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Extending Knowledge and 'Fruitful Concepts': Fregean Themes in the Foundations of Mathematics[1]

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As they are currently done, mathematical research and philosophical foundations of mathematics have little contact with one another; rarely does work in one area affect work in the other. It is natural to ask: is this to be embraced as a sign that mathematical and philosophical work are finally in the distinct spheres it is in their natures to occupy? Or is it to be deplored as a mark of stagnation—a sign that philosophical reflection on mathematics has become disengaged from the mathematical research that gives it life and substance?

This paper strives for a better sense of the issue by looking back to the days when the separation was less pronounced. The paper begins by developing a puzzling feature of Frege's *Grundlagen:* his notion of "fruitful" (*fruchtbar*) definitions, concepts and principles, which turns out to be bound up with his understanding of logical structure and its relation to mathematical structure, his early view of "extending knowledge" and aspects of the mathematical setting in which he worked. The picture of Frege that emerges contains a moral for current philosophical study of mathematics. We appear to have arrived at a stultifyingly narrow view of the scope and objectives of foundations of mathematics, a view we read back into Frege as if it could not but be Frege's own. In the final sections of the paper I attempt to draw out the independent interest of Frege's richer conception of foundational activity.

I What are Fruitful Definitions and why do they Matter to Frege?

The crucial remarks (henceforth called the "focal passage") occur in *Grundlagen* #88:

> [Kant] seems to think of concepts as defined by giving a simple list of characteristics in no special order; but of all ways of forming concepts, that is one of the least

> fruitful. If we look through the definitions given in the course of this book, we shall scarcely find one that is of this description. The same is true of the really fruitful definitions in mathematics, such as that of the continuity of a function. What we find in these is not a simple list of characteristics; every element is intimately, I might almost say organically, connected with the others...If we represent the concepts (or their extensions) by figures or areas in a plane, then the concept defined by a simple list of characteristics corresponds to the area common to all the areas representing the defining characteristics; it is enclosed by segments of their boundary lines. With a definition like this...we...use the lines already given in a new way for demarcating an area. [note: Similarly, if the characteristics are joined by "or".] Nothing essentially new, however, emerges in the process. But the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. What we shall be able to infer from it, cannot be inspected in advance; here we are not simply taking out of the box again what we have just put into it. The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic.(p. 100–101 [FA])

It is not obvious what these comments should be taken to mean. How are we to understand the contrast between lists of characteristics and properties with organic connections and why is continuity one of the latter? What are we to make of the metaphor of 'drawing new boundaries' and how does it fit with the analytic/synthetic contrast? How is it that most definitions in *Grundlagen* and some mathematical concepts underwrite judgments that are both analytic and ampliative in virtue of being fruitful in this lyrically specified way?

The issue is especially serious because the focal passage ties the fruitfulness of concepts/definitions to a central point of *Grundlagen*. Frege says that it is because of fruitful concepts that new knowledge is potentially available from analytic judgments. Indeed, the passage suggests that *only* via "drawing new boundary lines" can "anything essentially new [emerge]." This is the only place—prior to the introduction of the notion of sense—where Frege confronts the question of how analytic judgments extend knowledge. And outside *Grundlagen* there is only one place the issue is even mentioned: the beginning of [SM], where the *Begriffsschrift* is criticized.[2]

Frege speaks of fruitful definitions/concepts/claims in the unpublished essay [BLC] written between 1880 and 1881 and in three other places in *Grundlagen:*

> [I too believe] that definitions must show their worth by their fruitfulness: it must be possible to use them for constructing proofs. ([FA] p. ix)[3]

> *Our definition completed and its worth proved*
> #70. Definitions show their worth by proving fruitful. Those that could just as well be omitted and leave no link missing in the chain of our proofs should be rejected as completely worthless.

Let us try, therefore, whether we can derive from our definition of the number which belongs to the concept F any of the well-known properties of numbers. ([FA] p.81)

All identities would then amount to simply this, that whatever is given to us in the same way is to be reckoned as the same. This, however, is a principle so obvious and so [unfruitful] as not to be worth stating. We could not, in fact, draw from it any conclusion which was not the same as one of our premisses.([FA] p.79) [4]

Although "fruitful" lacks a technical term's sharpness of definition, these remarks (plus, as we will see, the use of "fruitful" in [BLC]) indicate that the expression has a stable use. The remarks reiterate the focal passage claim that fruitful definitions are distinctively valuable and that fruitfulness is connected to the potential for extending knowledge. Also ([FA] p.ix) and ([FA] p.81) have been the focus of a broader dispute over Frege's objectives.

The thesis of Paul Benacerraf [1981] comes in mild and extra spicy flavors. The mild thesis defended here is that in addition to the philosophical aims for *Grundlagen,* Frege had mathematical objectives which interact with the philosophical ones to contribute a unique richness to the work. The extra-spicy thesis, apparently to Benacerraf's taste, is that *Grundlagen* is a philosophical work only to the extent that philosophical problems have been reconstrued so the mathematical results Frege seeks will answer them.

Joan Weiner, in [1984] and [1990] opposes even the mild thesis, arguing that the motivation for *Grundlagen* was *solely* philosophical. Frege may not have taken it to be, but, she suggests, what he took to be motives consonant with the mathematical trends of his day were not in fact mathematical motives.[5]

Both Weiner and Benacerraf consider the term 'fruitful' specifically as applied to *definitions,* as a fulcrum. Benacerraf takes the apparent requirement that definitions be fruitful to support his view of Frege as pursuing a more mathematical project than twentieth-century efforts at logicist reduction; Frege appears to say fruitful definitions are those supporting mathematical proofs, and to require that definitions be fruitful. (One would expect—and Frege confirms—that he takes *Grundlagen*'s definition of number to be fruitful.[6]) *Grundlagen* "definitions are not *simply* conventions of abbreviation, for if they were the requirement of fruitfulness...would make little sense" (Benacerraf [1981] p.28) Weiner agrees on the conditional, but maintains definitions *are* just conventions of abbreviation, so the fruitfulness requirement must be trivial. On her view a definition is fruitful if it *can* be used in a proof and the passages ([FA] p.ix, p.81) aim only to separate definitions from illustrative examples.[7]

One of Weiner's arguments turns on the focal passage's example of continuity. She takes Frege to mean Cauchy's definition (f(x) is continuous at a iff $\lim_{x \to a} f(x) = f(a)$)[8] and argues that non-trivial fruitfulness cannot be a necessary condition because any occurrence of "f(x) is continuous at a" may be replaced by "$\lim_{x \to a} f(x) = f(a)$".[9] ([FA] p. 81) is then read in this light: "...the proofs in which the notion of continuity figures do depend on this definition. If the definition were omitted without any restatement of the relevant theorems, there would be gaps in all such proofs."(Weiner [1984] p.67)

Weiner's challenge to display the more substantive conception Frege intends is pressing, and facing it will teach us a few things about what Frege was trying to do. But her interpretation of fruitfulness is untenable for the simple reason that on it, the focal passage just makes no sense.[10] If fruitfulness only separates definitions from examples, definitions by lists should be no less fruitful than those displaying "organic connections". We have not yet figured out what "more fruitful" definitions are, but we know what lists of characteristics are, and they obviously can pass the muster of the trivial condition. Also the trivial notion will not support a distinction between analytic judgments which do, and do not, extend knowledge.[11]

The key to the metaphors in the focal passage is [BLC], which is largely devoted to a list of concepts (including continuity) explicitly marked out as fruitful. Frege uses the images of the focal passage in a way that makes it apparent they mark a distinction between definitions essentially involving (as we would now put it) Boolean combinations of properties and those involving Boolean combinations *and* quantifiers.[12] Of particular interest is p. 33–35 of [BLC]. [13] Of the list of definitions, he writes:

> All these concepts have been developed in science and have proved their fruitfulness. For this reason what we may discover in them has a far higher claim on our attention than anything that our everyday trains of thought might offer. For fruitfulness is the acid test of concepts, and scientific workshops the true field of study for logic. ([BLC] p.33)

Frege then considers Boolean combinations of properties, representing intersection and union by means of (what we now call) Venn diagrams. He remarks:

> If we look at what we have in the diagrams we notice that the boundary of the concept whether it is formed by [intersection or union] is made up of parts of the boundaries of the concepts already given. This holds for any concept formation that can be represented by the Boolean notation. ... In this sort of concept formation, then one must assume as given a system of concepts or, speaking metaphorically, a network of lines. These really already contain the new concepts: all one has to do is to use the lines that are already there to demarcate complete surface areas in a new way. ([BLC] p.33–34)

This so closely matches the focal passage in its central metaphors that one may be sure both express the same ideas. Here too, the potential for advances in knowledge is tied in:

> It is the fact that attention is principally given to this sort of formation of new concepts from old ones, while other more fruitful ones are neglected which surely is

responsible for the impression one easily gets in logic that for all our to-ing and fro-ing we never really leave the same spot. ([BLC] p.34)

But what contrast is being drawn? Now that we know what the unfruitful notions are, what are the fruitful ones? In the next paragraph it becomes clear:

> If we compare what we have here with the definitions contained in our examples, of the continuity of a function and of a limit [and the ancestral of a relation], we see that there's no question there of using the boundary lines of concepts we already have to form the boundaries of the new ones. Rather, totally new boundary lines are drawn by such definitions—and these are the scientifically fruitful ones. Here too, we use old concepts to construct new ones, but in so doing we combine the old ones together in a variety of ways by means of the signs for generality, negation, and the conditional. ([BLC] p.34)

Frege identifies quantifiers as the crux of the matter in the same passage: "[Boole's system is inadequate because] the lack of a representation of generality corresponding to mine would make a true concept formation—one that didn't use already existing boundary lines—impossible." ([BLC] p.35) Reconsider continuity in this light. In [BLC] Frege clearly means Weierstrass' definition, not Cauchy's: f is *continuous* at a if $(\epsilon > 0)(E\delta > 0)(x)(\text{if } |x\text{-}a| < \delta \text{ then } |f(x)\text{-}f(a)| < \epsilon)$.[14] This definition contributes explicit logical structure to proofs in a way Cauchy's does not. To illustrate, another quantifier-laden definition is useful: a sequence of functions $\{f_i(x)\}_{1 \leq i < \infty}$ *converges uniformly* to f(x) on an interval U if $(\epsilon>0)(EN)(x \text{ in } U)(n)(\text{if } n > N \text{ then } |f(x)\text{-}f_n(x)| < \epsilon)$.[15] A key theorem is due to Weierstrass: if $\{f_i(x)\}_{1 \leq i < \infty}$ is a uniformly convergent sequence of continuous functions (on U), and f(x) is the limit of $\{f_i(x)\}_{1 \leq i < \infty}$ then f(x) is continuous (on U).

This is the kind of result Frege had in mind when describing the definition of continuity as fruitful, as a closer look at [BLC] reveals.[16] He explicitly renders not only the Weierstrass definition of continuity into *Begriffsschrift* notation but also that of uniform convergence.[17] The Weierstrass theorem follows by simple arithmetic and quantifier inferences from these definitions, so it is a prime example of what is said to extend knowledge in the focal passage. That is, the definition of continuity is fruitful because some of the inferences involving it extend knowledge in virtue of the explicit quantifier structure the definition incorporates.

Of course for Kant, the fact that mathematical judgements can extend knowledge was a linchpin in his argument that they are founded in intuition. So if we show that the Weierstrass theorem is a logical result, we face Kant's question in a new form: how can theorems like this be the extensions of knowledge they clearly are, if they are not based on intuition? Though Frege directly addresses this question only with the terse remarks of the focal passage, the passage points to an extended discussion with much detail scattered throughout his writings.

Frege did not hold that the Weierstrass definition is creative, nor did he deny that in some sense it is a "mere abbreviation". But this should not incline us to a trivial interpretation: some punch is packed into the phrase "extends knowledge". To extend knowledge a logical proof has to involve quantifier inferences. Definitions contributing such quantifier structure to proofs that extend knowledge are "more fruitful" in the sense of the focal passage.

