Proof theory for reasoning with Euler diagrams: a Logic Translation and Normalization

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August, 2011

Abstract

Proof-theoretical notions and techniques, which are developed based on sentential/symbolic representations of formal proofs, are applied to Euler diagrams. A translation of an Euler diagrammatic system into a natural deduction system is given, and the soundness and faithfulness of the translation are proved. Some consequences of the translation are discussed in view of the notion of free ride, which is one of the most basic properties of diagrams that is mainly discussed in the literature of cognitive science as an account of inferential efficacy of diagrams. The translation enables us to formalize and analyze the free ride in terms of proof theory. The notion of normal form of Euler diagrammatic proofs is investigated, and a normalization theorem is proved. Some consequences of the theorem are further discussed: in particular, an analysis of the structure of normal diagrammatic proofs; a diagrammatic counterpart of the usual subformula property; and a characterization of diagrammatic proofs compared with natural deduction proofs.

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1 Introduction

Proof theory in logic has traditionally been developed based on sentential representations of logical proofs. Formal proofs are defined as chains of sentences, and other forms of representations, such as diagrams or graphs, are regarded not as components of formal proofs, but only as auxiliary tools to construct formal proofs or to explain ideas.

However in [1], Barwise questioned the logocentricity of logical studies by examining some examples, and claimed that diagrams and other forms of visual representation can be essential and legitimate components of formal proofs. E.g., a combination of both geometric manipulation of a diagram and algebraic manipulation of non-diagrammatic symbols in a proof of the Pythagorean Theorem; diagram chasing in category theory; the use of Euler and Venn diagrams to illustrate syllogistic reasoning. In particular, Shin [16] showed that Venn diagrams have their own syntax and semantics, and that logical studies such as soundness and completeness can be extended to reasoning using Venn diagrams. More recently, Howse et al. [6, 17] extended such results to Euler diagrams and more expressive diagrams.

Supported by Shin's and others work (cf. [1]), Barwise attempted to extend the traditional logical framework, in which the linguistic/sentential form of representation is presupposed, to a framework independent of representational form. He further tried to establish Heterogeneous Logic, which is partly implemented in the computer program *Hyperproof*, where both first-order formulas and diagrams are adopted to reason about blocks worlds (see articles of Barwise and Etchemendy in [1]).

In the semantic framework of Barwise and Shin, however, proof theory of diagrammatic reasoning has not been that well developed. In this paper, we show that proof-theoretical notions and techniques are not restricted to symbolic/sentential representations, but they can be extended to diagrammatic representations, in particular to Euler diagrams. In our framework, we are able to formalize, by using proof-theoretical notions, characteristic properties of diagrams, that have been discussed mainly in the literature of cognitive science.

Our study here is based on the Euler diagrammatic inference system introduced by Mineshima-Okada-Takemura [8, 9]. This inference system, in contrast to previous semantic studies (cf. [17]), is formalized in the manner developed in Gentzen's natural deduction [5], and designed to be as natural as possible to reflect intuitive manipulations of Euler diagrams. In [8], the system is shown to be sound and complete with respect to a set-theoretical semantics. Furthermore, reference [9] discusses how this Euler diagrammatic system is translatable into a natural deduction system. (See [9] for our discussion on the contrast between the traditional semantic framework of Venn diagrams and our proof-theoretical framework of Euler diagrams.) We review the Euler diagrammatic system of [8, 9] in Section 2. By extending the results in [8, 9], we discuss the following issues in this paper.

In Section 3, we investigate, in more detail, the above-mentioned translation of the Euler diagrammatic system into a natural deduction system. Translations between logical systems are one of the basic methods in proof theory, and applied to various systems for a variety of purposes. See, for example, [10] for an abstract formalization of logic translations. In Section 3.2, we present our translation to investigate how rules are rendered from one system to the other. We show soundness (Theorem 3.3) and faithfulness (Theorem 4.3) of the translation.

We then discuss, in Section 3.3, some consequences of the translation in view of the notion of free ride, which is named by Shimojima [15], and is mainly discussed in the literature of cognitive science as an account of inferential efficacy of diagrams. Free ride occurs when, by adding a piece of information to a diagram, the resulting diagram somehow comes to

present pieces of information not contained in the given diagram, nor in the original piece of information. We show that our translation is an instance of Shimojima and Barwise-Seligman [2]'s channel theoretic formulation of free ride in Section 3.3.2. We further discuss, in Section 3.3.3, how our translation permits an analysis of how many and what kind of pieces of information are obtained in a free ride under each application of an Euler diagrammatic inference rule, and how many inference steps are required to derive such pieces of information in terms of natural deduction inference rules.

In Section 4, we investigate notions of normal form and normalization of Euler diagrammatic proofs. Normalization theorem has played a central role in the development of proof theory, and much of the proof-theoretical results depend on this theorem. In particular, normal proofs have an essential property called the subformula property that gives a basis of theorem proving. Furthermore, the notion of normal form enables us to analyze the structure of proofs in a formal system. Cf. [13, 14, 11].

Our notion of normal diagrammatic proof is introduced on the basis of a correspondence, discussed in Section 4.2, between a class of Euler diagrammatic proofs and the class of normal natural deduction proofs. In Section 4.3, we define such a class as normal diagrammatic proofs. Then we show normalization (Theorem 4.7) of our Euler diagrammatic system in Section 4.3.1, and investigate some consequences of the theorem. In Section 4.3.2, we analyze the structure of normal diagrammatic proofs (Proposition 4.9). In Section 4.3.3, we discuss the diagrammatic counterpart of the usual subformula property in symbolic logical systems (Proposition 4.11). In Section 4.3.4, we give a proof-theoretical formalization of a difference between structures of Euler diagrammatic proofs and natural deduction proofs (Proposition 4.12).

In Section 5, we summarize our framework, and discuss our future work.

2   Euler diagrammatic system

Mineshima-Okada-Takemura introduced a proof-theoretical framework for Euler diagrams in [8]. The Euler diagrammatic system has the following features: (1) Euler diagrams are studied in terms of inclusion and exclusion relations holding on each diagram between circles and points, which clarify the correspondence between Euler diagrams and implicational formulas in symbolic logic; (2) Inference rules are decomposed into primitive rules that characterize intuitive manipulations on diagrams as formal inference rules; (3) Their proof-theoretical formalization permits the application of well-developed proof-theoretical techniques to diagrammatic reasoning studies.

We briefly review the syntax of reference [8] and set-theoretical semantics of Euler diagrams in Section 2.1, and the inference system in Section 2.2.

2.1   Syntax and semantics of Euler diagrams

Our Euler diagram is defined as a plane with named circles (simple closed curves) and points. Each diagram is specified by topological (inclusion and exclusion) relations maintained between circles and points. Thus diagrams are syntactically equivalent when the same relations hold on each. Based on the interpretation of circles (resp. points) as subsets (resp. elements) of a certain set-theoretical domain, each diagram is interpreted in terms of relations that hold on it.

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Definition 2.1 (EUL-diagram) An EUL-*diagram* is a plane ($\mathbb{R}^2$) with a finite number, at least two, of *named simple closed curves* (simply called *named circles*, and denoted by $A, B, C, \ldots$) and *named points* (denoted by $a, b, c, \ldots$), where no two named circles and points are completely concurrent, and no two named circles and points have the same name.

Named circles and points are collectively called *(diagrammatic) objects*, and denoted by $s, t, u, \ldots$. We use a rectangle to represent the plane for an EUL-diagram. EUL-diagrams are denoted by $\mathcal{D}, \mathcal{E}, \mathcal{D}_1, \mathcal{D}_2, \ldots$.

Definition 2.2 (Minimal diagram) An EUL-diagram consisting of only two objects is called a *minimal diagram*. Minimal diagrams are denoted by $\alpha, \beta, \gamma, \ldots$.

Our EUL-diagrams are investigated in terms of the following topological relations between diagrammatic objects.

Definition 2.3 EUL-*relations* are the following reflexive asymmetric binary relation $\sqsubset$, and irreflexive symmetric binary relations $\vdash$ and $\bowtie$:

$\quad A \sqsubset B \quad$ "the interior of A is *inside of* the interior of B,"
$\quad A \vdash B \quad$ "the interior of A is *outside of* the interior of B,"
$\quad A \bowtie B \quad$ "there is at least one *crossing* point between A and B,"
$\quad b \sqsubset A \quad$ "b is *inside of* the interior of A,"
$\quad b \vdash A \quad$ "b is *outside of* the interior of A,"
$\quad a \vdash b \quad$ "a is *outside of* b (i.e. a is not located at the point of b)."

Proposition 2.4 *Let $\mathcal{D}$ be an EUL-diagram. For any pair of distinct objects s and t of $\mathcal{D}$, exactly one of the EUL-relations $s \sqsubset t, t \sqsubset s, s \vdash t, s \bowtie t$ holds.*

Observe that, by Proposition 2.4, the set of EUL-relations holding on a given EUL-diagram $\mathcal{D}$ is uniquely determined. We denote the set by $\mathsf{rel}(\mathcal{D})$. We also denote by $pt(\mathcal{D})$ the set of named points of $\mathcal{D}$, by $cr(\mathcal{D})$ the set of named circles of $\mathcal{D}$, and by $ob(\mathcal{D})$ the set of objects of $\mathcal{D}$. As an illustration, for the diagram $\mathcal{D}_1$ of Fig. 1 below, we have $pt(\mathcal{D}_1) = \{a\}$, $cr(\mathcal{D}_1) = \{A, B, C\}$, and $\mathsf{rel}(\mathcal{D}_1) = \{A \bowtie B, A \bowtie C, B \bowtie C, a \vdash A, a \sqsubset B, a \vdash C\}$. In the description of a set of relations, we usually omit the reflexive relation $s \sqsubset s$ for each object s.

We consider the equivalence class of diagrams in terms of the EUL-relations in the following.

