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A Theory of Infinitary Relations Extending Zermelo's Theory of Infinitary Propositions

Abstract. An idea attributable to Russell serves to extend Zermelo's theory of systems of infinitely long propositions to infinitary relations. Specifically, relations over a given domain \mathfrak{D} of individuals will now be identified with propositions over an auxiliary domain \mathfrak{D}^* subsuming \mathfrak{D} . Three applications of the resulting theory of infinitary relations are presented. First, it is used to reconstruct Zermelo's original theory of urelements and sets in a manner that achieves most, if not all, of his early aims. Second, the new account of infinitary relations makes possible a concise characterization of parametric definability with respect to a purely relational structure. Finally, based on his foundational philosophy of the primacy of the infinite, Zermelo rejected Gödel's First Incompleteness Theorem; it is shown that the new theory of infinitary relations can be brought to bear, positively, in that connection as well.

Keywords: Definable set, Incompleteness Theorem for Peano Arithmetic, Infinitary relation, Russell's substitutional theory, Zermelo set theory.

1. Introduction

Zermelo is widely recognized for having introduced, *early* in his career, persuasive axioms describing the realm of urelements and sets that had been investigated by Cantor and Dedekind. Quite unrecognized, in contrast, is Zermelo's theory of infinitely long propositions, the focus of three papers published at the *end* of his career. The latter theory, like the axiomatization, was to make a difference for the foundations of mathematics, as Zermelo saw things. That said, Zermelo is in need, in both contexts, of a theory of relations but, in fact, has no such theory, early or late. This deficiency is one we seek to remedy in this paper. Moreover, we shall accomplish this in a manner that, perhaps surprisingly, unifies Zermelo's contributions, early and late, to the foundations of mathematics. The enabling element here will be an idea, due to Russell, that we will now describe briefly.

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Russell's ultimate solution to the paradoxes was the ramified theory of types of *Principia*. Along the way, however, he considered, and soon rejected, other ideas, among them a nominalistic theory known in the literature as the "substitutional theory" (see [10]). Russell's goal was to get the effect of classes or concepts while working exclusively with propositions and their constituents. (Russell held that Socrates, the man, is a constituent of the proposition that Socrates is mortal.) Where a is Socrates and p is the proposition that Socrates is mortal, Russell writes " $p/a; b$ " for the result of substituting b for constituent a within p . So " $b \in \{x \mid x \text{ is mortal}\}$ " then asserts only that proposition $p/a; b$ is true. While it may be useful heuristically to regard nondenoting complex " p/a " as a name for the class of mortals, the mortality concept, or the propositional function " x is mortal," Russell's theory posits no such entities and is variable-free, as he noted.

We shall draw inspiration, and substance, from Russell's substitutional theory as we proceed to extend Zermelo's theory of infinitely long propositions, described in Sect. 2, to a theory of infinitary relations, set out in Sect. 3. Three applications of the new theory then follow. First, in Sect. 4, it is used to reconstruct Zermelo's early axiomatization of the theory of urelements and sets, wherein an informal notion of "definite property" figures prominently and problematically. Second, a concise characterization of parametric definability with respect to a purely relational structure is presented in Sect. 5, after which we turn our attention to arithmetic.

The fresh account of relations yields an alternate description, in Sect. 6, of the set of infinitely long propositions that are expressible in the language of first-order Peano arithmetic. That then leads to a third application of our theory, in Sect. 7, where it is argued that said set of propositions constitutes the appropriate context in which to assess the significance, for infinitary logic, of Gödel's First Incompleteness Theorem. Our argument is offered as a corrective in light of Zermelo's unhappy framing of that result.

Zermelo's theory of infinitely long propositions is variable-free, and we follow him, and Russell, in this regard as we extend that theory to relations. Others have sought to develop mathematical logic in the absence of variables (see [12] and the literature regarding combinators). Even a fervent believer in variables may ask, Can a theory of relations manage without them? As Russell showed, the answer is yes: Variables themselves are not essential, whereas some notion of substitution is key.

2. Zermelo's Systems of Infinitely Long Propositions

We let a_1, a_2, \dots (or a, b, c, \dots) be arbitrary elements of nonempty domain $\mathfrak{D} = \{a, b, c, \dots\}$. (Thus a, b, c, \dots are particular domain individuals, whereas a, b, \dots are generic individuals.) We let catalogue K ("kappa") be p -tuple $\langle R_1, \dots, R_p \rangle$, where $p \geq 1$ and *fundamental predicate* R_ℓ with $1 \leq \ell \leq p$ is n_ℓ -adic with finite degree $n_\ell \geq 0$. We emphasize that R_ℓ is not a predicate *symbol* but, rather, an abstract, degree-and-order-indicating concept or relation analogous to "is a line," "is a point," and "lies between" in informal expositions of plane geometry.

Zermelo base $\mathfrak{G}_{\mathfrak{D},K}$ is the set of all atomic propositions over domain \mathfrak{D} and catalogue K . For example, proposition Rab is in $\mathfrak{G}_{\mathfrak{D},K}$ if $a, b \in \mathfrak{D}$ and dyadic R is a member of K . Domain elements a and b , as well as R , are *constituents* of Rab , and the former are understood to be intrinsically ordered within it—first a and then b .

Zermelo system $\mathcal{H}_{\mathfrak{D},K}$ is the well-founded hierarchy built up from $\mathfrak{G}_{\mathfrak{D},K}$ by closing under negation and infinitary disjunction (see [19]). Specifically, where ψ is the unary cardinal-valued function defined by

$$\psi(\zeta) =: \begin{cases} 2 & \text{if } \zeta = 0 \\ 2^{\psi(\xi)} & \text{if } \zeta = \xi + 1 \\ \bigcup_{\xi < \zeta} \psi(\xi) & \text{otherwise,} \end{cases}$$

we define level P_α for arbitrary ordinal α by writing

$$P_\alpha =: \begin{cases} \mathfrak{G}_{\mathfrak{D},K} & \text{if } \alpha = 0 \\ \{\neg \mathcal{A} \mid \mathcal{A} \in P_\beta\} \cup \{\bigvee J \mid J \subseteq P_\beta \wedge |J| \leq \psi(\beta)\} & \text{if } \alpha = \beta + 1 \\ \bigcup_{\beta < \alpha} P_\beta & \text{otherwise.} \end{cases}$$

Then $\mathcal{H}_{\mathfrak{D},K} =: \bigcup_{\alpha \text{ in On}} P_\alpha$ is the Zermelo system comprising all *propositions over domain \mathfrak{D} and catalogue K* . We write $\mathcal{A} \vee \mathcal{B}$ or $\mathcal{B} \vee \mathcal{A}$ indifferently for $\bigvee \{\mathcal{A}, \mathcal{B}\}$ and identify $\bigvee \{\mathcal{A}\}$ with \mathcal{A} so that $\mathcal{H}_{\mathfrak{D},K}$ constitutes a cumulative hierarchy. Conjunction $\bigwedge J$ is defined as $\neg \bigvee \{\neg \mathcal{B} \mid \mathcal{B} \in J\}$.

If $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},K}$ is $\neg \mathcal{B}$, then \mathcal{B} is an *immediate subproposition* of \mathcal{A} . If \mathcal{A} is $\bigvee J$, then each member of J is an *immediate subproposition* of \mathcal{A} . A set of propositions is said to be *transitive* if it is closed under the immediate subproposition relation. We then write $\text{subprop}(\mathcal{A})$ for the smallest transitive set containing \mathcal{A} so that $\text{subprop}(\mathcal{A}) = \{\mathcal{A}\}$ if \mathcal{A} is in $\mathfrak{G}_{\mathfrak{D},K}$.

With $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},K}$ we set $\text{rank}(\mathcal{A}) =: \bigcup \{\text{rank}(\mathcal{B}) + 1 \mid \mathcal{B} \in \text{subprop}(\mathcal{A}) \setminus \{\mathcal{A}\}\}$. Empty disjunction \perp and any member of $\mathfrak{G}_{\mathfrak{D},K}$ are then of rank 0, and empty conjunction \top is of rank 1.

Since $\mathcal{H}_{\mathfrak{D},K}$ is as big as the class of all ordinals, we shall emphasize an unbounded sequence of limited perspectives. Thus, the (strongly) inaccessible initial segment $\mathcal{H}_{\mathfrak{D},K}^\theta =: \bigcup_{\alpha < \theta} P_\alpha = P_\theta$ of system $\mathcal{H}_{\mathfrak{D},K}$ will be the *Zermelo logic of characteristic θ over \mathfrak{D} and K* . Any logic strictly above $\mathcal{H}_{\mathfrak{D},K}^\omega$ contains propositions of infinite length. (Like Zermelo, we count ω as the least inaccessible.) On the other hand, if propositions are represented as rooted trees with interior nodes labeled by \neg and \bigvee , then any proposition is well-founded in the sense that all paths are finite. Ensuing definitions and theorems will presuppose nonempty domain \mathfrak{D} , catalogue $K = \langle R_1, \dots, R_p \rangle$ with $p \geq 1$, and, as structural parameter, some fixed inaccessible θ . Accordingly, one can easily prove the fundamental claim formulated neatly as

THEOREM 2.1. *If $J \subseteq \mathcal{H}_{\mathfrak{D},K}^\theta$, then $\bigvee J \in \mathcal{H}_{\mathfrak{D},K}^\theta$ if and only if $|J| < \theta$.*

We write $\mathfrak{D}^n 2$ for the set of all n -ary Boolean-valued functions with arguments in \mathfrak{D} . In particular, $\text{Id}_{\mathfrak{D}}^{(2)} \in \mathfrak{D}^2 2$ is the binary function mapping $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathfrak{D}^2$ to 1 just in case $\mathbf{a} = \mathbf{b}$. We let $\mathfrak{T}_{\mathfrak{D}}^{n_1 \dots n_p}$ (or $\mathfrak{T}_{\mathfrak{D}}^{\vec{n}}$) denote $\mathfrak{D}^{n_1} 2 \times \dots \times \mathfrak{D}^{n_p} 2$.

DEFINITION 2.2. Suppose that $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},K}^\theta$ and let $\mathcal{M} =: \langle f_1^{(n_1)}, \dots, f_p^{(n_p)} \rangle \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}}$. Then we define the *semantic value* $\text{val}(\mathcal{A}, \mathcal{M})$ of \mathcal{A} in \mathcal{M} as follows:

1. if \mathcal{A} is atomic $R_\ell \mathbf{a}_1 \dots \mathbf{a}_{n_\ell}$ with $1 \leq \ell \leq p$ and $\mathbf{a}_1, \dots, \mathbf{a}_{n_\ell} \in \mathfrak{D}$, then $\text{val}(\mathcal{A}, \mathcal{M}) = \text{true}$ if $f_\ell^{(n_\ell)}(\mathbf{a}_1, \dots, \mathbf{a}_{n_\ell}) = 1$ and $\text{val}(\mathcal{A}, \mathcal{M}) = \text{false}$ otherwise
2. if \mathcal{A} is $\neg \mathcal{B}$ with $\mathcal{B} \in \mathcal{H}_{\mathfrak{D},K}^\theta$, then we have that $\text{val}(\mathcal{A}, \mathcal{M}) = \text{true}$ if and only if $\text{val}(\mathcal{B}, \mathcal{M}) = \text{false}$
3. if \mathcal{A} is $\bigvee J$, where $J \subseteq \mathcal{H}_{\mathfrak{D},K}^\theta$ with $|J| < \theta$, then $\text{val}(\mathcal{A}, \mathcal{M}) = \text{true}$ if and only if $\text{val}(\mathcal{B}, \mathcal{M}) = \text{true}$ for some $\mathcal{B} \in J$.

(We identify truth-values false and true with subsets of $\{\{\}\}$.)

We write $\mathcal{M} \models \mathcal{A}$, and say that *model \mathcal{M} satisfies proposition \mathcal{A}* , if $\text{val}(\mathcal{A}, \mathcal{M}) = \text{true}$. We set $\text{Mod}(\mathcal{A}) =: \{\mathcal{M} \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}} \mid \mathcal{M} \models \mathcal{A}\}$ and, with $J \subseteq \mathcal{H}_{\mathfrak{D},K}^\theta$, we let $\text{Mod}(J) =: \bigcap \{\text{Mod}(\mathcal{B}) \mid \mathcal{B} \in J\}$. (Hence $\text{Mod}(\emptyset) = \mathfrak{T}_{\mathfrak{D}}^{\vec{n}}$.) Proposition $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},K}^\theta$ is a *semantic consequence* of $J \subseteq \mathcal{H}_{\mathfrak{D},K}^\theta$ if $\text{Mod}(J) \subseteq \text{Mod}(\mathcal{A})$. Propositions $\mathcal{A}, \mathcal{B} \in \mathcal{H}_{\mathfrak{D},K}^\theta$ are *equivalent* if $\text{Mod}(\mathcal{A}) = \text{Mod}(\mathcal{B})$.

