Classification of non-well-founded sets and an application

Takashi Nitta
Dept. of Education, Mie University,
Kamihama, Tsu, 514-8507, Japan
E-mail: nitta@edu.mie-u.ac.jp

Tomoko Okada
Graduate school of Mathematics,
Nagoya University, Chikusa-ku,
Nagoya, 464-8602, Japan
E-mail: m98122c@math.nagoya-u.ac.jp

Athanassios Tzouvaras[1]
Dept. of Mathematics, Univ. of Thessaloniki,
540 06 Thessaloniki, Greece
E-mail: tzouvara@auth.gr
(Current address:
Dept. of Mathematics and Statistics,
Univ. of Cyprus, P.O. Box 20537
1678 Nicosia, Cyprus
E-mail: tzouvara@ucy.ac.cy)

[1]Corresponding author

Classification of non-well-founded sets and an application

Abstract

A complete list of Finsler, Scott and Boffa sets whose transitive closures contain 1, 2 and 3 elements is given. An algorithm for deciding the identity of hereditarily finite Scott sets is presented. Anti-well-founded (awf) sets, i.e., non-well-founded sets whose all maximal $\in$-paths are circular, are studied. For example they form transitive inner models of ZFC minus foundation and empty set, and they include uncountably many hereditarily finite awf sets. A complete list of Finsler and Boffa awf sets with 2 and 3 elements in their transitive closure is given. Next the existence of infinite descending $\in$-sequences in Aczel universes is shown. Finally a theorem of Ballard and Hrbáček concerning nonstandard Boffa universes of sets is considerably extended.

Mathematics Subject Classification: 03E30, 03E65.

Keywords: Anti-foundation axiom, non-well-founded set, Aczel, Finsler, Scott, Boffa set, anti-well-founded set.

1 Introduction

As is well known the foundation (or regularity) axiom says that the relation $\in$ is well-founded, i.e., there is no infinite descending $\in$-sequence

$$\cdots \in x_2 \in x_1 \in x_0.$$

Depicting $\in$ by an arrow $\leftarrow$, one turns elementhood relations into directed graphs. For instance the set $1 = \{0\}$ is depicted by the picture $1 \to 0$, or more abstractly, by the graph $a \to b$. In terms of graphs, well-foundedness says that there is no path of the form

$$x_0 \to x_1 \to x_2 \to \cdots.$$

Let ZFC$^-$ be ZFC minus foundation. An *anti-foundation* axiom is a principle which is added to ZFC$^-$ to fill the gap left by the missing foundation, and which, on the one hand postulates the existence of certain non-well-founded sets, and on the other controls their identity. (Remember that the ordinary extensionality axiom is often unable to determine identity of non-well-founded sets.) In 1988, Aczel ([1]) treated in a unified way a host of anti-foundation axioms that had been considered in isolation by several authors along several decades of set theory. These are Aczel's, Scott's, Finsler's and Boffa' axioms, which entail corresponding extensions **A**, **S**, **F** and **B**, respectively, of the standard ZF universe V, such that

$$V \subseteq \mathbf{A} \subseteq \mathbf{S} \subseteq \mathbf{F} \subseteq \mathbf{B}.$$

In the next section instead of formulating the anti-foundation axioms themselves, we shall describe directly the classes **A**, **S**, **F** and **B**.

The representation of sets by graphs allows one to refer to the elements (of the transitive closure) of a set as "nodes". So an n-node set x is a set whose graph consists of n nodes, or equivalently, $|TC(\{x\})| = n$.

In 1962, Richard Peddicord ([10]) computed the number of Zermelo-Fraenkel sets of finite nodes. In 1990, Booth ([5]) counted Finsler 1-node, 2-node and 3-node sets. In 1998, Milito and Zhang ([9]) proposed an algorithm for classifying Aczel sets, and and found an error in Booth's list of 3-node sets.

In section 2 of this paper, we provide the complete list of these sets. In section 3 we give an algorithm for identifying Scott sets, and obtain the number of Scott sets with one, two and three nodes. As a direct consequence we show that Scott sets and Finsler sets coincide with each other in the case of one and two nodes, and show that only two Finsler sets are not Scott sets in the case of three nodes. Generally speaking, it is interesting to construct Finsler sets that are not Scott sets. Dougherty found the first example of a Finsler set with nine nodes and 26 edges that is not a Scott set ([1], p. 55). Later, Moss found a simple example with only three nodes and five edges([1], p. 54). We construct examples of Finsler sets of any number of nodes that are not Scott sets. In particular, we obtain a new example with four nodes and eight edges.

In section 4 we show the existence of Aczel sets with infinite descending $\in$-sequences of any ordinal length, either circular or non-circular. This result is optimal since, as shown in ([12]), there are no infinite descending $\in$-sequences with length On in Aczel universe.

In section 5 we focus on a particular kind of non-well-founded sets, the *anti-well-founded* (awf) ones, which stand quite opposite to the well -founded sets. These are non-well-founded sets whose all maximal $\in$-paths are circular. It is shown that they form transitive inner models of ZFC minus foundation and empty set, and they include uncountably many hereditarily finite awf sets. A complete list of Finsler and Boffa awf sets with 2 and 3 elements in their transitive closure is given.

In section 6 we work with Boffa sets. For these sets Ballard and Hrbáček ([2]) developed a nonstandard universe in a class of urelements which satisfies an extension principle. In this paper we generalize their work to a larger class of sets, which we call "linear" and denote it by g_x. Furthermore we introduce an equivalence relation $\sim$ in a class of linear sets. In particular, we show the following:

EXTENSION PRINCIPLE: *Let U be a universe and κ an infinite cardinal number. Then there exists a κ-saturated universe W and an elementary embedding $F : U \to W$. Moreover, if g_x is a linear set equation of circular type and $(A_{g_x}/\sim) - (A_{g_x} \cap U/\sim)$ is a proper class, then one can assume that $F(x)$ is equal to x for all $x \in A_{g_x} \cap U$, and $A_{g_x} - W$ is a proper class.*

The results of sections 3, 4 and 6 are due to the first two authors. Section 5 is due to the third author[2].

Acknowledgement. The first two authors would like to thank Y. Yonezawa and Y. Yoshinobu for useful discussions, and also L. Weng for correcting an earlier draft of their manuscript.

[2]The first two authors submitted a manuscript to MLQ in February 2001 and in revised form in December 2001. The third author submitted independently another manuscript in November 2001. The two manuscripts happened to contain substantial overlaps, so, upon the Editor's request, we decided to rework them into a joint paper – the present one.

2 Preliminaries

2.1 Non-well-founded set theories

2.1.1 Graphs and systems

In this subsection we recall basic definitions and facts from [1].

A *directed graph* G is a pair $(G, \to)$, where G is a set of *nodes* and $\to$ is a binary relation on G, the set of *edges* of G. We usually write $(a \to b) \in G$, or just $a \to b$, instead of $(a, b) \in \to$. *Paths* are sequences of consecutive edges

$$a_0 \to a_1 \to \cdots \to a_n.$$

A *cycle* in the graph G is a path of the form

$$a_0 \to a_1 \to \cdots \to a_{n-1} \to a_0$$

(an *n-cycle*, i.e., a cycle with n nodes and n edges), or

$$a_0 \to a_1 \to \cdots \to a_n \to \cdots,$$

(an *infinite cycle*).

An *accessible pointed graph*, or *apg* for short, is a triple $(G, \to, a)$, where $(G, \to)$ is a directed graph and a is a distinguished node, the *point* of G, such that any other node of G is connected to a by a finite path.

Given G and $a \in G$, we set:

$a_G = \{b \in G : (a \to b) \in G\}$ (the set of children nodes of a in G),

$G_a =$ the apg with point a and nodes and edges those of G lying on paths starting from a.

A graph G is said to be *extensional* if

$$a_G = b_G \Rightarrow a = b.$$

An apg is said to be *well-founded* if it contains no circular path.

Let V be the set universe. Throughout the letters $a, b, c, \ldots$ are used as labels of nodes of directed graphs, while $x, y, z, \ldots$ range over sets.

A *decoration* of an apg G is a mapping $d : G \to V$ such that for any node $b \in G$, $d(b) = \{d(c) : (b \to c) \in G\}$. An apg G with point a is a *a picture* of a set x, if there is a decoration d of G such that $d(a) = x$. A decoration d of

G is *injective* if it is 1-1. The apg G is said to be an *exact picture* if it has an injective decoration.