Definition 2.5 (Equivalence among EUL-diagrams) Any pair of EUL-diagrams $\mathcal{D}$ and $\mathcal{E}$ are *syntactically equivalent* if $\mathsf{rel}(\mathcal{D}) = \mathsf{rel}(\mathcal{E})$.

Example 2.6 (Syntactic equivalence of diagrams) For example, diagrams $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, and $\mathcal{D}_4$ of Fig. 1 are equivalent since $\mathsf{rel}(\mathcal{D}_1) = \mathsf{rel}(\mathcal{D}_2) = \mathsf{rel}(\mathcal{D}_3) = \mathsf{rel}(\mathcal{D}_4)$.

Fig. 1 Syntactic equivalence of EUL-diagrams

On the other hand, $\mathcal{D}_1$ and $\mathcal{D}_5$ (resp. $\mathcal{D}_1$ and $\mathcal{D}_6$) are not syntactically-equivalent since different EUL-relations hold on each of these: $A \sqsubset C$ holds on $\mathcal{D}_5$ in place of $A \bowtie C$ of $\mathcal{D}_1$ (resp. $C \sqsubset A$ and $C \sqsubset B$ hold on $\mathcal{D}_6$ in place of $A \bowtie C$ and $C \bowtie B$ of $\mathcal{D}_1$).

In [7], the system is extended by introducing intersection, union, and complement regions, respectively, as diagrammatic objects, and the diagrams $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, and $\mathcal{D}_4$ are distinguished.

In what follows, the diagrams which are syntactically equivalent are identified, and they are referred by a single name.

Each EUL-diagram is interpreted in terms of EUL-relations that hold on it. To interpret the EUL-relations $\sqsubset$ and $\vdash$ uniformly as the subset relation and the disjointness relation, respectively, we regard each point as a special circle that does not contain, nor cross, any other object.

Definition 2.7 (Model) A *model* M is a pair (U, I), where U is a non-empty set (the domain of M), and I is an interpretation function which assigns to each named circle or point a non-empty subset of U such that $I(a)$ is a singleton for any named point a, and $I(a) \neq I(b)$ for any points a, b of distinct names.

Definition 2.8 (Truth conditions) Let $\mathcal{D}$ be an EUL-diagram. $M = (U, I)$ is a *model of* $\mathcal{D}$, written as $M \models \mathcal{D}$, if the following *truth-conditions* hold: For all objects s, t of $\mathcal{D}$,
 (1) $I(s) \subseteq I(t)$ if $s \sqsubset t$ holds on $\mathcal{D}$,   and   (2) $I(s) \cap I(t) = \emptyset$ if $s \vdash t$ holds on $\mathcal{D}$.

Remark 2.9 (Semantic interpretation of $\bowtie$-relation) By Definition 2.8, the EUL-relation $\bowtie$ does not contribute to the truth-condition of EUL-diagrams. Informally speaking, $s \bowtie t$ may be understood as $I(s) \cap I(t) = \emptyset$ *or* $I(s) \cap I(t) \neq \emptyset$, which is true in any model.

The semantic consequence relation, $\models$ between EUL-diagrams is defined as usual in symbolic logic. (See [8] for a detailed description.)

2.2   Euler diagrammatic inference system GDS

We next review the Euler diagrammatic inference system of [8], called Generalized Diagrammatic Syllogistic inference system GDS. GDS consists of two kinds of inference rules: Deletion and Unification. Deletion allows us to delete a diagrammatic object from a given EUL-diagram. Unification allows us to unify two EUL-diagrams into one diagram in which the semantic information is equivalent to the conjunction of the original two diagrams.

To motivate our definition of unification, let us consider the following question: Given the following diagrams $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ of Fig. 2, what diagrammatic information on A, B and c can be obtained? Fig. 2 represents a way of solving the question.

Fig. 2   Unification

Fig. 3   Inconsistency

In Fig. 2, at step one, the two diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ are unified to obtain $\mathcal{D}_1 + \mathcal{D}_2$, where point c in $\mathcal{D}_1$ and $\mathcal{D}_2$ are identified, and B is added to $\mathcal{D}_1$ so that c is inside of B and B overlaps with A without any implication of a specific relationship between A and B. We formalize such cases, where two given diagrams share one object, by unification rules U1–U8. (A complete list of these unification rules can be found in the Appendix A.) At step two, $\mathcal{D}_1 + \mathcal{D}_2$ is combined with a third diagram $\mathcal{D}_3$ to obtain $(\mathcal{D}_1 + \mathcal{D}_2) + \mathcal{D}_3$. Note that the diagrams $\mathcal{D}_1 + \mathcal{D}_2$ and $\mathcal{D}_3$ share two circles A and B: $A \bowtie B$ holds on $\mathcal{D}_1 + \mathcal{D}_2$ and $A \sqsubset B$ holds on $\mathcal{D}_3$. Since the semantic information of $A \sqsubset B$ on $\mathcal{D}_3$ is more accurate than that of $A \bowtie B$ on $\mathcal{D}_1 + \mathcal{D}_2$, according to our semantics of Section 2.1 (recall that $A \bowtie B$ means just "true" in our semantics), one keeps the relation $A \sqsubset B$ in the unified diagram $(\mathcal{D}_1 + \mathcal{D}_2) + \mathcal{D}_3$. We formalize such cases, where two given diagrams share two objects, by U9–U10 rules. Observe that the unified diagram $(\mathcal{D}_1 + \mathcal{D}_2) + \mathcal{D}_3$ of Fig. 2 represents the information of these diagrams $\mathcal{D}_1, \mathcal{D}_2$, and $\mathcal{D}_3$, that is, their conjunction.

Two kinds of constraint are imposed on unification. One is the *constraint for determinacy*, which blocks the disjunctive ambiguity with respect to locations of named points. For example, unification of the two diagrams $\mathcal{D}_2$ and $\mathcal{D}_3$ in Fig. 2 is not permitted because the location of point c is not determined (it can be inside A or outside A). The other is the *constraint for consistency*, which blocks representing inconsistent information in a single diagram. For example, the diagrams $\mathcal{D}_4$ and $\mathcal{D}_5$ (resp. $\mathcal{D}_6$ and $\mathcal{D}_7$) in Fig. 3 can not be unified because the associated relations contradict each other based on the semantics of Section 2.1. The unification rules are defined by requiring that one of the unified diagrams be a *minimal diagram*. This restriction to unification clarifies its operational meaning (see [8]). Our completeness ([8]) ensures that any (not restricted to being minimal) diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$ may be unified, under the requirements for determinacy and consistency, into one diagram whose semantic information is equivalent to the conjunction of those of $\mathcal{D}_1, \ldots, \mathcal{D}_n$.

Each inference rule is described by specifying (i) premise diagrams; (ii) the constraints that the premise diagrams should satisfy; and (iii) diagrammatic operations to introduce a new object into, or to rearrange a configuration of objects of, one of the premise diagrams. We also specify the set of EUL-relations $\mathsf{rel}(\mathcal{D} + \alpha)$ of the unified diagram, which is essential for our translation of the inference rules in Section 3.2. We also give schematic illustrations and concrete examples of applications of rules. An implementation of our diagrammatic operations described in unification rules is given by Stapleton et al. [18] based on graph theory.

Of the eleven unification rules, we describe here only rules U5 and U7. The full list of inference rules of GDS is found in Appendix A and [8].

Definition 2.10 (Inference rules of GDS)

$\boxed{\text{U5 rule}}$ **Premises:** A minimal diagram α on which $A \sqsubset B$ holds; and a diagram $\mathcal{D}$ such that $B \in cr(\mathcal{D})$.

Constraint for determinacy: $x \vdash B$ holds for all $x \in pt(\mathcal{D})$.

Operation: Add the circle A to $\mathcal{D}$ (with preservation of all relations on $\mathcal{D}$) so that the following conditions are satisfied on $\mathcal{D} + \alpha$: (1) $A \sqsubset B$ holds; (2) $A \bowtie X$ holds for all circles X ($\not\equiv B$) such that $X \sqsubset B$ or $X \bowtie B$ holds on $\mathcal{D}$.

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The set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ of the unified diagram is specified as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid X \sqsubset B \text{ or } X \bowtie B \in \mathsf{rel}(\mathcal{D}), X \not\equiv B\}$$
$$\cup \{A \sqsubset X \mid B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \cup \{X \vdash A \mid X \vdash B \in \mathsf{rel}(\mathcal{D})\} \cup \{x \vdash A \mid x \in pt(\mathcal{D})\}$$

Schema of U5   Example of U5

U7 rule   **Premises:** A minimal diagram α on which $A \vdash B$ holds; and a diagram $\mathcal{D}$ such that $B \in cr(\mathcal{D})$.

Constraint for determinacy: $x \sqsubset B$ holds for all $x \in pt(\mathcal{D})$.

Operation: Add the circle A to $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D} + \alpha$: (1) $A \vdash B$ holds; (2) $A \bowtie X$ holds for all circles X ($\not\equiv B$) such that $B \sqsubset X$ or $B \vdash X$ or $B \bowtie X$ holds on $\mathcal{D}$.

The set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ of the unified diagram is specified as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid B \sqsubset X \text{ or } B \vdash X \text{ or } B \bowtie X \in \mathsf{rel}(\mathcal{D}), X \not\equiv B\}$$
$$\cup \{X \vdash A \mid X \sqsubset B \in \mathsf{rel}(\mathcal{D})\} \cup \{x \vdash A \mid x \in pt(\mathcal{D})\}$$

Schema of U7   Example of U7

     The notion of *diagrammatic proof* (or, *d-proof* for short) is defined inductively as tree structures consisting of Unification and Deletion steps. (Cf. Fig. 12 in Example 4.6.) The provability relation between EUL-diagrams is defined as usual in symbolic logic. GDS is shown to be sound and complete with respect to the semantics of the previous section. (Note that, to avoid introducing the diagrammatic counterpart of the so-called absurdity rule in our system, we impose, in the formulation of our completeness below, a consistency condition on the set of premise diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$, i.e., it has a model. See [8] for a detailed discussion.)

