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LOGICS OF REJECTION: TWO SYSTEMS OF NATURAL DEDUCTION

Allard M. TAMMINGA

0. *Introduction*

This paper contains two systems of natural deduction for the rejection of non-tautologies of Classical Propositional Logic. The first system is correct and complete with respect to the body of all non-tautologies, the second system is correct and complete with respect to the body of all contradictions. The second system is a subsystem of the first. We begin with an historical synopsis of the development of the theories of rejection for the classical logic of propositions, taking our starting-point from the theories of their 'founding father', Jan Łukasiewicz. Subsequently, the systems of natural deduction are set forth and their correctness and completeness is showed. We shall conclude with an interesting 'Theorem of Inversion'.

1. *Łukasiewicz's theories of rejection*

Jan Łukasiewicz (1878-1956) was among Kazimierz Twardowski's pupils at Lwów University. In 1910 and 1911 Łukasiewicz participated in a seminar, presided by Alexius Meinong (1853-1921), at Graz University. Both Meinong and Twardowski, but also Edmund Husserl (1859-1938), studied with Franz Brentano (1838-1917). In 1911 Łukasiewicz was appointed as an extraordinary professor at Lwów University. In 1915 he was offered a chair in philosophy at Warsaw University, where he lectured until 1939. After the Second World War Łukasiewicz held a position as professor of mathematical logic at the Royal Irish Academy of Science in Dublin. Łukasiewicz owes his present-day fame to his research into multiple-valued logics, to his research into propositional logic, to the so-called 'Polish notation', and to his studies of the logic of Aristotle and of Stoic logic.

In this paper Łukasiewicz's work is of importance because he was the first to introduce the concept of 'rejection' into formal logic. According to Jerzy Słupecki[1], Łukasiewicz's idea to introduce the concept of 'rejection'

[1] Słupecki (1970); p. ix.

into formal logic, besides Frege's concept 'Anerkennung der Wahrheit', was first set forth in the paper 'Logika dwuwartościowa', which appeared in 1921. It will be recommendable to sweep the dust from this paper.

In the introduction of the paper 'Two-valued Logic', the English translation of 'Logika dwuwartościowa', Łukasiewicz writes: 'In adding 'rejection' to 'assertion' I have followed Brentano.'[2] Brentano in his *magnum opus*, his *Psychologie vom empirischen Standpunkt* which was published in 1874, defines the concept of 'Urteilen' as follows:

> Unter dem *Urteilen* verstehen wir [...] ein (als wahr) Annehmen oder (als falsch) Verwerfen.[3]

Brentano's lectures on logic, posthumously published as *Die Lehre vom richtigen Urteil*, also contain some passages on the concept of 'Verwerfung', that fall in with Łukasiewicz's ideas on the concept of 'rejection'. Brentano here describes the notion of an 'Urteil' in the following way:

> wo immer etwas anerkannt oder verworfen, bejaht oder verneint wird, haben wir ein Urteil vor uns[4]

and

> Wer urteilt, stellt das, was er beurteilt, vor [...] Doch eine zweite neue Beziehung zum Gegenstand kommt zu der im Vorstellen selbst gegebenen hinzu, sie ist charakterisiert durch das Anerkennen oder Verwerfen, Bejahen oder Verneinen[5]

and

> ein Urteil [ist] immer dann gegeben [...], wenn etwas anerkannt oder verworfen, bejaht oder verneint wird und die Kategorien wahr und falsch anwendbar sind[6]

2 Łukasiewicz (1921); p. 89. In (1951); p. 94, Łukasiewicz reports on the distinction between 'assertion' and 'rejection': 'I owe this distinction to Franz Brentano, who describes the acts of believing as *anerkennen* and *verwerfen.*'

3 Brentano (1874); p. 34.

4 Brentano (1956); p. 32.

5 Brentano (1956); p. 33.

6 Brentano (1956); p. 97.

The following passage most clearly presents Brentano's conceptions. Besides this, Brentano is the first to propound a pair of symbols to distinguish between the two kinds of judgment:

> Da jedem Urteil eine Vorstellung zugrunde liegt, so wird die Aussage als Ausdruck des Urteils notwendig einen Namen enthalten. Dazu wird aber noch ein anderes Zeichen kommen müssen, das demjenigen inneren Zustand entspricht, den wir eben Urteilen nennen, d. h. ein Zeichen, das den bloßen Namen zum Satz ergänzt. Und da dieses Urteilen von doppelter Art sein kann, nämlich ein Anerkennen oder Verwerfen, so wird auch das Zeichen dafür ein doppeltes sein müssen, eines für die Bejahung und eines für die Verneinung. Für sich allein bedeuten diese Zeichen nichts [...], aber in Verbindung mit einem Namen sind sie Ausdruck eines Urteils. Das allgemeinste Schema der Aussage lautet daher: A ist (A +) und A ist nicht (A –). Auf dieses Schema muß sich jedes *einfache* Urteil zurückführen lassen, d. h. jedes Urteil, in welchem wirklich nur eine Bejahung oder eine Verneinung vorkommt, gleichgültig, ob die Materie einfach oder zusammengesetzt ist, denn diese Ausdrucksform enthält alles, was zu einem einfachen Urteil gehört: einen Namen, der das Beurteilte nennt und ein Zeichen, welches zu erkennen gibt, ob das Beurteilte anzuerkennen oder zu verwerfen sei.[7]

Brentano also gives some examples of an 'Anerkennung' and of a 'Verwerfung'. The judgment 'Eine Million Menschen ist' must be classified as an 'Anerkennung', the judgment 'Ein weißer Rabe ist nicht' as a 'Verwerfung'. In this way, Brentano tries to build up a *non-propositional* theory of judgment, because that which is rejected or asserted in these examples is not a proposition, but the content of an idea ('Vorstellung'). In these examples, names of the concerning contents are 'Eine Million Menschen' and 'Ein weißer Rabe'.

With respect to the concepts 'assertion' and 'rejection' Łukasiewicz, in the paper 'Two-valued Logic', writes:

> I do not define these terms, and by *assertion* and *rejection* I mean the ways of behaviour with respect to the logical values, the ways known to everyone from his own experience. I wish to assert truth and only truth, and to reject falsehood and only falsehood. The words 'I assert' are denoted by *U*, and the words 'I reject' by *N*. I consider the sentences:

[7] Brentano (1956); p. 97-98.

U: $\top$, N: $\bot$,

which are read: 'I assert truth' and 'I reject falsehood', respectively, to be the fundamental principles of two-valued logic, although I do not quote them anywhere. These propositions are also read: 'I assert that truth is' and 'I reject that falsehood is'.[8]

Although Woleński[9] and Słupecki[10] disregard the difference, this concept 'rejection' is *not* identical with the concept 'rejection' which comes to the fore in Łukasiewicz (1951), (1952) and (1953). While Łukasiewicz in (1921) explicitly demands the falsity of a proposition as a condition for its rejection, in (1951) a proposition must be rejected if there is at least one distribution of truth-values over the propositional variables in which the proposition under consideration is false. On the other hand, nothing changes with respect to the conditions under which a proposition can be asserted:

> Of two intellectual acts, to assert a proposition and to reject it, only the first has been taken into account in modern formal logic. Gottlob Frege introduced into logic the idea of assertion, and the sign of assertion ($\vdash$), accepted afterwards by the authors of *Principia Mathematica*. The idea of rejection, however, so far as I know, has been neglected up to the present day.
>
> We assert true propositions and reject false ones. Only true propositions can be asserted, for it would be an error to assert a proposition that was not true. An analogous property cannot be asserted of rejection: it is not only false propositions that have to be rejected. It is true, of course, that every proposition is either true or false, but there exist propositional expressions that are neither true nor false. Of this kind are the so-called propositional functions, i.e. expressions containing free variables and becoming true for some of their values, and false for others. Take, for instance, p, the propositional variable: it is neither true nor false, because for $p/1$ it becomes true, and for $p/0$ it becomes false. Now, of two contradictory propositions, α and $\neg\alpha$, one must be true and the other false, one therefore must be asserted and the other rejected. But neither of the two contradictory propositional functions, p

[8] Łukasiewicz (1921); p. 91. I have adapted the notation.

and $\neg p$, can be asserted, because neither of them is true: they both have to be rejected.[11]

Notwithstanding Łukasiewicz's remark in his 'A System of Modal Logic':

> The idea of rejection was introduced into logic by myself in 1951[12]

the formal apparatus set forth in 'Two-valued Logic' suffices to formulate the later concept of 'rejection' adequately. The heart of the matter can be met in terms of Łukasiewicz's definition of (propositional) quantifiers.

In his 'Two-valued Logic' Łukasiewicz supposes that implication ($\rightarrow$) conforms to the following principles:

Z_1	U: $\bot \rightarrow \bot$
Z_2	U: $\bot \rightarrow \top$
Z_3	N: $\top \rightarrow \bot$
Z_4	U: $\top \rightarrow \top$

Further, he introduces the symbols ϕ, ψ, χ as variables, ranging over the logical values $\top$ and $\bot$. Finally, Łukasiewicz introduces the quantifiers Π and Σ with adjacent subscripts ϕ, ψ, χ. In this way, $\Pi\,\phi$ means 'for every ϕ' and $\Sigma\,\psi$ means 'for at least one ψ'. For the universal quantifier Łukasiewicz formulates the following truth-condition: 'I assert any expression containing variables with universal quantifiers which yields only asserted expressions on the replacement of the variables by the values $\bot$ and $\top$.'[13] Łukasiewicz does not formulate a truth-condition for the introduction of an existential quantor. Nevertheless, I trust that the following truth-condition would be entirely in accordance with Łukasiewicz's ideas in 'Two-valued Logic': 'I assert any expression containing variables with existential quantifiers which yields at least one asserted expression on the replacement of the variables by the values $\bot$ and $\top$.'

This language enables Łukasiewicz to formulate the following proposition:

[9] Woleński (1989); p.86.

[10] Słupecki (1970); p. ix.

[11] Łukasiewicz (1951); p. 94-95.

[12] Łukasiewicz (1953); p. 353.

[13] Łukasiewicz (1921); p. 99.

$$U: \Pi\phi[\phi \to \phi]$$

By substitution of the logical values $\top$ and $\bot$ for ϕ, we obtained the aforementioned principles Z_1 en Z_4.

We can also formulate the following:

$$U: \Sigma\psi[\psi]$$

For, when ψ is replaced by $\top$, we obtain U: $\top$, which was accepted above.

Using this terminology, the notion of 'satisfiability' can be formulated thus:

1.1 *Definition* Let $\phi_1, \ldots, \phi_n$ be the constituent variables of a formula Ξ. Then:
Ξ is *satisfiable* $\Leftrightarrow$ $U: \Sigma\phi_1 \ldots \Sigma\phi_n[\Xi]$.

Now, we are in a position to define the concept of '*x* is rejected':

1.2 *Definition* Let $\phi_1, \ldots, \phi_n$ be the constituent variables of a formula Ξ. Then:
Ξ is *rejected* $\Leftrightarrow$ $U: \Sigma\phi_1 \ldots \Sigma\phi_n[\neg\Xi]$.[14]

As a consequence, the next theorem can easily be established:

1.3 *Theorem* Let $\phi_1, \ldots, \phi_n$ be the constituent variables of a formula Ξ. Then:
Ξ is *rejected* $\Leftrightarrow$ $N: \Pi\phi_1 \ldots \Pi\phi_n[\Xi]$.