We write K_{\equiv} for *identity catalogue* $\langle R_1, \dots, R_p, I \rangle$ such that proposition $I \mathbf{a} \mathbf{b}$, for which we write $\mathbf{a} \equiv \mathbf{b}$, has semantic value true in arbitrary $\mathcal{M} =: \langle f_1^{(n_1)}, \dots, f_p^{(n_p)}, \text{Id}_{\mathfrak{D}}^{(2)} \rangle$ just in case $\text{Id}_{\mathfrak{D}}^{(2)}(\mathbf{a}, \mathbf{b}) = 1$. We write $\mathfrak{T}_{\mathfrak{D}}^{\vec{n}, \equiv}$ for the set of $(p + 1)$ -tuples of the form $\langle f_1^{(n_1)}, \dots, f_p^{(n_p)}, \text{Id}_{\mathfrak{D}}^{(2)} \rangle$. As in the case of Definition 2.2, we shall regularly omit explicit consideration of identity

catalogues. When talk of \equiv arises, the reader should understand an identity catalogue and obvious extensions of definitions.

The theory of systems of infinitary relations presented below, like the theory of systems of infinitely long propositions that it extends, assumes ordinal numbers and transfinite induction over them. We take the view that Cantor's ordinal numbers constitute an autonomous domain: They are not to be identified with sets but, rather, are merely represented by them. This seems to coincide with the view of Zermelo, late in his career, who urged "the existence of an unbounded sequence of [strongly inaccessible cardinals] as a new axiom for the metatheory of sets" ([18, p. 429]; see also [20, p. 591] for an analogous, if elliptical, remark). Incidentally, Zermelo's remark by itself implies no more than an ω -sequence of inaccessibles. In contrast, the stronger assumption known as "Tarski's Axiom of Inaccessibles" immediately yields, within any model of (first- or second-order) ZF plus the axiom, a transfinite, strictly increasing sequence of inaccessibles, one for each ordinal of the model.

3. A Substitutional Theory of Infinitary Relations

We next show how a domain-dependent notion of infinitary relation can be derived from Zermelo's domain-dependent characterization of infinitely long propositions as just described. To that end, we let *auxiliary domain* \mathfrak{D}^* be disjoint union $\mathfrak{D} \sqcup \theta$ so that $\mathfrak{D} \subsetneq \mathfrak{D}^*$ and $|\mathfrak{D}^* \setminus \mathfrak{D}| = \theta$. (Note that \mathfrak{D} and θ together fix \mathfrak{D}^* and that, possibly, $\theta \leq |\mathfrak{D}| \leq |\mathfrak{D}^*|$.) *Auxiliary (Zermelo) logic* $\mathcal{H}_{\mathfrak{D}^*, \mathbb{K}}^\theta$ will then be the hierarchy of "auxiliary propositions" built up from Zermelo base $\mathfrak{G}_{\mathfrak{D}^*, \mathbb{K}}$, as in Sect. 2. Auxiliary logic $\mathcal{H}_{\mathfrak{D}^*, \mathbb{K}}^\theta$, more properly its domain \mathfrak{D}^* , will possess a new, important feature, however.

Namely, we shall assume a well-ordered linearization $(a_\beta^*)_{\beta < \theta}$ of $\mathfrak{D}^* \setminus \mathfrak{D}$. (We thereby invoke Global Choice.) We write \prec for said ordering so that $a_0^* \prec a_1^* \prec a_2^* \prec \dots$. We write $\mathfrak{a}^*, \mathfrak{b}^*, \dots$ and the like for generic members of \mathfrak{D}^* . (Again, neither the terms of $(a_\beta^*)_{\beta < \theta}$ nor $\mathfrak{a}^*, \mathfrak{b}^*, \dots$ are symbolic constants; rather, they indicate members of $\mathfrak{D}^* \setminus \mathfrak{D}$, respectively \mathfrak{D}^* .)

As described below, propositions of auxiliary logic $\mathcal{H}_{\mathfrak{D}^*, \mathbb{K}}^\theta$ will do the work of level-1 relations over \mathfrak{D} and \mathbb{K} . (The relata of level-1 relations, in a sense suggested by that of Frege, are members of a fixed domain of individuals—*ground domain* \mathfrak{D} , in the present instance.) Members of $\mathfrak{D}^* \setminus \mathfrak{D}$ figuring in propositions of $\mathcal{H}_{\mathfrak{D}^*, \mathbb{K}}^\theta$ will constitute "unsaturated positions" available for uniform replacement by elements of \mathfrak{D}^* , in particular, by elements of \mathfrak{D} .

By a (*simultaneous*) *substitution* we shall mean any function $s : \mathfrak{D}^* \mapsto \mathfrak{D}^*$ fixing all elements of \mathfrak{D} . Given $\mathcal{A}^* \in \mathcal{H}_{\mathfrak{D}^*, \mathbb{K}}^\theta$ and substitution s , we define

substitution instance $s(\mathcal{A}^*)$ of \mathcal{A}^* inductively. If \mathcal{A}^* is atomic $R_\ell \mathbf{a}_1^* \dots \mathbf{a}_{n_\ell}^*$, then $s(\mathcal{A}^*)$ is $R_\ell s(\mathbf{a}_1^*) \dots s(\mathbf{a}_{n_\ell}^*)$. If \mathcal{A}^* is $\neg \mathcal{B}^*$ with $\mathcal{B}^* \in \mathcal{H}_{\mathfrak{D}^*, \mathbf{K}}^\theta$, then $s(\mathcal{A}^*)$ is $\neg s(\mathcal{B}^*)$. If \mathcal{A}^* is $\bigvee J$ with $J \subseteq \mathcal{H}_{\mathfrak{D}^*, \mathbf{K}}^\theta$ satisfying $|J| < \theta$, then $s(\mathcal{A}^*)$ is disjunction $\bigvee \{s(\mathcal{B}^*) \mid \mathcal{B}^* \in J\}$. In general, substitution instance $s(\mathcal{A}^*)$ of $\mathcal{A}^* \in \mathcal{H}_{\mathfrak{D}^*, \mathbf{K}}^\theta$ is a proposition of $\mathcal{H}_{\mathfrak{D}^*, \mathbf{K}}^\theta$. Should $s(\mathcal{A}^*)$ be in $\mathcal{H}_{\mathfrak{D}, \mathbf{K}}^\theta$, we speak of a *fully saturated substitution instance* of \mathcal{A}^* .

Where $\mathcal{A}^* \in \mathcal{H}_{\mathfrak{D}^*, \mathbf{K}}^\theta$, we let $S_{\mathcal{A}^*}$ be the set of all fully saturated substitution instances of \mathcal{A}^* , all of which are of one and the same rank within $\mathcal{H}_{\mathfrak{D}, \mathbf{K}}^\theta$, as shown by an easy inductive argument. (Two arbitrary members of $S_{\mathcal{A}^*}$ will, in general, be nonequivalent, however.) If, to take the simplest example, $R_\ell \mathbf{a}_1^* \dots \mathbf{a}_{n_\ell}^*$ and $R_\ell \mathbf{b}_1^* \dots \mathbf{b}_{n_\ell}^*$ are members of $\mathfrak{G}_{\mathfrak{D}^*, \mathbf{K}}$ such that both (1) \mathbf{a}_i^* is \mathbf{a}_j^* just in case \mathbf{b}_i^* is \mathbf{b}_j^* for $1 \leq i, j \leq n_\ell$ and (2) $\mathbf{a}_i^* \in \mathfrak{D}$ or $\mathbf{b}_i^* \in \mathfrak{D}$ implies that \mathbf{a}_i^* is \mathbf{b}_i^* for any $1 \leq i \leq n_\ell$, then $S_{R_\ell \mathbf{a}_1^* \dots \mathbf{a}_{n_\ell}^*} = S_{R_\ell \mathbf{b}_1^* \dots \mathbf{b}_{n_\ell}^*}$. (Thus $R_\ell \mathbf{a}_1^* \dots \mathbf{a}_{n_\ell}^*$ and $R_\ell \mathbf{b}_1^* \dots \mathbf{b}_{n_\ell}^*$ are “unifiable” in a well-known sense.)

With $\mathcal{A}^*, \mathcal{B}^* \in \mathcal{H}_{\mathfrak{D}^*, \mathbf{K}}^\theta$ let us write $\mathcal{A}^* \simeq \mathcal{B}^*$ (verbalized as “ \mathcal{A}^* and \mathcal{B}^* are similar”) whenever $S_{\mathcal{A}^*} = S_{\mathcal{B}^*}$. A first idea would be to identify relations over \mathfrak{D} and \mathbf{K} with equivalence classes modulo \simeq . However, fixing the degree of relations, so construed, turns out to require daunting machinery. We shall do something simpler and yet similarly nominalistic in spirit.

Namely, propositions of $\mathcal{H}_{\mathfrak{D}^*, \mathbf{K}}^\theta$ will themselves serve as relations over \mathfrak{D} and \mathbf{K} . This has, as a mildly jarring consequence, that two distinct, albeit similar, auxiliary propositions qua relations will do the very same work, so to speak. (This excess is an artefact, in a sense, of Russell’s theory also.) In any case, we write $\Gamma_{\mathfrak{D}, \mathbf{K}}^\theta =: \mathcal{H}_{\mathfrak{D}^*, \mathbf{K}}^\theta$ for the class of all *relations of characteristic θ over \mathfrak{D} and \mathbf{K}* . Constituents of \mathcal{A}^* in $\Gamma_{\mathfrak{D}, \mathbf{K}}^\theta$ that happen to be members of \mathfrak{D} are termed *parameters* of \mathcal{A}^* . (Constituents in $\mathfrak{D}^* \setminus \mathfrak{D}$ are *nonparametric*.)

Fixing the *ordinal degree* of relation \mathcal{A}^* in $\Gamma_{\mathfrak{D}, \mathbf{K}}^\theta$ is easy. First, let $\text{npconst}(\mathcal{A}^*)$ be the set of all members of $\mathfrak{D}^* \setminus \mathfrak{D}$ that are (**nonparametric**) **constituents** of \mathcal{A}^* . We then set $\text{deg}[\mathcal{A}^*]$ equal to the order-type of the sequence of ordinal indices of these constituents of \mathcal{A}^* . In symbols,

$$\text{deg}[\mathcal{A}^*] =: \text{otp} \left[\left(\{ \gamma \mid a_\gamma^* \in \text{npconst}(\mathcal{A}^*) \}, < \right) \right].$$

If $\text{deg}[\mathcal{A}^*] = \lambda$, then we say that \mathcal{A}^* is λ -*adic*. We have $\|\text{deg}[\mathcal{A}^*]\| = |\text{npconst}(\mathcal{A}^*)|$, although $\text{deg}[\mathcal{A}^*] \geq |\text{npconst}(\mathcal{A}^*)|$ in general. Note that $\mathcal{A}^* \simeq \mathcal{B}^*$ does not imply $\text{deg}[\mathcal{A}^*] = \text{deg}[\mathcal{B}^*]$ if the latter are nonfinite. With $\beta < \text{deg}[\mathcal{A}^*]$, we write $a_\beta^{\mathcal{A}^*}$ for the β th member of $(\text{npconst}(\mathcal{A}^*), <)$; in other words, $(a_\beta^{\mathcal{A}^*})_{\beta < \text{deg}[\mathcal{A}^*]}$ constitutes the sequence of nonparametric constituents of \mathcal{A}^* as determined by $<$. In the case of atomic $R\mathbf{a}^*\mathbf{b}^*$ with

$\mathfrak{a}^*, \mathfrak{b}^* \in \mathfrak{D}^* \setminus \mathfrak{D}$, we may have $a_1^{R\mathfrak{a}^*\mathfrak{b}^*} = \mathfrak{b}^*$ and $a_2^{R\mathfrak{a}^*\mathfrak{b}^*} = \mathfrak{a}^*$ so that the intrinsic ordering, within $R\mathfrak{a}^*\mathfrak{b}^*$, of nonparametric constituents does not coincide with \prec .