A *system* is a pair $(M, \to)$, where M and $\to$ are now *classes* of nodes and edges respectively, and for every $a \in M$, a_M is a set, i.e., each node has set many children. For instance $(V, \ni)$ is a system. For every system M and every node $a \in M$, clearly M_a is an apg.

Let V_0 be the class of all apg's. The elements of V_0 have the form G_a, where G is a graph and a is a node of G, and a is the point of G_a.

We may view the class V_0 as a system if we equip it with edges (G_a, G_b) whenever $a \to b$ is an edge in G.

The relationship between graphs and their decorations, or equivalently, between sets and their pictures, is a powerful tool for exploring the phenomenon of non-well-foundedness. For instance, every well-founded set has a picture which is a well-founded graph. And conversely (by Mostowski's Collapsing Lemma), every well-founded apg G has a unique decoration. This decoration is injective iff G is extensional. Therefore we may identify the universe WF of well-founded sets (which of course is the universe of ZF) with a certain subclass of V_0, namely

$$WF = \{G \in V_0 : \quad G \text{ is extensional and well-founded}\}.$$

2.1.2 Aczel sets

In 1988 Peter Aczel introduced the so-called Aczel's Anti-Foundation Axiom (AFA). This axiom claims that every graph has a unique decoration. AFA can be reformulated in terms of the notion of *system map.* A *system map* π from the system M to the system M' is a map such that for every $a \in M$, the set of children of $\pi(a)$ in M' is equal to the set $\{\pi(b) : b \text{ is a child of } a \text{ in } M\}$. I.e., $(\pi(a))_{M'} = \{\pi(b) : b \in a_M\}$.

Call a system M *strongly extensional* if for every graph G, there is at most one system map $\pi : G \to M$. It is proved (cf [1], p. 28) that AFA can be equivalently formulated as follows:

(AFA) An apg has an injective decoration iff it is strongly extensional.

Let

$$\mathbf{A} = \{G \in V_0 : G \text{ is strongly extensional}\}.$$

A is said to be the *Aczel universe* and we refer to the elements of **A** as *Aczel sets*.

We often write $x =_{Aczel} y$ to indicate that the sets x, y are equal in the sense of Aczel, i.e., they decorate the same graph. The unique set decorating the graph $a \to a$ is denoted by Ω.

2.1.3 Scott sets

In 1960, D. Scott ([11]), motivated by computer science considerations, provided another model of ZFC$^-$. To every apg G_a there corresponds an apg $(G_a)^t$ whose nodes are paths starting from the point a of G_a, and whose edges are pairs of paths of the form

$$(a \to \cdots \to b, a \to \cdots \to b \to b').$$

Let $\cong^t$ be the relation defined on V_0 as follows:

$$G_a \cong^t G'_{a'} \iff (G_a)^t \cong (G'_{a'})^t,$$

(where $\cong$ is the ordinary isomorphism between graphs).

A graph G is said to be *Scott-extensional* if it is $\cong^t$-extensional, i.e., if for any $b, c \in G$

$$G_b \cong^t G_c \;\Rightarrow\; b = c.$$

Let

$$\mathbf{S} = \{G \in V_0 : G \text{ is Scott-extensional}\}.$$

S is the *Scott universe* and the elements of **S** are referred to as *Scott sets*.

2.1.4 Finsler sets

In 1926, P. Finsler ([8]) proposed a group of three axioms as remedy of the paradoxes. The most remarkable of them says that isomorphic sets are equal. Roughly, the axiom is true in the system M, if M satisfies the following extensionality principle: For any $a, b \in M$,

$$M_a \cong M_b \;\Rightarrow\; a = b.$$

However this kind of extensionality does not imply ordinary extensionality, so P. Aczel weakened $\cong$ into a relation $\cong^*$, to the effect that if $a_M = b_M$ then $M_a \cong^* M_b$ (cf. [1], p. 57, for details).

A graph G is said to be *Finsler-extensional* if it is $\cong^*$-extensional i.e., if for all $a, b \in G$,

$$G_a \cong^* G_b \;\Rightarrow\; a = b.$$

Let

$$\mathbf{F} = \{G \in V_0 : G \text{ is Finsler-extensional}\}.$$

$\mathbf{F}$ is the *Finsler universe* and the elements of $\mathbf{F}$ are referred to as *Finsler sets.*

It is known that the Finsler sets constitute the largest universe defined by means of a "bisimulation" (cf. [1], Prop. 4.26 (2)). In particular we have

$$\mathbf{A} \subseteq \mathbf{S} \subseteq \mathbf{F}. \tag{1}$$

2.1.5 Boffa sets

In 1972, M. Boffa ([4]) proposed another type of non-well-founded set theory.

Definition 2.1.1 The system M is said to be a *transitive subsystem of the system M'*, abbreviated $M \prec M'$, if $M \subseteq M'$ and for every $a \in M$, the children of a in M and M' coincide, i.e., $a_M = a_{M'}$.

Boffa's antifoundation axiom is the following statement:

(BA) Given extensional graphs G_0 and G with $G_0 \prec G$ and an injective system map $G_0 \to V$, there is an injective system map $G \to V$ that makes the following diagram commute:

$$\begin{array}{ccc} G & & \\ \uparrow & \searrow & \\ G_0 & \longrightarrow & V \end{array}$$

(Equivalently, every exact decoration of a transitive subgraph of an extensional graph can be extended to an exact decoration of the whole graph.)

When working in Boffa universe, we usually need a strong axiom of global choice. The most suitable is von Neumann's axiom of choice $|V| = |On|$, saying that there is a bijection between V and On. So henceforth we fix a bijection $C : On \to V$.

In some places we do not need the full strength of the axiom BA but only "half" of it, namely the following consequence of BA:

(BA_1) An apg is an exact picture iff it is extensional.

BA_1 gives the most generous answer to the question "which apg's are exact pictures".

We denote by **B** a model of ZFC$^-$+BA or even of ZFC$^-$+BA_1 and we refer to elements of **B** as *Boffa sets*. For example, contrary to what happens in the universes **A**, **S** and **F**, in **B** there are class many distinct copies of the set $\Omega = \{\Omega\}$. More generally the following holds:

Lemma 2.1.2 *In* ZFC$^-$ + BA_1, *for every extensional graph G, there is a proper class of sets which are pictures of G.*

Proof. For any cardinal κ take a set of κ distinct copies of the graph G. These are easily made parts of an extensional apg E. By BA_1, there is an injective decoration of E. Thus we get κ distinct decorations for the copies of G. $\dashv$

2.2 Booth's classification of Finsler sets

Let $TC(x)$ denote the transitive closure of x. Following D. Booth, we call *level* of the set x the cardinality of $TC(x \cup \{x\})$. For $n \in \mathbb{N}$, clearly, x is of level n iff it decorates an apg of n nodes. Contrary to what happens with well-founded sets, a set may be hereditarily finite and of infinite level.

For any $n > 0$, let

$$S_n \text{ (resp. } F_n) = \{G \in \mathbf{S} \text{ (resp. } \mathbf{F}) : G \text{ is of level } n\}.$$

Let also

$$s_n = \sharp S_n \text{ and } f_n = \sharp F_n.$$

Theorem 2.2.1 (Booth ([5],[6]))

$$f_1 = 2, f_2 = 5, f_3 = 78.$$

As remarked in [9], Booth's classification method, proof and counting are correct, but there is an inaccuracy in Booth's list of F_3. Below, we make a correction. Our notation is the same as in [5] and [6]. Moreover in the following list, we identify sets of equations with graphs in the obvious way. For example, the equation $x = \{x\}$, is identified with $x \to x$.

Our list is as follows:

F_1: Sets of level one.

(1) $x = 0$; (2) $x = \{x\}$.

F_2: Sets of level two. First let x be the point.

(1) $x = \{y\}, y = 0$; (2) $x = \{x, y\}, y = 0$; (3) $x = \{x, y\}, y = \{y\}$.

Now let both x and y be points. It defines two sets.

$x = \{y\}, y = \{x, y\}$.

F_3: Sets of level three.
First, let all x, y, z be points.