In short, the 1921 paper 'Two-valued Logic' already contains all the ingredients necessary for a formal treatment of the concept of 'rejection'. A concept that will reach its full growth in Łukasiewicz's later works, starting with the 1951 monograph *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*[15]. Yet Łukasiewicz does not fully exploit the possibilities offered by the formal apparatus contained in 'Two-valued Logic'. Perhaps this can be explained by Łukasiewicz's strict adherence, in the

[14] Compare Łukasiewicz (1951); p. 95: 'By introducing quantifiers into the system we could dispense with rejection.' In the following Łukasiewicz presents two applications of Definition 1.2.

[15] Łukasiewicz (1951).

thirties, to Brentano's demand that falsity be a necessary condition for rejection. Therefore it seems more appropriate to interpret the operator '*N*:', as it is used in the paper 'Two-valued Logic', in strict analogy to the interpretation of '*U*:', as 'Anerkennung der Falschheit'. This would conform to Brentano's use of his concept of 'Verwerfung'. Łukasiewicz's use of the verbs 'to assert' and 'to reject' in his lecture 'O Determinizmie' ('On Determinism') offers additional support for this interpretation. In an attempt to refute an argumentation in favor of determinism, derived from Aristotle, he writes:

> On the assumption of John's presence at or absence from home tomorrow noon not yet being decided, the sentence 'it is true at the present instant that John will be at home tomorrow noon' can neither be accepted nor rejected, i.e. we cannot consider it either true or false.[16]

In brief, in order to be justified in saying 'I assert ϕ', we first have to *know* that a certain sentence ϕ is true, and likewise, in order to be justified in saying 'I reject ϕ', we first have to *know* that a certain sentence ϕ is false. As long as we do not (yet) possess the required knowledge, 'we can neither accept nor reject the sentence but should suspend our judgement'[17]. In order to escape from determinism, Łukasiewicz sees no other possibility than to reject the principle of bivalence and to introduce a *third* truth-value, besides the truth-values 'true' and 'false': 'there are propositions which are neither true nor false but *indeterminate*.'[18] According to Łukasiewicz's later, more formal, concept of rejection '$\dashv$', falsity of the proposition concerned is no longer a necessary condition for its rejection; rather, the one and only condition for '$\dashv \phi$' is that '$\vdash \phi$' ('ϕ is a tautology') does not hold.

Thus, *pace* Słupecki and Woleński, it is not quite correct to trace Łukasiewicz's later concept of rejection to his 1921 paper 'Two-valued Logic'. For, clearly, the concept of 'rejection' underwent a considerable change between 1921 and 1951.

Besides the operator decribed above, the paper 'Two-valued Logic' contains an interesting 'verbal rule'. Later on, we shall encounter a more precise formulation of this rule known as Łukasiewicz's 'Rule of

[16] Łukasiewicz (1922); p. 125.

[17] Łukasiewicz (1922); p. 124.

[18] Łukasiewicz (1922); p. 126.

Detachment'.[19] In 'Two-valued Logic', Łukasiewicz gives an embryonic formulation of this rule:

> 'I reject any expression which means the same as some rejected expression'[20]

In order to induce a better understanding of this rule, Łukasiewicz offers the following example.[21] Given the definition

$$U\colon\ \Pi\phi\Pi\psi\left[\phi \vee \psi \equiv_{Df} \neg\phi \rightarrow \psi\right]$$

we obtain by the substitutions $\perp$ for ϕ and $\perp$ for ψ:

$$(1)\quad U\colon \perp \vee \perp \equiv_{Df} \neg\perp \rightarrow \perp$$

Let it, moreover, be given that:

$$(2)\quad N\colon \neg\perp \rightarrow \perp$$

With the rule of rejection quoted above we finally obtain from (1) and (2):

$$N\colon \perp \vee \perp$$

These reflections on the paper 'Two-valued Logic' lead to the following conclusion: The paper does not contain the concept of rejection as formulated in Łukasiewicz's later works. It does, however, contain some seeds which were developed into what may count as Łukasiewicz's mature theory of rejection. In my opinion, the most important of these seeds is *not* the operator 'N:', but the rule of rejection quoted above.

Next we shall consider the mature concept of rejection as it is presented in Łukasiewicz's *Aristotle's Syllogistic*. In this monograph, Łukasiewicz dissociates himself from traditional views on Aristotle's syllogistic. According to Łukasiewicz, Aristotle's syllogistic is a system of theorems, *not* a system of rules of inference, concerning propositions in which *only* variables for general terms (like 'tree', 'man' and 'horse', but unlike 'Socrates' or 'Plato') occur. Further on Łukasiewicz, following Aristotle,

[19] Cf. Łukasiewicz (1951); p. 71: 'If the implication 'If α, then β is asserted, but its consequent β is rejected, then its antecedent α must be rejected too.'

[20] Łukasiewicz (1921); p. 99.

[21] Łukasiewicz (1921); p. 99-100.

establishes all the rules of conversion and all the 'affirmative moods' (true syllogistic forms) on the basis of four axioms, two definitions and the resources of the classical propositional logic (*CPL*). Łukasiewicz writes:

> Aristotle in his systematic investigation of syllogistic forms not only proves the true ones but also shows that all the others are false, and must be rejected.[22]

When a proof is required to show that a certain syllogistic form is *not* a theorem, Aristotle usually presents a counter-example. Let us consider the following syllogistic form:

> If every B is (an) A and no C is (a) B, then there is at least one C which is no A.

If we substitute the term 'animal' for A, the term 'man' for B and the term 'horse' for C, we obtain a false proposition. Therefore, the syllogistic form, of which this proposition is an instance, is not a theorem of the Aristotelian syllogistic.

Concerning this procedure Łukasiewicz writes:

> This procedure is correct, but it introduces into logic terms and propositions not germane to it. 'Man' and 'animal' are not logical terms, and the proposition 'All men are animals' is not a logical thesis. Logic cannot depend on concrete terms and statements.[23]

and

> Aristotle rejects most invalid syllogistic forms by exemplification through concrete terms. This is the only point where we cannot follow him, because we cannot introduce into logic such concrete terms as 'man' or 'animal'.[24]

Further on, Łukasiewicz even speaks of a 'flaw'[25] in Aristotle's explanation concerning the rejection of syllogistic forms by concrete terms.[26] Yet, this is not the whole story:

[22] Łukasiewicz (1951); p. 67.

[23] Łukasiewicz (1951); p. 72.

[24] Łukasiewicz (1951) p. 96.

[25] Łukasiewicz (1951); p. 131.

> There are, however, cases where he [Aristotle] applies a more logical procedure, reducing one invalid form to another already rejected. On the basis of this remark a rule of rejection could be stated [...]; this can be regarded as the commencement of a new field of logical enquiries and of new problems that have to be solved.[27]

What, then, is Aristotle's 'more logical procedure'? Łukasiewicz translates the crucial text in Aristotle's *Analytica Priora* (i. 5, 27^b12-27^b23) as follows:

> Let *M* belong to no *N*, and not to some *X*. It is possible then for *N* to belong either to all *X* or to no *X*. Terms of belonging to none: black, snow, animal. Terms of belonging to all cannot be found, if *M* belongs to some *X*, and does not belong to some *X*. For if *N* belonged to all *X*, and *M* to no *N*, then *M* would belong to no *X*; but it is assumed that it belongs to some *X*. In this way, then, it is not possible to take terms, and the proof must start from the indefinite nature of the particular premiss. For since it is true that *M* does not belong to some *X*, even if it belongs to no *X*, and since if it belongs to no *X* a syllogism is not possible, clearly it will not be possible either.[28]

Here, Aristotle tries to show the falsity of the implications (α') and (β') (consequently of (α) and (β)):

- (α) If no *N* is (a) *M* and at least one *X* is not (a) *M*, then every *X* is (a) *N*.
- (α') If no *N* is (a) *M* and at least one *X* is not (a) *M*, then at least one *X* is (a) *N*.
- (β) If no *N* is (a) *M* and at least one *X* is not (a) *M*, then no *X* is (a) *N*.
- (β') If no *N* is (a) *M* and at least one *X* is not (a) *M*, then at least one *X* is not (a) *N*.

In order to refute (α'), and consequently (α), Aristotle finds the terms 'black' for *M*, 'snow' for *N*, and 'animal' for *X*. In order to refute (β'), and consequently (β), Aristotle looks for terms making true all categorical sentences of (α). Further, Aristotle shows that no terms can be found

[26] I would like to attend the reader to the fact that Tarski's logical semantics enable us to construe counter-examples without any reference to concrete terms.

[27] Łukasiewicz (1951); p. 74.

[28] Łukasiewicz (1951); p. 70.

which satisfy both this demand and the dual condition that 'at least one X is (a) M' and 'at least one X is not (a) M' be true as well. In conclusion, Aristotle finds an other possibility: in order to refute (β) and (β') it is sufficient to refute the following syllogistic form:

(γ) If no N is (a) M and no X is (a) M, then at least one X is not (a) N.

For (β) implies (β') and (β') implies (γ). If we know that (γ) is rejected, then we are justified in making the *inference* that (β') must be rejected. Therefore (β) must be rejected too. In short, these results entitle us to conclude that no single syllogistic form with the antecedent 'if no N is (a) M and at least one X is not (a) M' and a consequent consisting of a subject X and a predicate N, can be a theorem.

This procedure, which Łukasiewicz characterized as 'more logical', can be summarized as follows:

(*c*) If the implication 'If α, then β' is asserted, but its consequent β is rejected, then its antecedent α must be rejected too.[29]

Immediately, Łukasiewicz formulates a second rule of rejection:

(*d*) If α is a substitution for β, and α is rejected, then β must be rejected too.[30]

In the chapter 'Aristotle's System in Symbolic Form' (*op. cit.*) Łukasiewicz constructs a formal system for Aristotelian syllogistic. He introduces four axiomatically asserted theses, two axiomatically rejected theses and the rules (*c*) and (*d*) mentioned above. This basis allows him to deduce all 232 false syllogistic forms. However, this fact does not imply that Łukasiewicz's system would be inconsistent: all theorems of his system are preceded by a number, and all non-theorems of the system are preceded by a number and an asterisk. Of course, no single formula is both a theorem and a non-theorem of the system. Unfortunately, it is not possible to deduce every formula, constructed out of the languages of propositional logic and syllogistic: not every formula of Aristotelian syllogistic can be either asserted or rejected. The system is Ł-undecidable[31] .

29 Łukasiewicz (1951); p. 71.

30 Łukasiewicz (1951); p. 72.

In 1948 Jerzy Słupecki showed that Łukasiewicz's system of Aristotelian syllogistic is powerful enough to deduce all true formulas of the system, but also that there is an infinity of false formulas that cannot be rejected within the system. In addition Słupecki showed that if a new rule of rejection were added to the system, all false formulas could be rejected within the system.[32] Thus the Ł-decidability of the extended version of Łukasiewicz's system for Aristotelian syllogistic was proved.