A more familiar notation for relations and their substitution instances will be useful. This will mean introducing a free-variable-like notation as an inessential, albeit highly convenient, *façon de parler* only. Namely, if $\text{deg}[\mathcal{A}^*] = \lambda$, then we shall frequently write $\mathcal{R}((\pi_\beta)_{\beta < \lambda}) \in \Gamma_\lambda$ to express the membership of \mathcal{A}^* , pseudonymously $\mathcal{R}((\pi_\beta)_{\beta < \lambda})$, in the class of λ -adic members of $\Gamma_{\mathfrak{D}, \mathfrak{K}}^\theta$, thereby assuming the entire context provided by \mathfrak{D} , \mathfrak{K} , and θ . If \mathcal{A}^* is of finite degree n , then we write $\mathcal{R}(\pi_1, \dots, \pi_n) \in \Gamma_n$, beginning with index 1 and making corresponding adjustments elsewhere. (See also the final sentence of the preceding paragraph.)

Letting $\mathcal{R}((\pi_\beta)_{\beta < \lambda})$ be \mathcal{A}^* and with $\gamma < \lambda$ and $\mathfrak{c} \in \mathfrak{D}$, we shall write either $[\mathcal{R}((\pi_\beta)_{\beta < \lambda})](\pi_\gamma/\mathfrak{c})$ or, more simply, $\mathcal{R}((\pi_\beta)_{\beta < \gamma}, \mathfrak{c}, (\pi_\beta)_{\gamma < \beta < \lambda})$ for $s(\mathcal{A}^*)$, where substitution $s : \mathfrak{D}^* \mapsto \mathfrak{D}^*$ is defined by writing

$$s(\mathfrak{b}^*) =: \begin{cases} \mathfrak{c} & \text{if } \mathfrak{b}^* \text{ is } a_\gamma^{\mathcal{A}^*} \\ \mathfrak{b}^* & \text{otherwise.} \end{cases}$$

(Thus the γ th nonparametric constituent of \mathcal{A}^* , as determined by \prec , is replaced by \mathfrak{c} ; the result $s(\mathcal{A}^*)$ may be of lower degree even if λ is nonfinite.)

As limiting case, $[\mathcal{R}((\pi_\beta)_{\beta < \lambda})](\pi_\beta/\mathfrak{c}_\beta)_{\beta < \lambda}$, or $\mathcal{R}((\mathfrak{c}_\beta)_{\beta < \lambda})$, where, again, $\mathcal{R}((\pi_\beta)_{\beta < \lambda})$ is \mathcal{A}^* , will be $t(\mathcal{A}^*)$ with substitution $t : \mathfrak{D}^* \mapsto \mathfrak{D}^*$ given by

$$t(\mathfrak{b}^*) =: \begin{cases} \mathfrak{c}_\beta & \text{if } \mathfrak{b}^* \text{ is } a_\beta^{\mathcal{A}^*} \text{ with } \beta < \lambda \\ \mathfrak{b}^* & \text{otherwise.} \end{cases}$$

(Thus t effects replacement of the entire λ -sequence $(a_\beta^{\mathcal{A}^*})_{\beta < \lambda}$ of nonparametric constituents of \mathcal{A}^* by parameter sequence $(\mathfrak{c}_\beta)_{\beta < \lambda}$.) Note that zero-adic $\mathcal{R}((\mathfrak{c}_\beta)_{\beta < \lambda})$ is a proposition of Zermelo logic $\mathcal{H}_{\mathfrak{D}, \mathfrak{K}}^\theta$. Also, whereas our notion of substitution itself does not assume \prec , our new notation *does*.

Three relation-forming operations over $\Gamma_{\mathfrak{D}, \mathfrak{K}}^\theta$ will be of interest. First, unary operator \sim will denote relational negation: If $\mathcal{R}((\pi_\beta)_{\beta < \lambda})$ is \mathcal{A}^* , then we write $\sim\mathcal{R}((\pi_\beta)_{\beta < \lambda})$ for $\neg\mathcal{A}^*$.

Second, we introduce a disjunction operation on small, indexed families of relations: If $\mathcal{R}_\iota((\pi_\beta)_{\beta < \lambda_\iota})$, of degree λ_ι , is \mathcal{A}_ι^* for each ι in I with $|I| < \theta$, then we shall write $\Upsilon\{\mathcal{R}_\iota((\pi_\beta)_{\beta < \lambda_\iota}) \mid \iota \text{ in } I\}$ for $\bigvee\{\mathcal{A}_\iota^* \mid \iota \text{ in } I\}$ of degree

$$\text{otp} \left[\left(\left\{ \gamma \mid a_\gamma^* \in \bigcup_{\iota \text{ in } I} \text{npconst}(\mathcal{A}_\iota^*) \right\}, \prec \right) \right] \geq \max\{\lambda_\iota \mid \iota \text{ in } I\}.$$

Third, if $|\mathfrak{D}| < \theta$, then, given $\ell \geq 1$ finite and m with $1 \leq m \leq \ell$, we have unary *existential generalization, modulo ℓ and m , over \mathfrak{D}* : If $\mathcal{R}(\pi_1, \dots, \pi_\ell)$ is \mathcal{A}^* , then

$$\bigvee_{\mathfrak{c} \text{ in } \mathfrak{D}} \mathcal{R}(\pi_1, \dots, \pi_{m-1}, \mathfrak{c}, \pi_{m+1}, \dots, \pi_\ell) \tag{*}$$

will be $(\ell - 1)$ -adic $\bigvee \{s_{\mathfrak{c}}(\mathcal{A}^*) \mid \mathfrak{c} \text{ in } \mathfrak{D}\}$, where, for each \mathfrak{c} in \mathfrak{D} , substitution $s_{\mathfrak{c}} : \mathfrak{D}^* \mapsto \mathfrak{D}^*$ is defined by

$$s_{\mathfrak{c}}(\mathfrak{b}^*) =: \begin{cases} \mathfrak{c} & \text{if } \mathfrak{b}^* \text{ is } a_m^{\mathcal{A}^*} \\ \mathfrak{b}^* & \text{otherwise.} \end{cases}$$

We emphasize that, by the likes of Theorem 2.1, (*) is in $\Gamma_{\ell-1}$ just in case $|\mathfrak{D}| < \theta$, which we do not assume, as a rule, our purpose being maximum generality.

In the case of any fully saturated substitution instance of a member of $\Gamma_{\mathfrak{D},\mathfrak{K}}^\theta$, we shall routinely replace relation-forming $\sim, \bigvee, \bigvee, \bigwedge, \bigwedge, \rightsquigarrow$ (the preceding three defined in terms of \sim and either \bigvee or \bigvee), and \leftrightarrow (defined in terms of \rightsquigarrow and \bigwedge) with proposition-forming $\neg, \bigvee, \bigvee, \bigwedge, \bigwedge, \rightarrow$, and \leftrightarrow , respectively, since said substitution instance is a proposition of Zermelo logic $\mathcal{H}_{\mathfrak{D},\mathfrak{K}}^\theta$. (We shall see this, in Sect. 4, when we present infinitary formulations of Zermelo’s axioms for the theory of urelements and sets.)

Relations constructible, in unspecified manner, over given \mathfrak{D} and \mathfrak{K} are termed “definite” in [16]. In the context of set theory, involving fundamental predicates “is an urelement” and “is a member of” only, being green-painted is not definite, Zermelo will later write (see [17, p. 363]). This is to underscore the dependence of the definiteness concept on catalogue \mathfrak{K} . To clarify the bounds of definiteness, we introduced a second consideration, namely, characteristic θ . One can show that any member of $\Gamma_{\mathfrak{D},\mathfrak{K}}^\theta$ is λ -adic with $\lambda < \theta$ and that the relational subcomponents of such a relation number fewer than θ , as do its parameters. So another way a relation may fail to be “definite,” as we shall understand this notion, is for it to be too big. Of course, a relation not definite with respect to θ may be definite relative to some larger characteristic, which is to invoke a dialectic featured in [18].

As a first application of our account of infinitary relations, we shall show that it serves as the basis for a compelling reconstruction of Zermelo’s theory of urelements and sets as first presented in [16]. As is well known, definiteness figured, problematically, in Zermelo’s Axiom of Separation:

Whenever the propositional function $\mathfrak{G}(x)$ is definite for all elements of the set M , M possesses a subset $M_{\mathfrak{G}}$ containing as elements precisely those elements x of M for which $\mathfrak{G}(x)$ is true. ([16, p. 195])

Unfortunately, Zermelo's antecedent characterization of definiteness assumes unidentified construction principles, citing only "laws of logic":

A question or assertion \mathfrak{G} is said to be *definite* if the fundamental relations of the domain, by means of the axioms and the universally valid laws of logic, determine without arbitrariness whether it holds or not. ([16, p. 193]; italics in original)

We shall be supplanting Zermelo's vague notion of definiteness with the class Γ_1 of monadic relations of characteristic θ over appropriate domain \mathfrak{D} and identity catalogue K_{\equiv} . That, in turn, will obviate the vexed question whether Zermelo's 1908 axiomatization, as opposed to its metatheory, is first- or second-order in spirit: If we are right, it is neither.

4. Zermelo Set Theory Reconstructed

Jeder mathematische Satz ist aufzufassen als eine Zusammenfassung von (unendlich vielen) Elementarsätzen durch Konjunktion, Disjunktion und Negation, und jede Ableitung eines Satzes aus anderen Sätzen, insbesondere jeder Beweis ist nichts anderes als eine Umgruppierung der zugrundeliegenden Elementarsätze.

—Ernst Zermelo *Thesen über das Unendliche in der Mathematik* (1921)

In presenting axioms for the theory of urelements and sets, Zermelo takes as his starting point some domain \mathfrak{D} , fixed at the very outset:

Set theory is concerned with a "domain" \mathfrak{D} of individuals [*Objekte*], which we shall call simply "objects" [*Dinge*] and among which are the "sets." If two symbols \mathfrak{a} and \mathfrak{b} denote the same object, we write $\mathfrak{a} \equiv \mathfrak{b}$, otherwise $\mathfrak{a} \neq \mathfrak{b}$. We say of an object \mathfrak{a} that it "exists" if it belongs to the domain \mathfrak{D} ; likewise we say of a class C of objects that "there exist objects of the class C " if \mathfrak{D} contains at least one individual of this class.

Certain "fundamental relations" of the form $\mathfrak{a} \in \mathfrak{b}$ obtain between the objects of the domain \mathfrak{D} . If for two objects \mathfrak{a} and \mathfrak{b} the relation $\mathfrak{a} \in \mathfrak{b}$ holds, we say "a is an *element* of the set \mathfrak{b} ," "b contains a as an

element,” or “ \mathfrak{b} possesses the element \mathfrak{a} .” An object \mathfrak{b} will be called a *set* if and—with a single exception—only if it contains another object \mathfrak{a} as an element.

The fundamental relations of our domain \mathfrak{D} are subject to [certain] “*axioms*” or “*postulates*.” ([16], pp. 191 and 193; italics in original but symbols changed throughout to match our own discussion)

Zermelo’s opening paragraph indicates that \mathfrak{D} constitutes part of the subject matter of set theory. Indeed, as mentioned in [5], two of Zermelo’s seven axioms make reference to \mathfrak{D} by name. Our reconstruction will preserve this domain-specific feature of Zermelo’s conception.

Reconstructing Zermelo set theory, using the account of definiteness given in Sect. 3, will mean first fixing Zermelo logic $\mathcal{H}_{\mathfrak{D}, \langle R_U, R_{\in}, I \rangle}^{\theta}$ of inaccessible characteristic θ , where (1) fundamental predicates R_U and R_{\in} are monadic and dyadic, respectively, (2) $|\mathfrak{D}| < \theta$, and (3) there exists $\langle f_U^{(1)}, f_{\in}^{(2)}, \text{Id}_{\mathfrak{D}}^{(2)} \rangle \in \mathfrak{T}_{\mathfrak{D}}^{1,2, \equiv}$ such that $\langle f_U^{(1)}, f_{\in}^{(2)}, \text{Id}_{\mathfrak{D}}^{(2)} \rangle \models -Z_{\mathfrak{D}}^{\theta}$, where $-Z_{\mathfrak{D}}^{\theta}$ is the theory described below. Constraints (2) and (3) together say what it is for \mathfrak{D} to be *combinatorially feasible modulo θ* . (Both concern $|\mathfrak{D}|$, as Theorems 4.1 and 4.2 will show.)