(i) $x = \{y\}, y = \{z\}, z = \{x, y\}$
(ii) $x = \{y\}, y = \{z\}, z = \{x, z\}$
(iii) $x = \{y\}, y = \{z\}, z = \{x, y, z\}$
(iv) $x = \{y\}, y = \{x, z\}, z = \{y, z\}$
(v) $x = \{y\}, y = \{x, z\}, z = \{x, y\}$
(vi) $x = \{y\}, y = \{x, z\}, z = \{x, y, z\}$
(vii) $x = \{y\}, y = \{y, z\}, z = \{x, y\}$
(viii) $x = \{y\}, y = \{y, z\}, z = \{x, z\}$
(ix) $x = \{y\}, y = \{y, z\}, z = \{x, y, z\}$
(x) $x = \{y\}, y = \{x, y, z\}, z = \{x, y\}$
(xi) $x = \{y\}, y = \{x, y, z\}, z = \{y, z\}$
(xii) $x = \{y\}, y = \{x, y, z\}, z = \{x, z\}$
(xiii) $x = \{y, z\}, y = \{x, y\}, z = \{x, y, z\}$
(xiv) $x = \{x, y\}, y = \{y, z\}, z = \{x, y, z\}$.

Then let x and y be points.

(1a) $x = \{y\}, y = \{x, z\}, z = 0$
(1b) $x = \{y\}, y = \{x, z\}, z = \{z\}$
(2a) $x = \{y\}, y = \{x, y, z\}, z = 0$
(2b) $x = \{y\}, y = \{x, y, z\}, z = \{z\}$
(3a) $x = \{y, z\}, y = \{x, y\}, z = 0$
(3b) $x = \{y, z\}, y = \{x, y\}, z = \{z\}$
(4a) $x = \{y, z\}, y = \{x, y, z\}, z = 0$
(4b) $x = \{y, z\}, y = \{x, y, z\}, z = \{z\}$
(5a) $x = \{x, y\}, y = \{x, y, z\}, z = 0$
(5b) $x = \{x, y\}, y = \{x, y, z\}, z = \{z\}$.

Finally, let x be the only point.

(1) $x = \{y\}, y = \{z\}, z = 0$
(2) $x = \{y\}, y = \{y, z\}, z = 0$
(3) $x = \{x, y\}, y = \{y, z\}, z = 0$
(4) $x = \{x, y\}, y = \{y, z\}, z = 0$
(5) $x = \{y\}, y = \{y, z\}, z = \{z\}$
(6) $x = \{x, y\}, y = \{y, z\}, z = \{z\}$
(7) $x = \{y, z\}, y = \{z\}, z = 0$
(8) $x = \{x, y, z\}, y = \{z\}, z = 0$
(9) $x = \{x, y, z\}, y = \{y, z\}, z = 0$
(10) $x = \{x, y, z\}, y = \{y, z\}, z = \{z\}$
(11) $x = \{y\}, y = \{z\}, z = \{y, z\}$
(12) $x = \{x, y\}, y = \{z\}, z = \{y, z\}$
(13) $x = \{x, z\}, y = \{z\}, z = \{y, z\}$
(14) $x = \{x, y, z\}, y = \{z\}, z = \{y, z\}$
(15) $x = \{y, z\}, y = \{y\}, z = 0$
(16) $x = \{x, y, z\}, y = \{y\}, z = 0$.

Booth's list differs from the preceding one with respect to items (15) and (16). In Booth's list item (15) is $x = \{x, y, z\}, y = \{y, z\}, z = \{0\}$, which coincides with (9) above, and item (16) is $x = \{x, y, z\}, y = \{y, z\}, z = \{y, z\}$, which is not a Finsler set.

3 Classification of Scott sets

In this section, we compare Scott and Finsler sets of level 1, 2 and 3. It is easy to see that for levels 1 and 2 $F_1 = S_1$ and $S_2 = F_2$. However $F_3 \neq S_3$. A simple example was provided by Moss and Johnson ([1], p. 55). This is the following:

$$x = \{y\}, y = \{x, z\}, z = \{x, y\}; \;\; x = \{x, y\}, y = \{z\}, z = \{y, z\}. \qquad (2)$$

These are items (v) and (12) in the list of the previous section. We shall prove below that these are the only sets in $F_3 - S_3$. In fact there is a general algorithm for checking whether an apg G of finite level corresponds to a Scott set. The algorithm is as follows:

Step 1: Check whether there are nodes p, q with same number of children. If there are no such nodes the apg is a Scott set. Hence the following apgs are Scott sets: (vi), (ix), (x), (xi), (xii), (1a), (2a), (4a), (4b), (5a), (5b), (2), (3), (7), (8), (9), (10), (14), (15) and (16).

Step 2: Suppose there are nodes p, q with same number of children. For each such pair let $p_1, p_2, \ldots, p_m$ be the children of p, and let $q_1, q_2, \ldots, q_m$ be the children of q. Let M_i, N_i be the number of children of p_i, q_i respectively for every $i = 1, \ldots, m$. Consider the following condition:

(*) There is a permutation $\sigma \in S(m)$ such that

$$(M_{\sigma(1)}, M_{\sigma(2)}, \ldots, M_{\sigma(m)}) = (N_1, N_2, \ldots, N_m).$$

If no pair p, q as above satisfies (*), then clearly the apg is a Scott set. For example, in (i) $x = \{y\}, y = \{z\}$ and $z = \{x, y\}$, x and y have the same number of elements. The element of x is y, the element of y is z, and y has one element, while z has two elements. Similarly the following apgs are Scott sets:

(i), (ii), (iii), (iv), (vii), (viii), (xiii), (xiv), (1b), (2b), (3a), (3b), (1), (4), (5), (6), (11) and (13).

Step 3: Suppose there are pairs of nodes p, q which satisfy condition (*). We examine them further as follows. Let the children of of p_i be

$p_{i1}, p_{i2}, \ldots, p_{in_i}$, and the elements of q be $q_{i1}, q_{i2}, \ldots, q_{in'_i}$, for $i \leq m$. Let also M_{ij}, N_{ij} denote the number of elements of p_{ij}, q_{ij} respectively.

Consider the following condition:

(**) There is a permutation $\sigma \in S(m)$ such that condition (*) above holds and moreover for every $i \leq m$, there is a permutation $\sigma_i \in S(n'_i)$ such that

$$(M_{\sigma(i),1}, M_{\sigma(i)2}, \ldots, M_{\sigma(i)n_{\sigma(i)}}) = (N_{i,\sigma_i(1)}, N_{i,\sigma_i(2)}, \ldots, N_{i,\sigma_i(n'_i)}).$$

If condition (**) is satisfied for no nodes p, q, then the apg is a Scott set. In the case of F_3, (v) and (12) satisfy the third step.

In general, the n-th step consists in formulating a condition $(* \cdots *)$ generalizing the preceding ones in the obvious (thought complicated) way. If an apg G with n nodes satisfies all conditions till the n-th step, then the apg is not a Scott set. In fact, since the apg is unfolded periodically with a period less than n, $(G_p)^t \cong (G_q)^t$ for some p, q. Hence (v) and (12) are not Scott sets. In this way we obtain:

Theorem 3.0.2 $s_1 = 2, s_2 = 5$ *and* $s_3 = 74$.

Using the examples (v) or (12) we construct Finsler sets of nodes $n \geq 3$ that are not Scott sets as follows:

$$x_i = \{x_{i+1}\}(1 \leq i \leq n-3),$$
$$x_{n-2} = \{x_{n-1}\}, x_{n-1} = \{x_{n-2}, x_n\}, x_n = \{x_{n-2}, x_{n-1}\}.$$

That is to say, we obtain the following:

Theorem 3.0.3 *For any* $n \geq 3$,

$$S_n \neq F_n.$$

We can illustrate the above algorithm by constructing examples of Finsler sets that are not Scott sets in the spirit of equations (2). In particular, as a new example we have in $F_4 - S_4$,

$$x = \{x, y\},\ y = \{x, t\},\ z = \{y, t\},\ t = \{z, t\}.$$

Remarks 3.0.4 Milito and Zhang ([9]) obtained an algorithm for deciding Aczel sets. In general, as commented in [9], it is difficult to construct an algorithm of deciding Finsler sets.

4 Existence of infinite descending $\in$-sequences

For any ordinal α, we shall prove that there exist both circular and non-circular paths of length α in Aczel set theory. Following [12], we define an α-path of a system as follows.

Definition 4.0.5 ([12]) Let X be a system and α be an ordinal. An *α-path* in X is a class of nodes Y well-ordered by an ordering $<$ such that:

a) For any $x \in Y, (x, x')$ is an edge of X, where x' is the immediate successor of x in $(Y, <)$.

b) If x is a limit point of $(Y, <)$, then there is a $y_0 \in Y$ such that $y_0 < x$ and (y, x) are edges of X for all $y \in Y$ with $y_0 < y < x$.

c) $ord(Y, <) = \alpha$.