When proving the Ł-decidability of the extended system (the system augmented with Słupecki's new rule of rejection), it turned out to be useful, in order to establish a certain lemma, to introduce 'rejection' for propositional logic as well. For *CPL* (Classical Propositional Logic) Łukasiewicz constructs the following system:

> We reject axiomatically the variable p, and accept the clear rules of rejection, (c) and (d).[33]

If we accept the symbols '$\vdash$' for 'assertion' and '$\dashv$' for 'rejection', symbols which Łukasiewicz for the first time uses in his paper 'A System of Modal Logic'[34], we can formulate this theory of rejection for *CPL* as follows:

axiom	$\dashv p$;
detachment	If $\vdash \phi \rightarrow \psi$ and $\dashv \psi$, then $\dashv \phi$;
substitution	If $\dashv \psi$ and ψ can be obtained out of ϕ by substitution, then $\dashv \phi$.

This system presupposes a complete system for the derivation of all theses of *CPL*. If ϕ is a thesis of *CPL*, we shall write: $\vdash \phi$.

31 A system is Ł-undecidable iff there is at least one formula, formulated in the language of the system, which can not be asserted or rejected with the available techniques of the system.

32 Cf. Łukasiewicz (1951); p. 103ff for details. Słupecki's rule of rejection is of no use within propositional logic.

33 Łukasiewicz (1951); p. 109. Cf. Łukasiewicz's description of this system in (1952); p. 333-334.

34 Łukasiewicz writes: 'The idea of assertion and its sign '$\vdash$' were introduced into logic by Frege in 1879, and afterwards accepted by the authors of the *Principia Mathematica*. In my previous papers I always omitted this sign, but here I am bringing it in because, besides assertion, I introduce rejection. [...] I denote rejection by an inverted sign of assertion following a suggestion of Ivo Thomas.' Łukasiewicz (1953); p. 352-353. In 'A System of Modal Logic' Łukasiewicz also uses the concepts 'assertion' and 'rejection' for a four-valued modal logic: the Ł-modal logic.

The formula $p \rightarrow q$ is not a thesis of *CPL*, for it is false if p is true and q is false. Accordingly, we may, using Łukasiewicz's theory of rejection for *CPL*, show that: $\dashv p \rightarrow q$[35].

(1) $\vdash ((p \rightarrow p) \rightarrow p) \rightarrow p$	$((p \rightarrow p) \rightarrow p) \rightarrow p$ is a thesis of *CPL*
(2) $\dashv p$	*axiom*
(3) $\dashv ((p \rightarrow p) \rightarrow p)$	*detachment*; by (1) and (2)
(4) $\dashv p \rightarrow q$	*substitution*; take in (4) $p \rightarrow p$ for p and p for q and we obtain (3)

Therefore $\dashv p \rightarrow q$.

The system that results out of a standard type of deductive system for *CPL*, augmented by this axiom and these rules of rejection, is both correct and complete with respect to the class of non-tautologies. In other words: exactly those formulas which are false according to at least one distribution of truth-values over the atoms can be rejected within this system. Klaus Härtig offers a short proof of this theorem.[36] Besides, Härtig shows that, if, instead of the axiom $\dashv p$ a contradiction is axiomatically rejected and the rule of substitution is cancelled, then the resulting system is correct and complete with respect to the class of contradictions.[37] Łukasiewicz considered his application of Aristotle's idea of 'rejection' to *CPL* as one of the most important contributions yielded by the formal part of his investigations in Aristotle's non-modal syllogistic.

2. *Theories of rejection for propositional logics after Łukasiewicz*

In the paper 'On Proofs of Rejection'[38] Walenty Staszek compares two definitions of a proof of rejection and, under certain conditions, establishes their equipollence. Łukasiewicz's proofs of rejection satisfy the first definition. In his paper 'Z badań nad klasyczna logika nazw'[39] Staszek sets forth another kind of proofs of rejection. These proofs satisfy the second

[35] Cf. Łukasiewicz (1951); p. 109.

[36] Härtig (1960); p. 241.

[37] Cf. Theorem 4.3.7 of this article.

[38] Staszek (1971).

[39] Staszek (1969) ('On the Classical Logic of Names'). The article is available only in Polish, therefore I was unable to consult it.

definition. However, the 1971 paper does not contain any example of a proof of rejection of this second kind. I shall attempt, however, a reconstruction of an example of a proof of a formula of *CPL* to illustrate Staszek's second definition of proofs of rejection.

Staszek notices that Łukasiewicz's rule of *detachment* and his rule of *substitution* have their analogues in proof theory. The rule of substitution in the usual proof theory can be formulated thus:

SUBSTITUTION If $\vdash\phi$ and ψ can be obtained out of ϕ by substitution, then $\vdash\psi$.

The rule of detachment in proof theory is the *modus ponens* rule:

DETACHMENT If $\vdash\phi \rightarrow \psi$ and $\vdash\phi$, then $\vdash\psi$.

Staszek maintains the usual notation for these BOLDFACE printed rules: If the rule of SUBSTITUTION is applied, he writes $\phi\vdash\psi$; if the rule of DETACHMENT is applied, he writes: $\phi \rightarrow \psi, \phi\vdash\psi$. The italic printed rules are submitted to the following convention: If the rule of *substitution* is applied, Staszek writes: $\psi\dashv\phi$; if the rule of *detachment* is applied, he writes: $\phi \rightarrow \psi, \psi\dashv\phi$. Thereupon, Staszek establishes the following relations between these pairs of rules of inference[40]:

(i) $\phi\vdash\psi \Leftrightarrow \psi\dashv\phi$

(ii) $\phi \rightarrow \psi, \phi\vdash\psi \Leftrightarrow \phi \rightarrow \psi, \psi\dashv\phi$

Staszek defines a sequence (α) as an arbitrary sequence of formulas $\phi_1, \phi_2, \ldots, \phi_n$. The definition of the concept of *i*-proof, which embodies the way Staszek himself conceives of a proof of rejection, runs as follows:

The sequence (α) is an *i*-proof of an expression ϕ if
1° the sequence (α) is a usual proof
and
2° $\phi_1 = \phi$ and ϕ_n is a rejected axiom.[41]

If we, following Łukasiewicz, axiomatically reject the propositional variable p, we may, using Staszek's method, prove the formula $p \rightarrow q$ as follows:

[40] Staszek (1971); p. 18.

[41] Staszek (1971); p. 19. I have slightly changed the notation.

(1) $p \rightarrow q$	premiss
(2) $(p \rightarrow p) \rightarrow p$	SUBSTITUTION; take in (1) $p \rightarrow p$ for p
and p	for q and we obtain (2)
(3) $((p \rightarrow p) \rightarrow p) \rightarrow p$	$((p \rightarrow p) \rightarrow p) \rightarrow p$ is a thesis of *CPL*
(4) p	DETACHMENT; by (2) and (3)

Therefore $\dashv p \rightarrow q$.

Further Staszek establishes, without any additional conditions, that for every Łukasiewicz-style proof of rejection of a formula ϕ there exists an *i*-proof of ϕ. If the logical systems under consideration satisfy certain conditions, the converse can be proved as well.

Working with the rule of *substitution* of Łukasiewicz's system of rejection for *CPL* can be rather difficult. Errors will easily be made. In 1961 Klaus Härtig faced this shortcoming of Łukasiewicz's system in his paper 'Zur Axiomatisierung der Nicht-Identitäten des Aussagenkalküls'. He replaced Łukasiewicz's rule of *substitution* by a set of more convenient rules. If we define '$\langle\phi\rangle$' as '$\{\psi: \psi$ is an atomic subformula of $\phi\}$', then Härtig's extended version of rejection for *CPL* can be formulated as follows:

Axiom 1	${}_H{\dashv}\phi$, if $\phi \in$PROPL[42];
Axiom 2	${}_H{\dashv}\neg\phi$, if $\phi \in$PROPL;
Detachment	If $\vdash \phi \rightarrow \psi$ and ${}_H{\dashv}\psi$, then ${}_H{\dashv}\phi$;
Disjunction	If ${}_H{\dashv}\phi$, ${}_H{\dashv}\psi$ and $\langle\phi\rangle \cap \langle\psi\rangle = \varnothing$, then ${}_H{\dashv}\phi \vee \psi$.

Härtig established that this extended system is correct and complete with respect to the class of non-theorems of *CPL*.[43] Thus, Härtig's system is equipollent with Łukasiewicz's system, although the former does not contain Łukasiewicz's rule of *substitution*. Let us illustrate the system of Härtig with the following example; a deduction of ${}_H{\dashv}p \rightarrow q$:

(1) $\vdash (p \rightarrow q) \rightarrow (\neg p \vee q)$	$(p \rightarrow q) \rightarrow (\neg p \vee q)$ is a theorem of *CPL*
(2) ${}_H{\dashv}q$	Axiom 1
(3) ${}_H{\dashv}\neg p$	Axiom 2
(4) ${}_H{\dashv}\neg p \vee q$	Disjunction; by (2), (3) and $\langle p\rangle \cap \langle q\rangle = \varnothing$
(5) ${}_H{\dashv}p \rightarrow q$	Detachment; by (1) and (4)

Therefore ${}_H{\dashv}p \rightarrow q$.

42 A definition of PROPL can be found in section 3.

43 Härtig (1960); p. 244.

In the paper under consideration, Härtig further investigates what conditions an axiom system of the non-theorems of *CPL* must meet in order to be complete. Lastly, Härtig points out several open problems for further investigation. For instance: 'Was entspricht Satz 1 [the completenesstheorem for Łukasiewicz's system of rejection for the propositional calculus] speziell im *intuitionistischen* Aussagenkalkül?'[44] Later, this problem was solved by Dutkiewicz (1989). Härtig concludes his paper as follows:

> Man kann *mit* der Einsetzungsregel [...] arbeiten oder sie ausschließen, und man kann auch versuchen, bei den Regeln den Prämissentyp (a) — also die *Bezugnahme auf die Eigenschaft* $\vdash$, *d.h. auf die gegebene deduktiv abgeschlossene Menge — zu vermeiden.* Schon für die Spezialfälle der klassischen Nicht-Identitäten und der intuitionistisch-ungültigen Ausdrücke wären möglichst einfache Axiomatisierungen, *in deren Regeln nicht* [...] *auf die Identitäten zurückgegriffen wird*, von Interesse.[45]

Further on, Härtig's system will play the leading part in the completeness proof of the system to be presented in § 3 of this paper.

Already in the paper 'On the Intuitionistic Theory of Deduction', Łukasiewicz has proposed to add a rule to his system of rejection for *CPL* in order to obtain a correct and complete system of rejection for the Intuitionistic Propositional Calculus (*IPL*). After a short description of his system of rejection for *CPL*, Łukasiewicz writes:

> If we add to these general rules a special rule of rejection which is valid according to Gödel in the intuitionistic system:
>
> (g) If α and β are rejected, then $\alpha \vee \beta$ must be rejected,
>
> we get, as far as I see, a categorical system in which all the classical theses not accepted by the intuitionists can easily be disproved.[46]

Thus, the system of rejection for *IPL*, envisaged by Łukasiewicz, must be as follows:

[44] Härtig (1960); p. 246.

[45] Härtig (1960); p. 247.

[46] Łukasiewicz (1952); p. 334. I slightly changed the notation.