As for the axioms of $-Z_{\mathfrak{D}}^{\theta}$, each proposition

$$\bigvee_{\mathfrak{b} \text{ in } \mathfrak{D}} \left[\neg R_U \mathfrak{b} \wedge \bigwedge_{\mathfrak{c} \text{ in } \mathfrak{D}} (R_{\in} \mathfrak{c} \mathfrak{b} \leftrightarrow [R_{\in} \mathfrak{c} \mathfrak{a} \wedge \mathcal{R}(\mathfrak{c})]) \right]$$

is an Axiom of Separation, where \mathfrak{a} is in \mathfrak{D} and $\mathcal{R}(\pi)$ is any member of Γ_1 . Given this reformulation of Separation, Zermelo’s demonstrations, in [16], that various propositional functions are definite may now be regarded as informal, finitary descriptions of members of Γ_1 that are well-founded albeit, in general, infinitary. Since we can let $\mathcal{R}(\pi)$ be $\bigvee_{\mathfrak{d} \text{ in } C} \pi \equiv \mathfrak{d}$, any subset C of set \mathfrak{a} is itself a set (cf. Theorem 5.3); such “domain closure under set-inclusion” was, most likely, Zermelo’s goal from the start (cf. [18, p. 403] and [1, p. 12]). We hasten to add that Zermelo’s brief remarks, in [16], regarding the semantic paradoxes may, but need not, be understood to *deny* closure. Should closure under set-inclusion *not* be Zermelo’s goal, this would be only a first way in which our reconstruction diverges from his conception.

In this connection, we note that, assuming closure, Choice is a consequence of Sum Set and Separation. That Choice nonetheless figures among the seven nonlogical axioms of [16] may then tell against our reconstruction. And it may not: Even if closure is what he intends, Zermelo may well

wonder whether he has achieved it, which might, in turn, urge listing of the crucial axiom as a cautious measure. By the way, Choice was to be picked up by the underlying logic of Zermelo's second axiomatization of the theory of urelements and sets, published in [18], from 1930; but that was not yet Zermelo's idea in 1908.

Each of the remaining nonlogical axioms is, in point of fact, the truth-functional expansion over \mathfrak{D} (cf. Definition 6.1) of an axiom of the theory commonly known as " Z^- with urelements," where Z^- is, in turn, the first-order set theory known as Z but without the Axiom of Foundation. (Expressing matters in this way is convenient but, of course, utterly anachronistic, since it is Zermelo's *informal descriptions*, in [16], of his axioms that decide membership in our $-Z_{\mathfrak{D}}^{\theta}$.) Thus Zermelo's Axiom of Extensionality becomes

$$\bigwedge_{\mathfrak{a} \text{ in } \mathfrak{D}} \bigwedge_{\mathfrak{b} \text{ in } \mathfrak{D}} \left([\neg R_U \mathfrak{a} \wedge \neg R_U \mathfrak{b}] \rightarrow \left[\bigwedge_{\mathfrak{c} \text{ in } \mathfrak{D}} (R_{\in} \mathfrak{c} \mathfrak{a} \leftrightarrow R_{\in} \mathfrak{c} \mathfrak{b}) \rightarrow \mathfrak{a} \equiv \mathfrak{b} \right] \right),$$

and Null/Pair Set, Sum Set, Power Set, and Infinity are likewise formulated *in expansione*. (We omit Choice.) Finally, as a characterization of urelements, we have the supplemental nonlogical axiom

$$\bigwedge_{\mathfrak{b} \text{ in } \mathfrak{D}} \left(R_U \mathfrak{b} \rightarrow \bigwedge_{\mathfrak{a} \text{ in } \mathfrak{D}} \neg R_{\in} \mathfrak{a} \mathfrak{b} \right).$$

In addition, we assume truth-functional versions of axioms ensuring that \equiv denotes an equivalence relation over \mathfrak{D} as well as a collection of logical axioms and rules of inference for $\mathcal{H}_{\mathfrak{D}, \langle R_U, R_{\in}, I \rangle}^{\theta}$. With respect to the latter, we can be guided by Karp's presentation of axioms and rules of inference for infinitary propositional logic [see [7], pp. 40–41]. It can be shown that the system $\mathfrak{S}_{\mathfrak{D}, \langle R_U, R_{\in}, I \rangle}^{\theta}$ comprising those axioms and rules of inference plus distributive laws is strongly complete in the sense that arbitrary $\mathcal{A} \in \mathcal{H}_{\mathfrak{D}, \langle R_U, R_{\in}, I \rangle}^{\theta}$ is a semantic consequence of arbitrary $J \subseteq \mathcal{H}_{\mathfrak{D}, \langle R_U, R_{\in}, I \rangle}^{\theta}$ just in case \mathcal{A} is derivable from J in the system $\mathfrak{S}_{\mathfrak{D}, \langle R_U, R_{\in}, I \rangle}^{\theta}$ plus distributive laws (see [7, Theorem 5.5.5]). Strong completeness then yields θ -compactness, since the length of any proof must be less than θ .

Of course, of greatest interest in the present context is the case where J , in the preceding paragraph, is $-Z_{\mathfrak{D}}^{\theta}$, and this leads us to an important remark: *Although we have written, to this point, of theory $-Z_{\mathfrak{D}}^{\theta}$ in the singular, our reconstruction of Zermelo's theory of urelements and sets comprises, in truth, an entire range of theories $-Z_{\mathfrak{D}}^{\theta}$ —one, and only one, for each \mathfrak{D} that is combinatorially feasible modulo θ .* Regarding the latter notion, one can, assuming Choice, prove

THEOREM 4.1. *A set \mathfrak{D} is a combinatorially feasible domain of “urelements” and “sets” in the sense of (2) and (3) provided that $|\mathfrak{D}| = \beth_\lambda(\mathfrak{n}) < \theta$ for some limit ordinal $\lambda > \omega$ and some cardinal number \mathfrak{n} .*

Zermelo, in [16], does not assume the Axiom of Foundation; consequently, we have not done so. Theorem 4.1 can be strengthened if we do assume Foundation, however. Let feasibility constraint (3') read: “There exists $\langle f_U^{(1)}, f_\epsilon^{(2)}, \text{Id}_{\mathfrak{D}}^{(2)} \rangle \in \mathfrak{T}_{\mathfrak{D}}^{1,2,\equiv}$ satisfying $Z_{\mathfrak{D}}^\theta$,” where $Z_{\mathfrak{D}}^\theta$ incorporates Foundation *in expansione*. Again assuming the Axiom of Choice in the metatheory, one can give a concise characterization of feasibility, in the new sense, by proving

THEOREM 4.2. *A set \mathfrak{D} is a combinatorially feasible domain of “urelements” and “sets” in the sense of (2) and (3') just in case $|\mathfrak{D}| = \beth_\lambda(\mathfrak{n}) < \theta$ for some limit ordinal $\lambda > \omega$ and some cardinal number \mathfrak{n} .*

The size of any feasible \mathfrak{D} , in the new sense, is given either by a singular cardinal or by a strongly inaccessible cardinal greater than ω .

We have taken feasibility as a starting point; without it, typical $\neg Z_{\mathfrak{D}}^\theta$ —for example, take \mathfrak{D} with $|\mathfrak{D}| < \beth_{\omega_2}(\mathfrak{n})$ with \mathfrak{n} finite—is unsatisfiable and, hence, inconsistent by the completeness of $\mathfrak{S}_{\mathfrak{D},(R_U, R_\epsilon, I)}^\theta$. On the other hand, requiring feasibility appears to undermine our reconstruction, given Zermelo’s high expectations regarding foundations: Any determination of which domains are feasible requires set theory. (The proofs of Theorems 4.1 and 4.2 involve also transfinite recursion, and hence Replacement, as well as either the assumption that domain \mathfrak{D} is a set or that inaccessible θ exists.) How does set theory provide a foundation for mathematics if its characterization assumes a model of a yet stronger theory? We put this vital issue to one side, momentarily, in order to first make the ultimate case for our reconstruction. Incidentally, Replacement is implicit in Zermelo’s use of transfinite induction in [16], as he himself acknowledges in a 1921 letter to Fraenkel (see [2, p. 137]); accordingly, Replacement was assumed in Sect. 2 at several points.

Zermelo writes within a philosophical tradition that harkens back, by way of Husserl, to Bolzano’s nonlinguistic conception of mathematical theories as systems of abstract propositions or *Sätze an sich*. Consequently, Zermelo must see the text of [16] as standing in for something, and the question then becomes, Standing in for what? Likely, Zermelo never answered this question satisfactorily for himself, although Skolem’s eventual answer, namely, “first-order Z^- with urelements,” can be excluded out of hand as a reading of Zermelo’s intentions (see below). Our reconstruction provides an answer to this question that achieves most, if not all, of Zermelo’s early aims

with respect to set theory. If correct, it clarifies what underlies the informal presentation of [16], what Zermelo's symbol-laden German sentences are vernacular surrogates *for*. Collectively, theories $\neg Z_2^\theta$ constitute the rightful axiomatization of Zermelo's 1908 theory of urelements and sets, or so we claim. As for relations, Zermelo clearly recognizes the need for them—thus his talk of “definite propositional functions”—but seems to hope that he can avoid any *theory* of relations.

Regarding the essential character of our reconstruction, it should strike no one as odd that infinitary logic, central to Zermelo's foundational philosophy late in his career, is here being used to clarify even his earliest ideas: When Zermelo, in his early defense of impredicativity, wrote that “every object [*Gegenstand*] can be determined in a wide variety of ways” ([15, p. 139]), he cannot have intended finitary determinations exclusively. (How could each and every real be determinable in such a manner?) Not so many years later, Zermelo came to write that “every genuinely mathematical proposition is to be considered a collection of infinitely many elementary propositions by means of conjunction, disjunction, and negation” and that, in particular, “the axioms of every mathematical theory must be infinitary” ([22, p. 307]). It seems likely that Zermelo held some such view already in the period around 1908.

Significantly, [16] manages to avoid any suggestion of strictly finitary logic. We read in Skolem's [11] from 1922, written in an attempt to clarify Zermelo's idea, that a definite proposition will now be a “finite expression” (*endlicher Ausdruck*) constructed from elementary propositions by finite iteration (*endlich viele Anwendungen*) of standard truth-functional and quantificational operations. In marked contrast, the adjective *endlich* is not introduced by Zermelo in *his* remarks, from 1908, concerning definiteness. This can only be deliberate: The portion of [16] in which the Axiom of Separation is presented itself contains references to “ultrafinite paradoxes,” “definability by means of a finite number of words,” and the “paradox of finite denotation,” so that the distinction finite/infinite is close at hand.

This encourages us to believe that, in 1908, Zermelo consciously shunned the words “finite” and “infinite” in connection with definiteness. To be sure, Zermelo's [17] from 1929, wherein finite iteration of logical operations *is* invoked, reads very differently—doubtless the result of Skolem's influence. However, our present goal is a reconstruction of Zermelo's initial 1908 axiomatization as presented in [16], written well before Skolem enters the picture and well before Skolem in effect causes Zermelo, at least through the twenties, to retreat from his initial, infinitary conception of the logic of mathe-

matics. (What will, soon enough, jolt Zermelo back to his earlier convictions is the shock of Gödel incompleteness [see Sect. 7 below].)

As Zermelo sees things, any mathematical discipline is characterized by a notion of definiteness: Certain “fundamental relations” are explicitly given and more complex relations are then constructed by means of logical operations. Zermelo neglects to indicate, in [16], how far one is permitted to go in iterating these logical operations, and our assumption of characteristic θ clarifies that one issue. Zermelo’s own reluctance here surely stems from his desire that definiteness be a notion devoid of any and all set-theoretic presuppositions. Of course, it is no such thing, at least if our ordinal-theoretic account of infinitary relations is to serve, and there is a problem. What, one wonders, can the foundational status of any $\neg Z_{\aleph}^{\theta}$ be, given that an ordinal-theoretic notion of a relation has been used to specify a family of Separation Axioms? (What was said above regarding the requirement of combinatorial feasibility gives rise to much the same sort of question, but we shall restrict our discussion to relations.)

We begin by recalling the situation that arises whenever one describes the syntax of the language of a given formal or semiformal theory. Namely, inductive definitions of a variety of notions, e.g., “term,” “formula,” and “free variable,” are presented. Typically, such definitions involve (implicit) iteration through the natural numbers only. Exceptions do occur, however, as in the case of infinitary logics, which may permit quantifier sequences, say, of uncountable length. The grammars of all such languages are inherently ordinal-theoretic, and this is no less true when the language in question is that of set theory. Set-theoretic notions play a role in any account of what qualifies as a well-formed formula; hence they figure in any determination of which formulæ are instances of axiom schemata. (There are type- and category-theoretic accounts of logic in which ordinals play no role, but they need not concern us.)

