Situations, Propositions, and Information States

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ABSTRACT. Two families of relational semantics for relevant logics, the ternary relation and the Fine-style, or operational-relational, semantics are compared on point of interpretation. Following Punčochář, it's noted that the former kind tend to be given *ontic* or *realist* styles of interpretation, whereas the latter tend to be given *epistemic* or *informational* styles. The equivalence between these semantic approaches means that we can have both in one setting (with one grounded in the other), but it's argued that, nonetheless, there are reasons to prefer a version which takes the realist interpretation as basic and the informational one as grounded in it. The resulting, layered, semantic picture is sketched, and an application to the Mares–Goldblatt interpretation of quantifiers is proposed.

Keywords. Philosophy of logic, Relational semantics, Relevant logic, Theory of meaning

1. Introduction

Relational semantics for relevant and substructural logics can be put into two camps: there is the *ternary relation* (TR) framework most famously studied by Sylvan (né Routley) and Meyer [39; 40] and the operational–relational, or *Fine-style* (F) framework most famously studied by Fine [20]. (For further details on the history of these developments, see Bimbó and Dunn [7]; Bimbó et al. [8].) Punčochář [33, §6], noting that these two frameworks are formally equivalent, suggests that the difference between them should be understood as having to do with the kind of explanation they tend to proffer for the meanings of the logical vocabulary. As he sees it, the TR framework tends to be understood *ontically*, as having to do with the real world and objects therein, while the F framework tends to be understood *epistemically*, or perhaps a better word is *informationally*, as having to do with the sorts of things grasped by agents, communicated by assertions, and which comprise theories.

While one should not put too much weight on the claim for TR semantics are read realistically and the F semantics otherwise (realist versions of F semantics have been given, for instance, by Jago [24], where the elements of an F model are taken to represent exact truthmakers), it does track a tendency. For example, Barwise and Perry [4], whose *situation theory* is often invoked as a philosophical story motivating TR semantics (for instance, in Restall [35]; Mares [30] and Tedder [42]), took the realism of their picture as one of its theoretical strengths. On the other hand, Logan

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[27; 28] has recently proposed a form of the F semantics, which takes the objects literally to be *theories* (i.e., deductively closed sets of sentences).

These two frameworks are formally equivalent in the sense that any model of one kind can be transformed into a model of the other kind satisfying all the same formulas as the original (this will be properly spelled out, and proved, in §4, building on [33]). Punčochář takes this fact to indicate that one is free to choose whichever framework one prefers for a particular application, and this is, of course, true. Indeed, there are mathematical reasons why one might prefer one to the other: the TR semantics is often simpler in a mechanical sense (there are fewer things in the frames and fewer constraints), whereas the F semantics is often simpler in a conceptual sense (binary operations and relations are more familiar as mathematical objects than are ternary relations).[1] So in a sense there is, and need be, no rivalry between these two approaches.

It has, however, been suggested for some time that the TR semantics of relevant logics are unmotivated, *ad hoc* formalisms that do not provide a true meaning theory for the logical vocabulary. Perhaps the most famous version of this line of criticism is due to Copeland [9; 10], and one gets the impression that the F semantics has usually been taken to be the more natural account, as providing a more natural interpretation of the central conditional connective.[2] So when the goal is to provide a philosophically significant theory of meaning of the relevant vocabulary, which is one of the things that one may take the relational semantics to be for, there is a serious question on the table, and a reason to want to take sides.

In this paper, I will harness the equivalence results to suggest that we approach this apparent distinction in a different way, by focusing on the fact that we can always construct one kind of model from another. The resulting picture is a *layered* semantics, where one kind of model is *grounded in* a model of the other kind. Rather than taking either an ontic or informational stand, *tout court*, I'll suggest that we can always have both, and that the question comes down to which we take to be *basic*. We can always capture the ontic flavor of the TR semantics and the informational flavor of the F semantics in one framework; the real question concerns the *direction of explanation*. Do we account the ontic properties of a TR model in terms of the informational properties of an underlying F model, or vice versa? I'll argue that, in general, an ontically-based presentation provides a more satisfying route for explanation of the facts to be accounted for by a semantics (namely, facts about *entailments*), and that therefore, if we take Punčochář's distinction seriously, we ought to prefer to take the TR semantics as basic. With this argument made, I'll briefly discuss an interesting upshot for the Mares and Goldblatt [32] semantics for quantifiers.

[1] It might be, cheekily, put that part of the miracle performed in Urquhart's [45] undecidability proof was simply in *finding* a ternary relation in the wild, in the form of co-linearity.