Axiom *i*	${}_i\dashv p$;
i-Detachment	If $\vdash_i \phi \to \psi$ and ${}_i\dashv \psi$, then ${}_i\dashv \phi$;
i-Substitution	If ${}_i\dashv \psi$ and ψ can be obtained out of ϕ by substitution, then ${}_i\dashv \phi$;
i-Disjunction	If ${}_i\dashv \phi$ and ${}_i\dashv \psi$, then ${}_i\dashv \phi \vee \psi$.

A deduction of the non-theorem $p \vee \neg p$ of *IPL* can be completed in the following way:

(1) ${}_i\dashv p$	Axiom *i*
(2) $\vdash_i ((p \to p) \to p) \to p$	$((p \to p) \to p) \to p$ is a theorem of *IPL*
(3) ${}_i\dashv (p \to p) \to p$	*i*-Detachment; by (1) and (2)
(4) ${}_i\dashv p \to q$	*i*-Substitution; take in (4) $p \to p$ for p and p for q and we obtain (3)
(5) $\vdash_i \neg p \to (p \to q)$	$\neg p \to (p \to q)$ is a theorem of *IPL*
(6) ${}_i\dashv \neg p$	*i*-Detachment; by (4) and (5)
(7) ${}_i\dashv p \vee \neg p$	*i*-Disjunction; by (1) and (6)

Łukasiewicz's conjecture that this system is correct and complete with respect to the class of non-theorems of *IPL* proved to be false. Admittedly, the addition of rule (g) to Łukasiewicz's classical system of rejection does not lead to the inconsistency of the resulting intuitionistic system. However, this resulting system is not complete with respect to the class of non-theorems of *IPL*.

In his paper 'The Method of Axiomatic Rejection for the Intuitionistic Propositional Logic'[47] Rafal Dutkiewicz presents an appropriate rule of rejection, which yields, added to Łukasiewicz's classical system of rejection, a correct and complete system to reject every non-theorem of *IPL*. However, the principal aim of Dutkiewicz's paper is to establish the Ł-decidability[48] of *IPL*. The added rule only serves to allow this proof of Ł-decidability; it is not meant for practical purposes. The application of this added rule is subjected to two complex conditions. In turn, these conditions presuppose a considerable technical apparatus.[49] Therefore, it will be desirable to construct a more convenient system to reject the non-theorems of *IPL*, the more so as Dutkiewicz showed that a complete and correct system for this task is possible.

[47] Dutkiewicz (1989).

[48] Cf. footnote 31.

[49] Cf. Dutkiewicz (1989); p. 455-456.

Almost all contemporary studies in the logic of rejection share the characteristic that they have been developed in order to prove some meta-logical theorems, for instance, the Ł-decidability of *CPL* or *IPL*. To the best of my knowledge, there have been so far only two attempts — those by Xavier Caicedo and by Valentin Goranko — to abandon axiomatic structure in order to obtain a more practicable system.

Caicedo is the first to present a self-sufficient system of rejection for *CPL*, i.e., a system which does *not* rely on the body of theorems of *CPL*. He does so in his paper 'A Formal System for the Non-Theorems of the Propositional Calculus'[50]. This formal system comprises two axioms and eight rules. The rules ensure that every formula occurring in a linear rejection proof is a non-tautology. Caicedo's system can be formulated as follows:

Axioms A_1 $\quad \phi \rightarrow \neg\phi$, if $\phi \in \mathrm{PROPL}$;
A_2 $\quad \neg\phi \rightarrow \phi$, if $\phi \in \mathrm{PROPL}$.

Rules R_1(a) $\dfrac{\psi}{\phi \rightarrow \psi}$, if i) $\phi \in \mathrm{PROPL}$; ii) $\phi \notin \langle \psi \rangle$.

R_1(b) $\dfrac{\psi}{\neg\phi \rightarrow \psi}$, if i) $\phi \in \mathrm{PROPL}$; ii) $\phi \notin \langle \psi \rangle$.

R_2 $\dfrac{\phi \rightarrow \psi}{\phi \rightarrow (\phi \rightarrow \psi)}$ $\qquad$ R_3 $\dfrac{\phi \rightarrow \psi}{(\chi \rightarrow \phi) \rightarrow \psi}$

R_4 $\dfrac{\neg\phi \rightarrow \psi}{(\phi \rightarrow \chi) \rightarrow \psi}$ $\qquad$ R_5 $\dfrac{\neg\phi \rightarrow \psi}{\phi}$

R_6 $\dfrac{\phi \rightarrow \psi}{\neg\neg\phi \rightarrow \psi}$ $\qquad$ R_7 $\dfrac{\phi \rightarrow (\psi \rightarrow \chi)}{\psi \rightarrow (\phi \rightarrow \chi)}$

R_8 $\dfrac{\phi \rightarrow \Xi \text{ and } \neg\psi \rightarrow \Xi}{\neg(\phi \rightarrow \psi) \rightarrow \Xi}$, where Ξ has the form indicated below.

[50] Caicedo (1978).

> The formula Ξ in R_8 must have the form $\Xi = \Xi_1$ or $\Xi = \Xi_1 \to (\Xi_2 \to \ldots (\Xi_{n-1} \to \Xi_n) \ldots)$, with $\Xi_i = \chi_i$ or $\Xi_i = \neg\chi_i$, $\chi_i \in \text{PROPL}, \chi_i \neq \chi_j$ for $i \neq j$, and $\langle \phi \to \psi \rangle \subseteq \{\chi_1, \chi_2, \ldots, \chi_n\}$.

Further on, Caicedo shows that his system is 'perfectly *unsound* and completely *antitautological*'[51], i.e. that his system is correct and complete with respect to the class of non-theorems of *CPL* with '$\neg$' and '$\to$' as the only logical connectives.[52]

According to the system described above, the following linear proof is a proof of ${}_c\dashv p \to q$:

(1)	$\neg q \to q$	A_2
(2)	q	R_5
(3)	$p \to q$	R_1(a)

Therefore ${}_c\dashv p \to q$.

Although Caicedo's system does not rely on the class of tautologies, its rules, especially R_8, lack an 'intuitive plausibility'. Goranko's remark concerning some of his own systems is also true of Caicedo's system of rejection:

> An obvious drawback in the sentential refutation systems [...] is that in most cases the specific refutation rules employed in them, although semantically well-formulated, are rather unhandy for practical purposes[53]

By and large, the extant systems of rejection verify this assertion. Apparently, the unwieldiness of these systems explains their relative obscurity outside the specialists.

Another objection, both theoretical and practical, which can be raised against most of the systems described above, is their dependence on another system to provide the theses (i.e., the tautologies), which are necessary for the system to work. With respect to this point Goranko writes (he uses *MT* to denote Łukasiewicz's *Rule of Detachment*):

[51] Caicedo (1978); p. 149.

[52] Caicedo (1978); p. 149-150.

[53] Goranko (1994); p. 313.

> Another feature of these systems is that the inference of the refutable formulae involves inference of the acceptable, i.e. provable ones, because of the rule *MT*. This makes the refutation systems inferior to the orthodox ones.[54]

Subsequent to his objections against the majority of the existing rejection systems, Goranko indicates two ways to counter them:

> A way to abolish this inequality is to construct refutation systems for refutable sequents rather than formulae[55]

Goranko opts for the construction of a system of rejection for *sequents*. Goranko is the first to present a system of rejection for *CPL* which is both practicable and independent of a system which provides the theses. His system, a system in the style of the sequent systems of Gerhard Gentzen, consists of an axiom scheme and seventeen rules. The system can be formulated as follows[56] :

Axiom scheme:

$\Gamma \dashv \Delta$, if i) $\Gamma \subseteq \text{PROPL}$ and $\Delta \subseteq \text{PROPL}$

ii) $\Gamma \cap \Delta = \varnothing$

Structural rules:

$$\frac{\Gamma, \phi \dashv \Delta}{\Gamma, \phi, \phi \dashv \Delta}\ (\text{Contr}^{-1}) \qquad \frac{\Gamma \dashv \Delta, \phi}{\Gamma \dashv \Delta, \phi, \phi}\ (\text{Contr}^{-1})$$

$$\frac{\Gamma, \phi \dashv \Delta}{\Gamma \dashv \Delta}\ (\text{Weak}^{-1}) \qquad \frac{\Gamma \dashv \Delta, \phi}{\Gamma \dashv \Delta}\ (\text{Weak}^{-1})$$

[54] Goranko (1994); p. 313.

[55] Goranko (1994); p. 314.

[56] Goranko (1994); p. 315.

Logical rules:

$$\frac{\Gamma, \phi, \psi \dashv \Delta}{\Gamma, \phi \wedge \psi \dashv \Delta}\ (l\wedge) \qquad \frac{\Gamma \dashv \Delta, \phi}{\Gamma \dashv \Delta, \phi \wedge \psi}\ (r\wedge) \qquad \frac{\Gamma \dashv \Delta, \psi}{\Gamma \dashv \Delta, \phi \wedge \psi}\ (r\wedge)$$

$$\frac{\Gamma, \phi, \dashv \Delta}{\Gamma, \phi \vee \psi \dashv \Delta}\ (l\vee) \qquad \frac{\Gamma, \psi \dashv \Delta}{\Gamma, \phi \vee \psi \dashv \Delta}\ (l\vee) \qquad \frac{\Gamma \dashv \Delta, \phi, \psi}{\Gamma \dashv \Delta, \phi \vee \psi}\ (r\vee)$$

$$\frac{\Gamma \dashv \Delta, \phi}{\Gamma, \phi \rightarrow \psi \dashv \Delta}\ (l\rightarrow) \qquad \frac{\Gamma, \psi \dashv \Delta}{\Gamma, \phi \rightarrow \psi \dashv \Delta}\ (l\rightarrow) \qquad \frac{\Gamma, \phi \dashv \Delta, \psi}{\Gamma \dashv \Delta, \phi \rightarrow \psi}\ (r\rightarrow)$$

$$\frac{\Gamma \dashv \Delta, \phi}{\Gamma, \neg\phi \dashv \Delta}\ (l\neg) \qquad \frac{\Gamma, \phi \dashv \Delta}{\Gamma \dashv \Delta, \neg\phi}\ (r\neg)$$

$$\frac{\Gamma \dashv \Delta}{\Gamma, \top \dashv \Delta}\ (\top) \qquad \frac{\Gamma \dashv \Delta}{\Gamma \dashv \Delta, \bot}\ (\bot)$$

The system described above is correct and complete with respect to the class of non-theses of *CPL*.[57] Thus, the formula $((A \rightarrow B) \wedge \neg A) \rightarrow \neg B$ can be derived with Goranko's system:

(1)	$B \dashv A$	Axiom
(2)	$B, B \dashv A$	Contr^{-1}
(3)	$A \rightarrow B, B \dashv A$	$l\rightarrow$
(4)	$A \rightarrow B \dashv A, \neg B$	$r\neg$
(5)	$A \rightarrow B, \neg A \dashv \neg B$	$l\neg$
(6)	$(A \rightarrow B) \wedge \neg A \dashv \neg B$	$l\wedge$
(7)	$\dashv ((A \rightarrow B) \wedge \neg A) \rightarrow \neg B$	$r\rightarrow$

Therefore $\dashv ((A \rightarrow B) \wedge \neg A) \rightarrow \neg B$.

