth-tables, we will refer to it as weak ST (wST, for short).

In their work, these scholars highlight the impossibility to understand this approach to logical consequence in terms of mere or *plain preservation* of some semantic status (or formally speaking, of some set of designated values). A technical upshot of this is the impossibility of representing wST using regular logical matrices. However, some interesting algebraic constructions called p-matrices can be used to represent entailment in our significance logic. These constructions designed by Szymon Frankowski in [15] and later studied, among others, by Yaroslav Shramko and Heinrich Wansing in [39], generalize regular logical matrices (meaning by that that the latter are special cases of the former).

As conceived by Frankowski, *p-matrices* are triples $\langle \mathbf{A}, D_1, D_* \rangle$ where $\mathbf{A}$ is an algebra, and D_1, D_* are distinguished subsets of A (the universe of the algebra) such that $\emptyset \neq D_1 \subseteq D_*$. As with regular logical matrices, the elements of the algebra in question are usually referred to as 'truth-values', while its operations are commonly represented in the form of the so-called 'truth-tables'. Valuations, again, refer to homomorphism from $\mathbf{FOR}(\mathcal{L})$ to A. Finally, a p-matrix $\mathbf{M} = \langle \mathbf{A}, D_1, D_* \rangle$ induces a p-consequence relation $\vDash_{\mathbf{M}}$ as follows, where $\Gamma \subseteq FOR(\mathcal{L})$ and $\varphi \in FOR(\mathcal{L})$:

$$\Gamma \vDash_{\mathbf{M}} \varphi \iff \text{for every valuation } v\text{: if } v[\Gamma] \subseteq D_1, \text{ then } v(\varphi) \in D_*$$

Thus, p-matrices generalize regular logical matrices, because when $D_1 = D_*$ the corresponding p-matrix is a regular matrix. Additionally, notice that if by

[14] Although the collaboration between these scholars is always producing new material, a to-this-day updated enumeration of their joint production revolving around ST must include [9], [8], [10], and [11], together with David Ripley's works [33] and [34].

$\overline{D_*}$ we refer to the complement of D_* relative to A (i.e., $A \setminus D_*$) the previous definition can be rephrased as follows:

$$\Gamma \vDash_{\mathbf{M}} \varphi \iff \text{ for every valuation } v\text{: if } v[\Gamma] \subseteq D_1, \text{ then } v(\varphi) \notin \overline{D_*}$$

This being said, our proposed logic to deal with nonsensical sentences can be seen as induced by a very simple p-matrix. Indeed, as is easy to corroborate, the logic wST can be understood as being induced by the p-matrix $\langle \mathbf{WK}, \{\mathbf{t}\}, \{\mathbf{t}, \mathbf{n}\}\rangle$, built on top of the weak Kleene algebra. Instantiating this last reformulation of the general definition, thus, gives us the following reading of logical consequence in wST—which appropriately formalizes the remarks and motivation present in the quote from Cobreros and his collaborators, appearing above:[15]

$$\Gamma \vDash_{\mathsf{wST}} \varphi \iff \text{ for every WK-valuation } v\text{: if } v[\Gamma] = \{\mathbf{t}\}, \text{ then } v(\varphi) \notin \{\mathbf{f}\}$$

Now, having defined what logical consequence amounts to in wST, let us proceed to show the *first result of this section.* That is, how our significance logic overcomes the difficulties suffered by the previously discussed accounts, allowing to refer to it as a classical logic of nonsense. For this purpose, let us remember that the significance logic embodying truth-preservation ($\mathsf{K}_3^{\mathsf{w}}$, that is) lacked tautologies, thus suffering the absence of all the tautologies of Classical Logic. On the other hand, the significance logic embodying non-falsity-preservation (PWK, that is) shared the tautologies of Classical Logic, but lacked some inferential principles dear to Classical Logic. Contrary to these limited frameworks, our favored logic wST not only has the same tautologies that Classical Logic, but also (as hinted by our previous remarks on its definition of logical consequence) has the same valid inferences that Classical Logic—as the following proof establishes.

