A Non-Standard Construction of Haar Measure and Weak König's Lemma

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Abstract

In this paper, we show within $\mathbf{RCA_0}$ that weak König's lemma is necessary and sufficient to prove that any (separable) compact group has a Haar measure. Within $\mathbf{WKL_0}$, a Haar measure is constructed by a non-standard method based on a fact that every countable non-standard model of $\mathbf{WKL_0}$ has a proper initial part isomorphic to itself [9].

1 Introduction

This paper is a contribution to *Reverse Mathematics*, an ongoing program to determine which set existence axioms are needed to prove particular theorems of ordinary mathematics ([7], [8]). Along this program, measure theory has been studied by Yu and Simpson [11], [12], but the right axioms for the existence of Haar measure had been unknown.

Haar measure has an important role in the foundations of real analysis, and also relates to a famous problem of Hilbert (i.e., the fifth of his twenty-three problems). The existence of Haar measure was first shown by Haar in 1933 for locally compact groups which are second countable, and subsequently by von Neumann for compact groups. As explained in a classical textbook [4] of Pontryagin, von Neumann's proof essentially depends on the Arzela-Ascoli lemma, which can not be proved within $\mathbf{WKL_0}$. Succeedingly, Weil, Cartan and others invented simpler proofs and yet for more general families of groups. Although some proofs do not use the Arzela-Ascoli argument, none seems to be free from the notion of sup or limit, which also requires et existence axioms beyond $\mathbf{WKL_0}$.

Later, Bishop [1] modifies Cartan's proof by a certain approximation trick to obtain his constructive version. By contrast, Hauser [3] and others simplify Weil's proof by way of non-standard analysis. Inspired by both of these disparate proofs (constructive and non-standard), we here manage to construct a Haar measure in $\mathbf{WKL_0}$.

The system $\mathbf{WKL_0}$ is a subsystem of second-order arithmetic obtained from $\mathbf{RCA_0}$ (= the axioms of ordered semirings, recursive comprehension and Σ^0_1 induction) by adding the weak König's lemma: *every infinite tree of sequences of 0's and 1's has an infinite path.* The first-order (arithmetical) part of $\mathbf{WKL_0}$ is the same as that of $\mathbf{RCA_0}$, or equivalently $\mathrm{I}\Sigma_1$, which is Π^0_2-conservative over $\mathbf{PRA}$. Thus, the consistency of $\mathbf{WKL_0}$ is as clear as that of $\mathbf{PRA}$, while $\mathbf{WKL_0}$ proves much more than $\mathbf{RCA_0}$ or $\mathbf{PRA}$, e.g., the Hahn-Banach theorem for separable Banach spaces [2], [5], the Cauchy-Peano theorem for ordinary differential equations [5], [6], [10].

Most of basic notions of analysis can be defined within $\mathbf{RCA_0}$. A *complete separable metric space* $\hat{A}$ is coded by its countable dense subset A together with a pseudo-metric d on it. Then $\hat{A}$ is said to be *compact* if there is an infinite sequence of finite subsets of A, $\langle A_n : n \in \mathbf{N} \rangle$, such that A_n is a 2^{-n}-net of $\hat{A}$ for each n. Let $\hat{G}$ be a compact group, i.e., a compact metric space with continuous group-operations. A *Haar measure* on $\hat{G}$ is a σ-additive left-invariant positive linear operator μ on the space $C(\hat{G})$ of continuous functionals on $\hat{G}$ such that $\mu(1) = 1$. More details of these definitions will be given in the next section.

Our main theorem of this paper is as follows:

Main Theorem 1 ($\mathbf{RCA_0}$) *The following are equivalent:*

(1) $\mathbf{WKL_0}$.

(2) Any compact group has a Haar measure.

In the next section, we define basic concepts of topological groups, and show in $\mathbf{RCA_0}$ that a kind of approximation to Haar measure exists. Section 3 is devoted to set up a non-standard method, by which we will construct a Haar measure in Section 4. It is also shown in Section 4 that $\mathbf{WKL_0}$ is necessary for the existence of Haar measure. Finally, in Section 5, we eliminate the non-standard argument from the previous construction of a Haar measure, and prove, for instance, that $\mathbf{WWKL_0}$, which is strictly weaker than $\mathbf{WKL_0}$, is necessary and sufficient to show that any compact group whose operations have a modulus of uniform continuity has a unique Haar measure. The work of Section 5 is essentially due to the second author, while the other parts are collaboration of the two authors.

2 Haar measure and its finite approximations

We are working within $\mathbf{RCA_0}$ unless otherwise stated. A *complete separable metric space* $\widehat{A}$ is coded by a set $A \subseteq \mathbf{N}$ together with a pseudo-metric $d : A \times A \to \mathbf{R}$. A *point* in $\widehat{A}$ is a sequence $\langle a_n : n \in \mathbf{N} \rangle$ from A such that $d(a_n, a_{n+i}) < 2^{-n}$ for each $n, i \in \mathbf{N}$. A complete separable metric space is *compact* if there exists an infinite sequence $\langle \langle a_{i,j} \in A : i \leq n_j \rangle : j \in \mathbf{N} \rangle$ of finite sequences of points in A such that for each j, $\langle a_{i,j} : i \leq n_j \rangle$ is a 2^{-j}-net, i.e., $\forall a \in \widehat{A} \exists i \leq n_j [d(a, a_{i,j}) < 2^{-j}]$. $\mathbf{RCA_0}$ proves that the unit interval $[0,1]$ is compact in this sense, but does not that $[0,1]$ has the Heine-Borel property.

A triple $b = \langle a, r, s \rangle \in A \times \mathbf{Q} \times \mathbf{Q}$ with $0 \leq s < r$ encodes a *basic function* $b : \widehat{A} \to \mathbf{R}$

defined by

$$b(x) = \begin{cases} 1 & \text{if } d(a,x) \leq s, \\ \dfrac{r - d(a,x)}{r - s} & \text{if } s < d(a,x) < r, \\ 0 & \text{if } d(a,x) \geq r. \end{cases}$$

Then a finite sequence $p = \langle\langle q_n, b_n\rangle : n \leq m\rangle$ encodes a *polynomial* $p(x) = \sum_{n=0}^{m} q_n b_n(x)$, where q_n's are rationals and b_n's are basic functions.

Let P be the set of all (codes for) polynomials. Assuming that $\widehat{A}$ is compact, P can be seen as a countable vector space over $\mathbf{Q}$ equipped with the sup-norm $||p|| = \sup\{|p(x)| : x \in \widehat{A}\}$. Finally, by $C(\widehat{A})$, we mean the separable Banach space $\widehat{P}$. A point in $C(\widehat{A})$ can be regarded as a continuous function $f : \widehat{A} \to \mathbf{R}$ in the obvious way, and moreover it has a modulus h of uniform continuity, i.e., such that for each n and each $x, y \in \widehat{A}$, $d(x,y) \leq 2^{-h(n)} \to |f(x) - f(y)| \leq 2^{-n}$.

Now we define a compact group as follows.