The reader has to keep in mind that Goranko's system is *not* a system of natural deduction: for Goranko's system consists exclusively of introduction rules, whereas a decent system of natural deduction should comprise both introduction rules and elimination rules. Besides, there is a close relationship between the semantic tableau method of Evert Willem Beth and Goranko's system for the rejection of non-theses of *CPL*. If we add to Beth's semantic tableau rules a new closure rule, which meets exactly all cases where the usual closure rule can not be applied, then we have found a rule which can directly be reformulated as Goranko's axiom scheme. Beth's method of semantical tableaus decomposes a given sequent in order

[57] Goranko (1994); p. 316-317.

to apply a closure rule, Goranko (following Gentzen) starts with the closure rule and tries to build up the intended sequent by means of his structural and logical rules. In this case, the distinction between a semantic and a syntactic treatment is not very clear indeed. Haskell B. Curry writes:

> In regard to the inferential methods, the ultimate source is Gentzen [...] Beth's semantic tableaux constitute, in some respects a refinement of the Gentzen rules.[58]

In the next section another system for the rejection of non-theses of *CPL* shall be set forth. This system differs from the systems of Łukasiewicz, Staszek, and Härtig, for it is, in contrast to them, *independent* of the class of theses of *CPL*. Next, Łukasiewicz's *rule of substitution* or Staszek's RULE OF SUBSTITUTION do not play a role in the system. Besides, the system to be expounded is constructed in the style of Stanisław Jaśkowski's systems of natural deduction. This means that, instead of the sequents of Goranko's system, formulas constitute the basic units of deductions. The system to be expounded shares this characteristic with the systems of Łukasiewicz, Staszek, Härtig and Caicedo. In contrast to the latter, I hope the reader will appreciate the practicability of my system. Besides, the system comprises, in contrast to Goranko's system, both introduction and elimination rules.

3. *A system of natural deduction*

Our syntaxis of classical propositional calculus will be defined according to the following definitions:
The alphabet of *CPL* consists of:

(i)	Propositional Constants	$P_1, P_2, P_3, \ldots$
(ii)	Logical Symbols	$\wedge, \vee, \rightarrow, \neg, \top, \bot$
(iii)	Auxiliary Symbols	(,).

The set of atomic formulas consists of all propositional constants and of $\top$ and $\bot$. The set of formulas is defined inductively as follows:

1) Every atomic formula is a formula;
2) If ϕ and ψ are formulas, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$ are formulas;

[58] Curry (1963); p. 25.

3) If ϕ is a formula, then $\neg\phi$ is a formula;
4) Formulas are only those expressions that are obtainable by 1)-3).

The following abbreviations are introduced for the sake of convenience:

ATOM denotes the set of all atomic formulas;
PROPL denotes the set of all propositional constants;
FORM denotes the set of all formulas.

In Example 3.1.4 the letters A en B are used as specific elements of PROPL, instead of the indexed letters P_1 and P_2. This is being done for the sake of legibility.

The rules of the system of natural deduction can be classified in groups, according to the starting-points which are required for the application of a rule.
Group I consists of the rules At, $\neg At$ and $\bot I$;
Group II_A consists of the rules $\vee E_1$, $\vee E_2$, $\rightarrow E_1$, $\rightarrow E_2$, $\wedge I_1$, $\wedge I_2$ and $\top E$;
Group II_B consists of the rules *Triv* en $\neg E$;
Group III consists of the rules $\neg I$ and *CRV* ('classical *reductio ad verum*');
Group IV consists of the rules $\vee I_1$, $\vee I_2$, $\rightarrow I_1$, $\rightarrow I_2$, and $\wedge E$.

The following points determine the use, the scope and the limits of the rules:[59]

1. Every natural deduction consists of a column of consecutively numbered formulas, accompanied, to the right of the column, by a justification of each formula and, to the left of the column, by a structure of vertical lines that indicate the scope of each hypothesis. This structure shows, for each formula, in which part of the deduction this formula is operative and in which part it is accessible (see point 6).
2. The hypothesis which was introduced last must be the first retracted. Thus, verticals indicating the scope of an hypothesis must not intersect.
3. A part of a natural deduction that belongs to one individual vertical indicating the scope of an hypothesis, will be called a 'subdeduction'.
4. Every rule of inference stipulates that if the column contains certain starting-points, then a certain formula may be written down on a newly introduced line at the bottom of the column. Starting-points can be both occurrences of formulas and occurrences of subdeductions.
5. The position and the order in which the starting-points appear is of the greatest importance. An application of one of the rules of Group III or

[59] I partly owe these points to Kradavis (1988); pp. 42-43.

Group IV requires that the conlusion that can be reached through application of the rule under consideration is written down on the *very next* line after the termination of the (last) relevant subdeduction.
Besides, an application of one of the rules of Group IV requires that the hypothesis of the (first) relevant subdeduction is introduced on the *very next* line after the formula which serves as starting-point for the application of the rule under consideration. An application of the rule $\wedge E$ requires that the hypothesis of the second relevant subdeduction is introduced on the *very next* line after the termination of the first relevant subdeduction.
The order of the relevant starting-points is irrelevant only whenever the rule $\neg E$ is applied.

6. A rule can be applied only if the required starting-points are *attainable*. We need some definitions.
   A formula ϕ_m on line m is *operative* at line n with $m \leq n$, if
   (i) ϕ_m occurs within the scope of an hypothesis which has not been retracted at line n;
   (ii) for any hypothesis, if line m is in its scope, so is line n.[60]
   Thus, outside the scopes of all hypotheses no formula is operative.
   A formula ϕ_m on line m is *accessible* at line n with $m < n$, if
   (i) both line m and line n lie outside the scope of all hypotheses.
   A formula ϕ_m on line m is *attainable* at line n with $m < n$, if
   (i) ϕ_m is operative at line n; or
   (ii) ϕ_m is accessible at line n.
   If a line n occurs under at least one hypothesis, then a formula ϕ_n on line n can be justified only by one of the rules of Group II_A or Group II_B by reference to a formula ϕ_m on line m (*and*, in case of the rule $\neg E$, a formula ϕ_l on line l), if ϕ_m (and ϕ_l) is (are) operative at line n.
   If line n lies outside the scope of all hypotheses, then a formula ϕ_n on line n can be justified only by one of the rules of Group II_A by reference to a formula ϕ_m on line m, if ϕ_m is accessible on line n.
   If a line n occurs within the scope an hypothesis, then a formula ϕ_n on line n can be justified only by one of the rules of Group IV by reference to a formula ϕ_m on line m and one (two) subdeduction(s) δ_1 $(, \delta_2)$, if ϕ_m is operative at line n.
   If line n lies outside the scope of all hypotheses, then a formula ϕ_n on line n can be justified only by one of the rules of Group IV by reference to a formula ϕ_m on line m and one (two) subdeduction(s) δ_1 $(, \delta_2)$, if

[60] A hypothesis ϕ_m on line m is within the scope of itself and, therefore, is operative at line m.

ϕ_m is accessible on line n *and* if ϕ_{m+1} on line $m+1$ is the hypothesis of the (first) relevant subdeduction.

The rules of Group III do not need such stipulations.

7. The double tilde ($\approx$) denotes parts of the deduction in which no hypothesis is retracted that was not introduced in the very same part of the deduction. This part may be empty as well.
8. The introduction and retraction of hypotheses is always permitted (but keep point 2 in mind). Note that the introduction of an hypothesis is useful only if we intend to apply a rule which requires a certain subdeduction as a starting-point.

3.1.1 *Definition* $\langle\phi\rangle = \{\psi\!: \psi$ is a subformula of ϕ and $\psi \in$ PROPL$\}$.

3.1.2 *Definition* The system of natural deduction for the rejection of non-theses of *CPL* exactly consists of *At*, $\neg$*At*, *Triv*, $\neg E$, $\vee E_1$, $\vee E_2$, $\vee I_1$, $\vee I_2$, $\rightarrow E_1$, $\rightarrow E_2$, $\rightarrow I_1$, $\rightarrow I_2$, $\wedge E$, $\wedge I_1$, $\wedge I_2$, $\neg I$, *CRV*, $\top E$ en $\bot I$.

$\approx$

n ϕ *At*, if i) $\phi \in$ PROPL;
ii) for every ψ which is operative at line n: $\phi \notin \langle\psi\rangle$.

$\approx$

n $\neg\phi$ $\neg$*At*, if i) $\phi \in$ PROPL;
ii) for every ψ which is operative at line n: $\phi \notin \langle\psi\rangle$.

$\approx$

m ϕ

$\approx$

n ϕ *Triv*, if i) ϕ on line m is operative on line n.

$$
\begin{array}{lll}
 & \approx & \\
k & \phi & \\
 & \approx & \\
m & \neg\phi & \\
 & \approx & \\
n & \top & \neg E,
\end{array}
$$

if i) ϕ on line k and $\neg\,\phi$ on line m are operative on line n.

$$
\begin{array}{ll}
\approx & \\
\phi \vee \psi & \\
\approx & \\
\phi & \vee E_1
\end{array}
\qquad\qquad
\begin{array}{ll}
\approx & \\
\phi \vee \psi & \\
\approx & \\
\psi & \vee E_2
\end{array}
$$

$$
\begin{array}{lll}
 & \approx & \\
i & \phi & \\
i+1 & \phi & \\
 & \approx & \\
n-1 & \psi & \\
n & \phi \vee \psi & \vee I_1
\end{array}
\qquad\qquad
\begin{array}{lll}
 & \approx & \\
i & \psi & \\
i+1 & \psi & \\
 & \approx & \\
n-1 & \phi & \\
n & \phi \vee \psi & \vee I_2
\end{array}
$$

$$
\begin{array}{ll}
\approx & \\
\phi \to \psi & \\
\approx & \\
\neg\phi & \to E_1
\end{array}
\qquad\qquad
\begin{array}{ll}
\approx & \\
\phi \to \psi & \\
\approx & \\
\psi & \to E_2
\end{array}
$$

$$
\begin{array}{lll}
 & \approx & \\
i & \neg\phi & \\
i+1 & \neg\phi & \\
 & \approx & \\
n-1 & \psi & \\
n & \phi \to \psi & \to I_1
\end{array}
\qquad\qquad
\begin{array}{lll}
 & \approx & \\
i & \psi & \\
i+1 & \psi & \\
 & \approx & \\
n-1 & \neg\phi & \\
n & \phi \to \psi & \to I_2
\end{array}
$$

$$
\begin{array}{lll}
i & \overset{\approx}{\phi \wedge \psi} & \\
i+1 & \quad \phi & \\
j & \quad \overset{\approx}{\chi} & \\
j+1 & \quad \psi & \\
n-1 & \quad \overset{\approx}{\chi} & \\
n & \chi & \wedge E
\end{array}
\qquad\qquad
\begin{array}{ll}
\overset{\approx}{\phi} & \\
\overset{\approx}{\phi \wedge \psi} & \wedge I_1
\end{array}
\qquad
\begin{array}{ll}
\overset{\approx}{\psi} & \\
\overset{\approx}{\phi \wedge \psi} & \wedge I_2
\end{array}
$$

$$
\begin{array}{lll}
 & \quad \overset{\approx}{\phi} & \\
n-1 & \quad \overset{\approx}{\top} & \\
n & \neg\phi & \neg I
\end{array}
\qquad\qquad
\begin{array}{lll}
 & \quad \overset{\approx}{\neg\phi} & \\
n-1 & \quad \overset{\approx}{\top} & \\
n & \phi & CRV
\end{array}
$$

$$
\begin{array}{ll}
\overset{\approx}{\top} & \\
\overset{\approx}{\phi} & \top E
\end{array}
\qquad\qquad
\begin{array}{ll}
\overset{\approx}{\bot} & \bot I
\end{array}
$$

3.1.3 *Definition* $\dashv \phi \iff$ There is at least one formal deduction of rejection for ϕ (according to the system of Definition 3.1.2).