The situation—whereby an ordinal-theoretic notion of a relation has a determinative role within Zermelo set theory, reconstructed as a range of theories $\neg Z_{\aleph}^{\theta}$ —is of a piece with the situation, just described, whereby ordinal-theoretic notions figure in the specification of formal theories generally. This sort of thing is not what Zermelo envisioned at the beginning of his career (see [13, Sect. 2]). However, it is a phenomenon that had won general acceptance not many years later. Set theories qua foundations elucidate the (interim) starting points of mathematics and make for universal modeling possibilities. They do not provide Cartesian certainty, however, and may themselves require ancillary support in the guise of something akin

to Tarski's Axiom, say. Accordingly, we claim only that the proposed range of theories $\neg Z_{\mathfrak{D}}^{\theta}$ realizes most, but not all, of Zermelo's early goals.

That said, why not formulate Zermelo's axioms using logic $\mathcal{L}_{\omega\theta}$ or the like? Our answer is this: Since quantifiers in the sense of Frege or Russell are not what is suggested by Zermelo's invocation of "conjunction, disjunction, and negation," cited a few paragraphs back, any such reconstruction would probably violate Zermelo's conception. Likewise, the generalization and instantiation rules associated with $\mathcal{L}_{\omega\theta}$ are not compatible with Zermelo's added remark that "every deduction and every proof is nothing but a re-grouping of the underlying elementary propositions" (see again [22, p. 307]). This is to take [22] very seriously; but there is every reason to do that, given what came later. As for his earlier publications, since Zermelo introduces no logical constants whatsoever in [16], always writing the German equivalents of "for all" and "there exists" to indicate generality, it is not obvious that the view expressed in [22], from 1921, was not already Zermelo's view in 1908. Moreover, the cited remark ("determined in a wide variety of ways") from [15], also from 1908, probably means that this *was* his early view. When Zermelo does, at the end of his career, introduce logical "quantification" and notation for it, he states explicitly that he intends domain-wide conjunctions and disjunctions (see [19, p. 545]).

We shift from set theory to model theory. A central task of any model theory is to say which subsets of given domain \mathfrak{D} —more generally, which subsets of the set \mathfrak{D}^{λ} of all λ -tuples of members of \mathfrak{D} —are definable relative to a given relational structure over \mathfrak{D} . We take up definability next and note the importance of not assuming, generally, that $|\mathfrak{D}| < \theta$. If we were to assume that, then the notion of parametric definability would be rendered trivial in the presence of identity, every domain subset being definable modulo any \mathfrak{D} -structure whatsoever (cf. Corollary 5.4). An alternative course of interest assumes $|\mathfrak{D}| < \theta$ but considers nonidentity catalogues only.

5. Definability

Again, we set $\mathfrak{D}^* =: \mathfrak{D} \sqcup \theta$ with θ strongly inaccessible and assume both Zermelo logic $\mathcal{H}_{\mathfrak{D},\mathbb{K}}^{\theta}$ and auxiliary Zermelo logic $\mathcal{H}_{\mathfrak{D}^*,\mathbb{K}}^{\theta}$ sharing catalogue $\mathbb{K} =: \langle R_1, \dots, R_p \rangle$. Also, where $\text{deg}[\mathcal{A}^*] = \lambda$, we continue to write $\mathcal{R}((\pi_{\beta})_{\beta < \lambda}) \in \Gamma_{\lambda}$ to express the membership of relation \mathcal{A}^* in the class of λ -adic members of $\Gamma_{\mathfrak{D},\mathbb{K}}^{\theta}$, thereby assuming context \mathfrak{D} , \mathbb{K} , and θ .

We begin by defining the extension $\mathcal{R}((\pi_\beta)_{\beta < \lambda})^{\mathcal{M}}$, relative to given model $\mathcal{M} =: \langle f_1^{(n_1)}, \dots, f_p^{(n_p)} \rangle$, of λ -adic relation $\mathcal{R}((\pi_\beta)_{\beta < \lambda})$.

DEFINITION 5.1. Let $\mathcal{R}((\pi_\beta)_{\beta < \lambda}) \in \Gamma_\lambda$ with $\lambda < \theta$. Then we define the extension $\mathcal{R}((\pi_\beta)_{\beta < \lambda})^{\mathcal{M}} \subseteq \mathfrak{D}^\lambda$ of $\mathcal{R}((\pi_\beta)_{\beta < \lambda})$ modulo $\mathcal{M} \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}}$ by writing

$$\mathcal{R}((\pi_\beta)_{\beta < \lambda})^{\mathcal{M}} =: \{(\mathfrak{c}_\beta)_{\beta < \lambda} \in \mathfrak{D}^\lambda \mid \mathcal{M} \models \mathcal{R}((\mathfrak{c}_\beta)_{\beta < \lambda})\}.$$

If $\lambda = 0$, then $\mathcal{R}((\pi_\beta)_{\beta < \lambda})^{\mathcal{M}}$ is one of two subsets of $\mathfrak{D}^0 = \{()\}$, identified with $\perp^{\mathcal{M}} = \emptyset = \text{false}$ and $\top^{\mathcal{M}} = \{()\} = \text{true}$. We write $\mathfrak{C}^{\mathcal{M}}$ for the set of extensions, modulo fixed $\mathcal{M} \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}}$, of members of $\mathfrak{C} \subseteq \Gamma_{\mathfrak{D}, \mathbb{K}}^\theta$. If \mathfrak{C} is $\Gamma_{\mathfrak{D}, \mathbb{K}}^\theta$ (or Γ_λ), we write $\Gamma^{\mathcal{M}}$ (or $\Gamma_\lambda^{\mathcal{M}}$) for $\mathfrak{C}^{\mathcal{M}}$. Finally, this is a good place to note that, due to the \prec -dependence of our variable-like notion, from $\mathcal{R}((\pi_\beta)_{\beta < \lambda}) =: \mathcal{A}^*$ and $\mathcal{S}((\rho_\beta)_{\beta < \lambda}) =: \mathcal{B}^*$ with $\mathcal{A}^* \simeq \mathcal{B}^*$ it does not follow that $\mathcal{R}((\pi_\beta)_{\beta < \lambda})^{\mathcal{M}} = \mathcal{S}((\rho_\beta)_{\beta < \lambda})^{\mathcal{M}}$; what *does* follow is that $\mathcal{R}((\pi_\beta)_{\beta < \lambda})^{\mathcal{M}}$ can be obtained from $\mathcal{S}((\rho_\beta)_{\beta < \lambda})^{\mathcal{M}}$ by uniform permutation of the latter's λ -tuple elements (see p. 7).

DEFINITION 5.2. Where $A \subseteq \mathfrak{D}$, we say that $C \subseteq \mathfrak{D}^\lambda$ is *A-definable modulo* $\mathcal{M} \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}}$ if there exists $\mathcal{R}((\pi_\beta)_{\beta < \lambda}) \in \Gamma_\lambda$ with parameters in A such that $C = \mathcal{R}((\pi_\beta)_{\beta < \lambda})^{\mathcal{M}}$.

THEOREM 5.3. Let $A \subseteq \mathfrak{D}$ and, with $\lambda < \theta$, let $C \subseteq A^\lambda$ satisfy $|C| < \theta$. Then C is *A-definable modulo arbitrary* $\mathcal{M} \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}, \equiv}$.

PROOF. Subset C is $\left[\bigvee_{(\mathfrak{c}_\beta)_{\beta < \lambda} \text{ in } C} \bigwedge_{\beta < \lambda} a_{\beta}^* \equiv \mathfrak{c}_\beta \right]^{\mathcal{M}}$. ■

Identifying $\mathfrak{a} \in \mathfrak{D}$ with $(\mathfrak{a}) \in \mathfrak{D}^1$, we then have

COROLLARY 5.4. Let A be an arbitrary subset of \mathfrak{D} with $|A| < \theta$. Then A is *A-definable modulo arbitrary* $\mathcal{M} \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}, \equiv}$.

By Theorem 5.3, given domain \mathfrak{D} and catalogue \mathbb{K}_{\equiv} , the members of $\Gamma_\lambda^{\mathcal{M}}$ for arbitrary $\mathcal{M} \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}, \equiv}$ include any small subset of \mathfrak{D}^λ . By closure of $\Gamma_\lambda^{\mathcal{M}}$ under complementation and sum sets of small families, any co-small subset of \mathfrak{D}^λ is in $\Gamma_\lambda^{\mathcal{M}}$ as is the sum set of any small family of small and co-small subsets. Converses of Theorem 5.3 and Corollary 5.4 require the assumption that $|\mathfrak{D}| < \theta$ but then hold trivially. (The converse of Corollary 5.4 is incompatible with $|\mathfrak{D}| \geq \theta$, since $\mathfrak{D} = \mathfrak{D}^1$ is \emptyset - and hence \mathfrak{D} -definable by dint of monadic $a_0^* \equiv a_0^*$.)

In general, not every subset of \mathfrak{D}^λ is definable modulo $\mathcal{M} \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}}$. On the other hand, Theorem 5.3 shows that, in the case of an identity catalogue, every small $C \subseteq A^\lambda \subseteq \mathfrak{D}^\lambda$ is *A-definable modulo arbitrary* $\mathcal{M} \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}, \equiv}$.

An intermediate state of affairs is possible as well. Namely, assuming that nonidentity catalogue K contains a nonmonadic element, that is, an element $R_\ell^{n_\ell}$ with $n_\ell > 1$, one can show that, although not every small $C \subseteq A^\lambda$ is A -definable modulo arbitrary $\mathcal{M} \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}}$, there does exist $\mathcal{M}' \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}}$ such that every small $C \subseteq A^\lambda$ is indeed A -definable modulo \mathcal{M}' .

As noted earlier, any \mathcal{B}^* in $\mathcal{H}_{\mathfrak{D}^*,K}^\theta$ involves fewer than θ parameters. So if relation \mathcal{B}^* is of degree $\lambda < \theta$, then we may regard the extension, modulo $\mathcal{M} \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}}$, of \mathcal{B}^* as an A -definable subset of \mathfrak{D}^λ , relative to \mathcal{M} , where A is the set comprising all and only the parameters of \mathcal{B}^* . (Definition 5.2 itself places no such restriction on $A \subseteq \mathfrak{D}$.) Accordingly, we shall refer to $\Gamma_\lambda^{\mathcal{M}}$ with $\lambda < \theta$ as the collection of subsets of \mathfrak{D}^λ (set-)parametrically definable modulo \mathcal{M} , and $\{\Gamma_\lambda^{\mathcal{M}}\}_{\lambda < \theta}$, in turn, is the hull of model \mathcal{M} .

Taking the analysis of finite definability put forward in [6] as prototype, we shall present, in Theorems 5.5 and 5.6 below, a succinct characterization of $\{\Gamma_\lambda^{\mathcal{M}}\}_{\lambda < \theta}$, beginning with the \emptyset -definables. Said characterization will be notable by virtue of its making no appeal to the notion of satisfaction in \mathcal{M} .

A first, preliminary goal is to define a binary collapse operation applicable to distinguished \mathfrak{D} -sequences. We say that ordinal sequence $(\xi_\gamma)_{\gamma < \mu'}$, assumed to be nonempty and strictly increasing, is μ -bounded if $\xi_\gamma < \mu$ for all $\gamma < \mu' \leq \mu$. Further, if ordinal sequence $(\xi_\gamma)_{\gamma < \mu'}$ is μ -bounded and \mathfrak{D} -sequence $(\mathbf{a}_\beta)_{\beta < \mu}$ is such that $\mathbf{a}_{\xi_\gamma} = \mathbf{a}_{\xi_\delta}$ for arbitrary $\gamma, \delta < \mu'$, then we say that $(\mathbf{a}_\beta)_{\beta < \mu}$ is $[(\xi_\gamma)_{\gamma < \mu'}]$ -identical. (By extension, we shall say that $C \subseteq \mathfrak{D}^\mu$ is $[(\xi_\gamma)_{\gamma < \mu'}]$ -identical provided that each of its members is.)