Though it is useful for exposition to focus on alternating quantifiers as in the Weierstrass theorem, reasoning that "extends knowledge" may turn on the use of one or more universal quantifiers either explicitly or implicitly as in an open sentence with free variables.[18] Dummett in effect notes this in his illuminating account of Frege's view of the value of deduction as exploiting 'new patterns' discerned by[19] replacing names with variables, thus decomposing a given content in different ways. [20] In [BLC] Frege develops this example:

> I only allow the formation of concepts to proceed from judgments. If, that is, you imagine the 2 in the content of the possible judgment $2^4 = 16$ to be replaceable by something else...which may be indicated by putting an x in place of the 2:
>
> $x^4 = 16$
>
> the content of possible judgment is thus split into a constant and a variable part. [This gives] the concept "4th root of 16" ...[If we replace the 4] we get the concept 'logarithm of 16 to the base 2':
>
> $2^x = 16$
>
> ....We may also regard the 16 in $x^4 = 16$ as replaceable in its turn, which we may represent, say, by $x^4 = y$. In this way we arrive at the concept of a relation, namely the relation of a number to its 4th power. ([BLC] p.16–17)

The idea of decomposition of contents incorporated in Frege's treatment of quantification has a philosophical edge in connection with the idea of "extending knowledge". Recall that in Kant's treatment of arithmetic, judgements were ampliative if they were supported by something—intuition—outside the concepts in the judgements themselves. So too on Frege's account, but the appeal to intuition drops away: a logical inference yields "new knowledge" if the concept attained by dissection is grafted to a fragment of some other content. The additional content plays the role of the "something extra" represented by intuition in Kant's scheme. Frege nods to how quantifier inferences which "extend knowledge" involve the decomposition of contents and the construction of new ones in two places in *Grundlagen.* In one he asks: "[If arithmetic is logic] how do the empty forms of logic come to disgorge so rich a content?"([FA] p.22). He initiates the response with:

> However much we disparage deduction, it cannot be denied that the laws established by induction are not enough. New propositions must be derived from them

> *which are not contained in any one of them by itself.* No doubt these propositions are in a way contained covertly in the whole set taken together, but this does not absolve us from the labor of actually extracting them and setting them out in their own right." ([FA] p. 23) (my italics)

A *single* proposition does not alone suffice, presumably because different starting points must be "pulled apart" to get different "radicals" to fit together. The lines just after the focal passage similarly stress "more than one":

> The truth is [the conclusions drawn from fruitful definitions] are contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house. *Often we need several definitions for the proof of some proposition, which consequently is not contained in any one of them alone, yet does follow purely logically from all of them together.* ([FA] p. 101) (my italics)

When Frege takes up the question "How do the empty forms of logic come to disgorge so rich a content?", he defers an answer until the focal passage by restructuring the question into that of how analytic judgements can give valuable knowledge. [21] The claim is not merely that general truths make reasoning easier for beings with psychological limitations like ours. In connection with the fact that a conditional can represent a deduction he remarks such general propositions are valuable because they present information structured in logical order:

> It is not impossible that the laws of number are of this type. This would make them analytic judgments, despite the fact that they would not normally be discovered by thought alone; for we are concerned here not with the way in which they are discovered but with the kind of ground on which their proof rests; or in Leibniz's words, "The question here is not one of the history of our discoveries, which is different in different men, but of the connection and natural order of truths, which is always the same." ([FA] p.23)

In [BLC] as in *Grundlagen,* concepts, definitions and principles are described as fruitful. I run these together here because Frege does: in particular, in the focal passage he uses "fruitful definition", "fruitful concept" and cognates interchangeably, in a way that makes clear the fruitful definitions are definitions of the fruitful concepts. [22] However, on Frege's account of content, the difference between concepts and definitions is not so great as one might otherwise suppose. Concepts, in the *Begriffsschrift* and [BLC] are arrived at by decomposing the content of judgments. The conceptual content of the sentence expressing a judgment is given by the logically valid inferences that sentence can enter into. In a logically perfect language, the logical grammar of a sentence is laid out in the syntax, so a concept defined within such a language is fixed by the logical consequences of the sentences containing the defined expression. One can get the flavor of Frege's early stance from:

> But arithmetic in the broadest sense also forms concepts—and concepts of such richness and fineness in their internal structure that in perhaps no other science are they to be found combined with the same logical perfection. And there are other judgments which arithmetic makes, besides mere equations and inequalities. [BLC] p.13

The rest of the passage makes clear that this is not just an elliptical manner of speaking: "concept formation" concerns the logical structure of the language of a given science. [23]

II Content and "Extending Knowledge"

As noted above, in the *Begriffsschrift* and *Grundlagen,* content is individuated by inferential consequence: [24]

> ...the contents of two judgments may differ in two ways: either the consequences derivable from the first, when it is combined with certain other judgments, always follow from the second [and conversely] or this is not the case. The two propositions "The Greeks defeated the Persians at Platea." and "The Persians were defeated by the Greeks at Platea." differ in the first way...I call that part of the content which is the *same* in both the *conceptual content.* [It] alone is of significance for our ideography...In a judgment I consider only that which influences its *possible consequences.* Everything necessary for a correct inference is expressed in full...; *nothing is left to guesswork* ([B] p.12 Italics his)

What is it to "leave...to guesswork"? Remarks in [BLC] and an early article explaining the *Begriffsschrift* cast light:

> [Unlike Boole] I had in mind *the expression of a content.* What I am striving after is a *lingua characterica* in the first instance for mathematics, not a *calculus* restricted to pure logic. But the content is to be rendered more exactly than is done by verbal language. For that leaves a great deal to guesswork, if only of the most elementary kind. There is only an imperfect correspondence between the way words are concatenated and the structure of the concepts. The words 'lifeboat' and 'deathbed' are similarly constructed though the logical relations of the constituents are different. So the latter isn't expressed at all, but is left to guesswork. ([BLC] p.12—13 italics his)[25]

> A strictly defined group of modes of inference is simply not present in [ordinary] language, so that on the basis of linguistic form we cannot distinguish between a "gapless" advance [in the argument] and an omission of connecting links....In [ordinary] language, logical relations are almost always only hinted at—left to guessing, not actually expressed. [SJCN] (p.85)

To express a content in a way that "avoids guesswork" is to display explicitly in the syntax of a sentence the potential the sentence has for supporting logical inferences. "Avoiding guesswork" requires a finer partition than the coarse notion of content gives, so it is misleading to say that content captures "all that is of significance to this ideography". Two sentences have the same content if they are logically equivalent. But the inference from a sentence S′ to another S need not be "gapless" even if S and S′ have the same content. To bring out the point, distinguish two ideas: S is an *immediate consequence* of S′ if the inference from S′ to S is "gapless", and S is an *ultimate consequence* of S′ if there is a proof of S from S′ consisting of only gapless inferences. The key notion to define content is ultimate consequence, while the treatment of "extending knowledge" exploits the finer notion of immediate consequence. Inferring one sentence from another with the same content can extend knowledge in Frege's sense if the best proof leading from one to the other involves quantifier inferences. The logical structure of a *Begriffsschrift* sentence reflects the structure of potential derivations and hence reflects the finer notion, [26] so two sentences with the same conceptual content *can* be such that inferring one from the other can do what *Grundlagen* calls "extending knowledge". This point is brought out indirectly by an insightful comment from Dummett:

> [The early account of content] has the consequence that any two analytically equivalent sentences will have the same conceptual content, a thesis completely out of accord with Frege's ideas even before he distinguished between sense and reference as is evident from the statement in *Grundlagen,* #91 that 'sentences which extend our knowledge may have analytic judgments as their contents' (Dummett [1981] p.300)

This neglects one nuance. The distinction between sense and reference need not have been independent of any changes in what counts as "extending knowledge". The thesis is "out of accord with Frege's ideas even before he distinguished between sense and reference" *if* the early writings tie content to extending knowledge as the later writings do. But they don't. The fine partition of sentences given by the under-articulated "drawing new boundaries" idea refines the coarse partition of sentences induced by the notion of conceptual content.

As several writers have noted, the notion of sense does several jobs simultaneously for Frege.[27] In particular senses/thoughts are bound up with epistemic novelty: If the inference from one sentence to another extends knowledge, the sentences have different senses. (The early notion of content does *not* do this job.) Recall (from the opening of [SM]):

> In my *Begriffsschrift* I assumed [that equality is a relation between names of objects]. The reasons which seem to favour this are the following: a=a and a=b are obviously statements of differing [knowledge value]; a=a holds *a priori* and, according to Kant, is to be labeled analytic, while statements of the form a=b often contain very valuable extensions of our knowledge and cannot always be established *a priori.* ([SM] p.157) [28]

This cuts across the corresponding categories in [FA]. Rather than "fruitful concepts/definitions" supporting transitions that "draw new boundaries" it is differences of "knowledge value" that establish which judgments extend knowledge. We all know what comes next: the notion of the sense of a sentence (a thought) is introduced to capture this difference in knowledge value. Thus we find in the concluding summation:

> If we found 'a=a' and 'a=b' to have differing [knowledge values], then the explanation is that for the purpose of acquiring knowledge, the sense of the sentence, viz., the thought expressed by it, is no less relevant than its meaning i.e. its truth-value. If now a=b, then indeed what is meant by 'b' is the same as what is meant by 'a' and hence the truth-value of 'a=b' is the same as that of 'a=a'. In spite of this, the sense of 'b' may differ from that of 'a' and thereby the thought expressed in 'a=b' differs from that of 'a=a'. In that case the two sentences do not have the same [knowledge value].([SM] p. 176–177)

This points to a reassessment of how the early and late writings coalesce. Frege rarely addresses how the early idea of content fits with sense/reference and he says nothing beyond: "I had not yet made the distinction between sense and meaning; and so, under the expression of 'a content of a possible judgment', I was combining what I now designate by...'thought' and 'truth-value'."([CO] p.187) With such words as a guide it has been widely assumed that nothing significant in the early writings depends on content in a way that precludes a fit with the later framework. Not so with the early treatment of "extending knowledge": sense and truth-value occupy territory that content leaves empty, leaving no room for a distinct "drawing new boundaries" story as the key to analytic judgments extending knowledge.

III Fruitfulness Elsewhere in Grundlagen and some Mathematical Motives

The preceding sections do not maintain that Frege puts forward the sharp notion refined from the focal passage and [BLC] as a *definition* of "fruitful". Rather he takes for granted an intuitive notion of "fruitful concept/fruitful definition" and he puts forward the sharp notion as part of a (fragmentary and incomplete) *substantive account* of the intuitive notion. The first part of this section will set aside the sharp notion and attempt to clarify how Frege understands the intuitive one. The second part develops some of the methodological significance Frege attaches to work like his account of the structure of fruitful concepts.

We get a sense of the loose, intuitive notion from the uses of "fruitful" elsewhere in *Grundlagen.* At ([FA] p.79) it is a principle which is described as unfruitful, though clearly to be unfruitful is to fail to support extensions of knowledge. ([FA] p.81) sets up the motivation for the definition of "the number belonging to F" by showing that it can be used in the proofs of theorems of antecedent interest. That Frege sees the need for this indicates fruitfulness cannot be just "not an illustrative example": that could have been established by examining the syntax. But does Frege see anything more to intuitive fruitfulness than just the brute fact that some concepts are often used in the proof of interesting theorems and others are not?[29]

One expects a richer notion of fruitfulness than that. We can assume Frege had not only logical acumen and philosophical gifts but also the judgment of the well-trained working mathematician he was about what is of value in mathematical practice. In this practice, significant conceptual advances typically involve a family of virtues we sometimes single out with praise like "seriousness" (to use G.H. Hardy's term). Though it is hard to nail down what such virtues are, the natural (perhaps naive) view is that there is more to them than just the bare fact that they generate proofs. The embroidery around the remarks at ([FA] p.ix) suggests Frege does have such a richer view:

> In this direction too I go, certainly, further than is usual. Most mathematicians rest content, in enquiries of this kind, when they have satisfied their immediate needs. If a definition shows itself tractable when used in proofs, if no contradictions are anywhere encountered, and if connections are revealed between matters apparently remote from one another, this leading to an advance in order and regularity, it is usual to regard the definition as sufficiently established, and few questions are asked as to its logical justification. This procedure at least has the advantage that it makes it difficult to miss the mark altogether. [I too believe] that definitions must show their worth by their fruitfulness: it must be possible to use them for constructing proofs. ([FA] p.ix)

Gauge carefully what Frege is and is not saying here. He decries taking a definition to be "sufficiently established" when certain mathematical virtues are displayed. But he doesn't want to say that it is bad to focus on these virtues. Rather he wants to say they are not enough. He emphasizes that they are to be sought, and indeed that pursuing them "...makes it difficult to miss the mark altogether." *Then* he adds the last sentence, which *grants* that constraining definitions by requiring that the result be an increase in order and regularity (etc.) is *not* misguided (though it does not by itself suffice).

It is hard to see how this sequence is to hang together if the last sentence just rules out illustrative examples. He is discussing a procedure whereby definitions are critically assessed. He says this procedure "makes it difficult to miss the mark altogether"—and he concurs that what it tracks is significant. The definitions he is concerned with are those which "increase order and regularity", "reveal connections between matters apparently remote from one another" etc. He is not agreeing with mathematicians (if there are any) who just lay down definitions and then use them blindly and unreflectively as if to display ability in brute combinatorics. So when Frege says "[I too believe] that definitions must be fruitful..." he is granting that the standard practice is tracking the right ideas, though not as acutely as they could be within the logical foundations he champions.

Two related perspectives on "fruitful" concepts have emerged: i) the intuitive notion involving mathematical virtues like "order", "regularity", "connections between matters apparently remote", etc. and ii) the refined notion in which quantifiers and logical structure give sharp content to the focal passage remark that "[in fruitful definitions] every element is intimately, one might almost say organically, connected with the others...". Frege is proposing the quantifier/"organic connections" story as part of an account of the structure of fruitful concepts as intuitively understood. It is important to recognize that he sees himself as actually laying bare the structure of mathematical notions. He states explicitly that one advantage of the *Begriffsschrift* is that it can "represent the formations of the concepts actually needed in science, instead of the relatively [unfruitful] multiplicative and additive combinations we find in Boole."([BLC] p.46)[30] and in *Grundlagen* he represents himself as displaying the structural complexity of the notion of number:"...the concept of number, as we shall be forced to recognize, has a finer structure than most of the concepts of the other sciences, even though it is one of the simplest in arithmetic."([FA]p.iv)

So the focal passage contains—among everything else—an attempt to marshal the *Begriffsschrift* analysis of quantification to give a logical/structural explanation of why the most fruitful mathematical notions have the fecundity they do. This is a solution to the problem of the value of deductive reasoning formulated in a particularly delicate way: if arithmetic and analysis are logic, we must explain how logical reasoning alone can support extensions of knowledge like the Weierstrass theorem. How can purely deductive reasoning be valuable *in this way*? I return to this in V, but first a bit more spadework.