We call α the *length* of Y. A path has always a first element but need not have a last one. If it does, and x, y are these elements, respectively, then we say that the path joins x and y. If Y is an X-path joining the nodes x and y, and (y, x) is an edge of X, then Y is said to be *circular*.

Theorem 4.0.6 *Let $\alpha \in On$. There are Aczel sets containing non-circular paths of length α, as well as Aczel sets containing circular paths of length α.*

Proof.

(1) Existence of non-circular paths.

For an arbitrary ordinal α, we define a graph G^α as follows:

For $\beta < \alpha$, let $G(\beta)$ be an apg identifying to β and let p_β be the point of $G(\beta)$.

Nodes: $\{(\alpha, \beta) \in \{\alpha\} \times \mathrm{On} : \beta < \alpha\} \cup \{\text{nodes of } G(\beta) : \beta < \alpha\}$;

Edges: $\{(\alpha, \beta) \to (\alpha, \gamma) : \beta < \gamma < \alpha\} \cup \{\text{edges of } G(\beta) : \beta < \alpha\} \cup \{(\alpha, \beta) \to p_\beta, \beta < \alpha\}$.

Clearly the graph G^α has point $(\alpha, 0)$.

Claim: the apg G^α is an Aczel set and it has a non-circular path of length α.

Proof of the claim: Suppose G^α is not an Aczel set, that is, there exist two nodes in G^α which are decorated by identical Aczel sets. Let da be a decoration of the node a. Then there exist ordinals β, γ such that $\beta < \gamma < \alpha$. Moreover, we have

(i) $d(\alpha, \gamma) =_{Aczel} \beta$ or (ii) $d(\alpha, \beta) =_{Aczel} \gamma$ or (iii) $d(\alpha, \beta) =_{Aczel} d(\alpha, \gamma)$.

Case (i) is impossible because $\beta < \gamma$ and $\gamma \in d(\alpha, \gamma)$.

Assume case (ii). Since $\gamma < \alpha$, there exists an ordinal δ such that $\gamma \leq \delta < \alpha$. Since $d(\alpha, \delta) \in d(\alpha, \beta)$ and $d(\alpha, \beta) =_{Aczel} \gamma$, $d(\alpha, \delta)$ is in γ. But $\delta \in d(\alpha, \delta)$ and $\gamma \leq \delta$ implies that $d(\alpha, \delta) \notin \gamma$. This is a contradiction.

Assume case (iii). Since β is an element of $d(\alpha, \beta)$ and children of (α, γ) are $d(\alpha, \delta)$ $(\gamma < \delta)$ and γ, there exists $\varepsilon \in On$ such that $\gamma < \varepsilon < \alpha$ and $\beta =_{Aczel} d(\alpha, \varepsilon)$. This is a contradiction by the same argument as in case (i).

Therefore G^α is an Aczel set. The following path of G^α is a non-circular path of length α:

$$(\alpha, 0) \to (\alpha, 1) \to (\alpha, 2) \to \cdots \to (\alpha, \omega) \to (\alpha, \omega + 1) \to (\alpha, \omega + 2) \to \cdots$$

$$\to (\alpha, \omega') \to (\alpha, \omega' + 1) \to (\alpha, \omega' + 2) \to \cdots \to (\alpha, \beta) \to (\alpha, \beta + 1) \to$$

$$(\alpha, \beta + 2) \to \cdots \to (\alpha, \beta') \to (\alpha, \beta' + 1) \to (\alpha, \beta' + 2) \to \cdots$$

where for a limit ordinal λ, the next limit ordinal is denoted by λ' , and $0, \omega, \omega', \ldots, \beta, \beta', \ldots$ is the sequence of limit ordinals of $\{\mu : \mu \in On, \mu < \alpha\}$.

(2) Existence of circular paths.

For an arbitrary ordinal α, we define a graph G'_α as follows.

Nodes: $\{(\alpha, \beta) \in \{\alpha\} \times \mathrm{On}, \beta \in \mathrm{On}, \beta \leq \alpha\} \cup \{\text{node of } G(\beta), \beta \in \mathrm{On}, \beta \leq \alpha\}$;

Edges: $\{(\alpha, \beta) \to (\alpha, \gamma) : \beta < \gamma \leq \alpha\} \cup \{\text{edges of } G(\beta) : \beta \leq \alpha\} \cup \{(\alpha, \beta) \to p_\beta : \beta \leq \alpha\}$.

As in (1), G'_α has a circular path of length α.

Theorem 4.0.7 *Let α be an ordinal and let $f : \alpha \to On$ be an increasing function. Then there are Aczel sets corresponding to f, containing both circular paths and non-circular paths of length α.*

Proof.

This may be done by replacing $G(\beta)$ by $G(f(\beta))$ in 4.0.6. $\dashv$

Corollary 4.0.8 *There exist uncountably many Aczel sets in which there are both circular and non-circular paths of length α.*

Proof.

There are uncountably many increasing functions $f : \alpha \to On$, so the claim follows from the previous theorem. $\dashv$

Corollary 4.0.9 *There exist uncountably many Aczel sets containing non-circular infinite descending $\in$-sequences.*

Proof.

We fix an increasing function $f : \omega \to On$, and define a graph G as follows:

$$G = \{(a_i \to a_{i+1}) : 0 \le i < \omega\} \cup \{(a_i \to p_i) : 0 \le i < \omega\} \cup (\cup_{i \in \mathbb{N}} G(f(i))),$$

where we denote the point of $G(f(i))$ by p_i. Then $a_0 \to a_1 \to a_2 \to \cdots$ is an infinite descending $\in$-sequence. Since increasing functions $f : \omega \to On$ are uncountably many, there exist uncountably many non-circular infinite descending $\in$-sequences. $\dashv$

Recall that x is hereditarily finite if $TC(x)$ is finite.

Corollary 4.0.10 *There exist uncountably many hereditarily finite Aczel sets, in which there are both circular and non-circular paths of length α.*

Proof.

As the proof of Corollary 4.0.9 above, just consider increasing functions $f : \omega \to \omega$. $\dashv$

5 Anti-well-founded sets

In this section we deal with a special kind of non-well-founded sets, which lie at the antipodes of well-founded ones. This is why we call them anti-well-founded.

Recall that an apg is well-founded if it contains no circular path. Otherwise it is said to be *circular*.

Definition 5.0.11 An apg G is said to be *totally circular* (t.c. for short) if every maximal path of G starting from its point is circular.

A set x is said to be *anti-well-founded* (awf for short) if it decorates a t.c. apg.

The simplest finite awf sets are those corresponding to the cyclic graphs C_i, $i \geq 0$, where C_i is the $(i+1)$-node cycle

$$a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_{i-1} \rightarrow a_0.$$

Let Ω_i be the awf (if it exists) whose picture is the graph C_i. In particular $\Omega_0 = \Omega$.

The following is easy.

Lemma 5.0.12 *x is an awf set iff $TC(x)$ contains neither $\emptyset$, nor urelements.*

As is well known all Aczel sets with the property of the above lemma are identical to Ω. Therefore there are no Aczel awf sets except Ω. So such entities live only in Scott, Finsler and Boffa universes. Especially in Boffa universes, sets come up (as we have seen) in proper classes of isomorphic copies. A *type* is a class of isomorphic sets. Each type also corresponds to a particular apg which is the exact picture of the members of the class. For example, to each graph C_i there corresponds the type Ω_i, of all sets decorating C_i, i.e.,

$$\Omega_i = \{x : x \text{ is a decoration of } C_i\}.$$

In particular $\Omega_0 = \Omega$. Note that, due to symmetry, every node of the graph C_i, can be taken as the point of C_i. Also, if d is an injective decoration of C_i, then for any two nodes a, b of C_i, $d(a) \cong^* d(b)$, i.e., $d(a), d(b) \in \Omega_i$.

Recall that **B** and **F** denote the Boffa and Finsler universes respectively. Let AWF^B and AWF^F be the classes of all Boffa and Finsler awf sets respectively. Obviously,

$$AWF^F \subset AWF^B.$$

Clearly, $\emptyset \notin AWF^B$. So, for any x let $\mathcal{P}^B(x) = \mathcal{P}(x) \cap AWF^B$ and $\mathcal{P}^F(x) = \mathcal{P}(x) \cap AWF^F$. $\mathcal{P}^B$, $\mathcal{P}^F$ are the powerset operations suitable for the classes AWF^B and AWF^F. For example the following is easy to check.

Lemma 5.0.13 *For every* $0 \leq i \leq \infty$, $\mathcal{P}^B(\Omega_i) = \{\Omega_i\}$.