Theorem 2.11 (Completeness of GDS [8]) *Let $\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{E}$ be EUL-diagrams, and $\mathcal{D}_1, \ldots, \mathcal{D}_n$ has a model. $\mathcal{E}$ is a semantically valid consequence of $\mathcal{D}_1, \ldots, \mathcal{D}_n$ ($\mathcal{D}_1, \ldots, \mathcal{D}_n \models \mathcal{E}$), if and only if, there is a diagrammatic proof of $\mathcal{E}$ from $\mathcal{D}_1, \ldots, \mathcal{D}_n$ ($\mathcal{D}_1, \ldots, \mathcal{D}_n \vdash \mathcal{E}$) in GDS.*

3 Translation of Euler diagrammatic system into Natural deduction system

In this section, we discuss the translation of the Euler diagrammatic inference system GDS into the natural deduction system. Translations between logical systems are one of the basic methods in proof theory, and are applied to various systems covering a variety of purposes; for example, purposes such as reducing the consistency of an original system (say, classical logic) to that of a translated system (intuitionistic logic), or studying logical connectives and inference rules of an original system (say, intuitionistic logic) in terms of those of a translated system (modal logic or linear logic). See, for example, [10] for an abstract formalization of logic translations.

In Section 3.2, we define our translation with the aim of investigating our Euler diagrammatic inference rules in terms of natural deduction inference rules. In Section 3.3, we discuss some consequences of the translation in view of the free ride notion of Shimojima [15].

3.1 Natural deduction

Natural deduction was introduced by Gentzen [5], and studied extensively by Prawitz [13], that being one of the major inference systems in proof theory. We remark that, natural deduction has some diagrammatic character, i.e., proofs are defined, not as linear successions of formulas, but as trees of formulas, and in fact, it helps the development of *structural proof theory*, which studies the general structures and properties of logical proofs (see [11]).

We consider natural deduction for propositional logic. Intuitively speaking, it might be appropriate to translate each named circle (resp. point) into a unary predicate (resp. constant) symbol of the first order language. However, our main purpose is to analyze the structure of diagrammatic proofs, and in our analysis quantifiers do not play any essential role. Thus, to avoid unnecessary complications, we concentrate on the propositional fragment.

We denote atoms (propositional variables) by $A, B, C, \dots$. Formulas are defined inductively as usual by using connectives $\wedge, \vee, \rightarrow, \neg, \bot$, and formulas are denoted by $\varphi, \psi, \theta, \dots$. We denote natural deduction proofs by $\pi, \pi', \pi_1, \pi_2, \dots$. In what follows, we consider the $\wedge$ connective as an n-ary connective for an appropriate n. Furthermore, we denote simply by a sequence (set) $\varphi_1, \dots, \varphi_n$ a conjunction $\varphi_1 \wedge \cdots \wedge \varphi_n$, where we assume all conjuncts are distinct. We also generalize the conjunction introduction ($\wedge I$) and elimination ($\wedge E$) rules of natural deduction to those for the n-ary $\wedge$ connective. (See [13, 14, 12] for natural deduction.)
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3.2 Translation of GDS

In general, diagrams correspond to formulas in symbolic logic, and diagram manipulations correspond to applications of inference rules in a certain logical system. In particular, our Euler diagrams specified in terms of EUL-relations correspond to implicational formulas, and inference rules of GDS correspond to natural deduction inference rules associated with the implicational connective.

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[1]In our Euler diagrammatic system, any named points are assumed to be distinct. Hence we assume, also in natural deduction, formulas of the form $a \rightarrow \neg b$ as axioms, where a and b are translations of two distinct named points. See Appendix A.

Definition 3.1 (Translation of EUL-diagrams) Each named circle or point is translated into an atom. Then each EUL-relation R is translated into a formula R° as follows:

$$(s \sqsubset t)^\circ := s \to t \qquad (s \vdash t)^\circ := s \to \neg t \qquad (s \bowtie t)^\circ := s \to s, t \to t$$

Let $\mathcal{D}$ be an EUL-diagram whose set of relations $\mathsf{rel}(\mathcal{D})$ is $\{R_1, \ldots, R_n\}$. The diagram $\mathcal{D}$ is translated into the conjunction $\mathcal{D}^\circ := R_1^\circ, \ldots, R_n^\circ$.

We give a translation of each rule into a combination of inference rules of natural deduction. Unification between $\mathcal{D}$ and α such that $\mathsf{rel}(\mathcal{D} + \alpha) = \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{R_1, \ldots, R_n\}$ is translated schematically as depicted in Fig. 4.

$$\cfrac{\mathcal{D}^\circ \quad \alpha^\circ \quad \cfrac{\begin{matrix}\mathcal{D}^\circ, \alpha^\circ \\ \vdots \\ R_1^\circ\end{matrix} \quad \cdots \quad \cfrac{\begin{matrix}\mathcal{D}^\circ, \alpha^\circ \\ \vdots \\ R_n^\circ\end{matrix}}{}}{\mathcal{D}^\circ, \alpha^\circ, R_1^\circ, \ldots, R_n^\circ}} {} \, \wedge I \qquad\qquad \left(\begin{matrix} \vdots \\ \varphi_n \end{matrix} \right)_n$$

<div align="center">Fig. 4   Fig. 5</div>

Thus, our translation is defined so that EUL-relations of the unified diagram $\mathcal{D} + \alpha$ (cf. Definition 2.10 in Section 2.2) are derived by using inference rules of natural deduction. In the following natural deduction proofs, $(\varphi_n)_n$ signifies the set of formulas $\varphi_1, \ldots, \varphi_n$. Furthermore, for each formula φ_n, the repetition of the same inference steps is expressed as the skeleton of a proof as in Fig. 5.

Definition 3.2 (Translation of GDS) Inference rules of GDS is translated into natural deduction as follows.

We give two interesting cases connected with rules U5 and U7 (cf. Definition 2.10); the other cases are treated in a similar way (see Appendix A for these). In the following translation, we omit for the sake of simplicity derivations for tautologies of the form $A \to A$. Each t_n is a named circle or point, and φ_n is t_n or $\neg t_n$.

U5 is translated as follows:

$$\cfrac{\mathcal{D}^\circ \quad \alpha^\circ \quad \left(\cfrac{\cfrac{[A]^1 \quad \cfrac{\alpha^\circ}{A \to B}}{B} \quad \cfrac{\mathcal{D}^\circ}{B \to \varphi_n}}{\cfrac{\varphi_n}{A \to \varphi_n} 1} \right)_n \quad \left(\cfrac{\cfrac{\cfrac{[A]^1 \quad \cfrac{\alpha^\circ}{A \to B}}{B} \quad \cfrac{[t_m]^2 \quad \cfrac{\mathcal{D}^\circ}{t_m \to \neg B}}{\neg B}}{\cfrac{\bot}{\neg A} 1}}{t_m \to \neg A} 2 \right)_m}{\mathcal{D}^\circ, \alpha^\circ, (A \to \varphi_n)_n, (t_m \to \neg A)_m}$$

U7 is translated as follows:

$$\cfrac{\mathcal{D}^\circ \quad \alpha^\circ \quad \left(\cfrac{\cfrac{\cfrac{[t_m]^2 \quad \cfrac{\mathcal{D}^\circ}{t_m \to B}}{B} \quad \cfrac{[A]^1 \quad \cfrac{\alpha^\circ}{A \to \neg B}}{\neg B}}{\cfrac{\bot}{\neg A} 1}}{t_m \to \neg A} 2 \right)_m}{\mathcal{D}^\circ, \alpha^\circ, (t_m \to \neg A)_m}$$

Definition 3.2 gives, by induction, a translation of any *diagrammatic proof* π of GDS into a natural deduction proof π°. Hence the following theorem is immediate:

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Theorem 3.3 (Soundness of translation) *Let $\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{E}$ be* EUL*-diagrams. If π is a diagrammatic proof of $\mathcal{E}$ from $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in* GDS*, then π° is a natural deduction proof of $\mathcal{E}^{\circ}$ from $\mathcal{D}_1^{\circ}, \ldots, \mathcal{D}_n^{\circ}$.*

Let us see the following example.

Example 3.4 (Barbara in GDS) The following diagrammatic proof on the left in Fig. 6, which expresses the famous valid syllogism called *Barbara*, is translated by Definition 3.2 into the natural deduction proof on the right. For the sake of simplicity, we omit tautologies of the form $A \to A$ in the proof:

Fig. 6 Barbara in GDS

3.3 Consequences of translation

In the following three subsections 3.3.1, 3.3.2, and 3.3.3, we investigate some consequences of our translation.

3.3.1 Minimal logic without disjunction NM

By Definition 3.2 of our translation (see also Appendix A), it is immediately seen that we do not need the full class of inference rules of classical logic, but only a particular class of rules for the translation of Euler diagrammatic inference rules. We need only the rules $\to I, \to E, \neg I, \neg E, \wedge I, \wedge E$, which form the most basic subsystem within classical logic, i.e., *minimal logic without disjunction* (cf. Prawitz [13]). We denote the system as NM.

Proposition 3.5 (NM) *The Euler diagrammatic inference system* GDS *is translated into the natural deduction system for minimal logic without disjunction* NM*.*

3.3.2 Free ride between GDS and NM

The **Free ride** property is one of the most basic properties of diagrammatic systems that is mainly studied in the literature of cognitive science as an account of inferential efficacy of diagrams. Let us begin by illustrating the notion of free ride by the following application of our unification rule U5 in Fig. 7.

The minimal diagram α has the information "A is inside B ($A \sqsubset B$)". By adding this piece of information to $\mathcal{D}$, i.e., by drawing the circle A inside the circle B of $\mathcal{D}$, we obtain the unified diagram $\mathcal{D} + \alpha$. From thus obtained diagram $\mathcal{D} + \alpha$, in virtue of the geometrical constraints

Fig. 7 Free ride

governing Euler circles, we can automatically read off pieces of information "A is inside E ($A \sqsubset E$)" and "A is outside C ($A \dashv C$)".