3.1.4 *Example*

1.	$\neg B$	$\neg At$
2.	$\neg B$	hyp.
3.	A	At
4.	$(A \to B) \wedge \neg A$	hyp.
5.	$A \to B$	hyp.
6.	B	$\to E_2$: 5
7.	$\top$	$\neg E$: 2, 6
8.	$\neg A$	hyp.
9.	$\top$	$\neg E$: 3, 8
10.	$\top$	$\wedge E$: 4, 5-7, 8-9
11.	$\neg((A \to B) \wedge \neg A)$	$\neg I$: 4-10
12.	$((A \to B) \wedge \neg A) \to \neg B$	$\to I_2$: 1, 2-11

4. *Soundness and completeness of the system*

4.1 *Semantics for CPL*

4.1.1 *Definition* A *classical model of CPL* is an ordered pair $\langle S, I\rangle$, with $S = \langle 1, 0\rangle$, and with I a function defined on a subset of PROPL with values in $\{1, 0\}$.

4.1.2 *Definition* Let M be a classical model of *CPL*. The *language of* $M = \langle S, I_M\rangle$ is the following set $\mathcal{L}_M$:
$\mathcal{L}_M = \{\phi : I_M(P_i)$ is defined for every $P_i \in \langle\phi\rangle\}$.

4.1.3 *Definition* Let M be a classical model of *CPL*. The *classical valuation of CPL based on M* is the following function, V_M, defined for every formula of $\mathcal{L}_M$ and with values in $\{1,0\}$:

Sem_{At} $V_M(\phi) = I_M(\phi)$, if $\phi \in$ PROPL;
$Sem_{\rightarrow}$ $V_M(\phi \rightarrow \psi) = 0 \Leftrightarrow V_M(\phi) = 1$ and $V_M(\psi) = 0$;
$Sem_{\wedge}$
$V_M(\phi \wedge \psi) = 0 \Leftrightarrow V_M(\phi) = 0$ and / or $V_M(\psi) = 0$;
$Sem_{\vee}$ $V_M(\phi \vee \psi) = 0 \Leftrightarrow V_M(\phi) = 0$ and $V_M(\psi) = 0$;
$Sem_{\neg}$ $V_M(\neg\phi) = 0 \Leftrightarrow V_M(\phi) = 1$;
$Sem_{\top}$ $V_M(\top) = 1$;
$Sem_{\bot}$ $V_M(\bot) = 0$.

4.2 *Soundness of the natural deduction for the logic of rejection*

4.2.1 *Definition* Let $\Gamma \subseteq$FORM. Then
$\dashv \Gamma \Leftrightarrow \exists_M(\forall_\phi \in \Gamma: (V_M(\phi) = 0))$.

The reader can easily prove the following:

4.2.2 *Lemma* Let $\Gamma \subseteq$FORM, $\Gamma_i \subseteq$FORM for i=1, 2 and ϕ, $\psi \in$FORM. Then

(i) $\dashv \Gamma_1$ and $\Gamma_2 \subseteq \Gamma_1 \Rightarrow \dashv \Gamma_2$
(ii) $\dashv \Gamma$ and $\phi \in \Gamma \Rightarrow \dashv \Gamma, \phi$
(iii) $\dashv \Gamma, \phi, \neg\phi \Rightarrow \dashv \Gamma, \phi, \neg\phi, \top$
(iv) $\dashv \Gamma, \phi \vee \psi \Rightarrow \dashv \Gamma, \phi \vee \psi, \phi$
(v) $\dashv \Gamma, \phi \vee \psi \Rightarrow \dashv \Gamma, \phi \vee \psi, \psi$
(vi) $\dashv \Gamma, \phi \rightarrow \psi \Rightarrow \dashv \Gamma, \phi \rightarrow \psi, \neg\phi$
(vii) $\dashv \Gamma, \phi \rightarrow \psi \Rightarrow \dashv \Gamma, \phi \rightarrow \psi, \psi$
(viii) $\dashv \Gamma, \phi \wedge \psi \Rightarrow \dashv \Gamma, \phi \wedge \psi, \phi$ and / or $\dashv \Gamma, \phi \wedge \psi, \psi$
(ix) $\dashv \Gamma, \phi, \psi \Rightarrow \dashv \Gamma, \phi, \psi, \phi \vee \psi$
(x) $\dashv \Gamma, \neg\phi, \psi \Rightarrow \dashv \Gamma, \neg\phi, \psi, \phi \rightarrow \psi$
(xi) $\dashv \Gamma, \phi \Rightarrow \dashv \Gamma, \phi, \phi \wedge \psi$
(xii) $\dashv \Gamma, \psi \Rightarrow \dashv \Gamma, \psi, \phi \wedge \psi$
(xiii) $\dashv \Gamma, \top \Rightarrow \dashv \Gamma, \top, \phi$
(xiv) $\dashv \Gamma \Rightarrow \dashv \Gamma, \bot$.

4.2.3 *Definition* Let $\Gamma \subseteq$FORM. Then
$\langle\Gamma\rangle = \{\phi : \phi \in \langle\psi\rangle \text{ and } \psi \in \Gamma\}$.

4.2.4 *Definition* Let $\Gamma \subseteq$FORM. Then a model $M = \langle S, I\rangle$ is Γ-*minimal*
$\Leftrightarrow$ i) $V_M(\chi) = 0$ for every $\chi \in \Gamma$;
ii) For every $\psi \in \langle\Gamma\rangle$: $I(\psi)$ is defined;
iii) For every $\psi \in$PROPL$\setminus\langle\Gamma\rangle$: $I(\psi)$ is undefined.

4.2.5 *Fact* $\dashv \Gamma \Leftrightarrow$ there is a Γ-minimal model.

4.2.6 *Lemma* Let $\Gamma \subseteq$FORM, $\phi \in$PROPL. Then
(i) $\dashv \Gamma$ and $\phi \notin \langle\Gamma\rangle \Rightarrow \dashv \Gamma, \phi$
(ii) $\dashv \Gamma$ and $\phi \notin \langle\Gamma\rangle \Rightarrow \dashv \Gamma, \neg\phi$

Proof: (ii) Suppose $\dashv \Gamma$ and $\phi \notin \langle\Gamma\rangle$ and $\phi \in$PROPL. Fact 4.2.5 implies that there is a Γ-minimal model. As ϕ is not a subformula of Γ, we can, since M is Γ-minimal and $\phi \in$PROPL, define a model $M' = \langle S, I'\rangle$ by extending M's interpretation function as follows: $I' = I \cup \{\langle\phi, 1\rangle\}$. Sem_{At} implies $V_{M'}(\phi) = 1$. Further, $Sem_{\neg}$ implies that M' is $\{\Gamma, \neg\phi\}$-mimimal. Finally, Fact 4.2.5 implies: $\dashv \Gamma, \neg\phi$.

(i) can be proved analogously. ●

4.2.7 *Lemma* Let Δ be a formal rejection deduction, running from line 1 to line k, in which a subdeduction δ occurs. This subdeduction runs from line h with a hypothesis ϕ_h to line j with a formula ϕ_j. The hypothesis ϕ_h has been retracted at line j+1. Let $\mathfrak{R}_n$ be defined for all n with $1 \leq n \leq k$ as follows:

$\mathfrak{R}_n = \{\phi : \phi$ is a formula occuring on a line m with $m < n$ which is operative at line n or ϕ is a hypothesis which is introduced at line $n\}$.[61]

Suppose $\dashv \mathfrak{R}_i \Rightarrow \dashv \mathfrak{R}_i, \phi_i$ for every i with $h \leq i \leq j$.
Further, suppose $\dashv \mathfrak{R}_h$.
Then $\dashv \mathfrak{R}_j, \phi_j$.

Proof: As every hypothesis that was introduced between ϕ_h and ϕ_j must have been retracted before line j, we may conclude that exactly the same hypotheses are operative at line j as those that are operative on line h. Since we are, in the course of this proof, interested only in line j, we can leave out of consideration any formula between lines h and j that occurs within the scope of one ore more additional hypotheses. Accordingly, we renumber the lines between h and j, skipping each formula that occurs within the scope of an additional hypothesis (introduced after line h). Line h gets number 1 and line j gets number r. We shall show that for every renumbered line q with $1 \leq q \leq r$ we have: $\dashv \mathfrak{R}_q, \phi_q$.

Induction hypothesis: Let the proposition to be proved be correct for every line p with $1 \leq p < q$: $\dashv \mathfrak{R}_p, \phi_p$.

Induction step: Consider ϕ_p.

If ϕ_p is a hypothesis, then $q = 1$ and $\phi_q = \phi_h$. We already know that $\dashv \mathfrak{R}_h$. As $\phi_h \in \mathfrak{R}_h$, Lemma 4.2.2(ii) implies: $\dashv \mathfrak{R}_h, \phi_h$. Therefore $\dashv \mathfrak{R}_q, \phi_q$.

If ϕ_p is not a hypothesis, then, according to the induction hypothesis, for $q-1$ it holds that $\dashv \mathfrak{R}_{q-1}, \phi_{q-1}$. Also $\mathfrak{R}_{q-1} \cup \{\phi_{q-1}\} = \mathfrak{R}_q$. Therefore $\dashv \mathfrak{R}_q$. Therefore, using the implication that is supposed to hold, $\dashv \mathfrak{R}_q, \phi_q$.

This implies: $\dashv \mathfrak{R}_r, \phi_r$. Therefore $\dashv \mathfrak{R}_j, \phi_j$. ●

61 $\phi_n \in \mathfrak{R}_n$ only if ϕ_n is a hypothesis which is introduced on line n.

4.2.8 *Soundnesstheorem* Let $\phi \in$FORM, then: $\dashv \phi \Rightarrow \Dashv \phi$.

Proof: Suppose that $\dashv \phi$. Then there must be a formal deduction of rejection, say Δ, with a final conclusion ϕ. Δ consists of a column of formulas $\phi_1, \phi_2, \ldots, \phi_k$ (with $\phi_k = \phi$), provided with verticals indicating the scope of each hypothesis and a correct annotation for every occurrence of a formula. The verticals and the annotation uniquely stipulate which formulas are operative at a certain line and which formulas are accessible at a certain line. Now we define for every n with $1 \leq n \leq k$.