Let ZFC^{--} be ZFC minus the foundation and empty set axioms. Let also

(BA$_1^c$) A t.c. apg is an exact picture iff it is extensional,

(FAFA^c) A t.c. apg is an exact picture iff it is $\cong^*$-extensional.

Theorem 5.0.14 *i)* AWF^B *is a transitive inner model of* ZFC^{--}+BA$_1^c$.
ii) AWF^F *is an inner model of* ZFC^{--}+AFA^c

Proof. Obviously AWF^B is a definable transitive subclass of **B**.

i) Extensionality is obvious.

2) Pairing: If $x, y \in AWF^B$, then clearly $\{x, y\} \in AWF^B$. Similarly,

3) Union: If $x \in AWF^B$ then $\cup x \in AWF^B$, and

4) Powerset: if $x \in AWF^B$ then $\mathcal{P}_B(x) = \mathcal{P}(x) - \{\emptyset\}$ and $\mathcal{P}^B(x) \in AWF^B$.

5) Infinity: Obvious since AWF^B contains proper classes of isomorphic sets, e.g. $\Omega_0 = \Omega, \Omega_1, \ldots$.

6) Separation: Clearly if $x \in AWF^B$ and $y \subseteq x$ and $y \neq \emptyset$, then $y \in AWF^B$.

7) Replacement: Let $\phi(x, y)$ be a relation such that $AWF^B \models (\forall x)(\exists! y)\phi(x, y)$ and let $z \in AWF^B$. Then clearly the set $u = \{y : (\exists x \in z)\phi(x, y)\}$ belongs to AWF^B.

8) Choice: Let $A \in AWF^B$ such that $x \in A \Rightarrow x \neq \emptyset$. By the choice of the ground model there is f such that $f(x) \in x$ for every $x \in A$. Since for every $(x, y) \in f$, both x, y are awf sets we easily see that f is awf, i.e., $f \in AWF^B$. Thus there is a choice function for A in AWF^B.

9) BA_1^c: Let G be an extensional t.c. apg in the sense of AWF^B. Then this is t.c. in the sense of **B**, hence, by BA_1, there is an injective decoration x. But then x is awf, hence $x \in AWF^B$. Conversely, if G has an injective decoration in AWF^B, this is an injective decoration in **B**, therefore G is extensional.

ii) Everything is as in (i) above except Infinity: Define the sets x_n as follows: $x_0 = \Omega$, $x_{n+1} = \{x_{n+1}, x_n\}$. For every n, $|TC(x_n)| = n + 1$, hence $x_n \neq x_m$ for $m \neq n$. Thus x_n, $n \in \mathbb{N}$, are distinct awf Finsler sets. $\dashv$

Because BA_1 produces types of isomorphic sets which are proper classes, when considering Boffa sets it would be better to switch from ZFC^-, to GBC^- (Gödel-Bernays theory of classes). Also, because most often we have to deal with representatives of these types, we need a strong axiom of choice SC enabling us to choose elements from classes in general instead only from sets. For example SC could be von Neumann's axiom of choice $|\mathbf{B}| = |On|$, or the principle $(\forall x)(\exists Y)\phi(x, Y) \Rightarrow (\exists Y)(\forall x)\phi(x, Y_{(x)})$. Due to such choice facilities, we can use the symbols C_i and Ω_i, $0 \leq i \leq \infty$, a bit vaguely, either to denote the corresponding types of objects or arbitrary representatives of them.

An apparent shortcoming of the classes AWF^B and AWF^F is that, in absence of $\emptyset$, they do not contain ordinary natural and ordinal numbers. However we might use convenient substitutes. The first thought is to define ordinals as usually, just replacing $0 = \emptyset$ by Ω. However it does not work, because the next ordinal $\{\Omega\}$ is identical to Ω.

One might also consider the awf sets Ω_i, $0 \leq i \leq \infty$, themselves as substitutes of natural numbers, and define $\Omega_n + \Omega_m = \Omega_{m+n}$ and $\Omega_n \cdot \Omega_m = \Omega_{m \cdot n}$. Putting for every $n \in \mathbb{N}$, $\overline{n} = \Omega_n$ (the natural numbers in the sense of AWF^B), we can provide substitutes α^* for all ordinals α, by setting $\overline{\omega} = \{\overline{0}, \overline{1}, \ldots\}$, and for all $\alpha \geq \omega$, $\overline{(\alpha + 1)} = \overline{\alpha}, \cup \{\overline{\alpha}\}$, $\overline{\alpha} = \cup \{\overline{\beta} : \beta < \alpha\}$. The ordering $<$ between ordinals is defined in the obvious way.

However $\Omega_i \notin AWF^F$, for $i > 0$, so the above definition does not work in Finsler universe. We may slightly alter our first attempt and define n^* as x_n in the proof of 5.0.14 (ii). Namely we set for every $n \in \mathbb{N}$:

$$0^* = \{0^*\} = \Omega, \qquad (n + 1)^* = n^* \cup \{(n + 1)^*\},$$

i.e., $n^* = \{0^*, 1^*, \ldots, n^*\}$ for every n.

Let $\omega^* = \{0^*, 1^*, \ldots\}$. We can see by induction that the graph of every n^* is $\cong^*$-extensional, therefore $\omega^* \subset AWF^F$ and also $n \neq m \Rightarrow n^* \neq m^*$.

Then we can continue our definition "classically", setting, for all $\alpha \geq \omega$, $\alpha^* = \{\beta^* : \beta \in \alpha\}$. Again inductively it is shown that α^* is $\cong^*$-extensional. If $On^* = \{\alpha^* : \alpha \in On\}$, then $On^* \subset AWF^F$ and also $\alpha \neq \beta \Rightarrow \alpha^* \neq \beta^*$.

In AWF^F the "ordinals" α^* are unique but in AWF^B, due to the existence of class-many copies of Ω, there are class many copies for each a^*.

5.1 The structure of Boffa and Finsler awf sets

Here we describe briefly the general method of producing all Boffa and Finsler awf sets. To gain intuition it is better to work with t.c. graphs rather than awf sets themselves. However the transition from the one to the other is straightforward.

The cycles C_i, $0 \leq i \leq \infty$, are, in a sense, the simplest non-reducible t.c. apg's. In order to find out the structure of all t.c. graphs we have to consider natural generalizations of them.

Let G be an apg. For any two nodes $a, b \in G$ we set $a \sim_G b$ if there is a path from a to b and a path from b to a. We can immediately check that $\sim$ is an equivalence relation. We can write just $\sim$ if there is no danger of confusion. A graph G is said to be a *generalized cycle* if for any two $a, b \in G$, $a \sim b$. Therefore, given G, the equivalence classes $[a]_\sim$ of G with respect to $\sim$ are maximal generalized cycles in G.

Generalized cycles may be either finite or infinite graphs. Note that if G is a generalized cycle, then every node of G defines a point, i.e., for every $a \in G$, G_a is an apg. In **B** we are interested in extensional such graphs, while in **F** we are interested in $\cong^*$-extensional such graphs. E.g. the cycles C_i exist in **B** but not in **F**. However **F** does contain generalized cycles.

Call an awf set x of AWF^B or AWF^F *cyclic* if its graph is a generalized cycle. In the next section we specify the number of cyclic sets x of AWF^B and AWF^F with $|TC(\{x\})| = 3$.

Let G be a graph. Given two classes $[a]$ and $[b]$ of G we write $[a] \preceq_G [b]$ if there is at least one path in G leading from some (and hence from every) node of $[a]$ to some (and hence to every) node of $[b]$. It is easy to see that $\preceq_G$ is a partial ordering. It suffices to check only that $[a] \preceq_G [b]$ and $[b] \preceq_G [a]$ implies $[a] = [b]$. Indeed if $[a] \preceq_G [b]$ and $[b] \preceq_G [a]$ there is a path from a to b and a path from b to a, therefore $a \sim b$ or, $[a] = [b]$.

Now the paths between two generalized cycles $[a]$, $[b]$ of G may be multiple and also of various lengths, subject only to the constraint of extensionality.

Definition 5.1.1 Given an apg G, the *extensional* (resp. $\cong^*$*-extensional) collapse of* G *is the apg* G' *resulting from* G *if we identify all the nodes* a, b *such that* $a_G = b_G$ *(resp. as well as the nodes* a, b *such that* $G_a \cong^* G_b$*).*

The above sum up to the following:

Theorem 5.1.2 *Every extensional (resp.* $\cong^*$*-extensional) totally circular graph is generated as follows: Take an ordering* $(X, x_0, \preceq)$ *with first element* x_0*. Replace every point* $x \in X$ *by an extensional (resp.* $\cong^*$*-extensional) generalized cycle* G_x*, or by a single node if* $x = x_0$*. Draw various paths from* G_x *to* G_y *iff* $x \preceq y$ *not forming new cycles. Then take the extensional collapse (resp.* $\cong^*$*-extensional) collapse of this graph.*

The above specify also the method for generating Boffa (resp. Finsler) awf set.