As illustrated in Fig. 7, by unifying diagrams, i.e., by adding a certain piece of information to a diagram, the resulting diagram somehow comes to present pieces of information not contained in the given premise diagrams. Shimojima [15] called such a phenomenon *free ride*, and analyzed its semantic conditions within the framework of Barwise-Seligman's channel theory [2]. In what follows, we review our slightly-restricted version of this formulation.[2]

Let $\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{D}_0$ be well-formed expressions having truth values (e.g., EUL-diagrams) in a logical system S (e.g., GDS), and $\varphi_1, \ldots, \varphi_m, \varphi_0$ be well-formed expressions (propositional formulas) in a logical system T (NM). We assume that S and T have a common semantic domain, and their respective interpretation functions are I_S and I_T. Between logical systems S and T, a *free ride* occurs when the following conditions (1) and (2) hold:

(1) $\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{D}_0$ are *projected* to $\varphi_1, \ldots, \varphi_n, \varphi_0$, i.e., for every $0 \leq i \leq n$, $I_\mathsf{S}(\mathcal{D}_i) \models I_\mathsf{T}(\varphi_i)$.

(2) A sequence $\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{D}_0$ is a *constraint* of S, i.e., $\mathcal{D}_0$ is provable from $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in S.

$$
\begin{array}{ccc}
 & (3) & \\
\varphi_1, \ldots, \varphi_n & (\vdash_\mathsf{T}) & \varphi_0 \\
(1) \Big\Updownarrow & & \Big\Updownarrow (1) \\
\mathcal{D}_1, \ldots, \mathcal{D}_n & \vdash_\mathsf{S} & \mathcal{D}_0 \\
 & (2) &
\end{array}
$$

In other words, a free ride occurs if we try to represent $\varphi_1, \ldots, \varphi_n$ (without φ_0) by $\mathcal{D}_1, \ldots, \mathcal{D}_n$ (1), then, both in a positive and negative way, φ_0 is automatically represented by $\mathcal{D}_1, \ldots, \mathcal{D}_n$ from the constraint in S (2). See [2] for a detailed explanation.

If the following condition also holds, free ride is *positive* (otherwise, negative):

(3) A sequence $\varphi_1, \ldots, \varphi_n, \varphi_0$ is a constraint of T, i.e., φ_0 is provable from $\varphi_1, \ldots, \varphi_n$ in T.

Observe that GDS and NM have a common set-theoretical semantics, and it is clear that $I_\mathsf{GDS}(\mathcal{D}) \models I_\mathsf{NM}(\mathcal{D}^\circ)$ holds for any $\mathcal{D}$. Thus our logical translation is an instance of a projection in channel theory, and Theorem 3.3 leads to the following proposition.

Proposition 3.6 (Free ride) *Positive free ride occurs between* GDS *and* NM.

---

[2]Our formulation is slightly restricted in comparison to the original one [2] in the following respects: (1) We assume our set-theoretical semantics as an instance of the *core channel* of [2], which is intended to be a system containing S and T as subsystems. (2) According to the restriction of the core channel, the notion of *projection* is also restricted to the semantic consequence relation.

3.3.3 Free ride in each inference rule

By slightly extending the notion of free ride of Shimojima [15], let us call pieces of information "free rides" those that are automatically represented in a diagram after a unification of given diagrams. In our framework, such pieces of information are explicated through our translation, or equivalently through EUL-relations holding on the diagram.

Fig. 8　Free rides of U5

Let us consider U5 rule. Recall that when we carry out the operation of U5 (cf. Definition 2.10), we only need to consider the relations between A and B, as well as between A and circles X such that $X \sqsubset B$ or $X \bowtie B$ holds on $\mathcal{D}$, and we do not need to take the other circles into account. Then the relations between A and the other circles are automatically determined by the geometrical constraints governing Euler circles. Thus the relations $\mathsf{rel}(\mathcal{D} + \alpha)$ holding on the unified diagram of U5 is divided into three parts: (1) the relations already holding on the premises; (2) those required to hold by the operation of U5; (3) those automatically represented by an application of U5, i.e., the "free rides" of U5.

$$
\overbrace{\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha)}^{(1)} \cup \overbrace{\{A \bowtie X \mid X \sqsubset B \text{ or } X \bowtie B \in \mathsf{rel}(\mathcal{D}), X \not\equiv B\}}^{(2)}
$$
$$
\underbrace{\cup \{A \sqsubset X \mid B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \cup \{X \vdash A \mid X \vdash B \in \mathsf{rel}(\mathcal{D})\} \cup \{x \vdash A \mid x \in pt(\mathcal{D})\}}_{(3)}
$$

Thus, we define the free rides of each Unification in terms of EUL-relations, by subtracting the relations of (1) and (2) from $\mathsf{rel}(\mathcal{D} + \alpha)$:

Definition 3.7 (Free rides) In an application U of Unification in GDS between $\mathcal{D}$ and α, the following set of relations is called the *free rides* of U:

$$
\mathsf{rel}(\mathcal{D} + \alpha) \setminus \big(\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{R \mid R \text{ is a relation specified in the operation of } \mathsf{U}\}\big)
$$

Thus, U7 has free rides $\{X \vdash A \mid X \sqsubset B \in \mathsf{rel}(\mathcal{D}), X \not\equiv B\} \cup \{x \vdash A \mid x \in pt(\mathcal{D})\}$. See Appendix A, in which free rides of each Unification are specified.

Note that the set of EUL-relations of a diagram is exactly the translation of the diagram, and by our translation, it becomes clear how many inference steps are required to derive free rides for each Unification in terms of natural deduction inference rules. Although we do not enter into details in this paper, this enables us to analyze free rides in terms of the well-developed studies on proof complexity, cf. e.g., [4].

4 Normalization in Euler diagrammatic system

In this section, we investigate notions of normal form and normalization of Euler diagrammatic proofs. In Section 4.1, we briefly review basic facts and terminology in the usual proof theory. By using the normalization theorem in natural deduction, we first investigate, in Section 4.2, the reverse of the translation (Definition 3.2) of Section 3.2, i.e., the translation of the natural deduction system into the Euler diagrammatic system. Then, in Section 4.3, we introduce a notion of normal form for Euler diagrammatic proofs, and show a normalization theorem. We further investigate some consequences of the normalization theorem.

4.1 Normal proof and normalization in proof theory

Normal proofs and normalization theorem play a central role in the development of proof theory, and much of the proof-theoretical results such as consistency proofs of various systems depend on the theorem. We here recall some basic facts and terminology concerning normalization.[3] Cf. [13, 14, 11]. A natural deduction proof, in general, may contain some redundant steps and formulas called **maximal formulas**, i.e., a formula stands at the same time as the conclusion of an introduction rule and as the major premise of an elimination rule. For example, the formula $\varphi_1 \wedge \varphi_2$ and the pair of applications of the $\wedge I$ and $\wedge E_1$ rules on the left in the following proof are redundant in the sense that without them we already have a proof π_1 of φ_1 as illustrated on the right.

$$\frac{\dfrac{\vdots \;\pi_1 \quad \vdots \;\pi_2}{\varphi_1 \qquad \varphi_2}}{\dfrac{\varphi_1 \wedge \varphi_2}{\varphi_1} \wedge E_1} \wedge I \qquad\qquad \begin{array}{c} \vdots \;\pi_1 \\ \varphi_1 \end{array}$$

A natural deduction proof is said to be in **normal form** when it does not contain any maximal formula. The following **normalization theorem** holds [13]: *If φ is provable from a set of formulas Γ, then there is a normal proof of φ from Γ.* [4] Normal proofs, particularly of NM, have an essential property called the **subformula property**: *A normal proof π of φ from Γ contains only subformulas of φ and Γ.* [5] In fact, this subformula property of normal proofs makes various proof-theoretical analyses possible. For example, the notion of normal proofs enables us to **analyze the structure of proofs** in a formal system. Prawitz [13, 14] shows that each normal proof consists of two parts: An analytical part in which assumptions are decomposed into their components by using elimination rules; A synthetic part in which the final components obtained in the analytical part are put together by using introduction rules. Our proof of the following Proposition 4.2 depends on this analysis.

4.2 Faithfulness of the translation

Now, let us investigate the reverse of translation presented in Section 3.2, i.e., the translation of natural deduction NM into our Euler diagrammatic system GDS. Note first that even if we

---

[3]Notions and properties related to normalization are more neatly formulated in Gentzen's *sequent calculus*, which can be seen as a formal representation of the derivability relation in natural deduction (cf. [5, 12]). The normalization theorem was formulated as *cut-elimination theorem* in sequent calculus, and proved by Gentzen [5].

[4]This theorem is sometimes called the **normal form theorem** to distinguish it from the following stronger form: *Any proof reduces to a normal proof.*

[5]The subformula property of natural deduction for classical logic is slightly more complicated (see [13, 11]).

restrict ourselves to the diagrammatic fragment of NM, i.e., given premises and a conclusion that are restricted to formulas translated from EUL-diagrams, inference rules of NM in general do not correspond to operations on diagrams. This is because a natural deduction proof may contain some inferences on complex formulas, say $(A \to B) \to C$, that no longer correspond to any EUL-diagrams. In such a case, normal natural deduction proofs play an essential role. By virtue of the subformula property, a normal proof contains only components of the given premises and conclusion, that are translated from diagrams, and hence, for such a proof, we are able to assign a corresponding diagrammatic proof.