Definition 1 ($\mathbf{RCA_0}$) *A compact metric space $\widehat{G}$ is called a* **compact group** *if it is equipped with an element $e \in G$ and continuous functions $^{-1} : \widehat{G} \to \widehat{G}$, $\cdot : \widehat{G} \times \widehat{G} \to \widehat{G}$ such that $(\widehat{G},\ e,\ ^{-1},\ \cdot)$ satisfies the axioms of groups.*

Let $\widehat{G}$ be a compact group. A *measure* μ on $\widehat{G}$ is defined to be a positive bounded linear functional on $C(\widehat{G})$ such that $\mu(1) = 1$. For each $f \in C(\widehat{G})$ and $s \in \widehat{G}$, let f^s denote the continuous function defined by $f^s(x) = f(sx)$. Then a measure μ on $\widehat{G}$ is called *left-invariant* if $\mu(f^s) = \mu(f)$ whenever $f, f^s \in C(\widehat{G})$. For example, the unit circle S^1 is regarded as a compact group with a left-invariant measure in $\mathbf{RCA_0}$. Finally, the countable additivity of measure is defined as usual. See [11].

Definition 2 ($\mathbf{RCA_0}$) *A measure μ on $\widehat{G}$ is called a* **Haar measure** *if μ is a countably additive left-invariant measure.*

We shall use the symbol $C(\widehat{G})^+$ informally to denote the set of positive functions in $C(\widehat{G})$. A standard construction of a Haar measure calls for the concept of least upper bound such as

$$(f:g) = \inf\{\sum_{i=0}^{n} a_i : f \leq \sum_{i=0}^{n} a_i g^{s_i} \text{ for some } s_i \in \widehat{G} \text{ and } a_i \geq 0\}$$

for $f,\, g \in C(\widehat{G})^+$. But, in $\mathbf{RCA_0}$ (or $\mathbf{WKL_0}$), the existence of "inf" cannot be guaranteed.

From now on, we assume that the group operations of $\widehat{G}$ are uniformly continuous. Then, an approximation to $(f:g)$ exists in $\mathbf{RCA_0}$ as follows.

Lemma 1 ($\mathbf{RCA_0}$) *Let $\widehat{G}$ be a compact group with uniformly continuous operations. Choose any $f,\, g \in C(\widehat{G})^+$. Then for each positive real $\varepsilon \in \mathbf{R}_{>0}$, there exists a finite sequence $\langle a_i : i \leq n\rangle$ of non-negative reals such that*

(1) *there exists a sequence $\langle s_i : i \leq n\rangle$ from G such that $f \leq \sum_{i=0}^{n} a_i g^{s_i}$, and*

(2) *if a sequence $\langle c_i : i \leq m\rangle$ of non-negative reals and a sequence $\langle v_i : i \leq m\rangle$ of points in $\widehat{G}$ satisfy $f \leq \sum_{i=0}^{m} c_i g^{v_i}$, then $\sum_{i=0}^{n} a_i \leq \sum_{i=0}^{m} c_i + \varepsilon$*

Proof. Working in $\mathbf{RCA_0}$. Take any $f,\, g \in C(\widehat{G})^+$. Since $g \neq 0$, there are $r > 0$ and $t \in G$ such that $2r < g(t)$. Since the operations are uniformly continuous, there exists $\delta_1 > 0$ such that $d(x,y) \leq \delta_1 \to r < g(tx^{-1}y)$.

Let $\langle t'_i : i \leq k\rangle$ be a δ_1-net of $\widehat{G}$, i.e., $\forall x \in \widehat{G} \exists i \leq k[d(t'_i, x) \leq \delta_1]$. Then for each $x \in \widehat{G}$, there exists $i \leq k$ such that $r < g(tt'^{-1}_i x)$. Now, we write t_i by tt'^{-1}_i. Then for each $x \in \widehat{G}$, there exists $i \leq k$ such that $r < g(t_i x)$. Hence we have,

$$\text{(i) } r < \sum_{i=0}^{k} g(t_i x).$$

Without loss of generality, we may assume that the t_i's are taken from G.

Choose a rational number M such that $||f|| < M$. So, we have

$$\text{(ii) } f(x) \leq r^{-1} M \sum_{i=0}^{k} g(t_i x).$$

Fix any $\varepsilon > 0$. Then take $\delta < (1 + r^{-2}M(k+1)^2)^{-1}\varepsilon$. Since $g(xy)$ is uniformly continuous and $\widehat{G}$ is compact, there exists a finite sequence $\langle u_i : i \leq l\rangle$ from G such that

$$\text{(iii) } \forall s \in \widehat{G} \exists i \leq l \forall x \in \widehat{G}[g(sx) \leq g(u_i x) + \delta].$$

Choose $J \in \mathbf{N}$ such that $J > \delta^{-1}(l+1)r^{-1}M(k+1)$. By bounded Π^0_1-comprehension, we define a set Φ as follows:

$$\Phi = \{\langle j_0, \ldots, j_l\rangle \in \{0, \ldots, J\}^{l+1} :$$
$$\forall x \in G[f(x) \leq \delta(l+1)^{-1}\sum_{i=0}^{l} j_i g(u_i x) + \delta r^{-2}M(k+1)\sum_{i=0}^{k} g(t_i x)]\}.$$

Then $\Phi \neq \phi$, since

$$\begin{aligned} f(x) &\leq r^{-1}M\sum_{i=0}^{k} g(t_i x) \text{ (by (ii))} \\ &\leq r^{-1}M(k+1)\sum_{i=0}^{l} g(u_i x) + \delta r^{-1}M(k+1) \text{ (by (iii))} \\ &\leq \delta(l+1)^{-1}\sum_{i=0}^{l} Jg(u_i x) + \delta r^{-2}M(k+1)\sum_{i=0}^{k} g(t_i x) \text{ (by (i) and the choice of } J \text{).} \end{aligned}$$

Choose $\langle j_i : i \leq l\rangle \in \Phi$ with the least $\sum_{i=0}^{l} j_i$. And, let

$$a_i = \begin{cases} \delta(l+1)^{-1} j_i, & \text{if } i \leq l, \\ \delta r^{-2}M(k+1), & \text{if } l+1 \leq i \leq k+l+1. \end{cases}$$

Then $\langle a_i : i \leq k+l+1\rangle$ satisfies property (1) of lemma 1 with $\langle s_i\rangle = \langle u_i\rangle^\frown\langle t_i\rangle$.