5.2 Hereditarily finite awf sets

D. Booth [5] provides some results concerning hereditarily finite Finsler sets. Among others, he specifies all sets whose transitive closures contain 2 and 3 elements. The corresponding problem here is to determine the isomorphism types of Boffa and Finsler awf sets with 2 and 3 sets in their transitive closure. We do it by an exhausting inspection of all t.c. apg's with 2 and 3 nodes.

Following D. Booth, we call *level* of the set x the cardinality of $TC(\{x\})$. For $n \in \mathbb{N}$, clearly, x is of level n iff it decorates an apg of n nodes. Contrary to what happens with well-founded sets, a set may be hereditarily finite and of infinite level.

Obviously Ω is the only awf set of AWF^F of level 1, and the only isomorphism type of awf sets of AWF^B of level 1.

Proposition 5.2.1 *i)* AWF^B *contains* 4 *isomorphism types of awf sets of level* 2*.*

ii) AWF^F *contains* 3 *awf sets of level* 2*.*

Proof. i) In AWF^B we have the following isomorphism types:

(1) Two types determined by the sets x, y defined by

$$x = \{x, y\}, \qquad y = \{x\}.$$

(2) One type determined by the set $z = \{z, \Omega\}$.
(3) One type determined by the equations

$$x = \{y\}, \qquad y = \{x\}.$$

x, y decorate the graph C_2, and determine the same type since $x \cong^* y$. It is easy to see that these are the only types possible.

i) In AWF^F we have only the first 3 sets of the above list. The graph C_2 is not $\cong^*$-extensional, so it is not decorated by Finsler sets. $\dashv$

Proposition 5.2.2 *i) AWF^B contains 74 isomorphism types of awf sets of level 3.*

ii) AWF^F contains 59 awf sets of level 3.

Proof. i) We have the following isomorphism types of the sets of level 3. We give the circular definitions of the sets. Besides each definition we give a triple of the form $k - l - m$, where $k, l, m \in \{0, 1, 2, 3\}$, which indicates that the corresponding graph contains k 3-cycles, l 2-cycles and m 1-cycles. E.g. the triple 0-2-3 means that we have no 3-cycles, two 2-cycles and three 1-cycles. The 74 isomorphism types of Boffa's awf sets are as follows:

(1) $x = \{y\}$, $y = \{y, \Omega\}$. (0-0-2. One type of set, x. y is of level 2.)

(2) $x = \{\Omega', \Omega''\}$. (0-0-2. One type. $\Omega' \cong^* \Omega''$ are distinct copies of sets of type Ω.)

(3) $x = \{x, y\}$, $y = \{y, \Omega\}$ (0-0-3. One type, x. y is of level 2.)

(4) $x = \{x, \Omega', \Omega''\}$ (0-0-3. One type, $\Omega' \cong^* \Omega'' \cong^* \Omega$.)

(5) $x = \{x, y, \Omega\}$, $y = \{y, \Omega\}$. (0-0-3. One type, x.)

(6) $x = \{y\}$, $y = \{z\}$, $z = \{y\}$. (0-1-0. One type, x. y, z are of level 2 and $y \cong^* z$.)

(7) $x = \{x, y\}$, $y = \{z\}$, $z = \{y\}$. (0-1-1. One type, x. $y \cong^* z$.)

(8) $x = \{y\}$, $y = \{z\}$, $z = \{y, z\}$. (0-1-1. One type, x.)

(9) $x = \{x, y, z\}$, $y = \{z\}$, $z = \{y\}$. (0-1-1. One type, x. $y \cong^* z$.)

(10) $x = \{y\}$, $y = \{x, \Omega\}$. (0-1-1). Two types.)

(11) $x = \{y, \Omega\}$, $y = \{x, \Omega\}$. (0-1-1. One type, $x \cong^* y$.)

(12) $x = \{x, y\}$, $y = \{y, z\}$, $z = \{y\}$. (0-1-2. One type, x.)

(13) $x = \{x, y\}$, $y = \{z\}$, $z = \{y, z\}$ (0-1-2. One type, x.)
(14) $x = \{x, y, z\}$, $y = \{z\}$, $z = \{y, z\}$. (0-1-2. One type, x.)
(15) $x = \{y\}$, $y = \{x, \Omega\}$. (0-1-2. Two types.)
(16) $x = \{x, y\}$, $y = \{x, \Omega\}$. (0-1-2. Two types.)
(17) $x = \{x, y, \Omega\}$, $y = \{x, \Omega\}$. (0-1-2. Two types.)
(18) $x = \{x, y\}$, $y = \{x, y, \Omega\}$. (0-1-3. Two types.)
(19) $x = \{y, z\}$, $y = \{y, x\}$, $z = \{x\}$. (0-2-1. Three types.)
(20) $x = \{x, y, z\}$, $y = \{x, y\}$, $z = \{x, z\}$. (0-2-2. Three types.)
(21) $x = \{y, z\}$, $y = \{x, y\}$, $z = \{x, z\}$. (0-2-2. Two types, $y \cong^* z$.)
(22) $x = \{x, y, z\}$, $y = \{x, y\}$, $z = \{x, z\}$. (0-2-3. Two types, $y \cong^* z$.)
(23) $x = \{y\}$, $y = \{z\}$, $z = \{x\}$. (1-0-0. One type, $x \cong^* y \cong^* z$.)
(24) $x = \{x, y\}$, $y = \{z\}$, $z = \{x\}$. (1-0-1. Three types.)
(25) $x = \{y\}$, $y = \{y, z\}$, $z = \{x, z\}$. (1-0.2. Three types.)
(26) $x = \{x, y\}$, $y = \{y, z\}$, $z = \{x, z\}$. (1-0-3. One type, $x \cong^* y \cong^* z$.)
(27) $x = \{y\}$, $y = \{z\}$, $z = \{x, y\}$. (1-1-0. Three types.)
(28) $x = \{y\}$, $y = \{y, z\}$, $z = \{x, y\}$. (1-1-1 Three types.)
(29) $x = \{y\}$, $y = \{z\}$, $z = \{y, z\}$. (1-1-1 Three types.)
(30) $x = \{xy\}$, $y = \{z\}$, $z = \{x, y, z\}$. (1-1-2. Three types.)
(31) $x = \{y\}$, $y = \{y, z\}$, $z = \{x, y, z\}$. (1-1-2 (iii). Three types.)
(32) $x = \{x, y\}$, $y = \{y, z\}$, $z = \{x, y, z\}$. (1-1-3. Three types.)
(33) $x = \{y, z\}$, $y = \{x, z\}$, $z = \{x\}$. (1-2-0. Three types.)
(34) $x = \{x, y, z\}$, $y = \{x, z\}$, $z = \{x\}$. (1-2-1. Three types.)
(35) $x = \{y, z\}$, $y = \{x, y, z\}$, $z = \{x\}$. (1-2-1. Three types.)
(36) $x = \{y, z\}$, $y = \{y, z\}$, $z = \{x\}$. (1-2-2. Three types.)
(37) $x = \{y, z\}$, $y = \{x, z\}$, $z = \{x, y\}$. (1-3-0. One type, $x \cong^* y \cong^* z$.)
(38) $x = \{x, y, z\}$, $y = \{x, z\}$, $z = \{x, y\}$. (1-3-1. Two types, $y \cong^* z$.)

ii) The awf sets of AWF^F result from the graphs of Appendix if we discard those which are not $\cong^*$-extensional. Equivalently, it suffices to discard clauses (2), (4), (6), (7), (9), (11), (21), (22), (23), (26), (37), (38) of the list of (i) above. These contain total 15 sets, therefore the distinct awf sets of AWF^F are 74-15=59. ⊣

The above 59 awf Finsler sets should be identical to those calculated by D. Booth ([5], Th. 15), if we drop from his list the Finsler sets whose transitive closure contains $\emptyset$. However there is some divergence. Booth's list contains

78 Finsler sets of which 16 involve $\emptyset$. Therefore his awf Finsler sets are 78-16=62. The divergence is due to the fact, already mentioned in section 2, that Booth's list contains certain improper sets, repetitions and omissions. Namely:

(a) He cites 15 circular triplets, defining 45 awf sets. However the triplet No (14) $x = \{y, z\}$, $y = \{x, z\}$, $z = \{y, z\}$ defines no sets, since it corresponds to a non extensional graphs. Therefore there are only 14 triplets defining 42 sets.