To Frege, the account of fruitfulness given in terms of quantification is fully engaged with, and indeed a part of, mathematical practice. To see this, let's look again at the first two sections of *Grundlagen,* bearing in mind a key observation of Benacerraf [1981]: those sections lay out what Frege characterizes as *mathematical* motivations. He takes his work to flow naturally from recent developments:

> Proceeding along these lines, we are bound eventually to come to the concept of number and to the simplest propositions holding of positive whole numbers, which form the foundation of the whole of arithmetic. Of course, numerical formulae like 7 + 5 = 12 and laws like the associative law of addition are so amply established by the countless applications of them that it may seem almost ridiculous to try to bring them into dispute by demanding a proof...([FA] p.2)

Why pursue such an inquiry? Well, "[It] is in the nature of mathematics to prefer proof, where proof is possible."([FA] p.2) Certainly the preference for proof is characteristic of mathematical practice. But say we ask in a fit of devilish advocacy: isn't this just a pedantic affectation? What reason—mathematical or otherwise—do we have for trying to prove the obvious? Frege answers this one a paragraph later:

> The aim of proof is, in fact, not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another. After we have convinced ourselves that a boulder is immovable, by trying unsucessfully to move it, there remains the further question: what is it that supports it so securely? The further we pursue these enquiries, the fewer become the primitive truths to which we reduce everything; and this simplification is in itself a goal worth pursuing. But there may be justification for a further hope: if, by examining the simplest cases, we can bring to light what mankind has there done by instinct, and can extract from such procedures what is universally valid in them, may we not thus arrive at general methods for forming concepts and establishing principles which will be applicable also in more complicated cases? ([FA] p.2)

Even by the standards of *Grundlagen,* this paragraph is uncommonly dense. Frege gives us *three* distinct reasons to seek proofs of obvious principles. We are told first that it is important to display logical dependency relations, second that it is valuable "in itself" to reduce the number of basic principles, and finally that there is "a further hope" that general principles of concept-formation and proof, "applicable also in more complicated cases" may be exposed. The last of these avowed mathematical motives (I'll call it "*the further hope*") is of special interest here. Presumably he is hoping to bring to light some of the principles serving up what the focal passage calls "the more fruitful type of definition". The point is that throughout *Grundlagen* Frege aims not merely to define the numbers and sketch proofs of elementary arithmetic, but also to reflect critically on what these definitions and proofs reveal about mathematical reasoning, with the aim of broader mathematical applications.

We are so accustomed to thinking of philosophical investigation of mathematics as mathematically inert that we tend to "blip over" remarks like the further hope: such claims are, we feel, just idle motivational blather occuring before the real argumentation takes place. But we miss a significant aspect of Frege's ambitions if we fail to recognize that he treated the further hope very seriously. Indeed, steps toward vindicating it are taken in *Grundlagen* itself. Note these remarks in the heart of the discussion of the analogy between number and direction:

> Thus we replace the symbol // by the more generic symbol =, through removing what is specific in the content of the former and dividing it between a and b. We carve up the content in a way different from the original way, and this yields us a new concept. ([FA] p.74–75)

The mode of expression suggests Frege took the definition of number to be fruitful,[31] and this is confirmed by the focal passage claim that looking through "definitions given in the course of this book, we shall scarcely find one that is [a list of features]." Remembering the further hope, we expect Frege to apply this method of attaining new contents to more complicated cases. (The payoff for laying bare what "mankind has [in simple cases] done by instinct".) And behold:

> We can obtain in a similar way from the parallelism of planes a concept corresponding to that of direction in the case of straight lines; I have seen the name "orientation" used for this. From geometrical similarity is derived the concept of shape, so that instead of "the two triangles are similar" we say that "the two triangles are of identical shape" or "the shape of one is identical with that of the other". It is possible to derive yet another concept in this way, to which no name has yet been given, from the collineation of geometrical forms. ([FA] p. 75)

The construction in question was well-known to mathematicians by 1884,[32] and Frege himself investigated the idea that the shape of a geometrical figure ("trianglar-ness") could be represented as an object (a complex number) in [LWC]. This suggests Frege regarded definitions of the sort he discusses in the above quote as having mathematical significance. In this instance he is trying to realize the further hope in his foundational work, with the prospect of applications in non-foundational mathematics.

This helps explain why Frege thinks that by singling out the virtues involved in the intuitive notion of fruitfulness and seeking out concepts with them, mathematicians "make it difficult to miss the mark altogether". The mathematical virtues exhibited by concepts with "every element intimately/organically connected to the others" are seen as arising out of reasoning which is clearly represented in the logical structure he has uncovered. In particular, he takes himself to have unearthed the logical basis of the fruitfulness of notions like continuity. In this case, he feels, the mathematician's pursuit of order, regularity, unsuspected connections etc. and the logician's pursuit of logical structure head for the same goal.

There is a straightforward and a more subtle point at issue. The straightforward point is captured in this call and response: "Is Frege's work mathematics or philosophy? Well, we have just seen a way that the same work is both." If this were everything at issue, the point might represent a pleasant enough diversion, but perhaps not so surprising. The more subtle point is that Frege's pursuit of the further hope displays aspects central to mathematical practice that ensure the boundary between mathematics and metatheoretic reflection/ foundational research will be blurred. The point will be taken up in V and VI after a detour to give the issue more substance in connection with the mathematics to which Frege alludes.

IV Projective Duality and the Mathematical Motivations of Grundlagen

I am suggesting that the account of fruitfulness based on the way quantification involves decomposition of contents is a natural first conjecture about what underwrites valuable mathematical reasoning, given Frege's mathematical environment. A widespread opinion about Frege's historical position can make this claim appear less plausible than it is: It often seems to be assumed that if Frege's aims differ from those of prominent nineteenth century analysts (as in some ways they clearly do) it shows that Frege was in no important way like the mathematicians around him.[33] To be sure, the evolution of rigor in analysis matches up well enough with Frege's project that the comparison can be enlightening.[34] But more than the rigorization of analysis was going on, and at times Frege clearly alludes to work in other neighborhoods (geometry, algebra and number theory, to name a few).

Frege speaks as if his effort to prove the basic properties of numbers has close relatives in other areas, though it is not obvious from his words alone what he has in mind. The purpose of this section is to detail some work Frege alludes to that naturally fits the "fruitful concepts are given by quantified expressions" picture. First let's ask what these words gesture toward:

> Later developments [in mathematics] have shown more and more clearly that in mathematics a mere moral conviction, supported by a mass of successful applications, is not good enough. Proof is now demanded of many things that formerly passed as self-evident. Again and again the limits to the validity of a proposition have been in this way established for the first time. ([FA] p.1)

Frege went out of his way to stress that "Again and again the limits to the validity of a proposition have been established" because "Proof is now demanded...". In the next paragraph it is one of just three trends singled out, and he repeats the point two pages later:

> In all directions these same ideals can be seen at work—rigor of proof, precise delimitation of extent of validity, and as a means to this, sharp definition of concepts. ([FA] p. 1)
>
> It not uncommonly happens that we first discover the content of a proposition and only later give the rigorous proof of it, on other and more difficult lines; and often this same proof also reveals more precisely the conditions restricting the validity of the original proposition. ([FA] p.3)

Frege speaks as if it was common for propositions that had been seen as self-evident, or "supported by moral conviction/successful applications" to be proven in a way that displays "the limits to the validity of [the proposition]."[35] After the first of the above quotes he notes function, continuity, limit, infinity, negative and irrational numbers as coming to be seen to need foundational scrutiny. He then sets up the above-quoted comparison of the work gestured at and his project: "the simplest propositions holding of positive whole numbers" are thought self-evident and supported by applications, but it's not enough:

> [It] is in the nature of mathematics to prefer proof, where proof is possible, to any confirmation by induction. Euclid gives proofs of many things which anyone would concede him without question. And it was when men refused to be satisfied even with Euclid's standards of rigour that they were led to the inquiries set in train by the axiom of parallels. ([FA] p.2)[36]

We can now produce conditions on an example of what Frege thinks is analogous to his foundations of arithmetic.

A) It must be a principle which was for a time, among some large group of mathematicians, accepted as basic and unprovable with "a mass of successful applications" and then proven for at least a limited domain.
B) The principle must have some limitations on the extent of its validity and its proof must have made the appropriate domain restriction apparent.
C) The principle must have incorporated mathematically pregnant notions which made possible inferences of the sort which "increase order and regularity", "reveal connections between matters apparently remote" etc.
D) The analysis of the structure of the concepts in virtue of which the principle may admit of proof must yield notions for which the "drawing new boundaries"/seeing patterns account is plausibly applied to analyze why proofs using the principle incorporating these concepts can "extend our knowledge".
E) It must be possible to show that Frege was thoroughly familiar with the principle and the incorporated notions to an extent that it, and they, could plausibly be seen as among the mathematical claims and ideas that Frege is alluding to in the remarks considered above.

The geometrical principle of projective duality meets these conditions.[37] For simplicity consider only the two-dimensional form, stating that the dual of every theorem of plane geometry holds. (In this unrestricted form it is false.) The dual is obtained by interchanging the expressions "point" and "line" everywhere (and "passes through" for "lies on" etc.).

Duality in some form was accepted as a basic principle of space by the synthetic geometers centered about Poncelet in France and Steiner in Germany.[38] Attitudes to the status of the principle and the prospects for proof differed. As one example: Poncelet thought it could be proven with techniques of polar reciprocation while Steiner held it to be more basic.[39]

The conceptual basis of Plücker's proof (in the 1830's) that duality holds in projective planes (though not in general) is very simple. In approaching this result, bear in mind one difference between analytic geometry up until the late nineteenth century and after. Today, a plane is taken to be a set of pairs and equations determine subsets. This is a fairly recent innovation; in the mid-nineteenth century the equations of analytic geometry were taken to pick out geometric entities—points, for example, or line segments—and equations were seen as complex terms referring to constructions out of these basic entities. Plücker saw that a space could be coordinatised either with points or lines, and anything provable analytically about the projective properties of a point (using line coordinates) could be proved about a line (the same proof but point coordinates). So long as the means of construction are restricted (in particular: no metric relations) the point and line coordinates will be symmetric.

Anyone studying geometry from Clebsch in the 1870's—as Frege did—would have been taught how line coordinates underwrite a proof of duality. Indeed we can get a decent sense of Frege's introduction to the topic from the first graduate lectures he took in geometry:[40]

> The introduction of precisely these principles of determining lines is connected with one of the most important principles of analytic geometry, *the Principle of Duality*... . If one then views at one time the points and at another time the lines of the plane as the basic elements out of which one constructs the configurations to be considered then there appears even in matters of detail a certain similarity of the two views. Between these two developments of geometry namely starting from the straight line and starting from points in addition to this analogy there is interchangeability in all particular respects.(Clebsch/Lindemann [1876] p. 28 italics as in the original)

Several such respects are listed and then the notes continue:

> *All of these relations we gather together in the principle of duality. It tells us that certain sentences which are valid for configurations of points can be transferred to configurations of lines. This transfer yields the connection of two points and the intersection of two lines as well as for all constructions which can be composed from these operations.* It no longer is valid if one applies additional devices for example it is no longer valid as soon as a metric is introduced. (Clebsch/Lindemann [1876] p. 28 italics as in the original)

Following these remarks, a now standard point is raised: Plücker's insight may be characterized in linguistic terms by considering the equation of a line (or point).[41] Say that a line is determined in a plane by $Ax + By + C = 0$.[42] From the form $ux + vy + C = 0$ alone it is not determined what is to be constant and what is to vary. Taking "u" and "v" (the "line places") to be constant fixes the basic entities:

> If we now introduce the coordinates of a straight line into its equation, the equation becomes:
>
> (1) $ux + vy + 1 = 0$
>
> The significance (*bedeutung*) of this equation can now be expressed better as follows: namely that it indicates the combination of the position of the point x,y with the line u,v; that is the configuration in which the point lies on the line or the line goes through the point, we take either the coordinates of the points or the coordinates of the line as variable. In the first case, what we have in (1) is the equation of a line,

> in the other case the equation of a point...through suitable choice of coordinates for a line one has obtained the equation which describes the combined configuration of the basic elements (*grundelementen*) with respect to one intuition and the basic elements with respect to the other. These basic elements appear in symmetric relation to one another.
>
> The point therefore plays the same role in the geometry of a line as the line plays in the geometry of points. (Clebsch/Lindemann [1876] p. 29)[43]

Consider the above in relation to the "extent of validity" of duality. In terms of the syntax of the equation: when no restrictions except the obvious grammatical type restrictions are placed on the "referents" of the variables the structure picked out is projective geometry. Duality holds because of what one may see as a linguistic symmetry in the most general formulation. Correspondingly we get a "limitation on the extent of validity": duality need not hold when more restrictions are placed on the range of the variables. To specify metric geometry (for example) one must restrict the range of the variables, and duality breaks down.

Frege explicitly states that duality holds in virtue of the generality of projective geometry and fails in the metric case because the requisite generality is lacking in his 1877 review [RGW].[44] Though Frege does not say so explicitly, he was surely aware of the statement of this generality in terms of unrestricted substitutions as in Clebsch/Lindemann.

> The authors show an insufficient insight into the respective positions of projective and metrical geometry. The true relationship may be intuited by means of the following picture. Projective geometry may be likened to a symmetrical figure where every proposition has a proposition corresponding to it by the principle of duality. If we cut out some arbitrary portion of the figure, the portion is in general no longer symmetrical. Metrical geometry may be likened to such a cut-out, or more precisely, to a cut-out which deserves consideration for some special reasons. To put it in non-pictorial terms, metrical geometry arises from projective geometry by specialization, and this is precisely why the principle of duality loses its validity. Thus if one is concerned about the principle of duality, the only reasonable thing to do is not to leave the field of general projective geometry...[The] principle of duality can never provide a reason for a specialization of general projective geometry; the consideration of metrical properties can be justified only by the special significance they have for our intuition. ([RGW] p.95)

Frege discusses duality in several other places [45] including his thesis ([GRI]). In most of the thesis, he studies metric geometry, though at the end he provides a "more general" (i.e. projective) representation via point and line coordinates in the style of Plücker; he cannot exploit the technique used in the bulk of the thesis because duality would fail:

> We have up to now represented an imaginary point by a straight line...And we have up to now represented an imaginary straight line by a pair of straight lines which had

to fulfill certain conditions...But in representing the two in these heterogeneous ways we have evidently failed to respect the principle of duality which holds between straight lines and points in the [projective] plane. ([GRI p.45–46)

Frege's thinking was engaged with line geometry in another respect: a repeated theme in his strictly mathematical work is the intuitive representation of constructions in algebraic geometry. The most renowned precursor to this work is Plücker's development of models for higher dimensional geometries in line coordinates.[46] Any of Frege's mathematical contemporaries would have recognized the following as a nod to this work, (and subsequent development by Lie and others) though Plücker is not mentioned explicitly:

One of the most far-reaching advances made by analytic geometry in more recent times is that it regards not only points but also other forms (e.g. straight lines, planes, spheres) as elements of space and determines them by means of coordinates. In this way we arrive at geometries of more than three dimensions without leaving the firm ground of intuition. ([LGP] p. 103) [47]

The principle of duality scores impressively on the *desiderata*.