Next, to show $\langle a_i\rangle$ satisfies the property (2), assume that $f(x) \leq \sum_{i=0}^{m} c_i g(v_i x)$. By (iii), there exists $\langle d_i : i \leq l\rangle$ such that $\sum_{i=0}^{m} c_i = \sum_{i=0}^{l} d_i$ and

$$\sum_{i=0}^{m} c_i g(v_i x) \leq \sum_{i=0}^{l} d_i g(u_i x) + \delta\sum_{i=0}^{l} d_i.$$

First consider the case $\sum_{i=0}^{m} c_i (= \sum_{i=0}^{l} d_i) \leq r^{-1}M(k+1)$. Since $1 < r^{-1}\sum_{i=0}^{k} g(t_i x)$ by (i),

$$f(x) \leq \sum_{i=0}^{l} d_i g(u_i x) + \delta r^{-2}M(k+1)\sum_{i=0}^{k} g(t_i x).$$

For each $i \leq l$, let $j'_i = \min\{j \in \mathbf{N} : j \leq J \wedge d_i \leq \delta(l+1)^{-1} j\}$. Clearly, $\langle j'_i : i \leq l\rangle \in \Phi$. So,

$$\begin{aligned} \sum_{i=0}^{k+l+1} a_i &= \delta(l+1)^{-1}\sum_{i=0}^{l} j_i + \delta r^{-2}M(k+1)^2 \\ &\leq \delta(l+1)^{-1}\sum_{i=0}^{l} j'_i + \delta r^{-2}M(k+1)^2 \end{aligned}$$

$$\leq \sum_{i=0}^{l}(d_i + \delta(l+1)^{-1}) + \delta r^{-2}M(k+1)^2 (\text{ since } \delta(l+1)^{-1}(j'_i - 1) < d_i \ )$$
$$\leq \sum_{i=0}^{l} d_i + \delta(1 + r^{-2}M(k+1)^2) \leq \sum_{i=0}^{m} c_i + \varepsilon.$$

Secondly, consider the case $r^{-1}M(k+1) \leq \sum_{i=0}^{m} c_i$. By (ii), $\langle c'_i : i \leq k\rangle = \langle r^{-1}M : i \leq k\rangle$ satisfies $f(x) \leq \sum_{i=0}^{m} c'_i g(t_i x)$ and $\sum_{i=0}^{m} c'_i = r^{-1}M(k+1)$. Hence, by the above argument, $\sum_{i=0}^{k+l+1} a_i \leq \sum_{i=0}^{m} c'_i + \varepsilon = r^{-1}M(k+1) + \varepsilon \leq \sum_{i=0}^{m} c_i + \varepsilon$. □

For each $\varepsilon \in \mathbf{R}_{>0}$ and each $f, g \in C(\widehat{G})^+$, $(f : g)^\varepsilon$ is defined to be $\sum_{i=0}^{n} a_i$ where a_i's are given in the above lemma. We may assume that $(f : g)^\varepsilon$ is rational.

Lemma 2 ($\mathbf{RCA_0}$) *Choose any f_1, f_2 and $g \in C(\widehat{G})^+$. Then, the following conditions hold:*

(1) *for each $\varepsilon > 0$, $(f_1 : g)^\varepsilon \geq \dfrac{||f_1||}{2||g||}$;*

(2) *if $f_1 \leq f_2$, then for each $\varepsilon > 0$, $(f_1 : g)^\varepsilon \leq (f_2 : g)^\varepsilon + \varepsilon$;*

(3) *for each $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$, $(f_1 : g)^\varepsilon \leq (f_1 : f_2)^{\varepsilon_1}(f_2 : g)^{\varepsilon_2} + \varepsilon$;*

(4) *for each $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$, $(f_1 + f_2 : g)^\varepsilon \leq (f_1 : g)^{\varepsilon_1} + (f_2 : g)^{\varepsilon_2} + \varepsilon$;*

(5) *for each $\varepsilon, \lambda > 0$, $|(\lambda f_1 : g)^\varepsilon - \lambda(f_1 : g)^\varepsilon| \leq (\lambda + 1)\varepsilon$;*

(6) *for each $\varepsilon > 0$ and each $s \in \widehat{G}$, $|(f_1^s : g)^\varepsilon - (f_1 : g)^\varepsilon| \leq \varepsilon$.*

Proof. (1) Since $f_1 \neq 0$, there exists $\alpha \in \widehat{G}$ such that $\dfrac{||f_1||}{2} < f(\alpha)$. If $f_1(x) \leq \sum_i a_i g(s_i x)$, then
$$\frac{||f_1||}{2} < f(\alpha) \leq \sum_i a_i g(s_i \alpha) \leq ||g|| \sum_{i=0}^{n} a_i.$$
Hence $(f_1 : g)^\varepsilon \geq \dfrac{||f_1||}{2||g||}$.

(2) is trivial. For (3), assume that $(f_1 : f_2)^\varepsilon = \sum_i a_i$ and $(f_2 : g)^{\varepsilon_2} = \sum_j c_j$ where $f_1(x) \leq \sum_i a_i f_2(s_i x)$ and $f_2(x) \leq \sum_j c_j g(t_j x)$. Then $f_1(x) \leq \sum_{i,j} a_i c_j g(s_i t_j x)$. Since $\sum_{i,j} a_i c_j \leq \sum_i a_i \sum_j c_j$, (3) holds. (4), (5) and (6) can be treated similarly. □

Now we define $I_g^\varepsilon(f)$ by $\frac{(f:g)^\varepsilon}{(1:g)^\varepsilon}$. We say g is *small of order c* if $g(x) = 0$ whenever $d(x,e) \geq c$. We are going to show that $I_g^\varepsilon(f)$ is "approximate" to the Haar measure when g is sufficiently small.

Lemma 3 ($\mathbf{RCA_0}$) *For each $\varepsilon \in \mathbf{R}_{>0}$ and each $f_0, \ldots, f_n \in C(\widehat{G})^+$, there exists $c \in \mathbf{R}_{>0}$ such that if $g \in C(\widehat{G})^+$ with $||g|| = 1$ is small of order c, then for each sufficiently small $\varepsilon' \in \mathbf{R}_{>0}$ and for each $0 \leq \lambda_i \leq 1$,*

$$\sum_{i=0}^n \lambda_i I_g^{\varepsilon'}(f_i) \leq I_g^{\varepsilon'}(\sum_{i=0}^n \lambda_i f_i) + \varepsilon.$$

Proof. Fix any $\varepsilon < \frac{1}{2}$. Let $h_j^\lambda = \frac{f_j}{\sum_{i=0}^n \lambda_i f_i + \varepsilon}$. It is easy to see that all the h_j^λ's have a common modulus of uniform continuity independent from the choice of j and $\lambda = \langle \lambda_i : i \leq n \rangle$ (with $0 \leq \lambda_i \leq 1$). Take $M > \sum_{i=0}^n ||f_i|| + 3$. Then there exists $c > 0$ such that for each λ such that $0 \leq \lambda_i \leq 1$,

$$(1)\ \forall j \leq n \forall s, x \in \widehat{G}(d(sx,e) < c \to |h_j^\lambda(s^{-1}) - h_j^\lambda(x)| < \frac{\varepsilon}{(n+1)M}).$$

Suppose that a $g \in C(\widehat{G})^+$ is small of order c and $||g|| = 1$. For each $\langle c_k \rangle$, $\langle s_k \rangle$ such that $\sum_{i=0}^n \lambda_i f_i + \varepsilon \leq \sum_{k=0}^m c_k g^{s_k}$, we have

$$\begin{aligned} f_j = h_j^\lambda \cdot (\sum_{i=0}^n \lambda_i f_i + \varepsilon) &\leq h_j^\lambda \sum_{k=0}^m c_k g^{s_k} \\ &\leq \sum_{k=0}^m c_k (h_j^\lambda(s_j^{-1}) + \frac{\varepsilon}{(n+1)M}) g^{s_k} \end{aligned}$$

by (1). Choose any $\varepsilon' < \frac{\varepsilon}{\max(M, n+1)}$. Then by Lemma 1 (2),

$$(f_j : g)^{\varepsilon'} \leq \sum_{k=0}^m c_k (h_j^\lambda(s_j^{-1}) + \frac{\varepsilon}{(n+1)M}) + \frac{\varepsilon}{(n+1)}.$$