(b) He cites 9 circular pairs defining 18 sets. However the pair No (9) $x = \{x, J\}$, $y = \{x, y, J\}$ (Booth writes J for Ω), defines only one set of level 3, since x is of level 2. Therefore there are only 17 sets of this kind.

(c) He includes as distinct the set $x = \{x, J_1, J\}$, where $J_1 = \{J_1, J\}$. But the latter is identical to $x = \{x, J\}$, therefore the set $y = \{x, y, J\}$ of (b) is no different from $x = \{x, J_1, J\}$.

(d) He includes the set $x = \{J, J_1\}$, which is just J_1, hence of level 2.

Therefore the true awf sets of level 3 contained in his list are 62-6=56 sets.

(e) On the other hand he omits from his list the sets defined by

$x = \{x, y\}$, $y = \{z\}$, $z = \{y, z\}$ (one set of level 3).

$x = \{x, y, \Omega\}$, $y = \{z, \Omega\}$ (two sets of level 3).

If we add to the 56 sets above the last 3 ones we find 59, which is exactly the number we found in 5.2.2.

Recall that an awf is said to be *cyclic* if its graph is a generalized cycle. In fact the majority of the Boffa and Finsler awf sets of level 3 cited above are cyclic. Namely:

Proposition 5.2.3 *i) There are* 51 *cyclic (isomorphism types of) sets of level* 3 *in* AWF^B.

ii) There are 42 *cyclic sets of level* 3 *in* AWF^F.

Proof. We just inspect which sets in the list of proposition 5.2.2 are cyclic.

i) The clauses of the above list which contain Boffa cyclic sets are (19)-(38). Their total number of sets is 51. ii) The clauses of the above list which contain Finsler cyclic sets are (19), (20), (24), (25) (27)-(36). Their total number of sets is 42. ⊣

Another result of [5] is that there are uncountably many hereditarily finite Finsler sets (Thm. 22) (see also [3]), p. 282). The proof is very simple: For every increasing mapping $g : \mathbb{N} \to \mathbb{N}$, consider the set x^g defined inductively by the sequence: $x^g = x^g_0$, $x^g_n = \{x^g_{n+1}, g(n)\}$. Then $g \neq f \Rightarrow x^g \neq x^f$. These sets are not awf. However we can easily convert this proof to one providing uncountably many hereditarily finite sets in AWF^F.

Proposition 5.2.4 *There are uncountably many hereditarily finite sets in AWF^F, hence in AWF^B.*

Proof. Simply consider the 1-1 mappings $g : \mathbb{N} \to \omega^*$, where ω^* is the class of finite ordinals in the sense of AWF^F, defined in the last section. If for each such g we define x^g as above, i.e., $x^g_n = \{x^g_{n+1}, g(n)\}$, clearly all x^g are distinct elements of AWF^F. ⊣

6 Nonstandard Boffa set theory

In this section we use the concept "linear set equation" to extend a result of Ballard and Hrbáček to the case of the solution space of a linear set equation. A *set equation* is just a quantifier-free formula of the language of set theory. In the sequel we feel free to interchange the arrow $\leftarrow$ of a graph with $\in$, and nodes a_i of G_{a_0} with variables x_i. If there is no danger of confusion, we write x instead of x_0. We denote the formula that defines the graph G_{a_0} by g_x. If a Boffa set s satisfies g_x, then we call *solution of* g_x and write $g_x(s)$. Let A_{g_x} be the set of solutions of g_x, i.e, $A_{g_x} = \{s \in \mathbf{B} \mid g_x(s)\}$.

Definition 6.0.5 (i) A set s *is linear* if each set in the transitive closure $TC(\{s\})$, has a unique element.

(ii) A set equation g_x is *transitive if* $g_x(s)$ and $t \in s$ implies $g_x(t)$.

(iii) For a finite number n, a linear set equation g_x is of *circular type of length* n if g_x is

$$x_1 \in x_n \in \cdots \in x_2 \in x_1 \ \ (\text{if } i \neq j, x_i \neq x_j).$$

(iv) A linear set equation g_x is of *non-circular type* if g_x is

$$\cdots \in x_j \in \cdots \in x_i \in \cdots \in x_2 \in x_1 \ \ (\text{if } i \neq j, x_i \neq x_j).$$

Accordingly we have two corresponding types of set equations.

Theorem 6.0.6 *Let G be an apg with corresponding set equation g_x. If A_{g_x} is a proper class, then each element of A_{g_x} is not a ZF set. Furthermore if g_x is transitive and an element of A_{g_x} is not a ZF set, then A_{g_x} is a proper class.*

Proof. If an element of A_{g_x} is a ZF set, then A_{g_x} is a set of one element because of the Extensionality Axiom. We assume that each element of A_{g_x} is not a ZF set and A_{g_x} is a set. Since g_x is transitive, A_{g_x} is transitive. Let $A^{\star}_{g_x}$ be a graph $A_{g_x} \cup G$ where we identify nodes of G that are decorated with well-founded sets, with well-founded sets of A_{g_x}, as ZF sets. Then A_{g_x} is transitive in an extensional $A^{\star}_{g_x}$. By (BA), there exists a Boffa set u and an isomorphism $A^{\star}_{g_x} \to u$. A_{g_x} is a proper subset in u. This contradicts the definition of the solution space A_{g_x}. HenceA_{g_x} is not a set. $\dashv$

Recall that a transitive proper class U is said to be a universe, if all axioms of ZF^- hold in U. The following lemma is a direct consequence of the global axiom of choice $|On| = |V|$.

Lemma 6.0.7 *For an arbitrary proper class A, there is a partition of A into two proper subclasses B and C, i.e., $A = B \cup C$ and $B \cap C = \emptyset$.*

In A_{g_x}, define $\sim$ as follows: For an element a and b in A_{g_x}, let

$$a \sim b \iff a \in TC(b) \text{ or } b \in TC(a).$$

Clearly $\sim$ is an equivalence relation. Let π be the projection of A_{g_x} onto $A_{g_x}/\sim$.

Lemma 6.0.8 *Let U be a universe. If $(A_{g_x}/\sim) - (A_{g_x} \cap U/\sim)$ is a proper class, there exists a proper class $A'_{g_x} \subseteq A_{g_x}$ such that $(A_{g_x} \cap U/\sim) \subseteq (A'_{g_x}/\sim)$, furthermore both classes $(A'_{g_x}/\sim) - (A'_{g_x} \cap U/\sim)$ and $(A_{g_x}/\sim) - (A'_{g_x}/\sim)$ are proper.*

Proof. By Lemma 6.0.7, $(A_{g_x}/\sim) - (A_{g_x} \cap U/\sim)$ has a decomposition into two proper subclasses D_1 and D_2. Let C_i be $\pi^{-1}(D_i)$ (i=1,2), and let $A'_{g_x} = (A_{g_x} \cap U) \cup C_1$. Then $(A'_{g_x}/\sim) - (A'_{g_x} \cap U/\sim) = D_1$, and D_2 is included in $(A_{g_x}/\sim) - (A'_{g_x}/\sim)$. $\dashv$

We are now ready to prove the following.