Theorem 6. *The logic* wST *has the same valid inferences that* CL*, i.e.,* $\Gamma \vDash_{\mathsf{wST}} \varphi$ *iff* $\Gamma \vDash_{\mathsf{CL}} \varphi$*.*

Proof. For the first equivalence, to prove the left to right direction, assume $\Gamma \nvDash_{\mathsf{CL}} \varphi$. Then, there is a Boolean valuation v such that $v[\Gamma] = \{\mathbf{t}\}$ and $v(\varphi) = \mathbf{f}$. But this trivially implies that there is a WK-valuation v^* such that $v^*[\Gamma] = \{\mathbf{t}\}$ and yet $v^*(\varphi) = \mathbf{f}$, whence $\Gamma \nvDash_{\mathsf{wST}} \varphi$. To prove the right to left direction, assume $\Gamma \nvDash_{\mathsf{wST}} \varphi$. Then, there is a WK-valuation v such that $v[\Gamma] = \{\mathbf{t}\}$ and yet $v(\varphi) = \mathbf{f}$. By the Principle of Component Homogeneity, we know not only that there is no propositional variable p in all of the $\psi \in \Gamma$ such

[15] Whereas instantiating the first formulation of the general definition for p-matrices gives us this alternative reading of logical consequence in wST:

$$\Gamma \vDash_{\mathsf{wST}} \varphi \iff \text{ for every WK-valuation } v\text{: if } v[\Gamma] = \{\mathbf{t}\}, \text{ then } v(\varphi) \in \{\mathbf{t}, \mathbf{n}\}$$

that $v(p) = \mathbf{n}$, but also that there is no propositional variable q in φ such that $v(q) = \mathbf{n}$. Hence, we are guaranteed that v is a Boolean valuation. Thus, we know that there is a Boolean valuation v^*, which is exactly like v, such that $v^*[\Gamma] = \{\mathbf{t}\}$ and yet $v^*(\varphi) = \mathbf{f}$, whence $\Gamma \not\models_{\mathsf{CL}} \varphi$. □

Observation 7. *The logic* wST *has the same tautologies that* CL*, i.e.,* $\models_{\mathsf{wST}} \varphi$ *iff* $\models_{\mathsf{CL}} \varphi$*.*

Proof. Recall that, in the context of a logic defined by a p-matrix $\langle \mathbf{A}, D_1, D_* \rangle$, for φ to be a tautology means that for every valuation v, $v(\varphi) \in D_*$. In the case of wST this means that for every valuation v, $v(\varphi) \in \{\mathbf{t}, \mathbf{n}\}$. It is easy to see that wST and CL share their tautologies by applying these considerations and assuming that $\Gamma = \emptyset$ in the proof of Theorem 6 above. □

As an anonymous reviewer highlights, the previous results show that there are ways of arriving at Classical Logic using p-matrices, instead of using the usual matrix-based semantics. But this is not the only advantage of our account. Our favorite significance logic has also an important saying in the truth-preservation versus non-falsity-preservation debate. In fact, the *second result of this section* consists in showing that embracing wST contributes to the dissolution of the truth versus non-falsity divide. We will establish this by reflecting upon the fact—noted, e.g., in [6, 2194, fn. 1]—that the definition of entailment that we are embracing can be paraphrased in a deeply illuminating and equivalent manner, as follows:

$$\Gamma \models_{\mathsf{wST}} \varphi \iff \text{ for every WK-valuation } v : \begin{cases} \text{if } v[\Gamma \cup \{\varphi\}] \subseteq \{\mathbf{t}, \mathbf{f}\}, \text{ then} \\ (\text{if } v[\Gamma] = \{\mathbf{t}\}, \text{ then } v(\varphi) \in \{\mathbf{t}\}) \end{cases}$$

In order to make our point, let us notice that being assigned a value in $\{\mathbf{t}, \mathbf{f}\}$ by a valuation v amounts to being meaningful or significant in the context of v. With this in mind, let us notice that the alternative definition of validity in wST that we just detailed makes it is reasonable to assert that logical consequence can be read in two equally satisfying manners within our significance logic.