As an illustration, let us consider the natural deduction proof given in Example 3.4 of Section 3.2 that is not in normal form since it contains a redundant step: without applying $\wedge I$ and $\wedge E$ rules, we already have a proof of $A \to C$. By reducing the proof, we obtain the left-most normal proof presented in Fig. 9.

$$\cfrac{\cfrac{\cfrac{[A]^1 \quad A \to B}{B} \to E \quad B \to C}{\cfrac{C}{A \to C} \to I, 1} \to E}{} \quad \rhd \quad \cfrac{\cfrac{\cfrac{A \to A \quad A \to B}{A \to B} \quad B \to C}{A \to C}}{A \to C} \quad \rhd \quad \cfrac{A \to B \quad B \to C}{A \to C}$$

Fig. 9 Normal proof for Barbara and derived rule

To see that the normal proof corresponds to the diagrammatic proof in Example 3.4, we modify the proof slightly as follows: We first add $A \to$ to formulas between the discharged hypothesis A and the application of $\to I$ rule of A as seen in the middle proof of Fig. 9. Then, by eliminating redundant steps, we obtain the right-most proof, which corresponds to the diagrammatic proof in Example 3.4. In the diagrammatic fragment of NM, such a rewriting of proofs is always possible, and natural deduction proofs in the fragment are considered as chains of the following derived rules as seen in the following proof of Proposition 4.2.

Lemma 4.1 ($\sqsubset$ and $\vdash$ rules) *The following $\sqsubset$ and $\vdash$ rules are derived rules in* NM.

$$\cfrac{A \to B \quad B \to C}{A \to C} \sqsubset \qquad\qquad \cfrac{A \to B \quad B \to \neg C}{A \to \neg C} \vdash$$

We first show faithfulness (Proposition 4.2) of the translation by restricting the given conclusion to the translation of a minimal diagram, and then we extend it to the general case in Theorem 4.3.

Proposition 4.2 (Faithfulness of translation in a minimal fragment) *Let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be a set of* EUL-*diagrams which has a model. Let α be a minimal diagram. Any natural deduction proof of α° from $\mathcal{D}_1^\circ, \ldots, \mathcal{D}_n^\circ$ in* NM *is transformed into a diagrammatic proof of α from $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in* GDS.

Proof. By the normalization theorem for NM, any proof of α° from $\mathcal{D}_1^\circ, \ldots, \mathcal{D}_n^\circ$ is transformed into a normal proof. Let π be such a normal proof. By virtue of the subformula property of NM, the premises $\mathcal{D}_1^\circ, \ldots, \mathcal{D}_n^\circ$ are decomposed by the $\wedge E$-rule in π into formulas corresponding to minimal diagrams. Thus, we assume, without loss of generality, that $\mathcal{D}_1, \ldots, \mathcal{D}_n$ form a set of minimal diagrams $\beta_1, \ldots, \beta_n$, denoted collectively by $\vec{\beta}$. For the sake of simplicity, we do not take tautologies of the form $A \to A$ into account, and we identify each minimal diagram with the non-reflexive EUL-relation holding on it.

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Note that the assertion is immediate, when there exists $1 \leq i \leq n$ such that $\beta_i^\circ \equiv \alpha^\circ$, or when $\alpha^\circ \equiv s \to \neg t$, and there exists $1 \leq i \leq n$ such that $\beta_i^\circ \equiv t \to \neg s$, because $\alpha \equiv \beta_i$ in these cases. Otherwise, we have the following two cases according to the form of α°.

(1) If $\alpha^\circ \equiv s \to t$, this is obtained by the $\to I$-rule since π is in normal form. Note that by our translation of diagrams, an atomic formula, other than $\bot$, is obtained by only the $\to E$-rule, and hence t is obtained by successive applications of this rule. Then π has the following form on the left, where the vertical dots signify successive applications of the $\to E$-rule. Note that all formulas $s \to s_1$, $s_i \to s_{i+1}$, and $s_n \to t$ are open assumptions, since π is in normal form, in which these cannot be obtained by the $\to I$-rule. Then π is transformed into the following π' on the right by using the $\sqsubset$-rule.

$$\pi \begin{cases} \dfrac{\dfrac{[s]^1 \quad s \to s_1}{s_1} {}_{\to E} \quad s_1 \to s_2}{s_2} {}_{\to E} \\ \vdots \\ \dfrac{\dfrac{s_n \qquad\qquad s_n \to t}{t} {}_{\to E}}{s \to t} {}_{\to I,\,1} \end{cases} \qquad \pi' \begin{cases} \dfrac{s \to s_1 \quad s_1 \to s_2}{s \to s_2} {}_{\sqsubset} \\ \vdots \\ \dfrac{s \to s_n \qquad\qquad s_n \to t}{s \to t} {}_{\sqsubset} \end{cases}$$

Observe that each application of the $\sqsubset$-rule on the left in Fig. 10 below is transformed into the diagrammatic proof on the right by using a pair of Unification and Deletion. Thus the above π' is transformed into a diagrammatic proof of α from $\vec{\beta}$.

$$\dfrac{A \to B \quad B \to C}{A \to C} \; \sqsubset$$

Fig. 10 $\sqsubset$-rule and d-proof

$$\dfrac{A \to B \quad B \to \neg C}{A \to \neg C} \; \vdash$$

Fig. 11 $\vdash$-rule and d-proof

(2) If $\alpha^\circ \equiv s \to \neg t$, this is obtained by the $\to I$-rule since π is in normal form. Note that $\neg t$ may be obtained by either the $\to E$ or $\neg I$ rule. If obtained through the $\to E$-rule, π has a similar form to case (1), else, $\neg t$ is obtained by the $\neg I$-rule, and π is shown to have the following form on the left, where the vertical dots signify successive applications of the $\to E$-rule, and each φ_j is an atom or its negation.

$$\pi \begin{cases} \dfrac{\dfrac{\dfrac{[s]^2 \quad s \to s_1}{s_1} {}_{\to E} \quad s_1 \to s_2}{s_2} {}_{\to E}}{\begin{array}{c} \vdots \\ u \end{array}} \qquad \dfrac{\dfrac{\dfrac{[t]^1 \quad t \to \varphi_1}{\varphi_1} {}_{\to E} \quad \varphi_1 \to \varphi_2}{\varphi_2} {}_{\to E}}{\begin{array}{c} \vdots \\ \neg u \end{array}} {}_{\neg E} \\ \dfrac{\dfrac{\bot}{\neg t} {}_{\neg I,\,1}}{s \to \neg t} {}_{\to I,\,2} \end{cases}$$

$$\pi' \begin{cases} \dfrac{s \to s_1 \quad s_1 \to s_2}{s \to s_2} \qquad \dfrac{t \to \varphi_1 \quad \varphi_1 \to \varphi_2}{t \to \varphi_2} \\ \dfrac{\begin{array}{c} \vdots \\ s \to u \end{array} \qquad\qquad \begin{array}{c} \vdots \\ t \to \neg u \end{array}}{s \to \neg t} \end{cases}$$

Note that, by the presupposition that $\vec{\beta}$ has a model, there are no cases where $s \equiv t$ and these are closed at the same time by the application of either $\neg I$ or $\to I$ (in such an application of $\to I$ or $\neg I$ the empty assumption could be closed). Furthermore, note that, since π is

in normal form, $\neg u$ cannot be obtained by the $\neg I$-rule, and hence, $\neg u$ is obtained by the $\rightarrow E$-rule. Thus π has the above form, and then is transformed into π' on the above right by using the $\vdash$- and $\sqsubset$-rules. Observe that each application of the $\vdash$-rule on the left in Fig. 11 above is transformed into the diagrammatic proof on the right by using a pair of Unification and Deletion. Note in particular that π with $s \equiv u$ is transformed into π' ending with the following form:

$$\pi' \left\{ \begin{array}{c} \vdots \\ \dfrac{t \rightarrow \neg u}{u \rightarrow \neg t} \end{array} \right.$$

Such an inference step collapses in GDS, because $t \vdash u = u \vdash t$. Thus π' is transformed into a diagrammatic proof of α from $\vec{\beta}$. ∎

Now, let us extend Proposition 4.2 to the general case, where the conclusion is no longer minimal. Given an EUL-diagram $\mathcal{E}$ and two objects, say s and t on $\mathcal{E}$, a minimal diagram is obtained from $\mathcal{E}$ by deleting all objects other than s and t. By Proposition 2.4, the set of such minimal diagrams is uniquely determined. The completeness proof of GDS in [8] gives a way to reconstruct $\mathcal{E}$ from the minimal diagrams obtained from $\mathcal{E}$ (cf. also Fig. 14 in Section 4.3.5). In other words, any diagram can be constructed from a set of minimal diagrams. Thus, by applying [8]'s construction of diagrams from minimal diagrams, Proposition 4.2 is naturally extended as follows.

Theorem 4.3 (Faithfulness of translation) *Let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be a set of EUL-diagrams which has a model. Let $\mathcal{E}$ be an EUL-diagram. Any natural deduction proof of $\mathcal{E}^\circ$ from $\mathcal{D}_1^\circ, \ldots, \mathcal{D}_n^\circ$ in NM is transformed into a diagrammatic proof of $\mathcal{E}$ from $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in GDS.*

Proof. By the normalization theorem for NM, any proof of $\mathcal{E}^\circ$ is transformed into a normal proof of $\mathcal{E}^\circ$. Let π be such a normal proof of $\mathcal{E}^\circ$ from $\mathcal{D}_1^\circ, \ldots, \mathcal{D}_n^\circ$, denoted by $\vec{\mathcal{D}}^\circ$. When $\mathcal{E}$ is minimal, the theorem is equivalent to Proposition 4.2. Therefore, we examine the case where $\mathcal{E}$ is not minimal. Let $\mathcal{E}^\circ$ be a conjunction $\alpha_1^\circ, \ldots, \alpha_m^\circ$. Then $\mathcal{E}^\circ$ is obtained by $\wedge E$ or $\wedge I$ rule.
(1) When $\alpha_1^\circ, \ldots, \alpha_m^\circ$ is obtained by applying $\wedge E$ to a conjunction $\alpha_1^\circ, \ldots, \alpha_m^\circ, \varphi_1, \ldots, \varphi_l$ for some $\varphi_1, \ldots, \varphi_l$, since π is in normal form, it is again obtained by $\wedge E$. In this way, π consists of successive applications of $\wedge E$ to some premise $\mathcal{D}_i^\circ$. Thus π is transformed into a d-proof of $\mathcal{E}$ from $\vec{\mathcal{D}}$ in GDS, where $\mathcal{E}$ is derived from $\mathcal{D}_i$ by successive applications of Deletion.
(2) When $\mathcal{E}^\circ$ is obtained by an application of the $\wedge I$-rule, without loss of generality, we assume that it is obtained by a single application of this rule, and that π is in the following form:

$$\begin{array}{ccc} \vec{\mathcal{D}}^\circ & & \vec{\mathcal{D}}^\circ \\ \vdots\ \pi_1 & & \vdots\ \pi_m \\ \alpha_1^\circ & \cdots & \alpha_m^\circ \\ \hline \multicolumn{3}{c}{\alpha_1^\circ, \ldots, \alpha_m^\circ} \end{array} \wedge I.$$