Therefore,

$$\sum_{i=0}^n \lambda_i (f_i : g)^{\varepsilon'} \leq \sum_{k=0}^m c_k (1 + \frac{\varepsilon}{M}) + \varepsilon.$$

Now take a $\langle c_j \rangle$ such that $(\sum_{i=0}^n \lambda_i f_i + \varepsilon : g)^{\varepsilon'} = \sum_j c_j$ and $\sum_{i=0}^n \lambda_i f_i + \varepsilon \leq \sum_j c_j g^{s_j}$ for some $\langle s_j \rangle$. So by Lemma 2,

$$\sum_{i=0}^n \lambda_i (f_i : g)^{\varepsilon'} \leq [(\sum_{i=0}^n \lambda_i f_i : g)^{\varepsilon'} + \varepsilon(1:g)^{\varepsilon'} + \varepsilon](1 + \frac{\varepsilon}{M}) + \varepsilon.$$

Dividing the both sides of the above inequality by $(1:g)^{\varepsilon'}$, we have

$$
\begin{aligned}
(2)\ \sum_{i=0}^{n}\lambda_i I_g^{\varepsilon'}(f_i) &\le (I_g^{\varepsilon'}(\sum_{i=0}^{n}\lambda_i f_i)+\varepsilon+\frac{\varepsilon}{(1:g)^{\varepsilon'}})(1+\frac{\varepsilon}{M})+\frac{\varepsilon}{(1:g)^{\varepsilon'}} \\
&\le I_g^{\varepsilon'}(\sum_{i=0}^{n}\lambda_i f_i)+\varepsilon(\frac{I_g^{\varepsilon'}(\sum_{i=0}^{n}\lambda_i f_i)}{M}+6)(\text{since } \frac{1}{2}\le(1:g)^{\varepsilon'}\text{by Lemma 2}).
\end{aligned}
$$

Since $\sum_{i=0}^{n}\lambda_i f_i \le \sum_{i=0}^{n}||f_i||$, by Lemma 2,

$$(\sum_{i=0}^{n}\lambda_i f_i : g)^{\varepsilon'} \le \sum_{i=0}^{n}||f_i||(1:g)^{\varepsilon'}+\varepsilon.$$

Then,

$$I_g^{\varepsilon'}(\sum_{i=0}^{n}\lambda_i f_i) \le \sum_{i=0}^{n}||f_i||+2\varepsilon \le M.$$

By (2) and the above inequality, we finally obtain

$$\sum_{i=0}^{n}\lambda_i I_g^{\varepsilon'}(f_i) \le I_g^{\varepsilon'}(\sum_{i=0}^{n}\lambda_i f_i)+7\varepsilon.$$

□

Lemma 4 ($\mathbf{RCA_0}$) *Let C be a finite subset of $C(\widehat{G})^+$. Given $\varepsilon \in \mathbf{R}_{>0}$, then there exists a $g \in C(\widehat{G})^+$ with $||g|| = 1$ such that for each $f_1, f_2 \in C$ and each sufficiently small $\varepsilon' \in \mathbf{R}_{>0}$,*

(1) *if $f_1 \le f_2$, then $I_g^{\varepsilon'}(f_1) \le I_g^{\varepsilon'}(f_2)+\varepsilon$;*

(2) $|I_g^{\varepsilon'}(f_1+f_2)-(I_g^{\varepsilon'}(f_1)+I_g^{\varepsilon'}(f_2))| < \varepsilon$;

(3) *if $f_1^s = f_2$ with $s \in \widehat{G}$, then $|I_g^{\varepsilon'}(f_1)-I_g^{\varepsilon'}(f_2)| < \varepsilon$;*

(4) *if $\lambda f_1 = f_2$ with $\lambda \in \mathbf{R}$, then $|\lambda I_g^{\varepsilon'}(f_1)-I_g^{\varepsilon'}(f_2)| < (\lambda+1)\varepsilon$.*

Proof. Fix any $\varepsilon \in \mathbf{R}_{>0}$. By Lemma 3, we can choose a $g \in C(\widehat{G})^+$ with $||g|| = 1$ such that for each sufficiently small $\varepsilon' \in \mathbf{R}_{>0}$ $(\varepsilon' < \frac{\varepsilon}{2})$,

$$I_g^{\varepsilon'}(f_1)+I_g^{\varepsilon'}(f_2) \le I_g^{\varepsilon'}(f_1+f_2)+\varepsilon$$

for each $f_1, f_2 \in C$. Since $\frac{1}{2} \le (1:g)^{\varepsilon'}$ by Lemma 2,

$$I_g^{\varepsilon'}(f_1+f_2) \le I_g^{\varepsilon'}(f_1)+I_g^{\varepsilon'}(f_2)+\varepsilon$$

Similarly, by (6) (resp.(5)) of Lemma 2, if $f_1^s = f_2$ for some $s \in \widehat{G}$ (resp. $\lambda f_1 = f_2$ for some $\lambda \in \mathbf{R}$),

$$|I_g^{\varepsilon'}(f_1) - I_g^{\varepsilon'}(f_2)| < \varepsilon \text{ (resp. } |\lambda I_g^{\varepsilon'}(f_1) - I_g^{\varepsilon'}(f_2)| < (\lambda+1)\varepsilon).$$

□

3 A non-standard method in $\mathbf{WKL_0}$

By $V = (M, S)$, we denote a structure of second-order arithmetic, where M is an ordered semiring and S consists of subsets of M. For an initial segment I of M, we put $S\lceil I = \{X \cap I : X \in S\}$ and $V\lceil I = (I, S\lceil I)$.

In [9], we have shown

Theorem 5 (the self-embedding theorem) *Let V be a countable non-standard model of* $\mathbf{WKL_0}$. *Then there exists a proper initial part $V\lceil I$ of V and an isomorphism $f : V \to V\lceil I$.*

Fix a countable non-standard model V of $\mathbf{WKL_0}$. By the above theorem, V has an initial part isomorphic to itself. Since the initial part and V are isomorphic to each other, they may exchange their roles, and thus we can say that V has an isomorphic extension ${}^*V = ({}^*M, {}^*S)$. We shall use ${}^*V$ as a non-standard universe.