On the one hand, it can be understood as requiring that if the premises and the conclusion are meaningful, then if the premises are true, the conclusion must also be true. This means that truth-preservation is the standard of wST validity—conditional on both premises and conclusion being meaningful. On the other hand, it can be understood as requiring that if the premises and the conclusion are meaningful, then if the premises are not false, the conclusion must also be not false. This means that non-falsity-preservation is also the standard of wST validity—granted both premises and conclusion are meaningful. This is the case, precisely, because truth and non-falsity coincide when meaningfulness is assumed. That is to say, conditional on the premises and conclusion being meaningful, truth- and non-falsity-preservation render the same results. Whence, given the definition of logical consequence in the context of wST can be presented in this conditional way—as above—both criteria are represented in a *qualified* sense within our definition of entailment. Therefore, we claim, embracing our logic makes it unnecessary to choose between these options.[16]

Before going any further, we should stop to notice that the previous reformulation of the definition of wST consequence ties it to another significance logic that we discussed earlier—Goddard and Routley's $\mathsf{S_0}$. Recall that such a logic was defined so that inferences were valid only if truth was preserved from premises to conclusion—only when premises and conclusions were guaranteed to be meaningful. This meant, in Goddard and Routley's cases, that this meaningfulness condition *had* to be respected. In other words, when either premises or conclusions were non-significant, validity was not properly defined. This is not to say that the validity judgment had a non-standard status, but validity was only defined for the fragment of the language for which it was guaranteed that sentences were always meaningful. As a result of such restrictions, the rule of substitution-invariance was deemed invalid. In contradistinction with Goddard and Routley's approach, ours does not fall prey of such inconveniences. In fact, wST *validates substitution-invariance*—as most people working with logical systems would like to do—without precluding the assessment of inferences with either meaningless premises or meaningless conclusions. This is possible, precisely, because we do not require that premises and conclusions are meaningful in order to evaluate the validity of the corresponding inference, but instead we evaluate the validity of the corresponding inferences conditional on premises and conclusions being meaningful. This need not be proved especially for the case of our logic, since it holds for all the systems induced by p-matrices—as proved in [16, Section 2], and as pointed out by an anonymous reviewer.

Moving forward, our *third result of this section* amounts to showing that within wST the linguistic divides referred in the previous sections can be dissolved. By these we mean, first, the dispute over the internal and the external logical vocabulary (due to Bochvar and Halldén) and, second, the division between the sentences built using only meaningful propositional variables and sentences built using unrestricted propositional variables (due to Goddard and Routley).

Regarding the former, we can happily show that working with wST the linguistic divide pertaining the internal versus external connectives debate can be straightforwardly dissolved. More concretely, by this we mean that the same inferences are valid in our significance logic, no matter which connectives are used to represent the logical notions. That is to say, without detriment to whether negation is expressed by the $\neg$ or the $\sim$ connective, or whether conjunction is expressed by the $\wedge$ or the $\sqcap$ connective, or whether disjunction is expressed by the $\vee$ or the $\sqcup$ connective. Similarly, without detriment of whether or not it is

[16] An anonymous reviewer inquires into the way in which this qualified property of truth-preservation, conditional on the premises and the conclusion being meaningful, is related to validity in $\mathsf{K_3^w}$ and PWK. To this extent, it should be said that when an inference is valid in any of these systems, then that inference enjoys the property in question. However, an inference enjoying this property is not sufficient for it to be valid in either $\mathsf{K_3^w}$ or PWK. To wit, consider the inference schemata $\psi \vDash \varphi \vee \neg\varphi$ and $\varphi, \neg\varphi \vDash \psi$, both of which preserve truth under the assumption that the formulae involved are meaningful, although the former is invalid in $\mathsf{K_3^w}$ and the latter is invalid in PWK.

assumed that all the propositional variables $p_1, \dots, p_2$ involved in the inference are meaningful, in the form of $\uparrow p_1, \dots, \uparrow p_n$.[17]

To give a rather formal expression to these ideas, consider the expansion of wST with all the external connectives $\sim, \sqcap, \sqcup, \uparrow$, etc. discussed in earlier sections, and let the resulting system be called $\mathsf{wST_E}$. Then, our claims can be formalized as follows.