Then, each proof π_i of α_i° for $1 \le i \le m$ is transformed, by Proposition 4.2, into a d-proof of α_i in GDS. By the presupposition that $\mathcal{E}^\circ$ is the translation of an EUL-diagram, the last application of the $\wedge I$-rule is transformed into some applications of Unification of GDS. A concrete procedure to transform the $\wedge I$-rule is given in the completeness proof of GDS in [8]. (See also Fig. 14 in Section 4.3.5.) Thus, we have a d-proof of $\mathcal{E}$ from $\vec{\mathcal{D}}$ in GDS. ∎

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4.3 Normal proof and normalization in GDS

Based on the discussion on the faithfulness of our translation in the previous section, in the class of Euler diagrammatic proofs, we are able to characterize the subclass to which normal natural deduction proofs correspond. We define this subclass as *normal diagrammatic proofs* in Section 4.3.1. Next we prove a normalization theorem in GDS, and discuss some consequences of the theorem, namely: an analysis on the structure of normal diagrammatic proofs (Section 4.3.2); a diagrammatic counterpart of the usual subformula property (Section 4.3.3); and a characterization of diagrammatic proofs compared with natural deduction proofs (Section 4.3.4).

4.3.1 Normal form

Let us define the notion of *normal diagrammatic proofs*. In what follows, for the sake of simplicity, we restrict ourselves to a **syllogistic fragment** of GDS, where the provability relation $\mathcal{D}_1, \ldots, \mathcal{D}_n \vdash \mathcal{E}$ is considered only for the following syllogistic diagrams.

Definition 4.4 (Syllogistic diagram) An EUL-diagram is called a *syllogistic diagram* if it takes one of the following forms:

Note that the universal syllogistic sentence *All A are B* (resp. *No A are B*) corresponds to the third (resp. fourth) minimal diagram; the particular sentence *Some A are B* (resp. *Some A are not B*) corresponds to the fifth (resp. sixth) diagram, of those the first two are component minimal diagrams. Thus we call the above diagrams syllogistic.

Observe that even if we restrict the premises and conclusion to being syllogistic, more complex diagrams might appear in a diagrammatic proof as seen in the following Fig. 12. The extension of the following analyses to the full fragment of GDS is discussed in Section 4.3.5.

In the syllogistic fragment of GDS, the notion of normal form is defined as follows.

Definition 4.5 (N-normal form) An *N-normal diagrammatic proof* is a diagrammatic proof, where each Unification is applied to two minimal diagrams.

Example 4.6 (General d-proof and N-normal d-proof) Fig. 12 are examples of a (general) diagrammatic proof and an N-normal diagrammatic proof having the following premises and conclusion:

By using the well-established normalization theorem for the natural deduction system NM, we derive a normalization theorem for GDS with respect to the N-normal form.

Theorem 4.7 (Normalization) *Let $\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{E}$ be syllogistic diagrams. If $\mathcal{E}$ is provable from $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in* GDS, *then there is an N-normal diagrammatic proof of $\mathcal{E}$ from $\mathcal{D}_1, \ldots, \mathcal{D}_n$.*

(general) d-proof **N-normal d-proof**

Fig. 12 General d-proof and N-normal d-proof (+ for Unification and − for Deletion)

Proof. Let $\vec{D} \vdash \mathcal{E}$ in GDS. Then, by the translation (Theorem 3.3), we have $\vec{\mathcal{D}^\circ} \vdash \mathcal{E}^\circ$ in NM. By the normalization theorem in NM, there is a normal natural deduction proof of $\mathcal{E}^\circ$ from $\vec{\mathcal{D}^\circ}$. We consider the following two cases depending on $\mathcal{E}$.

(1) If $\mathcal{E}$ is a minimal diagram, by Proposition 4.2, there is a d-proof π of $\mathcal{E}$ from $\vec{\mathcal{D}}$, which is in normal form as seen in Figs. 10 and 11 in Proposition 4.2.

(2) If $\mathcal{E}$ is not minimal, by Theorem 4.3, there is a d-proof π of $\mathcal{E}$ from $\vec{\mathcal{D}}$. Note that as evident in the proof of Theorem 4.3, the last rule of π is an application of Unification between two minimal diagrams because $\mathcal{E}$ is syllogistic, and that the other applications of Unification are on two minimal diagrams in the same way as (1). Thus π is in normal form. ∎

Strictly speaking, our normal diagrammatic proofs and normal natural deduction proofs do not correspond in a one-to-one way, because formulas of the forms $A \to \neg B$ and $\neg B \to A$ are required to collapse in GDS. However, by considering an equivalence class of normal natural deduction proofs with respect to the above set of formulas, we have a bijective correspondence as seen in Figs. 10 and 11 in Proposition 4.2.

Remark 4.8 (U9, U10, Point Insertion in N-normal form) When premise diagrams are restricted to two minimal diagrams, applications of rules U9, U10 (cf. Appendix A) should take the following form:

Since their conclusions are equivalent to one of their respective premises, these rules do not produce anything. Furthermore, under the same restriction, Point Insertion (cf. Appendix A) is equivalent to U3 or U4. Thus, we assume U9, U10, and Point Insertion do not appear in any N-normal diagrammatic proofs.

In subsequent sections, we examine some properties of N-normal diagrammatic proofs.

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4.3.2 Structure of normal Euler diagrammatic proofs

In the syllogistic fragment, as seen in Fig. 12, N-normal diagrammatic proofs have the following canonical structure.

Proposition 4.9 (Structure of N-normal proofs) *In the syllogistic fragment, applications of* Unification *and* Deletion *alternate in an N-normal diagrammatic proofs.*

Proof. We first consider a possible rule after an application of Unification in a given N-normal d-proof. Let $\mathcal{D}_1 + \mathcal{D}_2$ be obtained by Unification. After this rule, Unification is impossible, since it is only applied to two minimal diagrams in any normal form, and $\mathcal{D}_1 + \mathcal{D}_2$ is no longer minimal. (Only when U9 or U10 is applied, $\mathcal{D}_1 + \mathcal{D}_2$ may be minimal, but we do not need to take them into account as discussed in Remark 4.8.) Hence, only Deletion is possible after Unification.

We next consider a possible rule after an application of Deletion. Let $\mathcal{D} - t$ be obtained by Deletion. We observe that $\mathcal{D} - t$ should be minimal. Assume to the contrary that $\mathcal{D} - t$ is not minimal. Then $\mathcal{D} - t$ has more than 2 objects, and hence, $\mathcal{D}$ has more than 3 objects. Note that any diagram obtained by Unification has exactly 3 objects in any N-normal form. Hence $\mathcal{D}$ also should be obtained by an application of Deletion. In this way, $\mathcal{D} - t$ should be obtained by successive applications of Deletion, but it is impossible since premises are restricted to those that are syllogistic diagrams, which consists of at most 3 objects. Therefore, $\mathcal{D} - t$ should be minimal, and only Unification is applicable to this diagram. ∎

4.3.3 Subformula property in Euler diagrammatic system

To investigate a diagrammatic counterpart of the usual subformula property, we define the complexity of a diagram as follows.

Definition 4.10 (Complexity) The *complexity* of an EUL-diagram $\mathcal{D}$ is the number of objects (named circles and points) in $\mathcal{D}$.

Let us see the diagrammatic proofs of Fig. 12 in Example 4.6, for which the premises and conclusion have complexities of at most 3. On the one hand, the general diagrammatic proof contains the following diagram whose complexity is 5 (i.e., it consists of 4 circles A, B, D, E and 1 point c).

On the other hand, any diagram appearing in the N-normal diagrammatic proof of Fig. 12 has complexity of at most 3, and hence, the proof contains no more complex diagrams than its premises and conclusion. This property holds in general for N-normal diagrammatic proofs, and we obtain the following proposition, which is considered as the diagrammatic counterpart of the usual subformula property in symbolic logic.

Proposition 4.11 (Subformula property) *Let $\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{E}$ be syllogistic diagrams of which at least one is not minimal. If $\mathcal{E}$ is provable from $\mathcal{D}_1, \ldots, \mathcal{D}_n$, then there exists a diagrammatic proof of $\mathcal{E}$ in which any diagram is no more complex than $\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{E}$.*
If $\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{E}$ are all minimal, there is a diagrammatic proof of $\mathcal{E}$ in which any diagram has complexity at most 3.

Proof. Let $\vec{\mathcal{D}} \vdash \mathcal{E}$, of which one is not minimal. By normalization (Theorem 4.7), there is a normal d-proof π of $\mathcal{E}$ from $\vec{\mathcal{D}}$. We show that the normal d-proof π has the required property. Assume to the contrary that a more complex diagram than $\vec{\mathcal{D}}, \mathcal{E}$ appears in π, and let $\mathcal{F}$ be a most complex one. Then, since Deletion always reduces the complexity, $\mathcal{F}$ should be obtained by Unification. Note that in an N-normal d-proof, any diagram obtained by Unification has exactly 3 objects, and hence so does $\mathcal{F}$. However, it contradicts the assumption that $\mathcal{F}$ is more complex than $\vec{\mathcal{D}}, \mathcal{E}$, which have complexities at most 3.