Let f be a function from $\mathbf{N}$ to $\mathbf{R}$ in V. Rigorously, f is coded by its graph $F \subseteq \mathbf{N} \times \mathbf{R} \subseteq \mathbf{N} \times \mathbf{N} \times \mathbf{Q}$. Then, F must satisfy the following conditions: for each $m \in M$,

(1.1)$\forall i \le m \forall n \le m \exists q \in \mathbf{Q}(\langle i, n, q\rangle \in F)$;

(1.2)$\forall i \le m \forall n \le m \forall q_1, q_2 \le m(\langle i, n, q_1\rangle, \langle i, n, q_2\rangle \in F \to q_1 = q_2)$;

(1.3)$\forall i \le m \forall n_1, n_2 \le m \forall q_1, q_2 \le m(\langle i, n_1, q_1\rangle, \langle i, n_2, q_2\rangle \in F \wedge n_1 \le n_2$
$\to |q_1 - q_2| < 2^{-n_1})$,

where q_1, q_2 in the bounded quantifiers are treated as their codes. Let ${}^*\mathbf{Q}$ be the set of rationals in ${}^*V$. Take a set ${}^*F \in {}^*S$ such that $F = {}^*F\lceil M$. Fix a non-standard element $\alpha \in {}^*M$. Then, for each $m \in M$, it holds in ${}^*V$ that

$(2.1) \forall i \le m \forall n \le m \exists q \le \alpha (q \in {}^*\mathbf{Q} \wedge \langle i,n,q\rangle \in {}^*F);$

$(2.2) \forall i \le m \forall n \le m \forall q_1, q_2 \le m(\langle i,n,q_1\rangle, \langle i,n,q_2\rangle \in {}^*F \to q_1 = q_2);$

$(2.3) \forall i \le m \forall n_1, n_2 \le m \forall q_1, q_2 \le m(\langle i,n_1,q_1\rangle, \langle i,n_2,q_2\rangle \in {}^*F \wedge n_1 \le n_2$

$\to |q_1 - q_2| < 2^{-n_1}).$

Since (2.1), (2.2) and (2.3) are Σ^0_0, by overspill, there exists a non-standard element $\beta \in {}^*M$(with $\beta \le \alpha$) such that in ${}^*V$

$(3.1) \forall i \le \beta \forall n \le \beta \exists q \le \alpha (q \in {}^*\mathbf{Q} \wedge \langle i,n,q\rangle \in {}^*F);$

$(3.2) \forall i \le \beta \forall n \le \beta \forall q_1, q_2 \le \beta(\langle i,n,q_1\rangle, \langle i,n,q_2\rangle \in {}^*F \to q_1 = q_2);$

$(3.3) \forall i \le \beta \forall n_1, n_2 \le \beta \forall q_1, q_2 \le \beta(\langle i,n_1,q_1\rangle, \langle i,n_2,q_2\rangle \in {}^*F \wedge n_1 \le n_2$

$\to |q_1 - q_2| < 2^{-n_1}).$

For a set $X \in {}^*S$, let $X(m)$ denote $\{n \in X : n \le m\}$. Then put ${}^*F_0 = {}^*F \cap ({}^*\mathbf{M}(\beta) \times {}^*\mathbf{M}(\beta) \times {}^*\mathbf{Q}(\alpha))$. Since ${}^*F_0$ is a finite subset of ${}^*F$ in ${}^*V$, we can define ${}^*F_1 : {}^*\mathbf{M}(\beta) \times {}^*\mathbf{M}(\beta) \to {}^*\mathbf{Q}(\alpha)$ as follows:

$${}^*F_1(i,n) = \min\{q : {}^*V \models (i,n,q) \in {}^*F_0\},$$

where "min" means the minimum with respect to codes for rationals. Then we obtain a function ${}^*f$ from ${}^*\mathbf{M}(\beta)$ to ${}^*\mathbf{Q}(\alpha)$ (called an *extension* of f) such that ${}^*f(i) = {}^*F_1(i,\beta)$. Occasionally, we regard ${}^*f(i)$ as a ${}^*V$-finite sequence $\langle {}^*F_1(i,0), \ldots, {}^*F_1(i,\beta)\rangle$. By noticing that for each $i,n \in M$ and $q \in \mathbf{Q}$, ${}^*V \models {}^*F_1(i,n) = q$ iff $V \models (i,n,q) \in F$, we easily obtain the following lemma.

Lemma 6 *Let ${}^*f$ be an extension of a real-valued function f. Then for each $i \in M$ and $q, q' \in \mathbf{Q}$,*

(1) *if $V \models q < f(i) < q'$ then ${}^*V \models q < {}^*f(i) < q'$;*

(2) *Regarding ${}^*f(i)$ as a ${}^*V$-finite sequence, $f(i)$ is an initial segment of ${}^*f(i)$.*

Let $\widehat{A}$ be a complete separable metric space with metric d. Since d is a real-valued function on $A \times A$, d has an extension ${}^*d$. Moreover we can take β such that ${}^*d$ is a pseudo-metric on ${}^*A(\beta) \times {}^*A(\beta)$. We use ${}^*A$ in the place of ${}^*A(\beta)$ for simplicity. Then, we can think that the pseudo-metric space ${}^*A$ includes $\widehat{A}$ in the following sense: For each $x = \langle a_n \rangle \in \widehat{A}$, there exists ${}^*a \in {}^*A$ such that for each $n \in M$, ${}^*V \models |{}^*d(a_n, {}^*a)| < 2^{-n}$. Using this ${}^*d$ instead of $|\ |$ in (2.3) and (3.3), we can define an extension of a sequence from $\widehat{A}$.

Let Seq_2 be the set of finite sequences of 0's and 1's, and d_{Seq_2} be a metric on Seq_2 defined by $d_{Seq_2}(\sigma, \tau) = |\sum_{i<lh(\sigma)} \sigma(i)2^{-i-1} - \sum_{j<lh(\tau)} \tau(j)2^{-j-1}|$. The space $\widehat{Seq_2}$ can be regarded as a closed unit interval $[0,1]$. Then, for each function f from M to $[0,1]$, an extension ${}^*f$ is a function from a proper initial segment of ${}^*V$ to a set of ${}^*V$-finite 0-1 sequences. By adding suitably many 0's at the end of a sequence, we may suppose that every sequence in the range of ${}^*f$ has the same length $\beta \in {}^*M \setminus M$. Hence, ${}^*[0,1]$ is defined by a set of ${}^*V$-finite 0-1 sequences with length β. Similarly, with a metric $d_k \stackrel{\text{def}}{=} k d_{Seq_2}$, ${}^*[0,k]$ is defined by a set of ${}^*V$-finite 0-1 sequences.

Since a continuous function f from $\widehat{A}$ to $\widehat{B}$ is uniquely determined by a sequence $\langle f(a) : a \in A \rangle$, we define an *extension* ${}^*f$ of f to be ${}^*\langle f(a) : a \in A \rangle$. The extension of a continuous function plays a leading part in the arguments of next section.

4 Haar measure and $\mathbf{WKL_0}$

In this section, we describe a construction of Haar measure by a non-standard method within $\mathbf{WKL_0}$. Fix a countable non-standard model V of $\mathbf{WKL_0}$. Choose a compact group $\widehat{G}$ (in V). Then we have, as continuous functions, the norm $||\ ||$ on $C(\widehat{G})$, the operation $Ab : C(\widehat{G}) \to C(\widehat{G})$ defined by $Ab(f)(x) = |f(x)|$ and $L : C(\widehat{G}) \times \widehat{G} \to C(\widehat{G})$ by $L(f, s) = f^s$. Notice that any continuous function is uniformly continuous within $\mathbf{WKL_0}$. And a continuous function has an extension in ${}^*V$, which we shall often denote by the same symbol.