EXTENSION PRINCIPLE *Let U be a universe and κ an infinite cardinal number. Then there exist a κ-saturated universe W and an elementary embedding $F : U \to W$. Moreover, if g_x is a linear set equation of circular type and $(A_{g_x}/\sim) - (A_{g_x} \cap U/\sim)$ is a proper class, then one can assume that $F(x) = x$ for all $x \in A_{g_x} \cap U$, and $A_{g_x} - W$ is a proper class.*

Proof. Let U be a universe such that $(A_{g_x}/\sim) - (A_{g_x} \cap U/\sim)$ is a proper class. By Lemma 6.0.8, there exists a proper class $A_{g_x} \subseteq A_{g_x}$ such that

$$(A_{g_x} \cap U/\sim) \subseteq (A_{g_x}/\sim)$$

and both $(A'_{g_x}/\sim) - (A'_{g_x} \cap U/\sim)$ and $(A_{g_x}/\sim) - (A'_{g_x}/\sim)$ are proper classes. Let U_α be the transitive closure of $U \cap C[\alpha]$, where $C[\alpha] = \{C(\alpha) : \gamma < \alpha\}$. Let D be a κ-good ultrafilter (see e.g. [7] for the definition), let $I = \bigcup D$, and (A_α, E_α) be the ultraproduct of $(U_\alpha, \in_{U_\alpha})$ over D, and let $d_\alpha : U_\alpha \to A_\alpha$ be the natural elementary embedding. Since U_α is transitive, the inverse image of $(A_{g_x} \cap U_\alpha/\sim)$ under π is $(A_{g_x} \cap U_\alpha)$. Each (A_α, E_α) is extensional, and it is κ-saturated because D is κ-good. Let

$$A_{g_x}(A_\alpha) = \{f_\alpha \in A_\alpha \,|\, \{i \in I : f_\alpha(i) \in A_{g_x}\} \in D\}.$$

Now we use transfinite recursion. We divide A_0 into a disjoint sum

$$d_0(A_{g_x} \cap U_0),\ A_{g_x}(A_0) - d_0(A_{g_x} \cap U_0) \text{ and } A_0 - A_{g_x}(A_0),$$

all of which are transitive sets. Each element of $A_{g_x}(A_0)$ satisfies g_x. Since $A_{g_x}(A_0) - d_0(A_{g_x} \cap U_0)$ is a set and $A'_{g_x} - U$ is a class, there exists an injection

$$e_0 \,:\, A_{g_x}(A_0) - d_0(A_{g_x} \cap U_0) \to A'_{g_x} - U$$

preserving ${}^\star\!\in$ and $\in$-structures. Let W_0 be a disjoint union of $(A_{g_x} \cap U_0)$, the range of e_0 and $A_0 - d_0(A_{g_x}(A_0))$. We define $g_0 : A_0 \to W_0$ such that restrictions to $d_0(A_{g_x} \cap U_0)$, $A_{g_x}(A_0) - d_0(A_{g_x} \cap U_0)$ and $A_0 - d_0(A_{g_x}(A_0))$ give $g_0(d_0(x)) = x$, $g_0(y) = e_0(y)$ and $g_0(z) = z$, respectively. For all $\alpha < \beta$, given isomorphisms $g_\alpha : A_\alpha \to W_\alpha$ to transitive sets W_α such that $g_\alpha[A_{g_x}(A_\alpha)] \subseteq$

$A'_{g_x} \cap W_\alpha$, and $g_\alpha(d_\alpha(x)) = x$ for all $x \in A_{g_x} \cap U_\alpha$. If $\alpha < \alpha' < \beta$, $W_\alpha \subseteq W_{\alpha'}$ and $g_\alpha = g_{\alpha'} \circ i_{\alpha\alpha'}$, where $i_{\alpha\alpha'}$ is the inclusion map. Let

$$A'_\beta := \cup_{\alpha<\beta}\ i_{\alpha\beta}[A_\alpha], \qquad W'_\beta := \cup_{\alpha<\beta} W_\alpha,$$

and define

$$g' : A'_\beta \cup A_{g_x}(A_\beta) \to W'_\beta \cup A'_{g_x}$$

so that g' restricted to $i_{\alpha\beta}[A_\alpha]$ is $g_\alpha \circ i_{\alpha\beta}^{-1}$ and g' restricted to $A_{g_x}(A_\beta) \cap d_\beta[U_\beta]$ agrees with d_β^{-1} for each $\alpha < \beta$. Now g_x is linear, A_β is a set and

$$(A_{g_x}/\sim) - (A_{g_x} \cap U/\sim)$$

is a proper class. Hence g' restricted to $A_{g_x}(A_\beta) - (A'_\beta \cup d_\beta[U_\beta])$ can be defined as a one to one mapping into $A'_{g_x} - (U \cup W'_\beta)$ and preserves $\in$-structures. Clearly $dom(g')$ is transitive in A_β, $g'(d_\beta(x)) = x$ for all $x \in A_{g_x} \cap U_\beta$, and g' is an isomorphism of $(dom(g'), E_\beta \cap dom(g')^2)$ onto $(ran(g'), \in_{ran(g')})$, where $ran(g')$ is transitive. Then by (BA) there exist g_β and W_β such that $g' \subset g_\beta$, $ran(g') \subseteq W_\beta$, W_β is transitive, and g_β is an isomorphism between (A_β, E_β) and $(W_\beta, \in_{W_\beta})$. Next we show that $A_{g_x} \cap W_\alpha = A'_{g_x} \cap W_\alpha$ for each α. Let s be an element of $A_{g_x} \cap W_\alpha$. Then $g_\alpha^{-1}(s) \in A_{g_x} \cap A_\alpha$. We write $g_\alpha^{-1}(s)$ as s_1 and denote the length of g_x by n. Since s_1 is a solution of g_x, there exist elements $s_2, s_3, \ldots, s_n$ of A_α such that $s_1 \bar{\in} s_n \bar{\in} \cdots \bar{\in} s_2 \bar{\in} s_1$. Let

$$D_j(1 \le j \le n-1) = \{i \in I : s_{j+1}(i) \in s_j(i)\}$$

and

$$D_n = \{i \in I : s_1(i) \in s_n(i)\}.$$

Since

$$s_1(i) \in s_n(i) \in \cdots \in s_2(i) \in s_1(i)$$

for $i \in \cap_{1 \le j \le n}\ D_j$, $s_1(i)$ is an element of A_{g_x}. Hence $s_1 \in A_{g_x}(A_\alpha)$, that is,

$$g_\alpha^{-1}(A_{g_x} \cap W_\alpha) \subset A_{g_x}(A_\alpha).$$

Thus

$$A_{g_x} \cap W_\alpha \subset g_\alpha(A_{g_x}(A_\alpha)).$$

Since

$$g_\alpha(A_{g_x}(A_\alpha)) \subset A'_{g_x} \cap W_\alpha,$$

it follows that

$$A_{g_x} \cap W_\alpha = A'_{g_x} \cap W_\alpha.$$

Therefore

$$A_{g_x} - (A_{g_x} \cap W_\alpha) = A_{g_x} - (A'_{g_x} \cap W_\alpha).$$

Note that $(A_{g_x}/\sim) - (A'_{g_x}/\sim)$ is a proper class, so $A_{g_x} - A'_{g_x}$ is also proper. Finally since

$$A_{g_x} - A'_{g_x} \subset \cap_{\alpha \in On}(A_{g_x} - (A'_{g_x} \cap W_\alpha))$$

and

$$\cap_{\alpha \in On}(A_{g_x} - (A'_{g_x} \cap W_\alpha)) = A_{g_x} - W,$$

$A_{g_x} - W$ is a proper class. This completes the proof. $\dashv$

Remarks 6.0.9 Ballard and Hrbáček's result in [2] concerns the equations $g_x : x = \{x\}$. Our result works for every linear set equations, e.g. $x = \{y\}$ and $y = \{x\}$, etc.

References

[1] Aczel, P., *Non-Well-Founded Sets*, CSLI Lecture Notes, No. 14, Stanford, (1988).

[2] Ballard D. and Hrbáček, K., Standard Foundations for Nonstandard Analysis, *The Journal of Symbolic Logic*, **57** (1992), 741-748.

[3] Barwise, J. and Moss, L., *Vicious Circles*, CSLI Lecture Notes, No. 60, Stanford, (1996).

[4] Boffa, M., Forcing et négation de l' axiome de Fondement, *Memoire Acad. Sci. Belg.* tome **XL**, fasc. 7, (1972).

[5] Booth, D., Hereditarily Finite Finsler Sets, *The Journal of Symbolic Logic*, **55** (1990), 700-706.

[6] Booth, D. and Ziegler, R., (Eds), *Finsler Set Theory: Platonism and Circularity*, Birkhäuser Verlag, (1996).

[7] Chang, C.C. and Keisler H.J., *Model theory*, North-Holland, Amsterdam, (1973).

[8] Finsler, P., Über die Grundlagen der Mengenlehre, *I. Math. Zeitschrift*, **25** (1926), 683-713.

[9] Milito, E. and Zhang, L., Classification of Minimal Finite Hypersets, *Congressus Numerantium*, **133** (1998), 85-93.

[10] Peddicord, R., The Number of Full Sets with n Elements, *Proc. Amer. Math. Soc.*, **13** (1962), 825-828.

[11] Scott, D., A Different Kind of Model of Set Theory, Unpublished paper presented at the 1960 Stanford Congress of Logic, Methodology and Philosophy of Science.

[12] Tzouvaras, A., Non-Circular, Non-Well-Founded Set Universes, *Math. Log. Quarterly*, **39** (1993), 454-460.