The case in which all $\vec{D}, \mathcal{E}$ are minimal is shown in the same manner. ∎

Similarly to the way that the subformula property in symbolic logic gives a basis to theorem proving, the above property serves to define an effective strategy to construct a diagrammatic proof by restricting the number of objects appearing in such a proof.

4.3.4 Euler diagrammatic proofs and natural deduction proofs

As seen in Figs. 9, 10, and 11 of Section 4.2, normal Euler diagrammatic proofs reflect, to some extent, the structure of natural deduction proofs. Thus, by comparing structures of normal diagrammatic proofs and general diagrammatic proofs, we are able to characterize differences between natural deduction proofs and Euler diagrammatic proofs in the framework of diagrammatic system. We formalize here one such difference in respect to free ride.

Let us recall that, in N-normal diagrammatic proofs, each Unification is always applied to two minimal diagrams. Hence free ride in such a Unification, say U5, is explicated through the translation into natural deduction as seen in the following Fig. 13.

Fig. 13   Free ride in normal diagrammatic proof

On the one hand, in a general diagrammatic proof, each unification rule has several free rides occurring concurrently as seen in Fig. 8 of Section 3.3.3. On the other, in an N-normal diagrammatic proof, we have the following proposition as seen in the above Fig. 13.

Proposition 4.12 (Minimal free ride in N-normal form) *An application of* Unification *in any N-normal diagrammatic proof has* minimal free ride, *i.e., at most 1 free ride.*

In other words, while a general diagrammatic inference rule has several *concurrent free rides*, a diagrammatic inference rule corresponding to a natural deduction rule has *minimal free rides*. If we regard N-normal diagrammatic proofs as natural deduction proofs in view of the bijective correspondence between them, the above proposition is a proof-theoretical formalization of the following difference: On the one hand, an Euler diagrammatic rule has concurrently many consequences, and on the other, a natural deduction rule has a single consequence.

4.3.5   Extension to full GDS

Although our discussion has so far been restricted to the syllogistic fragment, it can be naturally extended to the full fragment of GDS. Based on the construction of *canonical proofs* given in the completeness proof of GDS [8], we extend the notion of normal form as follows.

An **N-normal diagrammatic proof** in the full fgrament of GDS consists from the top down of the following parts:

Deletion part: With the use of the Deletion rule, premises $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are decomposed into minimal diagrams.

N-normal part: With the use of the U1–U7 rules for minimal diagrams obtained in the deletion part, N-normal diagrammatic proofs are constructed for all pointed (resp. point-free) minimal diagrams in each of which a relation $a \sqsubset B$ or $a \vdash B$ (resp. $A \sqsubset B$ or $A \vdash B$) of the conclusion $\mathcal{E}$ holds.

Venn construction part: With the use of the U1, U2, and Point Insertion rules for minimal diagrams obtained in the N-normal part, a "Venn-like diagram" is constructed, in which $A \bowtie B$ holds for any pair of circles in it, and which consists of all points and circles of the conclusion $\mathcal{E}$. (The U8 rule is used when $\mathcal{E}$ has no point.)

Modification part: Using rules U9, U10, the conclusion $\mathcal{E}$ is constructed by unifying the Venn-like diagram and all point-free minimal diagrams obtained in the N-normal part.

Thus, each normal diagrammatic proof has a form as depicted in Fig. 14:

Fig. 14 Normal form in full GDS

The **normalization theorem** is extended with respect to the above notion of normal form in exactly the same way as Theorem 4.7.

We have the following correspondence between normal diagrammatic proofs and normal natural deduction proofs. (1) The deletion part corresponds to the $\wedge E$ rule of NM. (2) The N-normal part corresponds to a natural deduction proof by Proposition 4.2. (3) The Venn construction part corresponds to the $\wedge I$ rule of NM. (4) The modification part corresponds to the $\wedge I$ rule of NM.

As for the **subformula property** (Proposition 4.11), we observe that each diagram in a normal diagrammatic proof has the following complexity. (1) Any diagram in the deletion part has complexity at most those of each of the premises $\mathcal{D}_1, \ldots, \mathcal{D}_n$. (2) Any diagram in the N-normal part has complexity at most 3 as seen in Proposition 4.11. (3) Any diagram in the Venn construction part has complexity at most that of the Venn-like diagram $\mathcal{V}$ for which

the complexity is the same as the conclusion $\mathcal{E}$. (4) Any diagram in the modification part has complexity at most that of $\mathcal{E}$. Therefore, the same subformula property as Proposition 4.11 holds for full GDS.

For the **characterization of normal form in terms of free ride** (Proposition 4.12), let us check for free rides occurring in each part. (1) In the deletion part, no free ride occurs. (2) In the N-normal part, as seen in Proposition 4.12, at most one free ride occurs in each application of Unification. (3) In the Venn construction part, no free ride occurs in each application of U1, U2, U8 and Point Insertion as demonstrated in Appendix A. (4) In the modification part, depending on the order of applications of U9 and U10, concurrent free rides may occur. Note however that such free rides are inessential, since all relations (i.e., minimal diagrams) holding on $\mathcal{E}$ are already derived in the N-normal part. Hence, it is possible to arrange the order of applications of U9 and U10 in the modification part so that no free ride occurs in the following manner: (i) To the given Venn-like diagram $\mathcal{V}$, we first apply U10 successively. No free ride occurs by these applications of U10, since free rides occur for U10 only when $\sqsubset$-relations hold on $\mathcal{V}$, but no $\sqsubset$-relation holds on it. (ii) We then apply U9 successively in an appropriate order. As an illustration, let $\mathcal{F}$ be the following diagram of Fig. 15 obtained by (i), in which all of $\vdash\!\!\dashv$-relations of $\mathcal{E}$ hold. Note that, since $\vdash\!\!\dashv$-relations are already fixed, all possible free rides of U9 result from the transitivity of the $\sqsubset$-relation (cf. Appendix A). We can avoid such free rides as follows. By examining the transitivity relations holding on the conclusion $\mathcal{E}$, we obtain a chain of objects with respect to the $\sqsubset$-relation, i.e., $A \sqsubset B \sqsubset C$ in Fig. 15. Then we apply U9 in the order $A \sqsubset C, A \sqsubset B, B \sqsubset C$ as seen in Fig. 15, where no free ride occurs. In this way, we can apply U9 and U10 in the modification part without producing any free ride. Thus, we can extend Proposition 4.12 to full GDS.

Fig. 15

5 Conclusion and future work

Let us summarize our framework with the following Fig. 16.
(1) We have introduced a translation from the Euler diagrammatic inference system GDS of [8] into the natural deduction system for minimal logic without disjunction NM. The sound-

Fig. 16 Summary of our framework

ness (Theorem 3.3) of the translation implies that free ride occurs between GDS and NM (Proposition 3.6). Furthermore, our translation enables the proof-theoretical investigation of free rides in terms of natural deduction inference rules.

(2) By using the normalization theorem of natural deduction system, we have demonstrated the faithfulness of our translation (Theorem 4.3).

(3) Based on the normal natural deduction proofs, we have introduced the notion of *N-normal diagrammatic proofs*, and have presented the normalization theorem (Theorem 4.7). We have also investigated the structure of normal diagrammatic proofs (Proposition 4.9), and have proved the diagrammatic counterpart of the usual subformula property (Proposition 4.11).

(4) Finally, through the bijective correspondence between normal diagrammatic proofs and normal natural deduction proofs, we have introduced a proof theoretical method to compare structures of diagrammatic proofs and natural deduction proofs. We have characterized the difference between these in terms of concurrent and minimal free rides (Proposition 4.12).

Our framework as summarized above is not restricted to Euler diagrams, but is able to be applied to other visual representations such as Venn diagrams and graph theoretical representations. This enables proof-theoretical formalizations and analyses of properties of such diagrams and graphs.

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A   Euler diagrammatic inference system GDS

We give full descriptions of unification rules of GDS (Definition 2.10) (see [8] for axioms and deletion rule) and the translation Definition 3.1, i.e., the translation of each inference rule of GDS into a combination of inference rules of natural deduction. We also specify free rides of each rule.

Definition A.1 Axiom, unification, and deletion of GDS are defined as follows.

Axiom:

A1: For any pair of circles A and B, any minimal diagram where $A \bowtie B$ holds is an axiom.

A2: Any EUL-diagram that consists only of points is an axiom.

Unification: Unification rules are divided into three groups, Group (I), (II), and (III). The rules in Group (I) and (II) are classified according to the number and type of objects shared by a diagram $\mathcal{D}$ and a minimal diagram α. The rule in Group (III) is the Point Insertion rule, where neither of two premise diagrams is restricted to be minimal.

In what follows, in order to avoid notational complexity in a diagram, we express each named point, say $\overset{c}{\bullet}$, simply by its name c.

(I) The case $\mathcal{D}$ and α share one object:

U1 rule   **Premises:** $b \sqsubset A$ holds on α, and $b \in pt(\mathcal{D})$.

Constraint for determinacy: $pt(\mathcal{D}) = \{b\}$.

Operation: Add the circle A to $\mathcal{D}$ (with preservation of all relations on $\mathcal{D}$) so that the following conditions are satisfied on $\mathcal{D} + \alpha$: (1) $b \sqsubset A$ holds; (2) $A \bowtie X$ holds for all circles X of $\mathcal{D}$.

The set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ is specified as $\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\}$.

Schema of U1   Example of U1   Translation of U1

$$\frac{\mathcal{D}^\circ \quad \alpha^\circ}{\mathcal{D}^\circ, \alpha^\circ} \wedge I$$

U1 has no free ride.

$\boxed{\text{U2 rule}}$ **Premises:** $b \vdash A$ holds on α, and $b \in pt(\mathcal{D})$.

Constraint for determinacy: $pt(\mathcal{D}) = \{b\}$.