Theorem 8. $(\Gamma)^{\mathsf{ext}} \vDash_{\mathsf{wST_E}} (\varphi)^{\mathsf{ext}}$ *iff* $\Gamma \vDash_{\mathsf{wST}} \varphi$.

Proof. First, we notice that $(\Gamma)^{\mathsf{ext}} \vDash_{\mathsf{wST_E}} (\varphi)^{\mathsf{ext}}$ iff $(\Gamma)^{\mathsf{ext}} \vDash_{\mathsf{\Sigma}} (\varphi)^{\mathsf{ext}}$. This can be established by noticing that $\mathsf{wST_E}$ and $\mathsf{\Sigma}$ valuations are the same, and that although their definition of logical consequence is different, the only kind of WK-valuations that are $\mathsf{\Sigma}$ counterexamples but not $\mathsf{wST_E}$ counterexamples are those where $v[(\Gamma)^{\mathsf{ext}}] = \{\mathbf{t}\}$ and $v((\varphi)^{\mathsf{ext}}) = \mathbf{n}$. However, by its very construction we know $v((\varphi)^{\mathsf{ext}}) \in \{\mathbf{t}, \mathbf{f}\}$ and, thus, it is impossible to have these kind of counterexamples. Whence, the equivalence. Secondly, we recall from Observation 3 that $(\Gamma)^{\mathsf{ext}} \vDash_{\mathsf{\Sigma}} (\varphi)^{\mathsf{ext}}$ iff $\Gamma \vDash_{\mathsf{CL}} \varphi$. Thirdly, that $\Gamma \vDash_{\mathsf{CL}} \varphi$ iff $\Gamma \vDash_{\mathsf{wST}} \varphi$. Finally, by chaining these equivalences, we arrive at the desired result. □

Theorem 9. *If* $Var(\Gamma \cup \{\varphi\}) = \{p_1, \dots, p_n\}$, $\uparrow p_1, \dots, \uparrow p_n, \Gamma \vDash_{\mathsf{wST_E}} \varphi$ *iff* $\Gamma \vDash_{\mathsf{wST}} \varphi$.

Proof. Similarly to the previous proof, we first notice that $\uparrow p_1, \dots, \uparrow p_n, \Gamma \vDash_{\mathsf{wST_E}} \varphi$ iff $\uparrow p_1, \dots, \uparrow p_n, \Gamma \vDash_{\mathsf{C}} \varphi$. This can be established by noticing that $\mathsf{wST_E}$ and C valuations are the same, and that although their definition of logical consequence is different, the only kind of WK-valuations that are C counterexamples but not $\mathsf{wST_E}$ counterexamples are those where $v[\Gamma \cup \{\uparrow p_1, \dots, \uparrow p_n\}] = \{\mathbf{n}\}$ and $v(\varphi) = \mathbf{f}$. However, by its very construction we know $v[\Gamma \cup \{\uparrow p_1, \dots, \uparrow p_n\}] \subseteq \{\mathbf{t}, \mathbf{f}\}$ and, thus, it is impossible to have these kind of counterexamples. Whence, the equivalence. Secondly, we recall from Observation 4 that $\uparrow p_1, \dots, \uparrow p_n, \Gamma \vDash_{\mathsf{C}} \varphi$ iff $\Gamma \vDash_{\mathsf{CL}} \varphi$. Thirdly, that $\Gamma \vDash_{\mathsf{CL}} \varphi$ iff $\Gamma \vDash_{\mathsf{wST}} \varphi$. Finally, by chaining these equivalences, we arrive at the desired result. □

Now, we may also prove something similar regarding the division between sentences built using unrestricted and possibly meaningless propositional variables, and sentences built using restricted and forcefully meaningful propositional variables. In the context of the framework that we defend, this distinction is ineffective, as it induces no perceptible differences from a logical or inferential point of view. This implies that the same set of valid inferences will be valid within wST regardless of whether the involved sentences are part of the corseted language or not.