The needed collapse operation can now be defined. Namely, if (1) ordinal sequence $(\xi_\gamma)_{\gamma < \mu'}$ is μ -bounded, (2) \mathfrak{D} -sequence $(\mathbf{a}_\beta)_{\beta < \mu}$ is $[(\xi_\gamma)_{\gamma < \mu'}]$ -identical, and (3) $\text{otp}[(\mu \setminus \{\xi_\gamma \mid 0 < \gamma < \mu'\}, <)] = \lambda$, then

$$\text{collapse}[(\mathbf{a}_\beta)_{\beta < \mu}, (\xi_\gamma)_{\gamma < \mu'}] \tag{*}$$

will denote the λ -sequence that is the result of eliminating from $(\mathbf{a}_\beta)_{\beta < \mu}$ all but the first term ("0 < γ ") of identical subsequence $(\mathbf{a}_{\xi_\gamma})_{\gamma < \mu'}$. (Note that if $\mu' = 1$, then (*) is first operand $(\mathbf{a}_\beta)_{\beta < \mu}$ itself.)

Next, we shall need a notion of infimum for infinite sequences of sets of progressively collapsing, yet never vanishing, \mathfrak{D} -sequences. If ordinal sequence $(\xi_\gamma)_{\gamma < \mu'}$ is μ -bounded, then $(\mathbf{a}_\gamma)_{\gamma < \mu'} \in \mathfrak{D}^{\mu'}$ is said to be the $[(\xi_\gamma)_{\gamma < \mu'}]$ -subsequence of $(\mathbf{b}_\delta)_{\delta < \mu} \in \mathfrak{D}^\mu$ provided that $\mathbf{a}_\gamma = \mathbf{b}_{\xi_\gamma}$ for all $\gamma < \mu'$. Further, sequence $\sigma \in \mathfrak{D}^{\mu'}$ is a (nonempty) subsequence of sequence $\tau \in \mathfrak{D}^\mu$, and we write $\sigma \triangleleft \tau$, if σ is the $[(\xi_\gamma)_{\gamma < \mu'}]$ -subsequence of τ for some μ -bounded ordinal sequence $(\xi_\gamma)_{\gamma < \mu'}$. If $(\sigma_\xi)_{\xi < \zeta}$ is a sequence of

\mathfrak{D} -sequences such that, for all $\gamma < \delta < \zeta$, we have $\sigma_\delta \leq \sigma_\gamma$, then we write $\inf_{\xi < \zeta} \sigma_\xi$ for the longest \mathfrak{D} -sequence τ such that $\tau \leq \sigma_\xi$ for all $\xi < \zeta$. (If ζ is a [nonzero] limit ordinal, then this “sequential infimum” exists by Sum Set and Separation.)

Given μ -bounded ordinal sequence $(\xi_\gamma)_{\gamma < \mu'}$, we say that “ $S \subseteq \mathfrak{D}^\mu$ extends $S' \subseteq \mathfrak{D}^{\mu'}$ pointwise with respect to $(\xi_\gamma)_{\gamma < \mu'}$,” and write $S' \sqsubseteq_{(\xi_\gamma)_{\gamma < \mu'}} S$, provided that there exists (unique) bijection $h_{(\xi_\gamma)_{\gamma < \mu'}} : S' \mapsto S$ such that $h_{(\xi_\gamma)_{\gamma < \mu'}}$ takes $(\mathbf{a}_\gamma)_{\gamma < \mu'} \in S'$ to $(\mathbf{b}_\delta)_{\delta < \mu} \in S$ just in case $(\mathbf{a}_\gamma)_{\gamma < \mu'}$ is the $[(\xi_\gamma)_{\gamma < \mu'}]$ -subsequence of $(\mathbf{b}_\delta)_{\delta < \mu}$. (Hence $S' \sqsubseteq_{(\xi_\gamma)_{\gamma < \mu'}} S$ if and only if each μ' -tuple in S' is the $[(\xi_\gamma)_{\gamma < \mu'}]$ -subsequence of a unique μ -tuple in S and each μ -tuple in S has a unique μ' -tuple in S' as $[(\xi_\gamma)_{\gamma < \mu'}]$ -subsequence.) Set $S \subseteq \mathfrak{D}^\mu$ uniformly extends $S' \subseteq \mathfrak{D}^{\mu'}$ pointwise with $0 < \mu' \leq \mu$, and we write $S' \sqsubseteq S$, if $S' \sqsubseteq_{(\xi_\gamma)_{\gamma < \mu'}} S$ for some μ -bounded ordinal sequence $(\xi_\gamma)_{\gamma < \mu'}$. The reader will wish to note that relation \sqsubseteq is transitive and that $S' \sqsubseteq S$ implies $|S'| = |S|$.

Finally, suppose $(S_\xi)_{\xi < \zeta}$ is a sequence of sets of \mathfrak{D} -sequences such that $\emptyset \neq S_0 \subseteq \mathfrak{D}^\mu$ with $\mu > 0$ and $S_\delta \sqsubseteq S_\gamma$ for all $\gamma < \delta < \zeta < \theta$. (Thus lengths of \mathfrak{D} -sequences neither increase nor vanish as we proceed through $(S_\xi)_{\xi < \zeta}$.) Further, for any $\gamma < \delta < \zeta$, let $h_{\gamma, \delta} : S_\delta \mapsto S_\gamma$ designate the bijection witnessing $S_\delta \sqsubseteq S_\gamma$. Then, where $\sigma \in S_0$, we define function φ_σ , taking ordinals less than ζ to \mathfrak{D} -sequences, by writing

$$\varphi_\sigma(\delta) =: \begin{cases} \sigma & \text{if } \delta = 0 \\ h_{\gamma, \delta}^{-1}(\varphi_\sigma(\gamma)) & \text{if } \delta = \gamma + 1 \\ \inf_{\gamma < \delta} \varphi_\sigma(\gamma) & \text{otherwise.} \end{cases}$$

The “pointwise infimum of sequence $(S_\xi)_{\xi < \zeta}$ ” is then defined by

$$\inf_{\xi < \zeta} S_\xi =: \left\{ \inf_{\xi < \zeta} \varphi_\sigma(\xi) \mid \sigma \in S_0 \right\}. \tag{*}$$

Definition (*) presupposes a ζ -sequence of extension-witnessing bijections. That is no less true of the fourth case at (‡) below, where the bijections involved are fixed by the collapse operations figuring in that definition.

Assume both $\emptyset \neq A \subseteq \mathfrak{D}^\mu$ with $\mu > 0$ and sequence $\{(\xi_\alpha^\gamma)_{\alpha < \mu'}\}_{\gamma < \zeta}$ of ordinal sequences with $0 < \zeta < \theta$. Then we define, by simultaneous ordinal recursion, unary functions $v : \zeta + 1 \mapsto \mu + 1$ and F , whereby $F(\delta)$ with $\delta < \zeta + 1$ is a subset of $\mathfrak{D}^{v(\delta)}$, by writing both

$$v(\delta) =: \begin{cases} \mu & \text{if } \delta = 0 \\ \text{otp}[(v(\gamma) \setminus \{ \xi_\alpha^\gamma \mid 0 < \alpha < \mu'_\gamma \}, <)] & \text{if } \delta = \gamma + 1 \text{ and } \mathbf{both} \\ & (\xi_\alpha^\gamma)_{\alpha < \mu'_\gamma} \text{ is } v(\gamma)\text{-bounded and} \\ & F(\gamma) \text{ is } [(\xi_\alpha^\gamma)_{\alpha < \mu'_\gamma}]\text{-identical} \\ & \text{(henceforth “} \otimes \text{ holds”)} \\ v(\gamma) & \text{if } \delta = \gamma + 1 \text{ but } \otimes \text{ fails to hold} \\ \bigcap_{\gamma < \delta} v(\gamma) & \text{otherwise} \end{cases} \quad (\dagger)$$

and

$$F(\delta) =: \begin{cases} A & \text{if } \delta = 0 \\ \left\{ \text{collapse} [(\mathbf{a}_\beta)_{\beta < v(\gamma)}, (\xi_\alpha^\gamma)_{\alpha < \mu'_\gamma}] \mid (\mathbf{a}_\beta)_{\beta < v(\gamma)} \in F(\gamma) \right\} & \text{if } \delta = \gamma + 1 \text{ and} \\ & \otimes \text{ holds} \\ F(\gamma) & \text{if } \delta = \gamma + 1 \text{ but} \\ & \otimes \text{ fails to hold} \\ \inf_{\gamma < \delta} F(\gamma) & \text{otherwise.} \end{cases} \quad (\ddagger)$$

We are now ready to formulate

THEOREM 5.5. *Let $\mathcal{M} =: \langle f_1^{(n_1)}, \dots, f_p^{(n_p)}, \text{Id}_{\mathfrak{D}}^{(2)} \rangle \in \mathfrak{T}_{\mathfrak{D}}^{\bar{n}, \equiv}$. Suppose that, for each $\lambda < \theta$, we have that $\mathfrak{B}_\lambda \subseteq \wp(\mathfrak{D}^\lambda)$ and that $\{\mathfrak{B}_\lambda\}_{\lambda < \theta}$ is the smallest family $\{\mathfrak{C}_\lambda\}_{\lambda < \theta}$ satisfying*

- (i) (Fundamental Predicates) *pre-image $[f_\ell^{(n_\ell)}]^{-1}(1)$ is in \mathfrak{C}_{n_ℓ} for $1 \leq \ell \leq p$, and $[\text{Id}_{\mathfrak{D}}^{(2)}]^{-1}(1)$ is in \mathfrak{C}_2*
- (ii) (Identification) *for arbitrary λ with $1 < \lambda < \theta$, we have $\{(\mathbf{b}_\alpha)_{\alpha < \lambda} \in \mathfrak{D}^\lambda \mid \mathbf{b}_\gamma = \mathbf{b}_\delta\}$ in \mathfrak{C}_λ for any fixed $\gamma < \delta < \lambda$*
- (iii) (Complementation) *for arbitrary $\lambda < \theta$, if A is in \mathfrak{C}_λ , then so is $\mathfrak{D}^\lambda \setminus A$*
- (iv) (Sum Sets of Small Families) *for arbitrary $\lambda < \theta$, if each member A_ι of family $\{A_\iota\}_{\iota \text{ in } I}$ with $|I| < \theta$ is in \mathfrak{C}_λ , then so is $\bigcup_{\iota \text{ in } I} A_\iota$*
- (v) (Cylindrification) *if A is in \mathfrak{C}_{λ_2} with $\lambda_1 + \lambda_2 + \lambda_3 = \lambda < \theta$, then $\{\tau_1 \hat{\cap} \tau_2 \hat{\cap} \tau_3 \in \mathfrak{D}^\lambda \mid \tau_1 \in \mathfrak{D}^{\lambda_1} \text{ and } \tau_2 \in A \text{ and } \tau_3 \in \mathfrak{D}^{\lambda_3}\}$ is in \mathfrak{C}_λ*
- (vi) (Collapse) *if nonempty A is in \mathfrak{C}_μ with $0 < \mu < \theta$ and each member of sequence $\{(\xi_\alpha^\gamma)_{\alpha < \mu'_\gamma}\}_{\gamma < \zeta}$, with $0 < \zeta < \theta$ fixed, is itself an ordinal*

sequence, then we have that $F(\delta)$, as defined at (\ddagger) , is in $\mathfrak{C}_{v(\delta)}$ for all $0 < \delta < \zeta + 1$, where ordinal function v is defined as at (\dagger) .

Then \mathfrak{B}_λ is $\text{Def}_\lambda^\emptyset =: \{C \subseteq \mathfrak{D}^\lambda \mid C \text{ is } \emptyset\text{-definable modulo } \mathcal{M}\}$ for each $\lambda < \theta$.

PROOF. Although differing considerably in their details, the structure of the proof of Theorem 5.5, and that of Theorem 5.6 below, is precisely that of [9, Proposition 1.3.4]. ■

Preliminary to the characterization of parametric definability in Theorem 5.6, we define a ternary interpolation operation that is the inverse, in a stronger sense, of the collapse operation defined earlier. Suppose that $(\xi_\alpha)_{\alpha < \mu'}$ is a μ -bounded sequence of ordinals and let $\text{otp}[(\mu \setminus \{\xi_\alpha \mid \alpha < \mu'\}, <)] = \lambda$. Then, with $(\mathbf{a}_\beta)_{\beta < \lambda} \in \mathfrak{D}^\lambda$ and $(\mathbf{b}_\alpha)_{\alpha < \mu'} \in \mathfrak{D}^{\mu'}$ given,

$$\text{interpolate}[(\mathbf{a}_\beta)_{\beta < \lambda}, (\mathbf{b}_\alpha)_{\alpha < \mu'}, (\xi_\alpha)_{\alpha < \mu'}]$$

is defined to be the sequence $(\mathbf{c}_\beta)_{\beta < \mu} \in \mathfrak{D}^\mu$ given by

$$\mathbf{c}_\beta =: \begin{cases} \mathbf{a}_\gamma & \text{if } \beta \text{ is the } \gamma\text{th term of ordinal sequence } (\mu \setminus \{\xi_\alpha \mid \alpha < \mu'\}, <) \\ \mathbf{b}_\alpha & \text{if } \beta = \xi_\alpha \text{ for some (necessarily unique) } \alpha < \mu' \end{cases}$$

for all $\beta < \mu$. (Thus the terms of sequence $(\mathbf{b}_\alpha)_{\alpha < \mu'}$ are interpolated, within sequence $(\mathbf{a}_\beta)_{\beta < \lambda}$, as directed by ordinal sequence $(\xi_\alpha)_{\alpha < \mu'}$ so as to obtain a new sequence $(\mathbf{c}_\beta)_{\beta < \mu} \in \mathfrak{D}^\mu$.)