Operation: Add the circle A to $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D} + \alpha$: (1) $b \vdash A$ holds; (2) $A \bowtie X$ holds for all circles X of $\mathcal{D}$.

$\mathsf{rel}(\mathcal{D} + \alpha) = \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\}$.

Schema of U2   Example of U2   Translation of U2

$$\frac{\mathcal{D}^\circ \quad \alpha^\circ}{\mathcal{D}^\circ, \alpha^\circ} \wedge I$$

U2 has no free ride.

$\boxed{\text{U3 rule}}$ **Premises:** $b \sqsubset A$ holds on α, and $A \in cr(\mathcal{D})$.

Constraint for determinacy: $A \sqsubset X$ or $A \vdash X$ holds for all circles X of $\mathcal{D}$.

Operation: Add the point b to $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D} + \alpha$: (1) $b \sqsubset A$ holds; (2) $b \vdash x$ holds for all points x such that $x \sqsubset A$ holds on $\mathcal{D}$.

$\mathsf{rel}(\mathcal{D} + \alpha) = \mathsf{rel}(\mathcal{D}) \ \cup \ \mathsf{rel}(\alpha) \cup \ \{b \sqsubset X \mid A \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \cup \ \{b \vdash X \mid A \vdash X \in \mathsf{rel}(\mathcal{D})\} \cup \ \{b \vdash x \mid x \in pt(\mathcal{D})\}$

Note that we assume $x \to \neg y$ being an axiom of our NM for any distinct points x, y.
U3 has free rides $\{b \sqsubset X \mid A \sqsubset X \in \mathsf{rel}(\mathcal{D}), X \not\equiv A\} \cup \{b \vdash X \mid A \vdash X \in \mathsf{rel}(\mathcal{D})\} \cup \{b \vdash x \mid A \vdash x \in \mathsf{rel}(\mathcal{D})\}$.

U4 rule **Premises:** $b \vdash A$ holds on α, and $A \in cr(\mathcal{D})$.

Constraint for determinacy: $X \sqsubset A$ holds for all circles X of $\mathcal{D}$.

Operation: Add the point b to $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D} + \alpha$: (1) $b \vdash A$ holds; (2) $b \vdash x$ holds for all points x such that $x \vdash A$ holds on $\mathcal{D}$.

$$\mathsf{rel}(\mathcal{D} + \alpha) = \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{b \vdash X \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{b \vdash x \mid x \in pt(\mathcal{D})\}.$$

U4 has free rides $\{b \vdash X \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D}), X \not\equiv A\} \cup \{b \vdash x \mid x \sqsubset A \in \mathsf{rel}(\mathcal{D})\}$.

U6 rule **Premises:** $A \sqsubset B$ holds on α, and $A \in cr(\mathcal{D})$.

Constraint for determinacy: $x \sqsubset A$ holds for all $x \in pt(\mathcal{D})$.

Operation: Add the circle B to $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D} + \alpha$: (1) $A \sqsubset B$ holds; (2) $B \bowtie X$ holds for all circles $X (\not\equiv A)$ such that $A \sqsubset X$ or $A \vdash X$ or $A \bowtie X$ holds on $\mathcal{D}$.

$$\mathsf{rel}(\mathcal{D} + \alpha) = \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{X \bowtie B \mid A \sqsubset X \text{ or } A \vdash X \text{ or } A \bowtie X \in \mathsf{rel}(\mathcal{D}), X \not\equiv A\}$$
$$\cup \{X \sqsubset B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{x \sqsubset B \mid x \in pt(\mathcal{D})\}$$

U6 has free rides $\{X \sqsubset B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D}), X \not\equiv A\} \cup \{x \sqsubset B \mid x \in pt(\mathcal{D})\}$.

U8 rule **Premises:** $A \bowtie B$ holds on α, and $A \in cr(\mathcal{D})$.

Constraint for determinacy: $pt(\mathcal{D}) = \emptyset$.

Operation: Add the circle B to $\mathcal{D}$ so that $B \bowtie X$ holds for all circles X of $\mathcal{D}$.

$$\mathsf{rel}(\mathcal{D} + \alpha) = \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{B \bowtie X \mid X \in cr(\mathcal{D})\}.$$

U8 has no free ride.

(II) When $\mathcal{D}$ and α share two circles, these may be unified into $\mathcal{D} + \alpha$ by the following U9 and U10 rules.

U9 rule **Premises:** $A \sqsubset B$ holds on α, and $A \bowtie B$ holds on $\mathcal{D}$.

Constraint for consistency: There is no object s such that $s \sqsubset A$ and $s \vdash B$ hold on $\mathcal{D}$.

Operation: Modify all circles X (including A) of $\mathcal{D}$ such that $X \sqsubset A$ holds so that the following conditions are satisfied on $\mathcal{D} + \alpha$: (1) $X \sqsubset B$ holds; (2) $X \square t$ holds with $\square \in \{\sqsubset, \sqsupset, \vdash, \bowtie\}$ for all object t of $\mathcal{D}$ such that $t \sqsubset A, X \square t \in \mathsf{rel}(\mathcal{D})$.

$$\mathsf{rel}(\mathcal{D} + \alpha) = \big(\mathsf{rel}(\mathcal{D}) \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } B \sqsubset Y \in \mathsf{rel}(\mathcal{D})\} \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \vdash B \in \mathsf{rel}(\mathcal{D})\}\big)$$
$$\cup \{X \sqsubset Y \mid X \sqsubset A \text{ and } B \sqsubset Y \in \mathsf{rel}(\mathcal{D})\} \cup \{X \vdash Y \mid X \sqsubset A \text{ and } Y \vdash B \in \mathsf{rel}(\mathcal{D})\}$$

U9 is translated as follows: φ_m is Y_m or $\neg Y_m$.

U9 has free rides $\{X \sqsubset Y \mid X \sqsubset A \text{ and } B \sqsubset Y \in \mathsf{rel}(\mathcal{D}), X \not\equiv A, Y \not\equiv B\} \cup \{X \vdash Y \mid X \sqsubset A \text{ and } B \vdash Y \in \mathsf{rel}(\mathcal{D})\}$.

U10 rule **Premises:** $A \vdash B$ holds on α, and $A \bowtie B$ holds on $\mathcal{D}$.

Constraint for consistency: There is no object s such that $s \sqsubset A$ and $s \sqsubset B$ hold on $\mathcal{D}$.

Operation: Modify all circles X (including A) and Y (including B) of $\mathcal{D}$ such that $X \sqsubset A$ and $Y \sqsubset B$, respectively hold on $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D} + \alpha$: (1) $X \vdash B$ holds; (2) $X \square t$ holds with $\square \in \{\sqsubset, \sqsupset, \vdash, \bowtie\}$ for all object t of $\mathcal{D}$ such that $t \sqsubset A, X \square t \in \mathsf{rel}(\mathcal{D})$; (3) $Y \vdash A$ holds; (4) $Y \square s$ holds with $\square \in \{\sqsubset, \sqsupset, \vdash, \bowtie\}$ for all object s of $\mathcal{D}$ such that $s \sqsubset B, Y \square s \in \mathsf{rel}(\mathcal{D})$.

$$\mathsf{rel}(\mathcal{D} + \alpha) = \big(\mathsf{rel}(\mathcal{D}) \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \sqsubset B \in \mathsf{rel}(\mathcal{D})\}\big) \cup \{X \vdash Y \mid X \sqsubset A \text{ and } Y \sqsubset B \in \mathsf{rel}(\mathcal{D})\}$$

U10 has free rides $\{X \vdash Y \mid X \sqsubset A \text{ and } Y \sqsubset B \in \mathsf{rel}(\mathcal{D}), X \not\equiv A, Y \not\equiv B\}$.

(III) Neither of two premise diagrams is restricted to be minimal.

$\boxed{\text{Point Insertion}}$ **Premises:** $X \square Y \in \mathsf{rel}(\mathcal{D}_1)$ *iff* $X \square Y \in \mathsf{rel}(\mathcal{D}_2)$ holds for any circles X, Y with $\square \in \{\sqsubset, \sqsupset, \vdash, \bowtie\}$, and $pt(\mathcal{D}_2) = \{b\}$ such that $b \notin pt(\mathcal{D}_1)$.

Operation: Add the point b to $\mathcal{D}_1$ so that the following conditions are satisfied on $\mathcal{D}_1 + \mathcal{D}_2$: (1) $b \square t$ of $\mathsf{rel}(\mathcal{D}_2)$ holds for all objects t; (2) $b \vdash x$ holds for all $x \in pt(\mathcal{D}_1)$.

$\mathsf{rel}(\mathcal{D}_1 + \mathcal{D}_2) = \mathsf{rel}(\mathcal{D}_1) \cup \mathsf{rel}(\mathcal{D}_2) \cup \{b \vdash x \mid x \in pt(\mathcal{D}_1)\}$.

$$\frac{\mathcal{D}_1^\circ \quad \mathcal{D}_2^\circ \quad (b \to \neg x_l)_l}{\mathcal{D}_1^\circ, \mathcal{D}_2^\circ, (b \to \neg x_l)_l} \wedge I$$

Point Insertion has no free ride.

$\boxed{\text{Deletion}}$ **Premise:** $\mathcal{D}$ contains an object s.

Constraint: $\mathcal{D}$ is not minimal.

Operation: Delete the object s from $\mathcal{D}$.

The set of relations $\mathsf{rel}(\mathcal{D} - s)$ of the unified diagram is specified as follows:

$$\mathsf{rel}(\mathcal{D}) \setminus \{s \square t \mid t \in ob(\mathcal{D}), \square \in \{\sqsubset, \sqsupset, \vdash, \bowtie\}\}$$

Deletion is translated as follows:

$$\frac{\mathcal{D}^\circ}{\mathcal{D}^\circ \setminus \{\varphi \mid \varphi \text{ is an implicational formula containing } s\}} \wedge E$$

Deletion has no free rides.