$\mathfrak{R}_n = \{\phi{:}\phi$ is a formula occurring on a line m with $m<n$ which is operative at line n or ϕ is a hypothesis which is introduced at line $n\}$.[62]

We shall show for every n with $1 \leq n \leq k$:

★ $\Dashv \mathfrak{R}_n \Rightarrow \Dashv \mathfrak{R}_n, \phi_n$

This will be sufficient, as we have for the case n=k:

$\Dashv \mathfrak{R}_k \Rightarrow \Dashv \mathfrak{R}_k, \phi_k$

Because on the last line of a natural deduction for the logic of rejection all hypotheses have been retracted, we have $\mathfrak{R}_k = \varnothing$. Therefore we have $\Dashv \varnothing \Rightarrow \Dashv \phi_k$ Because every model falsifies every formula of $\varnothing$ this implies $\Dashv \phi_k$. Further, $\phi_k = \phi$. Therefore $\Dashv \phi$.

The proof for ★ can be carried out by a course-of-values induction.

Induction hypothesis: Let the proposition to be proved be correct for every line m with $m < n$: $\Dashv \mathfrak{R}_m \Rightarrow \Dashv \mathfrak{R}_m, \phi_m$.

Induction step: We distinguish cases according to the annotation on line n. Suppose that $\Dashv \mathfrak{R}_n$.

Case 1 — $\neg E$ ϕ_n is the formula $\top$ on line n which has been derived with $\neg E$. Then there must be lines i and j with i<n and j<n with formulas ϕ_i and ϕ_j such that $\phi_i = \neg\phi_j$, whilst both formulas ϕ_i and ϕ_j are operative at line n. Therefore $\phi_i \in \mathfrak{R}_n$ and $\phi_j \in \mathfrak{R}_n$. Therefore $\Dashv \mathfrak{R}_n, \phi_i, \phi_j$, and also $\Dashv \mathfrak{R}_n, \neg\phi_j, \phi_j$. With Lemmas 4.2.2(iii) and (i) we obtain: $\Dashv \mathfrak{R}_n, \top$. Therefore $\Dashv \mathfrak{R}_n, \phi_n$.

Case 2 — $\vee E_1$ ϕ_n is a formula on line n which has been derived with $\vee E_1$. Then we must have one of the two following cases:

i) There is a line m with m<n with a formula ϕ_m such that $\phi_m = \phi_n \vee \psi$, while $\mathfrak{R}_n \neq \varnothing$. Then ϕ_n can be derived out of ϕ_m, only if ϕ_m is operative at line n. Therefore

62 $\phi_n \in \mathfrak{R}_n$ only if ϕ_n is a hypothesis which is introduced on line n.

$\phi_m \in \mathfrak{R}_n$. With Lemmas 4.2.2(i), (ii) and (iv) we obtain: $\dashv \mathfrak{R}_n, \phi_n$.

ii) There is a line m with $m<n$ with a formula ϕ_m such that $\phi_m = \phi_n \vee \psi$, while $\mathfrak{R}_n = \varnothing$. Then ϕ_n can be derived out of ϕ_m, only if $\mathfrak{R}_m = \varnothing$. With our induction hypothesis we have: $\dashv \mathfrak{R}_m \Rightarrow \dashv \mathfrak{R}_m, \phi_m$., As $\mathfrak{R}_m = \varnothing$, this implies $\dashv \phi_m$. With Lemmas 4.2.2(i) and (iv) we obtain: $\dashv \phi_n$. As $\mathfrak{R}_n = \varnothing$, we also have: $\dashv \mathfrak{R}_n, \phi_n$.

The cases $\vee E_2$, $\rightarrow E_1$, $\rightarrow E_2$, $\wedge I_1$, $\wedge I_2$ and $\top E$ can be proved analogously.

Case 3 — $\vee I_1$ ϕ_n is a formula on line n which has been derived with $\vee I_1$, while $\phi_n = \phi_i \vee \phi_{n-1}$. Then there must be a line i with $i<n$ with a formula ϕ_i and a subdeduction δ, running from line $i+1$ to $n-1$. This subdeduction starts with the hypothesis ϕ_{i+1}, such that $\phi_i = \phi_{i+1}$, and ends on line $n-1$ with formula ϕ_{n-1}. Immediately after line $n-1$ the hypothesis ϕ_{i+1} is retracted. If $\mathfrak{R}_n \neq \varnothing$, then ϕ_i is operative at line n, therefore $\phi_i \in \mathfrak{R}_n$. Therefore, using Lemma 4.2.2(ii), we obtain: $\dashv \mathfrak{R}_n, \phi_i$. If $\mathfrak{R}_n = \varnothing$, then surely $\mathfrak{R}_i = \varnothing$, hence $\dashv \mathfrak{R}_i$. As $i<n$, we can apply the induction hypothesis to get: $\dashv \phi_i$. Therefore $\dashv \mathfrak{R}_n, \phi_i$. Both cases lead to the same result: $\dashv \mathfrak{R}_n, \phi_i$.

Because it is impossible to introduce any formula between lines $n-1$ and n, we have for every $\phi \in \mathfrak{R}_n$ that there must be a line m with $m < i+1$ with a formula ϕ_m, such that $\phi = \phi_m$ and ϕ_m is operative at line n. Therefore, every formula $\phi \in \mathfrak{R}_n$ is operative at line i+1. Hence, for every $\phi \in \mathfrak{R}_n$, we have $\phi \in \mathfrak{R}_{i+1}$. Therefore $\mathfrak{R}_n \subseteq \mathfrak{R}_{i+1}$. As $\phi_i = \phi_{i+1}$ and as $\phi_{i+1} \in \mathfrak{R}_{i+1}$ (ϕ_{i+1} is a hypothesis), we have: $\mathfrak{R}_n \cup \{\phi_i\} \subseteq \mathfrak{R}_{i+1}$.

Conversely, suppose that $\phi \in \mathfrak{R}_{i+1} \setminus \{\phi_{i+1}\}$. Then there must be a line m with $m<i$+1 and a formula ϕ_m, such that $\phi = \phi_m$ and ϕ_m is operative at line i+1. This formula ϕ_m must also be operative at line n. Therefore $\mathfrak{R}_{i+1} \setminus \{\phi_{i+1}\} \subseteq \mathfrak{R}_n$. As $\phi_i = \phi_{i+1}$, we have: $\mathfrak{R}_{i+1} \subseteq \mathfrak{R}_n \cup \{\phi_i\}$. Combining this with the conclusion of the preceding paragraph we obtain: $\mathfrak{R}_{i+1} = \mathfrak{R}_n \cup \{\phi_i\}$. Hence $\dashv \mathfrak{R}_{i+1}$.

It may now be checked that, if we read number h for number i+1, and number j for number $n-1$, every condition of Lemma 4.2.7 holds. For one thing, the induction hypothesis holds for every line in the subdeduction under consideration. Using Lemma 4.2.7, we obtain: $\dashv \mathfrak{R}_{n-1}, \phi_{n-1}$. As it is impossible to introduce any formula between lines $n-1$ and n, we have $\mathfrak{R}_n \subseteq \mathfrak{R}_{n-1}$. We also have $\phi_i = \phi_{i+1} \in \mathfrak{R}_{n-1}$. Therefore $\mathfrak{R}_n \cup \{\phi_i, \phi_{n-1}\} \subseteq \mathfrak{R}_{n-1} \cup \{\phi_{n-1}\}$ With Lemma 4.2.2(i) we obtain: $\dashv \mathfrak{R}_n, \phi_i, \phi_{n-1}$. Using Lemmas 4.2.2(i) en (ix) we obtain: $\dashv \mathfrak{R}_n, \phi_i \vee \phi_{n-1}$. As $\phi_n = \phi_i \vee \phi_{n-1}$, we have: $\dashv \mathfrak{R}_n, \phi_n$.

The cases $\vee I_2$, $\rightarrow I_1$ and $\rightarrow I_2$ can be proved analogously.

The remaining cases are left as exercises (Use Lemma 4.2.6!).●

4.3 *Completeness of the natural deduction for the logic of rejection*

In this section we shall prove the completeness of our system of natural deduction for the logic of rejection, by showing that we can simulate the axioms and the rules of derivation of the system of Klaus Härtig within our system of natural deduction. Härtig's system (Härtig, 1960) is complete.

As we have seen in section 2, Härtig's system can be formulated thus:

Axiom 1	${}_{H}\dashv \phi$, if $\phi \in$PROPL
Axiom 2	${}_{H}\dashv \neg\phi$, if $\phi \in$PROPL
Rule of Detachment	$\vdash \phi \rightarrow \psi$ and ${}_{H}\dashv \psi \Rightarrow {}_{H}\dashv \phi$
Rule of Disjunction	${}_{H}\dashv \phi$ and ${}_{H}\dashv \psi$ and $\langle\phi\rangle \cap \langle\psi\rangle = \varnothing \Rightarrow {}_{H}\dashv \phi \vee \psi$

4.3.1 *Definition* ${}_{\alpha}\dashv \phi \iff$ There is at least one formal deduction of rejection for ϕ (according to the system of Definition 3.1.2) in which no occurrence of a formula is justified with one of the atomic rules *At* or $\neg$*At*.

Remark A deduction of rejection Δ_1 with a conclusion ϕ in which no occurrence of a formula is justified with an atomic rule, has a pleasant quality, namely, this deduction of rejection Δ_1 with conclusion ϕ can, without causing any problems, be fitted in within the scope of hypotheses of an arbitrary deduction of rejection Δ_2.

4.3.2 *Lemma* Let ϕ, $\psi \in$FORM, then
${}_{\alpha}\dashv \neg(\phi \rightarrow \psi)$ and ${}_{\alpha}\dashv \neg\phi \Rightarrow {}_{\alpha}\dashv \neg\psi$.

Proof:

1.	ψ	hyp.
	Δ_1	
i.	$\neg(\phi \to \psi)$	Given
	Δ_2	
j.	$\neg\phi$	Given
j+1.	ψ	*Triv*: 1
j+2.	ψ	hyp.
j+3.	$\neg\phi$	*Triv*: *j*
j+4.	$\phi \to \psi$	$\to I_2$: $j+1$, $j+2$ - $j+3$
j+5.	$\top$	$\neg E$: i, $j+4$
j+6.	$\neg\psi$	$\neg I$: 1- $j+5$

As in the course of the deduction of $\neg\psi$ we have not made an appeal to one of the atomic rules, we have ${}_{\alpha}\dashv \neg\psi$. ●

4.3.3 *Lemma* Let $\phi \in$FORM, then:
$\vdash \phi \Rightarrow {}_{\alpha}\dashv \neg\phi$.[63]

Proof: First, we prove the negations of every axiom of an axiom system for *CPL*, for instance the system $K\top\bot ax$[64]. These deductions are left as exercises to the reader.