Lemma 7 *Let $V = (M, S)$ be a countable non-standard model of $\mathbf{WKL_0}$ and $\widehat{G}$ be a compact group (in V). Then there exists $I : P \to \mathbf{R}_{\geq 0}$ such that:*

1. *for each non-negative* $f = \langle p_i : i \in \mathbf{N} \rangle \in C(\widehat{G})$ *with* $||f|| \leq 1$, $I(f) \stackrel{\mathrm{def}}{=} \lim_i I(p_i)$ *exists.*

2. *for* $f_1, f_2, f_3 \in C(\widehat{G})^+$ *with* $||f_i|| \leq 1$ $(i = 1, 2, 3)$,

   (1) *if* $f_3 = f_1 + f_2$, *then* $I(f_3) = I(f_1) + I(f_2)$;

   (2) *if* $f_1^s = f_2$ *with* $s \in \widehat{G}$, *then* $I(f_1) = I(f_2)$;

   (3) *if* $\lambda f_1 = f_2$ *with* $\lambda \in \mathbf{R}_{>0}$, *then* $\lambda I(f_1) = I(f_2)$;

   (4) $I(1) = 1$.

Proof. We first define a Σ_0^0-formula $\varphi(\sigma, m)$ with parameters from ${}^*V$ which roughly means that σ is a 2^{-m}-approximation of Haar measure on $\{p \in P(m) : p \text{ is positive}, ||p|| < 2\}$. More precisely, φ asserts the following: σ is a finite sequence from ${}^*[0, 2]$ with length m and, for each $p_1, p_2, p_3 \in {}^*P(m)$ with $||p_i|| < 2$ and each $l \leq m$,

(i) if $||Ab(p_3) - (Ab(p_1) + Ab(p_2))|| < 2^{-l}$, then $|\sigma(p_3) - (\sigma(p_1) + \sigma(p_2))| < 6 \cdot 2^{-l}$;

(ii) if $||Ab(L(p_1, s)) - Ab(p_2)|| < 2^{-l}$ with $s \in {}^*G(m)$, then $|\sigma(p_1) - \sigma(p_2)| < 5 \cdot 2^{-l}$;

(iii) if $||Ab(rp_1) - Ab(p_2)|| < 2^{-l}$ with $r \in {}^*\mathbf{Q}_{>0}(m)$, then $|r\sigma(p_1) - \sigma(p_2)| < (r + 5)2^{-l}$;

(iv) $\sigma(1) = 1$.

Similarly to the proof of Lemma 1, the following claim can be shown in $\mathbf{WKL_0}$: Given any $g \in C(\widehat{G})^+$ and any finite sequence $\langle f_i : i \leq n \rangle$ from $C(\widehat{G})^+$, for each $\varepsilon \in \mathbf{R}_{>0}$, there exists a rational sequence $\langle q_i : i \leq n \rangle$ such that $q_i = (f_i : g)^\varepsilon$ for each $i \leq n$.

Fix any $m \in M$. Letting $C = \{Ab(p) : p \in P(m) \text{ and } ||p|| < 2\}$ and $\varepsilon = 2^{-m}$ in Lemma 4, we take g and ε' to satisfy the assertion of the lemma. Then there exists a finite sequence σ of length m such that $\sigma(p) = I_g^{\varepsilon'}(Ab(p))$ if $p \in P(m)$ and $||p|| < 2$, and $\sigma(p) = 1$ otherwise. We shall see that σ satisfies $\varphi(\sigma, m)$.

To show that the condition (i) of $\varphi(\sigma, m)$ holds, assume that $p_1, p_2, p_3 \in {}^*P(m)$ with $||p_i|| < 2$ and $||Ab(p_3) - (Ab(p_1) + Ab(p_2))|| < 2^{-l}$ $(l \leq m)$. By Lemma 6, the same assertion holds in V. Then using Lemma 4, the following two inequalities hold in V:

$$|I_g^{\varepsilon'}(Ab(p_1) + Ab(p_2)) - (I_g^{\varepsilon'}(Ab(p_1)) + I_g^{\varepsilon'}(Ab(p_2)))| < 2^{-m},$$

$$|I_g^{\varepsilon'}(Ab(p_3)) - I_g^{\varepsilon'}(Ab(p_1) + Ab(p_2))| < 2^{-l} + 2^{-m+2}.$$

Then,

$$|I_g^{\varepsilon'}(Ab(p_3)) - (I_g^{\varepsilon'}(Ab(p_1)) + I_g^{\varepsilon'}(Ab(p_2)))| < 6 \cdot 2^{-l}.$$

Again by Lemma 6, the last inequality holds in ${}^*V$. Similarly, (ii) to (iv) hold.

Therefore, for each $m \in M$,

$${}^*V \models \exists \sigma \varphi(\sigma, m).$$

By overspill, there exists $\gamma \in {}^*M \setminus M$ and σ_0 such that $\varphi(\sigma_0, \gamma)$ holds in ${}^*V$.

Let $I = \sigma_0 \lceil M$. Since ${}^*[0,2]$ is regarded as a set of ${}^*V$-finite 0-1 sequences, I is a $[0,2]$-valued function defined all over P. $I(1) = 1$ is trivial. By the definition of φ and Lemma 6, for any positive $p_1, p_2, p_3 \in P$ with $||p_i|| < 2$ and each $l \in M$,

(a) if $||p_3 - (p_1 + p_2)|| < 2^{-l}$, then $|I(p_3) - (I(p_1) + I(p_2))| < 6 \cdot 2^{-l}$;

(b) if $||p_1 - p_2^s|| < 2^{-l}$ with $s \in G$, then $|I(p_1) - I(p_2)| < 5 \cdot 2^{-l}$;

(c) if $||rp_1 - p_2|| < 2^{-l}$ with $r \in \mathbf{Q}_{>0}$, then $|rI(p_1) - I(p_2)| < (r+5)2^{-l}$.

Fix $f \in C(\widehat{G})^+$ with $||f|| \leq 1$. Since f can be taken as a sequence $\langle p_i : i \in M \rangle$ of positive polynomials with $||p_i|| < 2$, $\lim_i I(p_i) = \langle I(p_{i+4})_{i+4} : i \in M \rangle$ is a real by (a). The other properties of I can be easily shown by (a) to (c). $\square$

From the above result, we can easily obtain the following theorem.

Theorem 8 ($\mathbf{WKL_0}$) *Any compact group has a Haar measure.*

Proof. Fix a countable non-standard model V of $\mathbf{WKL_0}$ and fix a compact group $\widehat{G}$ (in V). For each $f \in C(\widehat{G})$, $f^+ = \max(f, 0)$ (resp. $f^- = -\min(f, 0)$) is a non-negative point of $C(\widehat{G})$. Moreover, we can obtain functions: $f \mapsto f^+$, $f \mapsto f^-$. Therefore, using I of Lemma 7, we can define a left-invariant measure $\mu : P \to \mathbf{R}$ as follows:

$$\mu(f) = (||f|| + 1)(I(\frac{f^+}{||f|| + 1}) - I(\frac{f^-}{||f|| + 1})).$$

The countable additivity of a measure is provable in $\mathbf{WWKL_0}$ which is a weaker subsystem of $\mathbf{WKL_0}$ [11]. Thus the proof is completed.□

Now, we have our main results.