A) Duality was indeed accepted as basic and unprovable by some. Even among those who thought it might admit of proof, it was applied widely and successfully before it was proven.
B) The principle that was ultimately proven was a considerably restricted version of that accepted by—for example—Poncelet. The analytic proof made the limitation clear.
C) Duality is a stellar example of a principle that "increases order and regularity" and "reveals connections between matters apparently remote from one another" The definitions of point and line coordinates that underwrite the proof in the projective case do the same.
D) The proof—especially the syntactic variant in which the form of the equation supports the reasoning—is a striking example of reasoning which is valuable because it involves the decomposition and reconstitution of sentences in such a way that patterns are revealed and "new boundaries drawn". One has two equations—already reached by dissection—$Ax + By + C = 0$ and $ua + vb + C = 0$. Both of these—through a further dissection—give the same general form $ux + vy + C = 0$ and any conclusion which depends *simply* on one of the two available types of substitution will also hold (*mutatus mutandis*) if the other is performed.
In the early writings Frege explicitly nods toward this kind of linguistic duality:

Since the sign Φ occurs in the expression $\Phi(A)$ and since we can imagine that it is replaced by other signs, Θ or X, which would then express other function of the argument A, *we can also regard $\Phi(A)$ as a function of the argument Φ*.([B] p. 24 italics his)

E) Given the extent duality fits the allusions Frege makes to what is happening in mathematics around him, it is reasonable to take it as one of the things he is alluding to when he says: "....later developments have shown that a mere moral conviction, supported by a mass of successful applications, is not enough. Proof is now demanded...". Note especially that he was familiar with this "limitation of the extent of validity" not just as a matter of passing interest, but as central to his early geometrical research.[48]

It is worth stressing, since our understanding of Frege's mathematical setting has suffered through a reliance on an overly narrow range of examples from analysis, that projective duality has further similarities to Frege's project of defining the numbers and proving their basic properties. To mention just two (neither to be pursued further here): the relation between syntax and "basic elements" and the importance of generality considerations, are themes that recur throughout Frege's work.[49] If analogies like this are part of the motivation for the search for fruitfulness in the basic notions of number, we should look closer to see if it holds clues to what Frege was trying to find. Also one might ask—setting aside historical issues—is what Frege sought worth seeking?

V Fruitfulness—why Frege was trying to understand it (and why we should too)

Why does good mathematics help us understand the world? What does it tell us about good mathematics that it confers understanding the way it does? The thesis of this section is that Frege's remarks on fruitfulness contain a fragmentary answer—an account of the value of applied mathematics broadly speaking—with a philosophical dimension missing from most current work in the philosophy of mathematics. ("Broadly speaking" because the concern is not just with mathematics as applied directly to the physical world, but also with its use to illuminate questions within mathematics.[50]) Furthermore, on Frege's account, questions about the aspects of mathematical knowledge that account for its value to us have answers that are as objective and non-psychologistic as questions about logical consequence and consistency.

First a few words to get clear on the issue before elaborating Frege's distinctive response. One reason mathematics is valuable to us is that it provides us with knowledge of deductive relations. Is that all? Hartry Field has suggested that it is: mathematical knowledge is just logical knowledge of what follows from what and which sets of claims are consistent. [51] But now, not all mathematical knowledge is equally useful—some is illuminating and some is logically correct but pointless. What kind of fact supports this distinction?

Views like Field's have a long history, and throughout it they have prompted variants on Frege's: "How do the empty forms of logic disgorge so rich a content?" Poincaré asked the early logicists a variation: how can mere tautologies yield such surprises and difficulties? This version invites an answer in terms of the psychological limits of beings who reason as we do. A classic statement is:

> Indeed, at first glance it is difficult to believe that the whole of mathematics, with its theorems that it cost such labour to establish, with its results that so often surprise us, should admit of being resolved into tautologies. But there is just one point which this argument overlooks: it overlooks the fact that we are not omniscient. An omniscient being, indeed would at once know everything that is contained in the assertion of a few propositions...We ourselves, however, first have to make ourselves conscious of [consequences] by successive tautological transformations, and hence it may prove quite surprising to us that in asserting a proposition we have implicitly also asserted a proposition which seemingly is entirely different...(Hahn [1959] p.159)[52]

Hahn's point is that any value logic and mathematics has to us is a psychological fact of no significant relevance to epistemology. Presumably this is Field's position as well. I will call this the "psychological advantage" view: that people who think like us have cognitive limits, and the value of mathematical reasoning for broader intellectual endeavors is to be explained *solely* in terms of how it allows us to cope with these limits. I will here sketch out an opposing position—the "conceptual setting" view—that takes there to be objective aspects of mathematical patterns of reasoning (many of them difficult to nail down precisely, of course) that are responsible for the value of good mathematics.[53] What distinguishes good mathematics from mathematics that is logically acceptable but unilluminating is in part that the good mathematics isolates the "right" concepts, classifications and theoretical contexts, where this "rightness" is not to be understood just in terms of our psychological quirks and shortcomings.

The subtle discussion in Dummett [1973] helps bring into relief how easy it is to fail to distinguish the two approaches. Dummett develops the inherent tension between the value and the cogency of deductive reasoning:[54]

> When we contemplate the simplest basic forms of inference, the gap between recognizing the truth of the premisses and recognizing that of the conclusion seems infinitesimal; but, when we contemplate the wealth and complexity of number-theoretic theorems which, by chains of such inferences, can be proved from the...Peano axioms, we are struck by the difficulty of establishing them and the surprises they yield.(Dummett [1973] p.297)[55]

These remarks echo Hahn's: the informativeness of deductive conclusions turns on "the difficulty of establishing them and the surprises that they yield". Frege's understanding of the problem is different: the issue he addresses, and the solution he provides, should not be understood in this psychological way. Rather, he takes there to be a fact about what the structure of potentially enlightening inferences are, which his logical system captures.

To better characterise the conceptual setting view, it is helpful to set up a caricature and indicate one absence. Say that mathematics works as a set of black boxes that generate answers to well-posed problems. In the simplest case, one might have a complicated set of conditions relating the values of x and y. What is y if x is n? The right box—a formula computing an explicit solution, or a technique for approximating a solution, or something—will tell you. True, the problems one applies mathematics to often are more subtle and may require sequences of boxes. You want a solution to a polynomial over Q? Apply your Galois box to tell you if it has a solution in radicals and if it does, apply the solution in radicals box to generate the answer. Proof is just a way to ensure the boxes are reliable.

One thing the caricature leaves out is that we use mathematics to understand problems, not just to generate answers. Galois theory achieves more than just a way to identify polynomials whose solutions can be found through a straightforward procedure. It also allows us to understand why polynomials with rational coefficients admit the solutions they do. In the most famous special case, it tells us *why* not all fifth-degree polynomials with rational coefficients can be solved in radicals.[56]

The name "conceptual setting" is chosen—with a nod to Manders [1987]—to reflect a particularly striking way that such understanding can be accomplished.[57] One can come to understand a problem, or take significant steps toward understanding it, just by finding the right mathematical context to put it in. Finding the right conceptual setting is often partially or entirely *constitutive* of understanding a problem, as this cognitive achievement is assessed in mathematical practice. The conceptual setting theorist might hold the following facts to be as objective as questions about logical compulsion: that certain problems in number theory are best addressed in the context of the geometry of complex numbers, or that certain questions concerning the solutions to polynomials with rational coefficients are best addressed within Galois theory or that Riemann surfaces are, in Weyl's words, "the native land, the only soil in which [mereomorphic] functions grow and thrive."[58]

The Weirstrass theorem and projective duality as in Clebsch/Lindemann are compelling examples. The Plücker argument displays *why* duality holds, in a way one might describe as making the theorem obvious by capturing the "picture" Frege spells out in [RGW]. Much of the understanding is conveyed by formulating the issue in terms of polynomial structure then noting the obvious linguistic symmetries. By getting the definitions of coordinates right, projective duality is established in a way that reveals an apparently miraculous symmetry principle as unsurprising and natural. Plücker puts the problem into a more appropriate setting and the naturalness of projective duality is thereby revealed. Similarly the Weierstrass analysis of continuity effects a reconceptualisation through which the relations between continuity and types of convergence come to be not merely known, but laid bare. The understanding conferred by the Weierstrass theorem (formulated in quantification theory) is largely made up of the insight gained by setting the question in a context where the logical structure of quantifier relations can be laid out.

It is difficult to nail down exactly what kind of advance the Plücker proof is—apart from the acknowledged psychological fact that it renders matters comprehensible to folk who think like us—but we have not addressed a central aspect of mathematical practice unless we confront it.[59] Frege's approach emphasises the structural character of the decomposition and reconstitution involved, and takes just this type of insight as "extending knowledge" in the focal passage sense.

Continuity illustrates this point too, and gives further purchase to the metaphors of the focal passage. We may safely assume the Weierstrass definition is among those Frege takes to "increase order and regularity/reveal connections". These virtues bestow psychological blessings: certain proofs are easier for us to find, certain confusions are easier for us to avoid, and so on. But there is an objective ground for these virtues: the logical structure of continuity. The Weierstrass definition displays this structure (in terms that are further studied in the *Begriffsschrift*) thereby representing elements each of which is "intimately, one might almost say organically connected with the others." To use a well-worn metaphor: the Weierstrass definition and the duality of point and line coordinates are fruitful because they carve conceptual reality at the joints.

It is of course wrong to ignore psychology altogether when assessing what makes a given reconceptualisation "correct", but some care should be taken in delineating just how psychology fits in. Consider as an analogy, the classification of functions according to linear, polynomial, NP, etc. in computational complexity theory, or quantifier alternations in the arithmetical and analytical heirarchies. Many have taken these to deliver a classification of problems according to how "hard" or difficult to solve they are, and this is a helpful way to talk if we don't take it too seriously. Indeed, if, (as it may be) the sharp complexity classifications correspond roughly to what we find psychologically more and less difficult, it may be useful to take the psychological facts about what is found difficult or complicated to be (with appropriate reservations) responses to the properties captured in the sharp complexity classes. But to take this attitude is not to make the complexity classifications psychological categories rather than objective measures of computational properties.

The conceptual setting view of the transitions that represent real extensions of knowledge is analogous. There are many psychological states that typically correlate with the recognition of a valuable reconceptualisation. Not only are there the familiar "aha" states of awareness ("Aha! So that's how to do it!"/"How else could it be? How could I have thought otherwise?") there are functional states (increased problem-solving and theorem-proving ability in the given domain). But the observation that valuable mathematical proofs and definitions typically involve a certain kind of restructuring activity need not be taken, and on the conceptual setting view is not taken, to pertain only to these psychological phenomena. Rather it can be taken, and on the conceptual setting view is taken, to concern the inferential structure or family of structures to which these psychological phenomena can be taken to be loosely responding.

Frege's attitude to "extending knowledge" by "drawing new boundaries" is objective/non-psychological in this way. Straightforward psychological accounts would not yield the distinctions Frege draws between Boolean combinations and quantificational definitions. A complex Boolean combination of properties may be more difficult to process and support more surprising conclusions than a simple quantifier expression. Yet the second will support extensions of knowledge because inferences directly involving it "draw new boundary lines". What it is about a fruitful concept/definition/principle which yields "extensions of knowledge" is only incidentally connected with our apprehension of thoughts as psychologically novel. The Weierstrass theorem is a paradigmatic analytic judgment "extending knowledge" because of objective facts about the structure of its derivation, not because its proof is difficult or surprising or anything else turning on psychological quirks of thinkers like us. So far as the intuitive notion of fruitfulness is concerned projective duality is exemplary. The quantifier inferences involved are, however, quite simple. What makes Plücker's argument valuable is not its difficulty or the surprise it prompts, but rather the way it effects an illuminating reconceptualisation.

Let's look again at the focal passage. Quantifier inferences support analytic judgments that, Frege holds, "extend knowledge". Thus he has addressed the problem of explaining how arithmetic can yield extensions of knowledge if it is analytic, just as many logical positivists did. But Frege's way of posing the question, and the answer he arrives at, are only superficially similar to what came later. The account Frege proposes of the value of deduction has more subtlety and sensitivity to the mathematics he is engaging: how does *this* sort of theoretical virtue (proofs increasing order/regularity, revealing connections etc.) result from this sort of logical/inferential structure (*Begriffsschrift*)?

Today we know enough to suspect that the *Begriffsschrift* alone is inadequate to support an account of "fruitful concepts" in the intuitive sense. After one hundred years of research on logic we can recognize Frege's question of how analytic judgments can extend knowledge in the way important mathematical innovations do to be a question of staggering difficulty. But he has identified and essayed a useful first response to a question that remains worth addressing. The ideas in the connected family "extending knowledge"/"fruitful concept"/"drawing new boundaries"/increasing order and regularity"/"revealing connections" etc. neatly illustrate the conceptual setting view. And it is not as if anyone else has shed more light on it in the past hundred years.