THEOREM 5.6. (Hull Construction) *Let $\mathcal{M} \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}, \equiv}$. Suppose further that, for each $\lambda < \theta$, we have that $\mathfrak{B}_\lambda \subseteq \wp(\mathfrak{D}^\lambda)$ and that $\{\mathfrak{B}_\lambda\}_{\lambda < \theta}$ is the smallest family $\{\mathfrak{C}_\lambda\}_{\lambda < \theta}$ satisfying (i)–(vi) of Theorem 5.5 as well as*

- (vii) (Parametrization) *if nonempty A is in \mathfrak{C}_μ with $0 < \mu < \theta$ and $(\xi_\alpha)_{\alpha < \mu'}$ is a μ -bounded sequence of ordinals, then, with “parameter sequence” $(\mathbf{b}_\alpha)_{\alpha < \mu'} \in \mathfrak{D}^{\mu'}$ fixed and $\lambda =: \text{otp}[(\mu \setminus \{\xi_\alpha \mid \alpha < \mu'\}, <)]$, we have that \mathfrak{C}_λ contains*

$$\left\{ (\mathbf{a}_\beta)_{\beta < \lambda} \in \mathfrak{D}^\lambda \mid \text{interpolate}[(\mathbf{a}_\beta)_{\beta < \lambda}, (\mathbf{b}_\alpha)_{\alpha < \mu'}, (\xi_\alpha)_{\alpha < \mu'}] \in A \right\}.$$

Then the family $\{\mathfrak{B}_\lambda\}_{\lambda < \theta}$ is identical with the hull $\{\Gamma_\lambda^{\mathcal{M}}\}_{\lambda < \theta}$ of model \mathcal{M} .

We have succeeded in constructing the “definable hull” of model $\mathcal{M} \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}, \equiv}$ without appeal to the satisfaction relation. Absent a theory of infinitary relations, one would be unable to say the first thing about definability in the context of Zermelo’s theory of infinitely long propositions. That said, Theorems 5.5 and 5.6, formulated for maximum generality, transcend anything Zermelo himself might have envisioned. In particular, since

the only models he considered were models of *mathematical theories*, he routinely assumed that $|\mathfrak{D}| < \theta$, lest the domain-wide conjunctions and disjunctions serving as axioms fail to be propositions of context-furnishing Zermelo logics $\mathcal{H}_{\mathfrak{D},\mathfrak{K}}^\theta$. Thus, whereas, in the present section, we have provided Zermelo with a model theory, in no sense should it be seen as a reconstruction. Indeed, Zermelo lacks even something so basic as our Definition 2.2 (cf. [20, p. 593]).

In Sect. 7 we shall offer a final application of our variable-free, ordinal-theoretic account of infinitary relations. The new application will concern Gödel's First Incompleteness Theorem, Zermelo's reaction to it, and philosophical issues concerning the foundations of mathematics. We first set the stage, in Sect. 6, by presenting an equivalence result linking our nonlinguistic theory of infinitary relations to the language of first-order arithmetic.

6. Arithmetic Expressibility and Infinitary Relations

Let \mathbb{N} be denumerable set $\{a_0, a_1, a_2, \dots\}$ and let Ar be identity catalogue $\langle R_1, R_2, R_3, I \rangle$, where R_1 is dyadic (as suggested by "is the successor of") and R_2 and R_3 are both triadic (as suggested by "is the sum (product) of ... and ..."). Let $\mathcal{L}_{\mathbb{N},\text{Ar}}$ be the first-order language comprising (1) n_ℓ -adic predicate constant \mathbf{R}_ℓ for each fundamental predicate R_ℓ of Ar as well as dyadic predicate constant $\dot{=}$, (2) individual constants $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots$ denoting a_0, a_1, a_2, \dots , respectively, (3) a countably infinite set of individual variables $\mathbf{x}_1, \mathbf{x}_2, \dots$, (4) connectives $\dot{\vee}$ and $\dot{\wedge}$ and quantifier \exists , and (5) comma and parentheses. A *term* is an individual constant or a variable, and formation rules are the usual ones. We let n_1, n_2, \dots and the like stand for generic elements of \mathbb{N} , and $\mathbf{n}_1, \mathbf{n}_2, \dots$ will indicate individual constants denoting them. We write $\vec{\mathbf{x}}$ for variable sequence $\mathbf{x}_1, \dots, \mathbf{x}_k$ and $\vec{\mathbf{n}}$ for constant sequence $\mathbf{n}_1, \dots, \mathbf{n}_k$. As usual, $\Phi(\vec{\mathbf{x}})$ is a formula of $\mathcal{L}_{\mathbb{N},\text{Ar}}$ whose free variables (if any) are among $\mathbf{x}_1, \dots, \mathbf{x}_k$, and $\Phi(\vec{\mathbf{n}})$ will be the result of replacing each and every occurrence of variable \mathbf{x}_i , with $1 \leq i \leq k$, by constant \mathbf{n}_i throughout $\Phi(\vec{\mathbf{x}})$.

Modulo saturation by individual constants, formulæ of $\mathcal{L}_{\mathbb{N},\text{Ar}}$ can be associated, in a natural way, with propositions of Zermelo logic $\mathcal{H}_{\mathbb{N},\text{Ar}}^\theta$ whereby $\theta > \omega$, as described in

DEFINITION 6.1. Where $\Phi(\vec{\mathbf{x}})$ is a formula of $\mathcal{L}_{\mathbb{N},\text{Ar}}$ and $n_1, \dots, n_k \in \mathbb{N}$, we define the *propositional expansion* $[\Phi(\vec{\mathbf{n}})]_{\mathbb{N}}^+$ of $\Phi(\vec{\mathbf{x}})$ modulo saturation by individual constants $\vec{\mathbf{n}}$ as follows:

1. if $\Phi(\vec{x})$ is $\mathbf{R}_\ell t_1 \dots t_{n_\ell}$ with $1 \leq \ell \leq 3$, then $[\Phi(\vec{n})]_{\mathbb{N}}^+$ is $R_\ell m_1 \dots m_{n_\ell}$, where, for all $1 \leq i \leq n_\ell$, we have

$$m_i = \begin{cases} n_j & \text{if term } t_i \text{ is variable } \mathbf{x}_j \text{ with } 1 \leq j \leq k \\ a_j & \text{if term } t_i \text{ is individual constant } \mathbf{j} \end{cases} \quad (*)$$

2. if $\Phi(\vec{x})$ is $t_1 \doteq t_2$, then $[\Phi(\vec{n})]_{\mathbb{N}}^+$ is $m_1 \equiv m_2$, where m_i with $1 \leq i \leq 2$ is defined as at (*)
3. if $\Phi(\vec{x})$ is $\dot{\neg}\Psi(\vec{x})$, where $\Psi(\vec{x})$ is a formula of $\mathcal{L}_{\mathbb{N},Ar}$, then $[\Phi(\vec{n})]_{\mathbb{N}}^+$ is $\neg[\Psi(\vec{n})]_{\mathbb{N}}^+$
4. if $\Phi(\vec{x})$ is $(\Psi(\vec{x}) \dot{\vee} \Xi(\vec{x}))$, where $\Psi(\vec{x})$ and $\Xi(\vec{x})$ are formulæ of $\mathcal{L}_{\mathbb{N},Ar}$, then $[\Phi(\vec{n})]_{\mathbb{N}}^+$ is $[\Psi(\vec{n})]_{\mathbb{N}}^+ \vee [\Xi(\vec{n})]_{\mathbb{N}}^+$
5. if $\Phi(\vec{x})$ is $\exists \mathbf{x}_{k+1}(\Psi(\vec{x}, \mathbf{x}_{k+1}))$, where $\Psi(\vec{x}, \mathbf{x}_{k+1})$ is a formula of $\mathcal{L}_{\mathbb{N},Ar}$, then $[\Phi(\vec{n})]_{\mathbb{N}}^+$ is $\bigvee \left\{ [\Psi(\vec{n}, \mathbf{m})]_{\mathbb{N}}^+ \mid \mathbf{m} \in \mathbb{N} \right\}$.

Since $\theta > \omega$, we have that $[\Phi(\vec{n})]_{\mathbb{N}}^+ \in \mathcal{H}_{\mathbb{N},Ar}^\theta$ invariably. We set $\mathbb{N}^* =: \mathbb{N} \sqcup \theta$ and assume a well-ordered linearization of $\mathbb{N}^* \setminus \mathbb{N}$.

DEFINITION 6.2. We say that $\mathcal{A} \in \mathcal{H}_{\mathbb{N},Ar}^\theta$ is (arithmetically) expressible, and write $\mathcal{A} \in \text{expr}_{\mathbb{N},Ar}$, provided that \mathcal{A} is $[\Phi(\mathbf{n}_1, \dots, \mathbf{n}_k)]_{\mathbb{N}}^+$ for some formula $\Phi(\mathbf{x}_1, \dots, \mathbf{x}_k)$ of $\mathcal{L}_{\mathbb{N},Ar}$ and some $n_1, \dots, n_k \in \mathbb{N}$.

Of course, $\text{expr}_{\mathbb{N},Ar}$ is a countable subset of uncountable $\mathcal{H}_{\mathbb{N},Ar}^\theta$, which will be important in Sect. 7.

Our goal in the present section is an alternate, nonlinguistic characterization of $\text{expr}_{\mathbb{N},Ar}$. To this end, we present

DEFINITION 6.3. The family $\{\mathcal{F}_\alpha\}_{\alpha < \omega_2}$ of first-order arithmetical relations (of finite character) is the smallest family $\{\mathcal{G}_\alpha\}_{\alpha < \omega_2}$ satisfying:

1. with $1 \leq \ell \leq 3$ and $\mathbf{a}_1^*, \dots, \mathbf{a}_{n_\ell}^* \in \mathbb{N}^*$, relation $R_\ell \mathbf{a}_1^* \dots \mathbf{a}_{n_\ell}^*$ is in \mathcal{G}_0 , as is $\mathbf{a}_1^* \equiv \mathbf{a}_2^*$
2. if relation $\mathcal{R}(\pi_1, \dots, \pi_k)$ with $k \geq 0$ is in \mathcal{G}_β with $\beta < \omega_2$, then we have that $\sim \mathcal{R}(\pi_1, \dots, \pi_k)$ is in $\mathcal{G}_{\beta+1}$
3. if relation $\mathcal{R}(\pi_1, \dots, \pi_k)$ is in \mathcal{G}_{β_1} and relation $\mathcal{S}(\rho_1, \dots, \rho_\ell)$ is in \mathcal{G}_{β_2} with $k, \ell \geq 0$ and $\beta_1, \beta_2 < \omega_2$, then $\mathcal{R}(\pi_1, \dots, \pi_k) \dot{\vee} \mathcal{S}(\rho_1, \dots, \rho_\ell)$ is in $\mathcal{G}_{\max(\beta_1, \beta_2)+1}$
4. $\mathcal{G}_\omega = \bigcup_{\alpha < \omega} \mathcal{G}_\alpha$

5. if relation $\mathcal{R}(\pi_1, \dots, \pi_k)$ with $k \geq 1$ is in \mathcal{G}_β with $\omega \leq \beta < \omega 2$, then existential generalization $\bigvee_{\mathbf{c} \text{ in } \mathbb{N}} \mathcal{R}(\pi_1, \dots, \pi_{j-1}, \mathbf{c}, \pi_{j+1}, \dots, \pi_k)$ is in $\mathcal{G}_{\beta+1}$ for any j with $1 \leq j \leq k$.

Thus the members of $\bigcup_{\alpha < \omega 2} \mathcal{F}_\alpha$, all of finite degree, are structurally simple: They are all and only those “infinitary” relations, over \mathbb{N} and Ar , built up from atomic relations, possibly with parameters in \mathbb{N} , by finitely many applications of the relation-forming operations defined in Sect. 3.