Theorem 9 ($\mathbf{RCA_0}$) *The following assertions are pairwise equivalent.*

(1) $\mathbf{WKL_0}$*;*

(2) *Any compact group has a Haar measure.*

Proof. Since (1)→(2) is shown by Theorem 8, we only need to prove (2)→(1). We reason in $\mathbf{RCA_0}$. Deny $\mathbf{WKL_0}$ and let $T \subset 2^{<\mathbf{N}}$ be an infinite tree with no path.

Let $G = \{\sigma \in T : \sigma = \emptyset \vee (\sigma \neq \emptyset \to \sigma(lh(\sigma) - 1) = 1)\}$ and $\widehat{G}$ be a complete separable metric space coded by G and d_{Seq_2}. (d_{Seq_2} was defined at the end of the last section.) If there were $x \in \widehat{G} \setminus G$, x would be a path through T. So, $\widehat{G}$=G. Take a bijection $h : \mathbf{Z} \to G$. Then it induces a group operation on G. Since $\widehat{G}$=G, the operation must be continuous on $\widehat{G}$. For each $n \in \mathbf{N}$, $G_n = \{\sigma \in G : \mathrm{lh}(\sigma) \leq n\}$ is 2^{-n}-net. Hence, $\widehat{G}$ is a compact group.

Now, if $\widehat{G}$ had a Haar measure μ, $\mu(\widehat{G}) = 0$ in the case that $\mu(\{e\}) = 0$, and $\mu(\widehat{G}) = \infty$ otherwise, either of which is a contradiction. So $\widehat{G}$ does not possess a Haar measure. □

5 Some variations

In this section, we eliminate the non-standard argument from the previous construction of Haar measure to obtain some refined assertion. Among others, we show within $\mathrm{RCA_0}$ the existence of a left-invariant measure on a compact group with a modulus of uniform continuity for its operations.

Lemma 10 (RCA$_0$) *Let $\widehat{G}$ be a compact group with a modulus of uniform continuity. Given $f \in C(\widehat{G})^+$ and $\varepsilon \in \mathbf{Q}_{>0}$, we can effectively find $c > 0$ such that if a positive $p \in P$ with $||p|| = 1$ is small of order c, there exist two finite sequences $\langle c_i : i \leq k\rangle$ and $\langle d_i : i \leq k\rangle$ of nonnegative rationals and a sequence $\langle s_i : i \leq k\rangle$ from G such that*

$$\max(||f - \sum_{i=0}^{k} c_i p^{s_i}||, ||1 - \sum_{i=0}^{k} d_i p^{s_i}||) < \varepsilon.$$

Proof. Fix any $f \in C(\widehat{G})^+$ and $\varepsilon \in \mathbf{Q}_{>0}$. We may suppose that $\varepsilon < \frac{1}{2}$. Then we can effectively find $c > 0$ such that $d(y^{-1}x, e) \leq c \to |f(x) - f(y)| \leq \frac{\varepsilon}{2}$. We shall see that this c satisfies the condition of lemma.

Assume that a positive $p \in P$ with $||p|| = 1$ is small of order c. Let $\tilde{p}(x) = p(x^{-1})$ and $\eta < \frac{\varepsilon}{6[(f : \tilde{p})^1 + 1]}$. Sinnce p is uniformly continuous, $c' > 0$ such that $d(y^{-1}x, e) \leq c' \to |p(x) - p(y)| \leq \eta$. We take a $\frac{c'}{2}$-net $\langle s_i : i \leq m\rangle$ from G and $h_0, \ldots, h_m \in C(\widehat{G})^+$ such that $\sum_{i=0}^m h_i = 1$, and $h_i(x) = 0$ wherever $c' \leq d(s_i x, e)$ (Lemma II.7.3 [7]. Then, for each $x, s \in \widehat{G}$,

$$\begin{aligned} h_i(s)f(s)(p(s^{-1}x) - \eta) &\leq h_i(s)f(s)p(s_i x) \\ &\leq h_i(s)f(s)(p(s^{-1}x) + \eta). \end{aligned}$$

Since p is small of order c,

$$(f(x) - \frac{\varepsilon}{2})p(s^{-1}x) \leq f(s)p(s^{-1}x) \leq (f(x) + \frac{\varepsilon}{2})p(s^{-1}x).$$

Therefore,

$$\begin{aligned} (f(x) - \frac{\varepsilon}{2})p(s^{-1}x) - \eta f(s) &\leq f(s)(p(s^{-1}x) - \eta) \\ &\leq \sum_{i=0}^{m} h_i(s)f(s)p(s_i x) \\ &\leq \sum_{i=0}^{m} h_i(s)f(s)(p(s^{-1}x) + \eta) \\ &\leq (f(x) + \frac{\varepsilon}{2})p(s^{-1}x) + \eta f(s). \end{aligned}$$

Using Lemma 2, for each positive $p' \in P$ and each $\varepsilon' > 0$,

$$(f(x) - \frac{\varepsilon}{2})(\tilde{p} : p')^{\varepsilon'} - \eta(f : p')^{\varepsilon'} \leq (\sum_{i=0}^{m} p(s_i x)h_i f : p')^{\varepsilon'} + (||f|| + 8 + \eta + \frac{\varepsilon}{2})\varepsilon'.$$

Similarly,

$$(\sum_{i=0}^{m} p(s_i x) h_i f : p')^{\varepsilon'} \le (f(x) + \frac{\varepsilon}{2})(\tilde{p} : p')^{\varepsilon'} + \eta(f : p')^{\varepsilon'} + (||f|| + 8 + \eta + \frac{\varepsilon}{2})\varepsilon'.$$

Therefore,

$$\begin{aligned}(1)\ f(x) - \frac{\varepsilon}{2} - \frac{\eta(f : p')^{\varepsilon'}}{(\tilde{p} : p')^{\varepsilon'}} - \frac{M_1 \varepsilon'}{(\tilde{p} : p')^{\varepsilon'}} \\ \le \frac{1}{(\tilde{p} : p')^{\varepsilon'}}(\sum_{i=0}^{m} p(s_i x) h_i f : p')^{\varepsilon'} \\ \le f(x) + \frac{\varepsilon}{2} + \frac{\eta(f : p')^{\varepsilon'}}{(\tilde{p} : p')^{\varepsilon'}} + \frac{M_1 \varepsilon'}{(\tilde{p} : p')^{\varepsilon'}},\end{aligned}$$

where $M_1 = ||f|| + 9$. Again using Lemma 2, since $0 \le p(s_i x) \le 1$,

$$(2)\ I_{p'}^{\varepsilon'}(\sum_{i=0}^{m} p(s_i x) h_i f) \le \sum_{i=0}^{m} p(s_i x) I_{p'}^{\varepsilon'}(h_i f) + \frac{(3m+2)\varepsilon'}{(1 : p')^{\varepsilon'}}.$$

By (1) and (2),

$$(3)\ f(x) - \frac{\varepsilon}{2} - \frac{\eta(f : p')^{\varepsilon'}}{(\tilde{p} : p')^{\varepsilon'}} - \frac{M_1 \varepsilon'}{(\tilde{p} : p')^{\varepsilon'}} \le \frac{1}{I_{p'}^{\varepsilon'}(\tilde{p})} \sum_{i=0}^{m} p(s_i x) I_{p'}^{\varepsilon'}(h_i f) + \frac{(3m+2)\varepsilon'}{(\tilde{p} : p')^{\varepsilon'}}.$$