Why should one accept any version of the conceptual setting view?[60] Two separate questions must be distinguished. First the question of descriptive adequacy: should it be granted that mathematical practice would not make sense without the norms separating good mathematical reasoning from that which is logically cogent but pointless? On the conceptual setting view, judgements about the extent that a conjecture, definition or proof possesses these features are so deeply embedded within mathematical practice that the practice cannot be understood without distortion if such judgements are not taken into account. The psychological value theorist would, I think, have a tough row to hoe were he to contest this, but I will leave a more detailed investigation for elsewhere.

More delicate questions arise if the psychological value theorist concedes the descriptive point but contests its philosophical significance. The suggestion would be that the ideas of "proper setting for a problem", "reason why a property holds" and the like do indeed guide mathematical research in practice. But, the response might continue, though perhaps accurate as descriptions of the phenomenology of mathematical activity, these are *just* descriptions of the phenomenology. (With the implication: It may be true, but it's not *philosophy*.) The psychological value theorist might thereby stick to his guns, and maintain that patterns of deductive justification are the only mathematical data epistemology must address.[61]

This issue is intricate, and here only one factor which tells in favour of the conceptual setting view will be considered. These aspects of mathematical activity should be part of an account of the theorems which issue from that activity because mathematical reasoning has a feature I will call "self-reflectiveness": that one piece of reasoning is mathematically valuable and another not, is itself studied within mathematics. There are—internal to the practice of mathematics—standards governing how assessments of "proper setting for a problem", "reason for a theorem holding", etc. are to be investigated, and hence the ideas emphasised by the conceptual setting view cannot be "just phenomenology": they are part of the subject matter of mathematical activity. The next section will be devoted to elaborating both this point and a more delicate one: that it is a mistake to regard "philosophy" and "mathematics" as sharply distinguishable here. Even uncontroversially "epistemological" issues like the nature of "self-evidence" turn out to be within the blurred mathematical/philosophical boundary.

VI Conceptual Settings and the Self-Reflectiveness of Mathematics

The "fundamental homomorphism theorem" for many significant classes of structures (groups, rings, fields, etc.) states that any homomorphism ϕ between structures in the class can be decomposed into two further homomorphisms to and from an intermediate structure defined in terms of ϕ. The two factor homomorphisms and the intermediate structure in this "triangle' have useful properties and—as the saying goes—"the diagram commutes". In certain special cases when homomorphisms can be counted on to factor in this way they are said to have "the universal property".[62]

That possession of the universal property facilitates some mathematical constructions could no doubt be investigated empirically as a psychological fact. But what matters here is how it could be investigated *mathematically:* the theory of classes of structures with the universal property can be developed, theorems holding in such classes can be taxonomised, at the abstract level of category theory the significance of the universal property can be contrasted with that of other factorization properties of maps, and so on. This is not merely the mathematical examination of a property which happens to be mathematically valuable—it is a mathematical examination of what is mathematically valuable about this property.

This sort of examination—in which one identifies and studies an idea as a central one mathematically—is ubiquitous in mathematical practice. Hence there is often an antecedently worked out mathematical answer to questions of the type "Why do problems of sort S tend to be more easily solved in presence of property P?" This sort of investigation is not restricted to particular narrowly specifiable properties like the universal property, duality, or continuity. Entire mathematical contexts can be studied in reference to whether or not they are the right setting for problems of a certain sort.[63]

One thing to mark about this self-reflective character of mathematical theorizing is that in many significant cases it is artificial to separate investigations into exclusive categories of "theoretical" and "metatheoretical". But there is a further, more delicate point that Frege's investigation of "fruitful concepts" brings out. Sometimes it is artificial to segregate work into the related dualism of: "mathematical activity" and "philosophical reflection".[64] Consider for example the way continuity is studied mathematically in Fregean foundations and in (general) topology. One might (accurately if tendentiously) describe the latter as the study of "what makes continuity fruitful"—a mathematical discipline to be sure, though this description suggests the "metatheoretic", reflective character associated with philosophical investigation. In the former the key notion is generality/universal quantification rather than open set, so the investigation has a more "philosophical" cast, but this should not lead us to overlook that, as a strictly mathematical investigation, the examination of continuity in terms of generality is motivated by the kind of factors which motivate the examination in terms of open sets.

Generality as expressed in terms of quantified variables is not only a feature whose representation happens to distinguish Frege's logic from Boole's. It is also a feature of mathematical significance whose presence or absence can be responsible for theorems holding in some domains and failing in others. (This point is implicit in Frege's discussion of duality and generality in [RGW].) This can be isolated and studied both within mathematics and within the philosophical logic tradition in ways that turn out to largely coincide.

Unearthing the central notions in a given mathematical domain and studying what makes them central takes mathematical/analytical work. This point is obscured in discussions of universals or "natural properties" which present it as a deliverance of science what the real ones are. Theories don't in practice come antecedently regimented into a first order theory, with the central notions to be read off the basic vocabulary. The central notions are themselves determined through mathematical investigation.[65]

This supports a point *ad homenim* against a naturalist position that would maintain that science tells us what the real universals are while dismissing the conceptual setting view as dealing with mere phenomenology. If there is to be an objective fact about what the right divisions in nature are that science is supposed to discover, then some notion of "right conceptual setting" will be crucial to an account of the conceptual divisions that make a property central, given the integral role of mathematical formulations of physical theory.

But there is a deeper, independent point. There is an instability in the notion of "evident" from one mathematical context to another. What is immediate in one setting may be derived in another. This gives the fruits of mathematical self-examination a distinctly philosophical cast; procedures internal to mathematical practice clarify an idea Frege points to in *Grundgesetze:* "being evident".

> Generally people are satisfied if every step in the proof is evidently correct, and this is permissible if one merely wishes to be persuaded that the proposition to be proved is true. But if it is a matter of gaining an insight into the nature of this "being evident", this procedure does not suffice; ([BLA] p.4–5)[66]

That one transition is evident to people like us and another not can admit of empirical study; but this is not Frege's concern.[67] Rather he is concerned with an investigation that can be seen as mathematical (studying reasoning in arithmetic as one might study the universal property) or philosophical (investigating the grounds for the self-evidence of certain mathematical inferences.) The point is, of course, not that we should waste time adjudicating artificial jurisdictional disputes but rather that much of the resonance of Frege's work is due to its engagement with both mathematical and philosophical practice.

VII Dissolving the Ladder

> ...any thorough investigation of the concept of number is bound always to turn out to be rather philosophical. It is a task which is common to mathematics and philosophy.
>
> It may well be that the cooperation between these two sciences, in spite of many demarches from both sides, is not so flourishing as could be wished and would, for that matter, be possible. ([FA] p.v)

At the outset the distinction between mathematical and philosophical projects was accepted for preliminary orientation, but it must now be set aside. Indeed one respect in which Frege's early work deserves notice is that it blurs this distinction. Frege imposed no Procrustean restrictions on the nature of his investigation. He was interested in certain questions about numbers—for philosophical reasons (concerns about *a priori* knowledge, for example), as well as mathematical ones, and for reasons like his investigation of 'fruitful concepts' which are not happily classified as exclusively either.

The need to recalibrate our view of Frege becomes more explicit in contrast to an attitude whose classic statement is in Putnam [1967]: mathematics does not need foundations. This point appears to have entered the basic repertoire of objections to Frege's work. One often encounters variations on: "Those wondering whether mathematics really needs the kind of foundation Frege attempts to provide will not be satisfied with [Dummett's volume]" (Crittendon [1993]). If foundational research has as its sole *raison d'être* the conferring of warranted certainty on non-foundational mathematics, Putnam's essay strikes at the heart of the Fregean program. Who would deny that elementary arithmetic warrants more confidence than any foundation could deserve?

This point gains its force from a picture of foundational research external to mathematical practice, tacked on as an afterthought to provide epistemological legitimacy. If this were accurate, we should indeed wonder how *Grundlagen* could supply such legitimacy if mathematics were to lack it at the outset. But the foundational project Frege is pursuing is internal to the practice, generated in part by the pressure that produces other areas of mathematical investigation. Mathematically, he addresses the phenomenon that certain concepts and proof methods (including continuity and—we may plausibly assume—the analytic approach to projective duality) are fruitful. He strives to explain this with an account of the structural basis of this fruitfulness. He expects (overoptimistically, to be sure) that this will facilitate proofs of less elemental mathematical problems. Philosophically, his account of what it is for a conclusion to "extend knowledge" incorporates this conceptual setting account. The key idea in both of these investigations—the idea of generality—is one he both represents in the *Begriffsschrift* and discusses as a feature of mathematical theories which can underwrite the mathematical properties they possess.

There is no temptation to approach "Why does analysis need topology?" as "Why does arithmetic need foundations?" has come to be approached. Analysts need—or at least profit from—topology because it develops the systematic interrelations of a few notions, like continuity, central to analysis. Through topological investigation, these notions come to be better understood, in the way that mathematical investigation at its best renders ideas clearer.

Why shouldn't we say the same of logical foundations? Of course, some of the reasons epistemologists appeal to logical foundations are uniquely philosophical. Frege himself adduces some of these. But others have the hybrid character of the investigation of "fruitful concepts": a project with philosophical ambitions—exploring the nature of arithmetical reasoning—employing mathematical techniques and generated by pressures internal to mathematical practice. Frege aims at a deeper understanding of arithmetic, not by reducing it to something different, but rather by displaying what number *is*—"achieving knowledge of [the] concept in its pure form, [by] stripping off the irrelevant accretions which veil it from the eyes of the mind." ([FA] p.vii) There are patterns of understanding that are characteristic of good mathematical investigation, and Frege's work provides a foothold for understanding mathematics, and an exemplar of the foundations of mathematics—according to them.

Appendix: The Wages of Sinn

This paper, unlike others treating Frege's idea of fruitfulness, considers almost exclusively the period before the sense/reference distinction was articulated. (The post-1884 writings and correspondence do not mention "fruitful definition", nor "fruitful concept".) This appendix aims to fit the late writings into the picture. This is especially pressing as the literature addressing fruitfulness often stresses a few late passages in which the focal passage seems to be openly contradicted. For example (in 1903):

> No definition extends our knowledge. It is only a means of collecting a manifold content into a brief word or sign, thereby making it easier for us to handle. This and this alone is the use of definitions in mathematics. ([FG] p.274)

Similar remarks may be found in 1914 lectures:

> In fact it is not possible to prove something by means of a definition alone that would be unprovable without it. When something that looks like a definition really makes it possible to prove something which could not have been proved before, then it is no mere definition but must conceal something which must either be proved as a theorem or accepted as an axiom. ([LM]p. 208).

From the *Begriffsschrift* on, Frege noted that definitions could replace an unwieldy expression with a simple one to facilitate the psychological process of thinking.[68] But by the time of [LM], this is stressed as the *sole* advantage:

> It appears from this that definition is, after all, quite inessential. In fact considered from a logical point of view it stands out as something wholly inessential and dispensable...[though one should note that]...To be without logical significance is by no means to be without psychological significance. ...So if from a logical point of view definitions are at bottom quite inessential, they are nevertheless of great importance for thinking as this actually takes place in human beings.([LM]p. 208–209)

Taking the issue to revolve about the apparent conflict between the focal passage and later passages like this, one might ask: is the contradiction merely apparent (Weiner), is there an unresolved tension between the early and late writings (Benacerraf), or did Frege change his mind after 1884 (Dummett [1981])?[69] In terms of this trichotomy, this paper indicates at least one change: the idea of 'extensions of knowledge' has been reorganized.

Also, it seems that a misleading style of expression was given up. Early on, Frege spoke of both concepts and the definitions marking them out as fruitful and supporting extensions of knowledge. Later he stopped speaking of definitions that way, though the research into functions in [FC] and elsewhere suggests (though the issue is complex) that post-1884 he held similar views about concepts and functions supporting inferences that extend knowledge.

Allowing for these two changes there need be no further conflict between the idea that definitions must be fruitful and the late remarks of the sort considered above.[70]

Note that Frege explicitly reaffirms in later writings much of what he said earlier about definition. In 1906: "The insight [definition] provides into the logical structure is not only valuable in itself but also is a condition for insight into the logical linkage of truths."([FG2] p. 302) In late notes: "A definition in arithmetic that is never adduced in the course of a proof, fails of its purpose."([NLD] p. 256)[71] And that any definition is an abbreviation in that the new expression must be eliminable is stated repeatedly, before and after *Grundlagen*.[72]

One difference between [LM] and the material up to 1884 is that Frege distinguishes two types of definitions.[73] Constructive definitions (for which he reserves the name "definition") introduce a new sign to have the same sense as a complex sign. Of analytic definitions, which result from analysis of signs with an antecedent sense he says:

> "That [the sense of the complex expression] agrees with the sense of the simple sign is not a matter of arbitrary stipulation, but can only be recognized by an immediate insight...But it is better to eschew the word 'definition' altogether in this case, because what we should like to call a definition is really to be regarded as an axiom." ([LM] p.210)

Some interpreters have fastened on this distinction to dissolve the apparent conflict, suggesting that the definitions described as fruitful are analytic definitions.[74] So a) late Frege never talks about fruitful definitions because his use of the word "definition" has been refined. b) The remarks apparently contradicting the focal passage concern constructive definitions. But this won't do: the discussion of "fruitful concept formation" in [BLC] makes it clear that the stipulative introduction of a newly defined notion can be fruitful. What makes a definition fruitful—quantifier structure—doesn't depend on whether the definiens is new or old.

Further elements of the late view of sense sit uncomfortably with the early "drawing new boundaries" idea. Sentences in indirect contexts refer to their ordinary senses, a sentence refers to its truth value, and compositionality of reference ties thoughts to the semantics of propositional attitudes. [75] Hence the sense that underwrites a claim that S extends our knowledge is exactly the thing that underwrites claims to the effect that S and S* differ in that "John believes that S." can be true when "John believes that S*." is false. For all that is said up to 1884 the partition of contents given by the "drawing new boundaries" criterion need have no systematic connection to attitude verbs.