With $\alpha < \omega 2$, we write $G_\alpha \subseteq \mathcal{H}_{\mathbb{N}, \text{Ar}}^\theta$ for the set of fully saturated substitution instances of members of \mathcal{F}_α . Our terminology notwithstanding, the family $\{G_\alpha\}_{\alpha < \omega 2}$ of fully saturated substitution instances of first-order arithmetical relations of finite character has been defined without reference to $\mathcal{L}_{\mathbb{N}, \text{Ar}}$. As a final technical offering, we bring two ideas together in

THEOREM 6.4. (Coincidence Theorem) $\bigcup_{\alpha < \omega 2} G_\alpha = \text{expr}_{\mathbb{N}, \text{Ar}}$.

PROOF. For one direction, one proves the strengthening: *If $\mathcal{R}(\pi_1, \dots, \pi_k)$ is a first-order arithmetical relation of finite character, then there exists a formula $\Phi(\mathbf{x}_1, \dots, \mathbf{x}_k)$ of $\mathcal{L}_{\mathbb{N}, \text{Ar}}$, unique up to ordering of disjuncts (if any), in which each of $\mathbf{x}_1, \dots, \mathbf{x}_k$ has at least one free occurrence such that, for any $n_1, \dots, n_k \in \mathbb{N}$, the fully saturated substitution instance $\mathcal{R}(n_1, \dots, n_k)$ of $\mathcal{R}(\pi_1, \dots, \pi_k)$ is expansion $[\Phi(\mathbf{n}_1, \dots, \mathbf{n}_k)]_{\mathbb{N}}^+$.* The proof proceeds by induction on the complexity of $\mathcal{R}(\pi_1, \dots, \pi_k)$.

In the other direction, also, one proves a stronger claim, namely: *If $\Phi(\mathbf{x}_1, \dots, \mathbf{x}_k)$ is a formula of $\mathcal{L}_{\mathbb{N}, \text{Ar}}$ and the members of $H_{\Phi(\vec{x})} \subseteq [1, k]$ are all and only the indices of variables among $\mathbf{x}_1, \dots, \mathbf{x}_k$ having free occurrences in $\Phi(\mathbf{x}_1, \dots, \mathbf{x}_k)$, then there exists first-order arithmetical relation $\mathcal{R}(\pi_1, \dots, \pi_j)$ of finite character, unique up to similarity, and unique bijection $\vartheta_{\mathcal{R}(\pi_1, \dots, \pi_j)}^{\Phi(\vec{x})} : H_{\Phi(\vec{x})} \mapsto [1, j]$ such that, for $n_1, \dots, n_k \in \mathbb{N}$ arbitrary, propositional expansion $[\Phi(\mathbf{n}_1, \dots, \mathbf{n}_k)]_{\mathbb{N}}^+$ is fully saturated substitution instance $\mathcal{R}(n_{[\vartheta_{\mathcal{R}(\pi_1, \dots, \pi_j)}^{\Phi(\vec{x})}]^{-1}(1)}, \dots, n_{[\vartheta_{\mathcal{R}(\pi_1, \dots, \pi_j)}^{\Phi(\vec{x})}]^{-1}(j)})$ of $\mathcal{R}(\pi_1, \dots, \pi_j)$.* The proof proceeds by induction on the complexity of $\Phi(\mathbf{x}_1, \dots, \mathbf{x}_k)$, where the more challenging subcase is that of disjunction. ■

Coincidence Theorem 6.4 in hand, we now turn to philosophy. Zermelo scorned Gödel's First Incompleteness Theorem, motivated ultimately by his own doctrine of the primacy of the infinite in logic and mathematics. More immediately, Zermelo adduced a simple cardinality argument purportedly unveiling the triviality of Gödel's result (see below). Yet rejection of that result is, truth be told, no inevitable consequence of Zermelo's foundational philosophy. Indeed, Coincidence Theorem 6.4 will be brought to bear so as

to cast Gödel's result in a favorable light, even with respect to the systems of infinitely long propositions that Zermelo preferred.

7. Incompleteness, Zermelo Logics, and Natural Kinds

Der Mensch trägt vielerlei Möglichkeiten in sich, und je nach der logischen Kette, die er um die Dinge wirft, kann er sich beweisen, daß sie gut oder schlecht sind.

—Hermann Broch *Esch oder die Anarchie*

By Peano arithmetic, or PA, we shall mean a well-known theory whose axioms are formulated in the first-order language $\mathcal{L}_{\mathbb{N}, \text{Ar}}$ described in Sect. 6. Provability in PA is related to Zermelo logic $\mathcal{H}_{\mathbb{N}, \text{Ar}}^\theta$ by way of

DEFINITION 7.1. Proposition $\mathcal{A} \in \text{expr}_{\mathbb{N}, \text{Ar}} \subseteq \mathcal{H}_{\mathbb{N}, \text{Ar}}^\theta$ is said to be *arithmetically decidable* if \mathcal{A} is the propositional expansion, modulo saturation, of a formula of $\mathcal{L}_{\mathbb{N}, \text{Ar}}$ that is either provable or refutable in PA.

Let h be the Gödel number of formula $\neg \exists \mathbf{x}_1 (\text{proof}_{\text{PA}}(\mathbf{x}_1, \mathbf{x}_2))$ of $\mathcal{L}_{\mathbb{N}, \text{Ar}}$. By Definition 6.1, the expansion $[\neg \exists \mathbf{x}_1 (\text{proof}_{\text{PA}}(\mathbf{x}_1, \mathbf{h}))]_{\mathbb{N}}^+$ (henceforth \mathcal{A}_g) of sentence $\neg \exists \mathbf{x}_1 (\text{proof}_{\text{PA}}(\mathbf{x}_1, \mathbf{h}))$ is in $\text{expr}_{\mathbb{N}, \text{Ar}}$. However, Gödel showed this sentence to be neither provable nor refutable in PA, assuming that theory consistent. Thus we have widely, if not quite universally, celebrated

THEOREM 7.2. (Gödel's First Incompleteness Theorem) *Assuming PA consistent, a member (namely, \mathcal{A}_g) of $\text{expr}_{\mathbb{N}, \text{Ar}}$ is arithmetically undecidable.*

Is Incompleteness Theorem 7.2 significant for Zermelo logics? Not according to Zermelo, who argued that Gödel's result is unsurprising:

From our point of view, there are no objectively undecidable propositions. On the other hand, Gödel has tried to prove the opposite. The only reason Gödel's proof works is because he applies the finitistic restriction only to the provable propositions of the system and not to all propositions belonging to the system. Thus only the former, but not the latter, form a countable set, and of course, when understood in this sense, there must exist undecidable propositions. ([19, p. 547])

Zermelo was unimpressed by Gödel's having *presented* an undecidable proposition. Cardinality is again the issue in a second, and final, letter to Gödel:

You obtain an uncountable system of possible propositions only a countable subset of which would be provable, and there would certainly have to exist undecidable propositions. ([21, p. 501])

The “uncountable system” of which Zermelo writes in both passages is Russell’s simple theory of types with natural numbers as individuals, which Gödel assumed, whereas “countable subset” refers to the collection of (propositions expressed by) sentences of that theory derivable from Peano’s axioms. It is convenient, and no great distortion, to translate Zermelo’s remarks so as to make them refer to Zermelo logic $\mathcal{H}_{\mathbb{N}, \text{Ar}}^\theta$, on the one hand, and, on the other, to the collection of arithmetically decidable propositions in the sense of Definition 7.1. Clearly, $\mathcal{H}_{\mathbb{N}, \text{Ar}}^\theta$, because uncountable, must contain propositions that are not in the merely countable subset of decidable ones. Zermelo concludes that Gödel’s result is redundant, telling us only what we already knew. However, it is wrong to focus on uncountable $\mathcal{H}_{\mathbb{N}, \text{Ar}}^\theta$, as does Zermelo. (That said, as we have reconstrued it, Zermelo’s misstep here is no mere willfulness: Lacking a theory of relations, what choice, compatible with his foundational philosophy, does Zermelo have for situating Gödel’s result within his own theory of infinitely long propositions?)

Gödel’s result is advantageously framed within Zermelo’s theory of infinitely long propositions, we would insist, only by identifying an appropriate *countable* collection of propositions containing undecidable \mathcal{A}_g . Further, such contextualization is ready to hand in the guise of $\text{expr}_{\mathbb{N}, \text{Ar}}$. Is Incompleteness Theorem 7.2, formulated in terms of countable $\text{expr}_{\mathbb{N}, \text{Ar}}$, not of considerable interest, given that cardinality alone does not predict it? Again, Zermelo, driven now by his doctrine of the primacy of the infinite, must answer no, since, by his lights, $\text{expr}_{\mathbb{N}, \text{Ar}}$ is not a natural collection:

Proceeding from the assumption that it should be possible to represent all mathematical concepts and theorems by means of a *fixed finite system of signs*, we inevitably run into the well-known “Richard antinomy,” recently resurrected in the form of Skolemism, i.e., the doctrine that every mathematical theory can be realized in a countable model. As is well known, everything can be proved from contradictory premises; but even the oddest conclusions Skolem and others have drawn from their basic assumption have apparently not sufficed to raise doubts about what has already attained the status of a dogma. But a healthy “metamathematics” will only become possible once we have definitively renounced that “finitistic prejudice.” Mathematics is not concerned with “combinations of signs” but with conceptually ideal relations among the elements of a conceptually posited infinite manifold. Our systems of signs are but *imperfect expedients* of our *finite* mind, which we employ in order to at least gradually get a hold on the infinite, which we can *neither* “survey” *nor* grasp *immediately* and

intuitively. [My goal is] to develop the foundations of a “mathematical logic” that is free from the “finitistic prejudice” and that permits mathematics’ further development *without arbitrary prohibitions and restrictions*. ([19, pp. 543, 545]; italics in original)

With respect to present concerns, we take Zermelo’s point here to be the following. Whereas finitude is a standing feature of human cognition, nonetheless, in mathematics, finitary methods are but stopgap measures. Consequently, $\text{expr}_{\mathbb{N}, \text{Ar}}$ is not a natural collection from the standpoint of mathematical science itself, according to Zermelo. This is because the essential character of $\text{expr}_{\mathbb{N}, \text{Ar}}$ is determined by finitary language $\mathcal{L}_{\mathbb{N}, \text{Ar}}$ and, hence, by accidental limitations of the human investigators of said science.

Before presenting our own contrasting view, it will be useful to summarize Sect. 6. First, $\text{expr}_{\mathbb{N}, \text{Ar}}$, a countable set of propositions, was defined, linguistically, in terms of propositional expansions of *formulae of language* $\mathcal{L}_{\mathbb{N}, \text{Ar}}$. Next, the family $\{G_\alpha\}_{\alpha < \omega_2}$ was defined. Our nonlinguistic characterization of that family made no reference to the language of arithmetic, appealing only to our nonlinguistic theory of infinitary relations, specifically to the relation-forming operations described in Sect. 3. Ultimately, Coincidence Theorem 6.4 asserted that $\text{expr}_{\mathbb{N}, \text{Ar}}$ is identical with $\bigcup_{\alpha < \omega_2} G_\alpha$. Thus we have two distinct characterizations—one linguistic and one nonlinguistic—of one and the same countable set of propositions of Zermelo logic $\mathcal{H}_{\mathbb{N}, \text{Ar}}^\theta$.

So is $\text{expr}_{\mathbb{N}, \text{Ar}}$, in the end, a natural collection of propositions? As we see things, the very existence of two alternate characterizations—only the first of which was available to Zermelo, incidentally—constitutes evidence, *pace* Zermelo, that “expressible proposition,” in the sense of Definition 6.2, is akin to a *natural kind*, a notion familiar from the Philosophy of Science. (Three examples from the world of experience would be “gold, and frankincense, and myrrh” as opposed to the random contents of Smith’s rental storage locker.) Meanwhile, Incompleteness Theorem 7.2 tells us something significant, and *surprising*, concerning the “arithmetically expressible” kind.

As for the worth of our own project, we would pose a final question. Absent a theory of infinitary relations, can one give an alternate characterization of $\text{expr}_{\mathbb{N}, \text{Ar}}$ like that derived from Definition 6.3, one that avoids reference to the language of PA and thereby buttresses our claim that $\text{expr}_{\mathbb{N}, \text{Ar}}$ is a natural collection? This time around, the answer seems to be no.

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