(i) ${}_{\alpha}\dashv \neg(\phi \to (\psi \to \phi))$

(ii) ${}_{\alpha}\dashv \neg((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))$

(iii) ${}_{\alpha}\dashv \neg((\phi \wedge \psi) \to \phi)$

(iv) ${}_{\alpha}\dashv \neg((\phi \wedge \psi) \to \psi)$

(v) ${}_{\alpha}\dashv \neg(\phi \to (\psi \to (\phi \wedge \psi)))$

(vi) ${}_{\alpha}\dashv \neg((\phi \to \chi) \to ((\psi \to \chi) \to ((\phi \vee \psi) \to \chi)))$

(vii) ${}_{\alpha}\dashv \neg(\phi \to (\phi \vee \psi))$

(viii) ${}_{\alpha}\dashv \neg(\psi \to (\phi \vee \psi))$

(ix) ${}_{\alpha}\dashv \neg(\neg\phi \to (\phi \to \bot))$

(x) ${}_{\alpha}\dashv \neg((\phi \to \bot) \to \neg\phi)$

(xi) ${}_{\alpha}\dashv \neg(\bot \to \phi)$

(xii) ${}_{\alpha}\dashv \neg(((\phi \to \bot) \to \phi) \to \phi)$

63 Cf. Härtig's less restrictive rule: $\vdash \phi \Rightarrow \dashv \neg\phi$. Härtig (1960); 244.

64 $K\top\bot ax$ is the system $K\wedge ax$ in Barth and Krabbe (1982); pp. 211-227, extended with one extra axiom (Ax $\top$): $\top$. The system $K\top\bot ax$ is complete for *CPL* in languages with both *vèrum* ($\top$) and *falsum* ($\bot$).

(xiii) $_{\alpha}\dashv\neg\top$

Next to this finite set of axioms, this system has one rule of inference: *modus ponens*. We complete the proof with induction over the number of lines occurring in the axiomatic derivation. Let it be given that $\vdash\phi$. Then there must be a derivation in $K\top\bot ax$, say A, with a conclusion ϕ. A consists of a column of formulas $\phi_1, \phi_2, ..., \phi_k$ (with $\phi_k = \phi$). Every occurrence of a formula is justified either with an appeal to one of the axioms of $K\top\bot ax$,, or with an appeal to *modus ponens*. Now, we prove by induction that for every line n with $1 \leq n \leq k$ we have: $_{\alpha}\dashv\neg\phi_n$.

Induction hypothesis: Let the proposition to be proved be correct for every line m with $m < n$: $_{\alpha}\dashv\neg\phi_m$.

Induction step: Consider ϕ_n. ϕ_n is justified either as an axiom, or by *modus ponens*.

i) Suppose ϕ_n is an axiom. Then we have from our list of proofs of negated axioms $_{\alpha}\dashv\neg\phi_n$.

ii) Suppose ϕ_n is justified by *modus ponens*. Then there must be two lines i and j with $i < n$ and $j < n$ with formulas ϕ_i and ϕ_j such that $\phi_j = \phi_i \rightarrow \phi_n$. Our induction hypothesis implies: $_{\alpha}\dashv\neg\phi_i \rightarrow \phi_n$ and $_{\alpha}\dashv\neg\phi_i$. Using Lemma 4.3.2 we have: $_{\alpha}\dashv\neg\phi_n$.

Therefore, for every n with $1 \leq n \leq k$ we have: $_{\alpha}\dashv\neg\phi_n$. Therefore $_{\alpha}\dashv\neg\phi$.●

4.3.4 *Lemma* Let $\phi \in$FORM, then $\dashv\!\dashv\phi \Rightarrow {}_{H}\dashv\phi$.

Proof: Cf. Härtig (1960).

4.3.5 *Lemma* Let $\phi \in$FORM, then $_{H}\dashv\phi \Rightarrow \dashv\phi$.

Proof: It will be sufficient to show that the two axioms and the two rules of inference of Härtig's system can be mimicked within our system of natural deduction for the logic of rejection.

Härtig's axioms can easily, without any need for adaptation, be simulated by the two atomic rules *At* en $\neg$*At* of our system of natural deduction.

Härtig's Rule of Detachment can be proved as follows:

Suppose: $\vdash\phi\rightarrow\psi$ and $\dashv\psi$.

By Lemma 4.3.3 we obtain: ${}_{\alpha}\dashv \neg(\phi \rightarrow \psi)$ and $\dashv \psi$.

	Δ_1	
$i.$	ψ	Given
i+1.	ψ	hyp.
	Δ_2	
$j.$	$\neg(\phi \rightarrow \psi)$	Given
j+1.	$\neg\phi$	hyp.
j+2.	$\neg\phi$	hyp.
j+3.	ψ	*Triv*: i+1
j+4.	$\phi \rightarrow \psi$	$\rightarrow I_1: j+1, j+2 - j+3$
j+5.	$\top$	$\neg E: j, j+4$
j+6.	ϕ	$CRV: j+1 - j+5$
j+7.	$\psi \vee \phi$	$\vee I_1: i, i+1 - j+6$
j+8.	ϕ	$\vee E_2: j+7$

Therefore $\dashv \phi$.

Härtig's Rule of Disjunction can be proved as follows: Suppose $\dashv \phi$ en $\dashv \psi$ en $\langle\phi\rangle \cap \langle\psi\rangle = \varnothing$. Then there are two natural deductions of rejection Δ_1 and Δ_2, such that Δ_1's conclusion is the formula ϕ and Δ_2's conclusion is the formula ψ. The formulas ϕ and ψ do not share any common propositional letter. It still remains possible that ϕ and the formulas occurring in the deduction Δ_2 have common propositional constants. Let $A = \{A_1, \ldots, A_m\}$ the set of propositional constants of ϕ. Let $B = \{B_1, \ldots, B_n\}$ the set of all propositional constants of formulas occurring in deduction Δ_2. A and B are finite sets of propositional constants. Therefore there is an infinity of propositional constants which are neither in A nor in B. Let $C_1, C_2, \ldots..$ be an enumeration of this infinite set of propositional constants. Let $C \in A \cap B$. Replace every occurrence of C in Δ_2 by C_1. The resulting deduction will be called ${}^1\Delta_2$ Note that the conclusion ψ cannot change by these procedure. Let B^i be the set of propositional constants of formulas occurring in deduction ${}^i\Delta_2$. We now repeat the procedure described above until $A \cap B^i = \varnothing$. Then ϕ and ${}^i\Delta_2$ do not have any common propositional constant. Deduction ${}^i\Delta_2$ can, without causing any problems, be fitted in within the scope of hypotheses of another deduction if, within this scope, only formulas consisting of propositional constants from A are operative. Therefore:

$$
\begin{array}{lll}
 & \Delta_1 & \\
i. & \phi & \text{Given} \\
i{+}1. & \phi & \text{hyp.} \\
 & {}^{i}\Delta_2 & \\
j. & \psi & \text{Given} \\
j{+}1. & \phi \vee \psi & \vee I_1 : i,\ i+1\text{-}j
\end{array}
$$

Therefore $\dashv \phi \vee \psi$. ●

4.3.6 *Completenesstheorem* Let $\phi \in$ FORM, then:

(i) $\models\!\dashv \phi \Rightarrow \dashv \phi$

(ii) $\models\!\dashv \phi \Leftrightarrow \dashv \phi$.

Proof:

(i) Directly from Lemma 4.3.4 and Lemma 4.3.5.

(ii) Directly from (i) and Theorem 4.2.8. ●

4.3.7 *Theorem* Let $\phi \in$ FORM, then:

${}_{\alpha}\!\dashv \phi \Leftrightarrow \phi$ is a contradiction.

Proof: $(\Rightarrow)$ If ${}_{\alpha}\!\dashv \phi$, then there is a deduction Δ_2 with a conclusion ϕ such that no single occurrence of a formula is justified by one of the atomic rules. Suppose: ϕ is not a contradiction, i.e. there is at least one model M such that $V_M(\phi)=1$. By $Sem_{\neg}$ we obtain that there is at least one model M such that $V_M(\neg\phi)=0$, in other symbols $\models\!\dashv \neg\phi$. By the Completenesstheorem we have $\dashv \neg\phi$. Therefore there must be a deduction Δ_1 with conclusion $\neg\phi$. We can, taking this deduction as our starting-point, proceed as follows:

$$
\begin{array}{lll}
 & \Delta_1 & \\
i. & \neg\phi & \text{Given} \\
i{+}1. & \neg\phi & \text{hyp.} \\
 & \Delta_2 & \\
j. & \phi & \text{Given} \\
j{+}1. & \top & \neg E : i+1,\ j \\
j{+}2. & \neg\phi \vee \top & \vee I_1 : i,\ i+1\text{-}\ j+1 \\
j{+}3. & \top & \vee E_2 : j+2
\end{array}
$$

Therefore $\dashv \top$. Using the Correctnesstheorem, we obtain: $\Dashv\top$, therefore, there must at least be one model M such that $V_M(\top)=0$. This is in contradiction with $Sem_{\top}$. Therefore ϕ is a contradiction.

($\Leftarrow$) If ϕ is a contradiction, then $\neg\phi$ is a tautology. By Lemma 4.3.3 we obtain: ${}_{\alpha}\!\dashv\neg\neg\phi$.

1.	$\neg\phi$	hyp.
	Δ_1	
i.	$\neg\neg\phi$	Given
i+1.	$\top$	$\neg E$: 1, i
i+2.	ϕ	CRV: 1-i+1

Therefore ${}_{\alpha}\!\dashv\phi$. ●

4.3.8 *Definition* Let $\phi, \psi \in$FORM, then:

$\psi\ {}_{\alpha}\!\dashv\phi \Leftrightarrow$ There is, taking premiss ψ as our starting-point, at least one deduction of rejection of ϕ, in which no single occurrence of a formula is justified by one of the atomic rules At or $\neg At$.

4.3.9 *Theorem of Inversion* Let $\phi, \psi \in$FORM, then:

$\phi \vdash \psi \Leftrightarrow \psi\ {}_{\alpha}\!\dashv\phi$.

Proof: We already know that:

$\phi \vdash \psi \quad \Leftrightarrow \quad \vdash \phi \to \psi$ (Deductiontheorem)

$\Leftrightarrow \quad {}_{\alpha}\!\dashv\neg(\phi \to \psi)$ (Theorem 4.3.7).

Therefore, it will be sufficient to show:

${}_{\alpha}\!\dashv\neg(\phi \to \psi) \Leftrightarrow \psi\ {}_{\alpha}\!\dashv\phi$.

($\Rightarrow$) Let it be given that: ${}_{\alpha}\!\dashv\neg(\phi \to \psi)$.

1.	ψ	prem.
2.	ψ	hyp.
	Δ_1	
i.	$\neg(\phi \to \psi)$	Given
i+1.	$\neg\phi$	hyp.
i+2.	$\neg\phi$	hyp.
i+3.	ψ	$Triv$: 2
i+4.	$\phi \to \psi$	$\to I_1: i+1, i+2-i+3$
i+5.	$\top$	$\neg E$: i, i+4
i+6.	ϕ	CRV: i+1-i+5
i+7.	$\psi \vee \phi$	$\vee I_1$: 1, 2 - $i+6$
i+8.	ϕ	$\vee E_2: i+7$

Therefore $\psi\ {}_{\alpha}\!\dashv\phi$.

($\Leftarrow$) Given: $\psi\ {}_{\alpha}\!\dashv \phi$.

1.	$\phi \rightarrow \psi$	hyp.
2.	ψ	$\rightarrow E_2 : 1$
	Δ_2	
i.	ϕ	Given
i+1.	$\neg\, \phi$	$\rightarrow E_1 : 1$
i+2.	$\top$	$\neg E: i, i+1$
i+3.	$\neg(\phi \rightarrow \psi)$	$\neg I: 1\text{-}i+2$

Therefore ${}_{\alpha}\!\dashv \neg(\phi \rightarrow \psi)$. ● [65]

University of Groningen

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