For the old and new to coalesce, it must be that inferences which, in the early account, "draw new boundaries" yield conclusions that differ in sense, and that inferences in which the premisses are equivalent to the conclusion and which exploit only propositional inferences do not.[76] This points to a potential tension in the later writings between the inferential role of sense captured in the older idea of content and "extending knowledge", and the new idea of sense as the bearer of the sort of information which can support propositional attitudes. The view of content as a function of possible derivations allows the incorporation of the fine structure reflecting whether or not a given definition/concept involves quantifiers; this information need not be reflected in the sense of an expression. Perhaps one can develop a notion of sense accounting for both propositional attitudes and "gapless" inference and yielding an idea of "extending knowledge" agreeing with the early one, but it is not obvious.

The recent Taschek [1993] charts a similar tension. Taschek identifies an inferential notion he calls "cognitive potential" plus a notion of semantic encoding and holds that Frege attempts (unsuccessfully) to develop a concept of sense to fill both roles. If so, this tension may represent part of an effort to preserve in the late writings some of the standpoint on content and inference that underwrote the idea of "fruitful concept". But this is a subject for a paper in its own right, rather than an appendix to a different one.

Notes

1A distant ancestor of this paper arose out of discussions in a seminar on Frege I co-taught with Jim Conant during the summer of 1992 at the University of Pittsburgh. Though I am indebted to all the seminar participants for much helpful debate, I should single out for special thanks Jim Conant, Steven Glaister, Phil Kremer, Madeline Larson, and John McDowell. Three superb papers—Bill Demopoulos' [1994], Jeff Horty's [1993] and Mark Wilson's [1992]—played a significant role in the early stages of composition, as did questions and comments from Charles Chihara, William Craig, Bill Demopoulos, Jan Alnes, Joan Weiner, Leon Henkin, Michael Idinopulos, Paddy Blanchette and the audience at the Logic and Methodology of Science colloquium at the University of California, Berkeley. My understanding of the issues discussed in section V owes a significant debt to the writings of Ken Manders and Georg Kreisel. Late drafts were improved (or at least changed) thanks to comments from Paul Benacerraf, Bob Brandom, Juliet Floyd, Richard Heck, Pen Maddy, Tom Ricketts, Gideon Rosen, Hans Sluga, Jack Silver, and the participants in a University of Pittsburgh philosophy faculty colloquium and a seminar at Princeton. I am particularly indebted to Mark Wilson and Ken Manders for comments and encouragement at each of the stages of development.

2Frege also talks of analytic judgments and nods at the idea that they can extend knowledge in a letter of 1882 ([Corr] p. 99–102) but he says nothing of interest there about the specific issue of how analytic judgments can extend knowledge, except to repeat the idea of the focal passage that Kant's mistake was to focus on overly simple examples.

3The Austin translation reads "Even I agree that definitions..." which has the misleading connotation—absent from the original—that Frege expects that there is something unexpected about this agreement or that the agreement is to be seen as a grudging admission.

4The square brackets are inserted to indicate a departure from the Austin translation of "unfruchtbar" as "sterile" rather than the more fruitful "unfruitful". I am indebted to Bob Brandom for drawing this passage to my attention.

5For example, in response to passages opening *Grundlagen* emphasized by Benacerraf, Weiner argues that the motivations displayed are not mathematical, despite what Frege says:

> Frege initially attempts to motivate his project by appealing to mathematical worries. It seems that, if we are to take Frege at his word, we should take seriously not only Frege's claims about the motivation of the project but also the fact that Frege begins with a mathematical rather than a philosophical story about the importance of his project. ...I will argue, however, that Frege's project is actually very different from most other, apparently similar, projects undertaken by mathematicians. Indeed, the mathematical motivations often attributed to Frege do not really provide motivations for his actual project. For, if these were his only motivations, Frege ought to have been content with available work by other mathematicians.(Weiner [1990] p. 18)

(Cf. Weiner [1990] p. 17—31) for more details of the historical account she marshals to support this thesis. Rather than address Weiner's broader historical thesis about nineteenth century mathematics I will try to bring out how Frege's mathematical motivations are natural ones. I consider the mathematical setting for Frege's early work in more detail in Tappenden [1994], where some discussion of Weiner's account is given.

6Direct textual evidence to this effect is considered in section III.

[7]Weiner also considers the possibility that the fruitfulness condition excludes not only illustrative examples but also hints and elucidations of the "don't begrudge me a pinch of salt" variety which are highlighted in the essay [CO]. Nothing in the present paper is affected if this nuance is ignored, which it is well to do since to consider it would introduce a motif which would lead far afield.

[8]For relevant reading from Cauchy's *Cours d'Analyse* cf. Birkhoff [1973] p.2

[9](Cf. p.90–91 Weiner [1990])

[10]One option for Weiner on the narrow issue might be to try to retain the interpretation of fruitfulness as a trivial condition by denying that "fruitful" is being used in the same sense in the focal passage as in the places where he seems to propose fruitfulness as an adequacy condition for definitions. Note however that this option would require that Weiner's use of "continuity" as an illustrative example of a fruitful notion be given up, as it is only in the focal passage, and obviously cognate passages in [BLC], where continuity is so described.

[11]Further textual argument that cuts against the trivial interpretation is better deferred to III, for the overall dialectic of the paper.

[12]It is important to remember that the propositional connectives are essential, in addition to the quantifier, in order for the full extent of Frege's innovation to be appreciated. The point is that not only the generality of a free-standing assertion, but also the generality of an embedded claim, can be represented, so that distinctions of the scope of a negation or quantifier can be reflected in the symbolism. This is more obvious today than it was: Schroeder failed to appreciate it in his review of the *Begriffsschrift*. He acknowledges that Frege has scooped him on the need for a sign for generality, and proposes a revision, which remains inadequate because it cannot mark distinctions of scope, as Frege notes in his reply [ACN] (p.99)

[13]To get the full effect of [BLC] as a decoding device, one should read all of p.33–35 in conjunction with the focal passage.

[14]Frege explicitly defines continuity at a point and on an interval at ([BLC] p.24–25), where he gives the Weierstrass versions in *Begriffsschrift* notation.

[15]Functions in this and similar contexts should be understood as real-valued functions of a real variable.

[16]This theorem was of central importance in the sequence of events which resulted in the rigorisation of analysis, and it is beyond doubt that Frege was familiar with it. For more detail on the historical significance of this and cognate discoveries cf. Grattan-Guinness [1970] p.118–130, Kitcher [1984] ch.10 or Bottozini [1986] ch.3. Demopoulos [1994] is an important exploration of some respects in which Frege's attempt to show that arithmetic arose from logic was engaged with the nineteenth century rigorization of analysis. Also of interest in this connection is Friedman [1992] ch. 1 and the remarks in the next footnote.

[17]We actually need a few manipulations for 21) on the list to yield a definition of uniform convergence. Actually the discussion in [BLC] comes even closer to a discussion of the Weierstrass theorem than the text indicates. A contrasting notion of convergence for functions is given by an interchange of quantifiers: a sequence of functions $\{f_i(x)\}_{1 \leq i < \infty}$ *converges pointwise* to f(x) on U if for every $\epsilon > 0$ and every x in U there is some N such that if $n > N$ then $|f(x)-f_n(x)| < \epsilon$. It was of considerable historical importance that this distinction be made and that the Weierstrass theorem hold only for uniform convergence. (In addition to the references in the last footnote, Grattan-Guinness [1975] described these historical events with special reference to the notion of quantification.) In [BLC] Frege spells out a pair of definitions—definitions 20) and 21)—from which the notions of uniform convergence and pointwise convergence are easily defined in turn. Furthermore the central feature distinguishing uniform and pointwise convergence—the quantifier interchange—is precisely the difference between 20) and 21).

Mark Wilson has pointed out in correspondence that the role of quantification in supporting these convergence properties supports an intriguing, fairly literal interpretation of Frege's metaphors of "drawing new boundaries". The point is that the existence of smooth curves to which a given family of smooth curves converges depends, in these cases, on logical properties rather than on intutition. If one takes classical synthetic methods in geometry to be identified both with "lines given in a diagram" and intuition, then the techniques for identifying new curves do indeed turn out to "draw new boundaries" in sense with a straightforward geometrical interpretation.

[18]This is especially worth noting in light of a common remark in reference to Frege's development of quantification theory—that in the late nineteenth century inferences involving "multiple generality" were becoming increasingly important. It should be noted—as the treatment of duality in IV

illustrates—inferences involving simple, non-iterated generality were also a focus of mathematical attention.

[19]Cf. Dummett [1991a] (p.38–46), from whom the expressions "patterns" and "dissection" were taken. The discussion of this and the following paragraph is deeply indebted to Dummett's treatment.

[20]Dummett has convincingly argued this in ([1991a] (p.38–46)) so I will refer the reader there for textual arguments.

[21]Here, and throughout the discussion of these sections, I am indebted to Dummett [1991a] p.36–46 and *passim.*

[22]Another pattern of "implicit equivalences" is obscured in the Austin translation of *Grundlagen.* In [BLC] the expression translated as "concept formation" is "*begriffsbildung*" which has an ambiguity like that of "construction" between a mentalistic and a non-mentalistic interpretation, though as in English one might feel that the mentalistic reading is the preferred one. In the focal passage, two different words are translated "concept formation": "*begriffsbildung*" and "*begriffsbistimmung*". The latter has the less mentalistic overtones of, say, "determination of a concept" or "laying out a concept". That Frege, in the focal passage, clearly regards "*begriffsbildung*" and "*begriffsbistimmung*" as interchangeable expressions supports the view that whatever "*begriffsbildung*/concept formation" is, it has only an incidental connection to mental acts. On this point I am indebted to Hans Sluga.

[23]We may distinguish two uses of "definition": one may use "definition" to mark out a sentence which fixes a simple sign and a complex one as definitionally equivalent through the use of a biconditional. Using the expression in this way, a definition is what, in *Begriffsschrift* syntax, could be written with the $\Vdash$ prefix. In the next paragraph we will use "definition$_S$" for a definitional sentence which is, in this sense, a definition. Alternatively "definition" can be used to mean the complex expression in a definition$_S$. Say for example we fix the meaning of "p&q" with $\Vdash((p\&q) \equiv \neg\mathfrak{X}(p \rightarrow \neg\mathfrak{X}q))$. It is acceptable in English to call the expression on the right hand side of the biconditional the definition of "p&q". Call a definition in this sense a definition$_C$. In the focal passage and [BLC], Frege not only uses definition in the sense of definition$_S$ and definition in the sense of definition$_C$ interchangeably, he also uses definition (apparently in the sense of definition$_S$) and "concept formation" interchangeably. But given that we are taking it that "arithmetic in the broadest sense forms concepts" this is no surprise—what would it be for a logical system in the abstract to "form concepts" but to have specified definition$_S$'s containing expressions for the concepts as definition$_C$'s? This simplifies matters, since we may then ignore the distinction of definition$_S$ and definition$_C$ (except for those cases where, as when a definition is described as "used in a proof" only a definition$_S$ understanding makes sense.)

[24]I am indebted to Bob Brandom for leading me to appreciate the connection between inference and content in Frege's early work.

[25]The remark about fulfilling the ideal of a "*lingua characterica*" should not be shrugged off as just a passing historical nod. (As Patzig notes, (Patzig [1969]) the idiosyncratic expression "*lingua characterica*" rather than the more common "*lingua characteristica*" is a borrowing from Trendelenburg.) In his early writings, Frege repeatedly invokes the Leibnizian ideal to set himself off from logicians of the Boole/Schroeder variety. A typical remark is:

> [Boole's language is inadequate because] the lack of a representation of generality corresponding to mine would make a true concept formation—one that didn't use already existing boundary lines—impossible. It was certainly also this defect which hindered Leibniz from proceeding further. (BLC p.35)

The different ideas of logic—universal characteristic and *calculus ratiocinator*—are explored with reference to Frege in the useful paper van Heijnoort [1967]. Additional historical detail on this Leibnizian tradition is in Sluga [1980], Sluga [1987], and Otte [1989]. Such references to Leibniz were common among mathematicians too: for example, in the introduction to the 1860 German translation of Salmon's standard text *Treatise on Conic Sections* the translator describes the invariant-theoretic approach to geometry as a fulfillment of Leibniz's ideas of a geometrical characteristic. (cf. Otte ([1989] p.33)

[26]What theoretical work *does* content do, if it doesn't underwrite "extending knowledge"? Well, one thing it does is support Frege's account of analyticity. In *Grundlagen,* a sentence is analytic if it admits of a proof relying ultimately only on logical laws and definitions. ([FA] p.4) So "distinctions

between...synthetic and analytic, concern, as I see it, not the content of the judgment but the justification for making the judgment."([FA] p.3) Also: to call something analytic is to make "a judgment about the ultimate ground upon which rests the justification for holding it true." ([FA] p.3) In a notorious footnote, he says he is not trying to "assign a new sense to these terms, but only to state accurately what other writers—Kant in particular—have meant by them."(FA p.3)

It is often held that this remark is a transparent dodge—an attempt to avoid open conflict with accepted Kantian usage. (So for example Dummett [1991a] (p.23) describes the remark as "somewhat disingenuous". Benacerraf [1981] p.25–26 suggests that Frege's connection between analyticity and justification starkly changes the subject, so far as the topic of analyticity is concerned—"footnote disclaimers to the contrary". (Benacerraf [1981] p.26) (See also (Katz [1992]).) On the other hand Weiner, in [1990], understands Frege's talk of analytic judgments in a way that is in some respects similar to that proposed here.) But with the notion of content Frege is working with, the distinction between notions of analyticity which turn upon content and those which turn upon justification blurs; it is not misleading for Frege to say that he is trying to lay out what Kant, etc. have meant all along. As we all know, Kant takes the notion of analytic judgment in two (not obviously compatible) ways: a judgment is analytic if the predicate is contained in the subject, and if its negation is self-contradictory. (Kant [1965] p.48–52)) The latter definition is the same as Frege's, if Frege's account of contradictoriness is taken over, as Frege notes:

> Kant obviously—as a result, no doubt, of defining them too narrowly—underestimated the value of analytic judgments, though it seems that he did have some inkling of the wider sense in which I have used the term. [footnote: On p.43 [B14] he says that a synthetic proposition can only be seen to be true by the law of contradiction, if another synthetic proposition is presupposed]([FA] (p.99–100))

With the early notion of content the Fregean definition also results naturally from the *first* Kantian definition by replacing the subject/predicate grammar with Frege's account of logical grammar. Frege does not merely avoid the subject/predicate division: he explicitly rejects it. "A distinction between *subject* and *predicate* does *not occur* in my way of representing a judgment." ([B]p.12, his italics). After this, he says the function/argument analysis is better because it reflects the logical structure of the judgment in the syntax, "leaving nothing to guesswork". With grammatical structure thus given by logical structure the analogue of "predicate contained in the subject" will be a notion of analytic according to which parts of a sentence combine to form a whole with a logical proof, because on the new analysis the grammar of a sentence depends on its logical properties. A sentence that is logically provable from definitions has the grammar it has precisely because it admits the proofs it does. On this account of logical grammar, the natural generalization of "predicate contained in the subject" really is "logically derivable".