Once more, to have a more concrete taste of what this amounts to, consider the division of propositional variables discussed earlier in §3. Thus, $FOR(\mathcal{L})$ is the set of well-formed formulae built using propositional variables in the sets

[17] In this regard, as with Observation 4 above, recall that this result says that all CL-valid inferences hold in wST if and only if all the propositional variables appearing in the formulae involved in the inference in question are *assumed* to be meaningful—taking this assumption as a working hypothesis, rather than an established fact.

Var or Var^*, while $FOR(\mathcal{L}^*)$ is the set of well-formed formulae built using only propositional variables in the set Var^*. Valuations over $\mathbf{FOR}(\mathcal{L})$ are weak Kleene valuations, whereas valuations over $FOR(\mathcal{L}^*)$ are Boolean. Let us refer to the logic induced by the last collection of valuations as wST^*. Then, our claim can be formalized as follows.

Theorem 10. *All the tautologies and valid inferences of* CL *are valid in* wST *regardless of whether all the propositional variables involved are meaningful, i.e.,* $\Gamma \vDash_{\mathsf{wST}} \varphi$ *iff* $\Gamma \vDash_{\mathsf{wST}^*} \varphi$.

Proof. We first notice that it immediately follows from the definitions that $\Gamma \vDash_{\mathsf{wST}^*} \varphi$ iff $\Gamma \vDash_{\mathsf{S}_0} \varphi$. For, when only Boolean valuations are around, defining logical consequence as truth-preservation or as the absence of a valuation that makes all the premises true but the conclusion false, is equivalent. Then, it all follows from the same kind of reasoning of the previous proofs. We know from Observation 5 that $\Gamma \vDash_{\mathsf{S}_0} \varphi$ iff $\Gamma \vDash_{\mathsf{CL}} \varphi$, and we know that $\Gamma \vDash_{\mathsf{CL}} \varphi$ iff $\Gamma \vDash_{\mathsf{wST}} \varphi$. Whence, the emergence of the desired result. □

Our claim, then, is that these three results (the fact that wST shares all of Classical Logic's valid inferences and tautologies without losing substitution-invariance, the fact that wST represents both truth- and non-falsity-preservation, and the fact that wST dissolves the internal versus external connectives divide, along with the need for any linguistic restrictions) make a strong case for our system to be a better significance logic than its contenders. Indeed, we see little room for improvement in what pertains to the logical properties of an ideal significance logic. Be that as it may, we believe to have accomplished what we aimed for: presenting a new logic where the divisions arising from the disputes appearing in the relevant literature were either overcome or dissolved, giving place to a better and more encompassing framework.

5 Conclusions

Between the mid 60s and the late 70s a series of articles (published mostly in the Australasian Journal of Philosophy) debated the legitimacy of even having a significance logic—that is, a logical framework capable of handling nonsensical though grammatical sentences. Among those who were for this project, there was no simple agreement concerning the way in which some of the technicalities should be handled. Some of the problems seemed to concern how to define logical consequence (either as truth-preservation or as non-falsity-preservation), and whether or not certain linguistic restrictions should be considered (either in the form of some external connectives, or in the form of some restricted propositional variables). At the time, these choices were perceived to be forced upon those scholars who wanted to work with significance logics behaving as similar to Classical Logic as possible. However, each of the alternatives had its shortcomings and none was ultimately satisfactory. In this article we presented what we take to be an overcoming proposal which has the same valid inferences and the same tautologies than Classical Logic, enjoys substitution-invariance, and within which the need for the aforementioned divisions is dissolved.

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