By Lemma 3, we take a sufficiently small ε' and a positive polynomial p with $||p|| = 1$ such that

$$(4)\ \sum_{i=0}^{m} p(s_i x) I_{p'}^{\varepsilon'}(h_i f) \le I_{p'}^{\varepsilon'}(\sum_{i=0}^{m} p(s_i x) h_i f) + \frac{\varepsilon}{6[(1 : \tilde{p})^1 + 1]}.$$

By (1) and (4),

$$(5)\ \frac{1}{I_{p'}^{\varepsilon'}(\tilde{p})} \sum_{i=0}^{m} I_{p'}^{\varepsilon'}(h_i f) p(s_i x) \le f(x) + \frac{\varepsilon}{2} + \frac{\eta(f : p')^{\varepsilon'}}{(\tilde{p} : p')^{\varepsilon'}} + \frac{M_1 \varepsilon'}{(\tilde{p} : p')^{\varepsilon'}} + \frac{\varepsilon}{6[(1 : \tilde{p})^1 + 1] I_{p'}^{\varepsilon'}(\tilde{p})}.$$

We may suppose that $\varepsilon' < \dfrac{\varepsilon}{12(M_1 + 3m + 2)}$.

Since $\dfrac{(f : p')^{\varepsilon'}}{(\tilde{p} : p')^{\varepsilon'}} \le (f : \tilde{p})^1 + 1$ and $\dfrac{(1 : p')^{\varepsilon'}}{(\tilde{p} : p')^{\varepsilon'}} \le (1 : \tilde{p})^1 + 1$ by Lemma 2,

$$|f(x) - \sum_{i=0}^{m} \frac{I_{p'}^{\varepsilon'}(h_i f)}{I_{p'}^{\varepsilon'}(\tilde{p})} p(s_i x)| < \varepsilon,$$

for all $x \in \widehat{G}$. Similarly, $||1 - \sum_{i=0}^{m} d_i p^{s_i}|| < \varepsilon$ for some $\langle d_i \rangle$. $\square$

Let $\dot{P} = \{\langle\langle q_i, b_i\rangle : i \leq m\rangle \in P : b_i$'s are basic functions and $q_i \in \mathbf{Q}_{\geq 0}\}$. Then we define a Σ^0_1-formula $\varphi(p, b, s, k, \sigma, \tau)$ to denote that $p \in \dot{P}$ and b is a basic function, two sequences σ, τ from $\mathbf{Q_{>0}}$ and one sequence s from G have all the same length and

$$\max(||p - \sum \sigma(i)b^{s(i)}||, ||1 - \sum \tau(i)b^{s(i)}||) < 2^{-k}.$$

Since $\forall p \in \dot{P}\forall k\exists b, s, \sigma, \tau\varphi(p, b, s, k, \sigma, \tau)$ by Lemma 10, there exists a function $F_1 : \dot{P} \times \mathbf{N} \to \mathbf{Q}$ such that

$$F_1 : \langle p, k\rangle \mapsto \frac{\sum_{i=0}^n c_i}{\sum_{i=0}^n d_i} \text{ such that } \varphi(p, b, s, k, \langle c_i : i \leq n\rangle, \langle d_i : i \leq n\rangle) \text{ holds for some } b, s.$$

We write $t^p_k = F_1(\langle p, k\rangle)$. Let F_2 be a function from $\dot{P} \times \mathbf{N}$ to $\mathbf{Q}$ such that

$$\langle p, k\rangle \mapsto t^p_{k+m_p+1},$$

where m_p is a number given effectively by $p \in \dot{P}$ such that $||p|| + 3 \leq 2^{m_p}$. The following lemma can be shown easily.

Lemma 11 (RCA$_0$) *For each $\varepsilon > 0$ and each $k \in \mathbf{N}$, there exists $c > 0$ such that if a basic function b is small of order c, then $|I^{\varepsilon'}_b(p) - t^p_{k+m_p+1}| \leq 2^{-k} + \varepsilon$ for each sufficiently small $\varepsilon' > 0$.*

Theorem 12 (RCA$_0$) *Let $\widehat{G}$ be a compact group with a modulus of uniform continuity. Then there exists a unique left-invarant measure on $\widehat{G}$.*

Proof. By Lemma 11, for each n, j $(n \leq j)$ and each $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that $|I^{\varepsilon'}_b(p) - t^p_{n+m_p+1}| \leq 2^{-n-1} + \varepsilon$ and $|I^{\varepsilon'}_b(p) - t^p_{j+m_p+1}| \leq 2^{-j-1} + \varepsilon$. Then, $|t^p_{n+m_p+1} - t^p_{j+m_p+1}| \leq 2^{-n} + 2\varepsilon$. Since ε is at random, $|t^p_{n+m_p+1} - t^p_{j+m_p+1}| \leq 2^{-n}$. Hence, $\langle F_2(p, k) : k \in \mathbf{N}\rangle$ is real. We define $F : \dot{P} \to \mathbf{R}$ by $F(p) = \langle F_2(p, k) : k \in \mathbf{N}\rangle$.

Since, for each $p \in P$, p can be expressed by $\langle\langle q_i, b_i\rangle : i \leq m\rangle$ such that $\forall i \leq l(0 \leq q_i)$ and $\forall i > l(q_i < 0)$, we define

$$\mu(p) = F(\langle\langle q_i, b_i\rangle : i \leq k\rangle) - F(\langle\langle -q_i, b_i\rangle : k < i \leq m\rangle).$$

Then μ is a left-invariant measure by Lemma 11.

Let μ' be another left-invariant measure. Fix any $p \in \dot{P}$ and any $k > 1$. We take finite sequences $\langle c_i \rangle$ and $\langle d_i \rangle$ of positive rationals such that $t_k^p = \frac{\sum_{i=0}^n c_i}{\sum_{i=0}^n d_i}$ with $\max(||p - \sum_i c_i b_p^{s_i}||, ||1 - \sum_i d_i b_p^{s_i}||) < 2^{-k}$. Then, $|\mu'(p) - \sum_i c_i \mu''(b_p)| < 2^{-k}$ and $|1 - \sum_i d_i \mu'(b_p)| < 2^{-k}$. Therefore,

$$t_k = \frac{\sum_{i=0}^n c_i}{\sum_{i=0}^n d_i} < \frac{\mu'(p) + 2^{-k}}{\mu'(b_p) \sum_{i=0}^n d_i} < \frac{\mu'(p) + 2^{-k}}{1 - 2^{-k}}.$$

Similarly, $\frac{\mu'(p) - 2^{-k}}{1 + 2^{-k}} < \frac{\sum_{i=0}^n c_i}{\sum_{i=0}^n d_i} = t_k$. Since $\mu(p) = \lim_k t_k$, $\mu'(p) = \mu(p)$. Then $\widehat{G}$ has a unique invariant-measure. □

By Theorem 1 in [12], Theorem 12 leads to the following corollary.

Corollary 13 (RCA$_0$) *the following assertions are pairwise equivalent.*

(1) WWKL$_0$*;*

(2) *Any compact group whose operations have a modulus of uniform continuity has a unique Haar measure.*

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