The result is compatible with the Kantian treatment exploiting non-contradiction. Indeed it is hard to see any other natural way to reconcile Kant's two specifications in a general setting. No wonder Frege took himself to have laid out what "Kant, and the others, have meant all along." By tying content and grammar to inference, he achieved a reconciliation of Talmudic proportions—perhaps the only natural reconciliation available—between two very different ideas that Kant marks out with one word.

[27]Also, it does several jobs in current work which adopts the [SM] framework. This point is made in different ways, with reference to different functions of the notion of sense, in Kaplan [1968], Burge [1977] and Salmon [1981]. A related point is explored in the appendix.

[28]The more common translation of Frege's expression "*erkenntniswerte*" is "cognitive value" though for present purposes "knowledge value" seems preferable. I am here indebted to a conversation with Jan Alnes.

[29]The idea that there is nothing more to the family of notions considered here than just the brute fact that some definitions are—as a matter of fact—often used in proofs and others are not seems to be Weiner's view. Consider for example this nod to the idea that proofs reveal mathematically valuable conceptual structure:

> In the last paragraph of section 2 Frege gives what looks to be a perfectly acceptable...mathematical motivation. He says "The aim of proof is, in fact, not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of

truths upon one another." It is important to remember that new proofs...are of mathematical interest. A series of proofs may well be evaluated...on grounds of how interesting they are or by how much insight they seem to provide. In this sense, Frege's project did have a mathematical motivation. But this is really not to say anything more about Frege's motivation than that it resulted in his proving theorems. We already know Frege proved theorems. (Weiner [1990] p. 27–28)

[30]Again "*ufruchtbar*" is here translated "unfruitful" rather than "sterile".

[31]That Frege describes the definition of number in this way is noted in Dummett ([1981] p. 332), who attributes the observation in turn to Gordon Baker.

[32]Wilson [1992] contains, *inter alia,* a superb discussion of this analogy and some of its historical antecedents in mid-nineteenth century geometry.

[33]This is the approach of Kitcher [1984] and Weiner [1990], as well as to some extent Currie [1982]. Note also Wagner [1992] who inserts the following aside into an otherwise critical discussion of Kitcher's work on Frege: "[Kitcher is right that] while difficulties in analysis prompted the foundational work of Cauchy, Weierstrass, Cantor and others, nothing of the kind moved Frege." (p.95–96) I discuss Frege's historical setting in more detail in Tappenden [1994].

[34]For other examples of the use of the history of analysis to illuminate the nature of Frege's project cf. Demopoulos [1994] and Burgess [1993].

[35]To be sure, a certain kind of mistake will leap into the mind of anyone who has made them often enough. One may discover, in the course of giving a proof, a limiting hypothesis one had neglected in the initial statement of the thesis. In some cases one might describe what is omitted as a "delimitation of the extent of validity"—for instance computing a power series and neglecting to specify the radius of convergence. But Frege clearly means to pick out more than just this tendency.

[36]It is important that Frege alludes to geometry as well as analysis and algebra as the example developed in the following is geometrical.

[37]It is perhaps worth emphasizing that this is to be considered an example of *one* of the theorems and principles Frege had in mind when he said some of the things he said. There are many others. I stress this both as a check against being understood as saying more than I believe—for example I do not believe Frege was driven by a singleminded desire to generalize the methods of the proof of duality and duality alone, and against being understood as saying less—for example I do not believe that there is only one compelling example of a principle satisfying conditions A)—E), or nontrivial subcollections of them, only that the example chosen here a particularly simple and clean one.

[38]Here I mean "synthetic" in the sense of "synthetic geometry" as opposed to analytic (i.e. coordinate) geometry. The mid-nineteenth century dispute over synthetic vs analytic methods is an important backdrop for assessing Frege's work, but space precludes exploring more than a few basic details here. I explore Frege's work in the context of nineteenth century geometry in Tappenden [1994]. See also Wilson [1992] on Frege and geometry. On nineteenth century geometry the comprehensive Kline [1972] is of course a handy overview. For more specific treatments of geometry in particular, useful sources include Coolidge [1940] and [1945], and Boyer [1956]. The philosophically engaged and insightful Nagel [1971] is also very useful if read with caution.

[39]Thus Steiner remarks: "...Dual relations in geometry appear simultaneously with the introduction of the fundamental elements, while the theory of reciprocal polars makes its appearance only subsequently, as a consequence of certain relations among the fundamental elements. [Duality] lies closer to the essence of things, and in this respect is prior to the theory of polars."(Steiner [1832] quoted in Nagel [1971] p.228)

[40]Of course, these lectures were not transcribed by Lindemann and preserved because they were the first graduate lecture courses Frege took in geometry but rather because they were the last lectures Clebsch gave. That Frege attended these lectures may be confirmed by comparing the list of lectures on which the notes were based (Clebsch/Lindemann [1876] p. V) with the list of lectures Frege is on record as having attended at Göttingen (cf. Bynum [1972]). (It should be noted though, that Lindemann is likely to have made some additions, so the record need not be an unaltered transcript of the lectures. Nothing much hangs on the fact that Frege attended these exact lectures, as he would have heard roughly the same thing from any other analytic geometer of the time in introductory lectures. But reading about projective duality more or less as Frege would have heard it from Clebsch does give it a special vivacity in this context.) All translations my own, with extensive help from Bill Craig, Hans Sluga and Jan Alnes.

[41]This linguistic aspect of Plücker's account of duality is mentioned by Wilson [1992] p.164–165. Wilson's remarks in this connection were an important impetus for this section. The discussion in Nagel [1971] also played a galvanic role.

[42]For this reasoning to be made completely rigorous, it would need to be recast so that the coordinates would be of the appropriate type (i.e. homogeneous coordinates) but this subtlety is incidental to the point made here.

[43]In connection with Frege's approach to ontology it is perhaps worth noting the way that this discussion treats the question of the fundamental elements ("*Grundelementen*") as turning about the type of terms which occupy a certain grammatical place in an equation, but this will not be explored further in this paper.

[44]There are several analogies between Frege's assessment of the generality of arithmetic and logic and his assessment of the generality of projective geometry which are skimmed over in the text. Some of these analogies are developed in Tappenden [1995]. Other useful treatments of Frege's view of generality of arithmetic and logic are in Ricketts [1986] and scattered through Dummett [1991a].

[45](For example ([FA] p.35–36). I explore in other work (Tappenden [1995]) reasons of a general historical nature why we may assume that the proof of projective duality, and Plücker's geometrical work generally, were well known to Frege. Here—apart from the above glimpse into Frege's first year graduate lectures—only those places where Frege explicitly discusses duality or Plücker coordinates in his writings up to and including the *Foundations* will be considered.

[46]The richest account of this work is Plücker [1868]. An extensive English-language account is Plücker [1865].

[47]To see that it is Plücker who would be taken to be the source of these innovations, compare how Lie (in 1870) assigns credit when originally introducing the "sphere mapping" to which Frege refers when noting that spheres can be elements of space and determined by coordinates: "The rapid development of geometry in the present century has been closely related to and dependent on the philosophic views of the nature of Cartesian geometry—views which have been set forth in their most general form by Plücker in his earliest works.

Those who have penetrated into the spirit of Plücker's works find nothing essentially new in the idea that one may employ as an element in the geometry of space any curve involving three parameters." (From the excerpt in Smith [1959] p.485–486)

[48]Note too that the issue of duality in connection with generalized coordinate schemes was connected with active research by mathematicians at Göttingen when Frege was a student there. Although the use of line and plane coordinates to settle the two and three dimensional cases was old news, Sophus Lie was at work on (what from one point of view could be counted as) a generalization of this work to higher-dimensional geometries in his treatment of "contact transformations" cf. Hawkins [1989] p.196

[49]I explore Frege's concern for generality further in Tappenden [1995]. Mark Wilson in [1992] has noted the way Plücker's work appears to reflect Frege's approach to concepts as involving the decomposition of antecedently fixed contents.

[50]What is set aside is its value as a source of aesthetic pleasure, or intellectual training, or the delight which often accompanies solving a difficult exercise in mental gymnastics. Though each of these is a reason to value mathematics, they are not under consideration here.

[51]Cf. Field [1984] ([1989] ch. 3) *passim* Field does allow that empirical knowledge (including knowledge about what is accepted in the mathematical community) will also be among the things that "separate those who know lots of mathematics from those who know only a little". (cf [1989] p. 113)

[52]Other examples of such statements of the issue are easy to find. As another example consider Carnap [1959] p.143

[53]The expression "conceptual setting" and the idea behind it is borrowed from Manders [1987].

[54]It is worth noting that Dummett alludes explicitly to the focal passage in the course of discussing Frege's attempt to bridge the gap, though he does not yet see how much detail is just below the surface.

Philosophers have traditionally stressed . . . the brevity of the gap between premisses and conclusion in a single inference. Frege, with his emphasis on the fruitfulness of deduction . . . was exceptional in stressing the contrasting feature of deductive inference. As Mill complains, however, few philosophers have made any serious attempt to relieve the tension between the two features: to resort to metaphor, as Frege did, and say that the conclusion is contained in the premisses "as plants are contained in their seeds, not as beams are contained in a house" is of no great help; we want to know how the metaphor is applied.(Dummett [1973] p. 300)

Dummett [1991a], on the other hand, contains an extensive and helpful discussion of what Frege is up to which effectively retracts the negative assessment in [1973].

[55]Dummett continues as follows: "We know, of course, that a man may walk from Paris to Rome, and yet that a single pace will not take him appreciably closer: but epistemic distance is more puzzling to us than spatial distance." It is worth noting that Frege is just as interested in extending knowledge with single inferences/single steps as with long proofs/walks to Rome. So the point that small advances can over the course of a long proof add up to big surprises does not exhaust the issue Frege takes himself to be addressing.

[56]The example of Galois theory is discussed in connection with the idea of mathematical explanation in Kitcher [1989] and in connection with the idea of conceptual setting in Manders [1987].

[57]Though I am deeply indebted to Manders [1987], I do not mean to suggest that Manders understands the notion of conceptual setting in exactly the way it is taken here.

[58]I borrow this last example from Wilson ([1992] p. 151)

[59]Consider an analogous question of assessing scientific explanation, which can be understood to have the same "semi-objective" status. We expect a good explanation to have virtues which pertain to the psychology of people like us: perhaps it renders an event comprehensible to us, or helps us to see an event within a more unified picture of the world. It is hard to imagine a good explanation which would not have some such psychological effect. But it may be that what it is to be a good explanation is not to be accounted for solely in terms of these psychological effects. Such questions pertaining to notions of explanation in connection with mathematics have been explored in the work of Mark Steiner. (See for example Steiner [1978], also Kitcher [1989]) Although I think that Steiner's work is very illuminating, I suspect that the notion of explanation is perhaps not the right one to appeal to in order to account for the aspects of mathematical practice that Steiner quite rightly calls attention to as worthy of study.

[60]This is, of course, a deep question, and the attention devoted here can represent only a preliminary response. I am currently preparing a series of papers that address the issue in more detail.

[61]Of course it would hardly pay to expend much energy disputing the application of a term of art like "epistemology", but it is worth noting that two different conceptions of the objectives of a study of knowledge seem to be presupposed by the disputants. On the conceptual setting view, the fact that such and such inferences give the practice its value for us should also be taken as basic and any account of mathematical knowledge and understanding must take this into account. If we are to reserve "epistemology" for the investigations with a more circumscribed focus on justification as a means of conferring warranted confidence, this should not prevent us from acknowledging the philosphical interest of the broader sort of investigation Frege clearly has in mind, in which knowledge and justification should be studied not just as more and less reliable, but also with reference to understanding.

[62]See for example Greub [1975] p.5

[63]Examples of this are more complicated to spell out, and so I must leave working out any in detail for work in progress. Several of the cases studied in Manders [1987] are helpful illustrations, especially his analysis of geometries in terms of model completion and his discussion of valuation theory.

[64]Again, this is not to say that there are not cases which are exclusively one or the other, but only that there is substantial overlap.

[65]For more arguments that tend in this direction, and some detailed and compelling case studies, see Wilson [1993] and [1994].

[66]The idea of logical analysis as an investigation of the idea of "self-evidence" is also brought up in [FA] (102–103).

[67]Of course, the idea that the foundations of mathematics should concern itself with the character of *evidence* (in the sense of "being evident" rather than "empirical evidence") is a recurrent motif in Kreisel's writings. See, for example, the extended reflection on the theme that the theory of proofs should concern itself with the nature of proofs rather than "merely the consequence relation" in Kreisel [1971]. The issues this paragraph raises are, of course, in need of further attention. I touch on some ways that Frege addresses such questions of evidence with specific reference to the distinction between geometrical and arithmetical reasoning in Tappenden [1995].

[68]See above cited bit on p. 55 of [B]; also [BLA] p. 2, etc.

[69]This formulation of the problem is adopted in Eva Picardi's helpful note Picardi [1988].

[70]This is not to suggest that there are not additional sources of tension between the early and late writings on definition. Quite the opposite: in at least one respect—the question of whether definitions require or admit justification—the early and late writings are difficult to reconcile. See for example [FA] p.ix in contrast to [LM].

[71]I am indebted to Weiner [1984] and [1990] on this point.

[72]For example, he says the following in the *Begriffsschrift* of a definition which is "fruitful" in the sense of [BLC] and [FA]:

But we can do without the notation introduced by this proposition and hence without the proposition itself as its definition; nothing follows from the proposition that could not also be inferred without it. Our sole purpose in introducing these definitions is to bring about an extrinsic simplification by stipulating an abbreviation.([B] p.55)

[73]A similar distinction is made in ([FG], p.302)

[74]This is the response of Grossmann [1969], and following him Proust [1989] (pp.122–133). Very early in my thinking about the focal passage I was inclined to see the constructive/analytic distinction as holding the key to the early remarks on definitions, until sharp and skeptical questioning from Steve Glaister led me to think otherwise.

[75](Cf. ([SM] p. 166–168) I am here indebted to Horty [1993].

[76]Of course, in cases where the conclusion is a weakening of the premisses, such as: A&B⊢ A the sense of premiss and conclusion will not be *identical*, but the sense of the conclusion must be "contained in" the premiss, in some way to be captured in the compositional principle.

References

Works of Frege:
(Date of publication (composition for unpublished writings) follows the references)

[ACN] "On the Aim of the 'Conceptual Notation'" in Bynum (1972) (1882)

[B] *Begriffschrift, a Formula Language, Modeled on that of Arithmetic, For Pure Thought* Stephan Bauer-Mengelberg, trans. in From Frege to Godel, van Heijenoort, J. (ed.) Cambridge: Harvard University Press 1967 (1879)

[BLA] *The Basic Laws of Arithmetic: Exposition of the System* Furth, M. trans. Berkeley: University of California Press 1964 (1893)

[BLC] "Boole's Logical Calculus and the Concept-Script" in *Posthumous Writings* (1880/81)

[CO] "On Concept and Object" in *Collected Papers* (1893)

[Corr] *Philosophical and Mathematical Correspondence* Gabriel, G., Hermes, H., Kambartel, F., Thiel, C. and Verhaart, A. Abridged from the German edition by McGuinness, B. Translated by Kaal, H. Chicago: University of Chicago Press, 1980

[FA] *The Foundations of Arithmetic* J.L.Austin, trans. 2nd rev. ed. Evanston: Northwestern University Press 1980 (1884)

[FC] "Function and Concept" in *Collected Papers* (1891)

[FG] "On the Foundations of Geometry: First Series" in *Collected Papers* (1903)

[FG2] "On the Foundations of Geometry: Second Series" in *Collected Papers* (1906)

[FTA] "On Formal Theories of Arithmetic" in *Collected Papers* (1885)

[GRI] "On a Geometrical Representation of Imaginary Forms in the Plane" in *Collected Papers* (1873)

[L2] "Logic" (second of two essays with that title) in *Posthumous Writings* (1897)

[LM] "Logic in Mathematics" in *Posthumous Writings* (1914)

[LGP] "Lecture on the Geometry of Pairs of Points in the Plane" *Collected Papers* (1883)

[LWC] "Lecture on a Way of Conceiving the Shape of a Triangle as a Complex Quantity" in *Collected Papers* (1878)

[MCE] "Methods of Calculation Based on an Extension of the Concept of Quantity" in *Collected Papers* (1874)

[NLD] "Notes for Ludwig Darmsteader" in *Posthumous Writings* (1919)
[PCN] "On Mr. Peano's Conceptual Notation and My Own" in *Collected Papers* (1897)
[RC] Review of H. Cohen, *Das Princip der Infinitesimal-Methode und Seine Geschichte* in *Collected Papers* (1885)
[RGW] Review of Gall and Winter: *Die Analytische Geometrie des Punktes und der Geraden und ihre Anwendung auf Aufgaben* in *Collected Papers* (1877)
[SJCN] "On the Scientific Justification of a Conceptual Notation" (Bynum, T. trans.) Reprinted in Bynum (1972) (1882)
[SM] "On Sense and Meaning" in *Collected Papers* (1892)
[Th] "Thoughts" in *Collected Papers* (1918)
[WF] "What is a Function?" in *Collected Papers* (1904) Collections referred to above: *Collected Papers on Mathematics, Logic and Philosophy* (1984) McGuinness, B. (ed.) Oxford: Basil Blackwell *Posthumous Writings* (1979) Hermes, H. Kambartel, F. and Kaulbach, F. (eds.) Oxford: Basil Blackwell

Other References:

Ayer, A. (1952). *Language, Truth and Logic* (2nd ed.) New York: Dover (1959). (ed.) *Logical Positivism* New York: The Free Press
Baker, G. and Hacker, P. (1984). *Frege: Logical Excavations* New York: Oxford University Press
Benacerraf, P. (1981). "Frege: The Last Logicist" in *Midwest Studies in Philosophy vol. 6*
French, P. Uehling, T. and Wettstein, H. (eds.) Minneapolis: University of Minnesota Press
Birkhoff, G. (1973). *A Source Book in Classical Analysis* Cambridge Ma: Harvard University Press
Bottozini, U. (1986). *The Higher Calculus: A History of Real and Complex Numbers from Euler to Weierstrass* New York: Springer
Boyer, C. (1956). *A History of Analytic Geometry* Princeton Junction, NJ: Scholar's Bookshelf
Burge, T. (1977). "Belief *De Re*" *Journal of Philosophy* 74 338–362
Burgess, J. (1984). Review of Wright (1983). *Philosophical Review* 93 638—640 (1993). "Hintikka et Sandu in re Arbitrary Functions" *Philosophica Mathematica* 50–65
Bynum, T. (1972). "On the Life and Work of Gottlob Frege" in *Conceptual Notation and Related Articles* Bynum T. trans. and ed. Oxford: Clarendon 1–54
Carnap, R. (1959). "The Old and the New Logic" Levi, I. (trans.) in Ayer (1959)
Clebsch A. and Lindemann F. (1876). *Vorlesungen über Geometrie* (lectures by Clebsch transcribed and edited by Lindemann) (vol. 1) Leipzig
Coolidge, J. (1924). *The Geometry of the Complex Domain* Oxford: Clarendon Press
(1940). *A History of Geometrical Methods* Oxford: Clarendon Press
(1945). *A History of the Conic Sections and Quartic Surfaces* Oxford: Clarendon Press
Crittendon, C. (1993). Review of Dummett (1991a). *Review of Metaphysics* XLVI #3 607—608
Currie, G. (1982). *Frege: An Introduction to his Philosophy* Totowa N.J.: Barnes and Noble Demopoulos, W. (1994). "Frege and the Rigorization of Analysis" *Journal of Philosophical Logic* 23 225–245
Dummett, M. (1973). "The Justification of Deduction" Reprinted in Dummett (1978)
(1978). *Truth and Other Enigmas* Cambridge Mass.: Harvard University Press
(1981). *The Interpretation of Frege's Philosophy* Cambridge Mass.: Harvard University Press
(1991a). *Frege: Philosophy of Mathematics* Cambridge Mass.: Harvard University Press
(1991b). "Frege and the Paradox of Analysis" in *Frege and other Philosophers* Oxford: Oxford University Press
(1992). Review of Weiner (1990). *Journal of Symbolic Logic*

Field H. (1984). "Is Mathematical Knowledge just Logical Knowledge?" reprinted in Field (1989). 79–124

(1989). *Realism, Mathematics and Modality* Oxford: Basil Blackwell

Friedman, M. (1992). *Kant and the Exact Sciences* Cambridge Mass.: Harvard University Press

Grattan-Guinness, I. (1970). *The Development of the Foundations of Analysis from Euler to Riemann* Cambridge Mass.: MIT Press

(1975). "Preliminary Notes on the Historical Significance of Quantification and of the Axioms of Choice in the Development of Mathematical Analysis" *Historia Mathematica* 2 475—488

Gray, J. (1989). "Algebraic Geometry in the late Ninteenth Century" in Rowe, D and McCleary, J. eds. *The History of Modern Mathematics vol.I* Boston: Academic Press 361–385

Greub, W. (1978). *Multilinear Algebra* (2nd ed.) New York: Springer

Grossmann, R. (1969). *Reflections on Frege's Philosophy* Evanston: Northwestern University Press

Hahn, H. (1959). "Logic, Mathematics, and Knowledge of Nature" (Arthur Pap, trans.) In Ayer, A. ed. (1959)

Hawkins, T. (1989). "Jacobi and the Birth of Lie's Theory of Groups" in *Archive for the History of the Exact Sciences* 187—276

(1989b). "Line Geometry, Differential Equations, and the Birth of Lie's Theory of Groups" in Rowe, D and McCleary, J. eds. *The History of Modern Mathematics vol.I* Boston: Academic Press 275—327

Horty, J. (1986). *Some Aspects of Meaning in Non-contingent Language* Phd thesis University of Pittsburgh

(1993). "Frege on the Psychological Significance of Definitions" *Philosophical Studies 72* 223–263

Kant, I. (1965). *Critique of Pure Reason* (Smith, N. Trans.) New York: St. Martin's Press

Kaplan, D. (1968). "Quantifying In" *Synthèse* 19 178–214

Katz, J. (1992). "The New Intensionalism" *Mind* 101 689 -719,

Kitcher, P. (1979). "Frege's Epistemology" *Philosophical Review* 88 235–262

(1984). *The Nature of Mathematical Knowledge* Oxford: Oxford University Press

(1989). "Explanatory Unification and the Causal Structure of the World" in P. Kitcher and W. Salmon (eds.) *Scientific Explanation* Minneapolis: University of Minnesota Press

Kreisel, G. (1967). "Informal Rigour and Completeness Proofs" in *Problems in the Philosophy of Mathematics* I. Lakatos, ed. Amsterdam: North-Holland

(1971). "Survey of Proof Theory II" in J.Fensted, ed. *Second Scandinavian Logic Symposium* Amsterdam: North-Holland

Manders, K. (1987). "Logic and Conceptual Relationships in Mathematics" in *Logic Colloquium '85* Amsterdam: Elsevior

Nagel, E. (1979). "The Formation of Modern Conceptions of Formal Logic in the Development of Geometry" in *Teleology Revisited and Other Essays in the Philosophy and History of Science* New York: Columbia University Press (originally published in 1939)

Otte, M. (1989). "The Ideas of Hermann Grassmann in the Context of the Mathematical and Philosophical Tradition since Leibniz" *Historia Mathematica* 16 p.1–35

Patzig, G. (1969). "Leibnitz, Frege, und die sogennannte '*lingua characteristica universalis*'" *Studia Leibnitiana Supplementa* 3 103–109

Picardi, E. (1988). "Frege on Definition and Logical Proof" in Cellucci C. and Sambin, G.(eds.) *Temi e Prospettive della Logica e della Filosofia della Scienza Contemporanee* vol. 1 Bologna 227–230

Plücker J. (1865). "On a New Geometry of Space" *Philosophical Transactions of the Royal Society of London* p.725—791 (1868)

(1868). *Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement* v.1 (A.Clebsch, ed.), (v.2 F.Klein, ed.)

Proust, J. (1989). *Questions of Form: Logic and the Analytic Proposition from Kant to Carnap* Brenner, A. (trans.) Minneapolis: University of Minnesota Press

Putnam, H. (1967). "Mathematics Without Foundations" reprinted in Putnam, H. *Mathematics, Matter, and Method* 43—60

Ricketts, T. (1986). "Generality, Meaning and Sense" *Pacific Philosophical Quarterly* 172–195

Salmon, N. (1981). *Reference and Essence* Princeton: Princeton University Press

Sluga, H. (1980). *Gottlob Frege* London: Routledge and Kegan Paul

(1987). "Frege Against the Booleans" *Notre Dame Journal of Formal Logic* 28 80—98

Smith, D. (1959). *A Source Book in Mathematics* Mineola NY: Dover Publications

Steiner, J. (1832). *Systematische Entwickelungen der Abhangigkeit Geometrische Geschicten* Berlin

Steiner M. (1978). "Mathematical Explanation" *Philosophical Studies* 34 135–151

Tappenden, J. (1995). "Geometry and Generality in Frege's Philosophy of Arithmetic" *Synthèse* 102, 319–361

Taschek, W. (1992). "Frege's Puzzle, Sense, and Information Content" *Mind* 101 767–791

Torretti, R. (1984). *The Philosophy of Geometry From Riemann to Poincaré* Dordrecht: D.Riedel

van Heijenoort, J. (1967). "Logic as Calculus and Logic as Language" *Synthèse* 17

Wagner, S. (1992). "Logicism" in M.Detlefsen, (ed.) *Proofs and Knowledge in Mathematics* London: Routledge 65–110

Weiner, J. (1984). "The Philosopher Behind the Last Logicist" In C. Wright ed. *Frege: Tradition and Influence* Oxford: Blackwell 57–79

(1990). *Frege in Perspective* Ithaca: Cornell University Press

Wilson, M. (1992). "Frege: The Royal Road from Geometry" *Noûs* 26:2 149–180

(1993). "Honorable Intensions" in Wagner, S. and Warner, R. (eds.) *Naturalism: A Critical Appraisal* Notre Dame: Notre Dame University Press 53–94

(1994). "Should we Trust Logical Form?" *Journal of Philosophy XCI* #10 p. 519–544