[2] It's been suggested to the author, in conversation with an interlocutor who shall remain nameless (you know who you are), that the TR truth condition is a *kludge*, trying, and failing, to capture the beauty and simplicity of the operational truth condition, due to Urquhart [44], and that the resulting framework is ugly and unmotivated.

2. Theories of Models and Theories of Entailment

The formal structure of the options for orders of explanation — either accounting for ontic features in terms of informational ones or vice versa — closely mirrors a related dispute in the philosophical discussion surrounding possible worlds and their use in frame semantics for modal logics. In that literature a salient distinction is that between grounding propositions in possible worlds and grounding possible worlds in propositions. Versions of these approach are discussed in Loux [29].

The dispute between these two approaches, as discussed in Divers [12], can be seen as circling around the question of how best to account for the modal properties of propositions. An account of possible worlds, and the relationship they bear to propositions, should provide an account of when propositions are necessary or possible. Ideally, it should give us insight into questions about which particular propositions are necessary or possible, and why. The account which the realist line on possible worlds gives of these is familiar, namely, that a proposition is necessary when it is true in every possible world and it is possible when true in some. So when we are tasked with accounting for, say, what makes it the case that a particular proposition is possible, if we take the realist line of explanation our task is to provide reasons to believe that there is a possible world which makes the proposition true.

Theories of relational semantics for relevant logics are not aimed at providing explanations for why propositions are necessary or possible, but rather are aimed at providing explanations for why certain propositions *entail* others. As Anderson and Belnap [1, §1] stress, entailment is the heart of logic. So if we aim to give an account of the meanings of the logical connectives in terms of a theory of models, a major part of our project will be to do so in a way that accounts for why, in general, some propositions entail others.[3] Furthermore, the account should provide us with the means to answer questions about which particular propositions entail which others. So when it comes to deciding between rival semantic theories, it will be in terms of their accounts of entailments, and the kinds of explanation they proffer for particular entailment facts, that I propose we make our decisions. One reason to prefer a realist picture, as opposed to one like that of [28], which takes the basic elements to be theories, is that the realist approach seems to stand a better shot at providing satisfying answers to questions of the form "why does such-and-so particular implication claim express an entailment (or not)?" In the case of a negative answer to such a question, the realist line would have us describe a situation which supports an instance of the antecedent but not the appropriate instance of the consequent. With such a countermodel in mind, one will have provided a strong reason to reject the claim that the implication in question is an entailment.[4] Positive answers will concern the properties of situations, and how they go about satisfying propositions. The account taking theories as basic seems hard-pressed to provide a similarly satisfying, non-circular explanation of entailment facts.

[3]Throughout, I'll discuss "explanation" in a *metaphysical* sense. In this sense, a fact explains another one when it features in an account of why the latter holds.

[4]One example of doing such a thing can be found in [42], where it is argued that implications of the form $(\mathcal{A} \to \mathcal{B}) \to ((\mathcal{B} \to \mathcal{C}) \to (\mathcal{A} \to \mathcal{C}))$ do not express entailments, by means of describing a concrete situation which would falsify this implication formula.

Having said this, note that even theories of meaning for the logical vocabulary which are formally equivalent may yet differ with respect to the explanations they offer of these facts. (I venture to suggest that non-realist approaches will generally fare worse on this front.) Consider the case of possible worlds again. Whether we take propositions to be composed of possible worlds or take possible worlds to be composed of propositions, we may wind up with the same collection of propositions being necessary. There is, nonetheless, still a debate between these about which approach provides the better explanation of those modal facts. How can such debates proceed?

2.1. **Data-Fit, Parsimony, and Explanatory Power.** One way to cash out such debates concerns the extent to which different proposals satisfy different principles of theory choice. One such principle, the fit to the data, does not decide between these: being equivalent, both theories will fit the data to the same extent. So the choice comes down to other principles, and for my purposes there seem to be two, which are most salient:

(1) Ontological Parsimony: One should choose a theory which, *ceteris paribus*, involves commitment to fewer kinds of entity.
(2) Explanatory Power: One should choose a theory which, *ceteris paribus*, provides a more satisfying explanation of the phenomena underlying the data.

In the case of modal theories, we can take the salient data to concern the modal status of propositions, and choose between the candidate theories based on (1) and (2). Following Lewis [25], there seem to be good reasons to think that a realist account performs better on (2), but non-realist accounts seem to fare better on (1). The question then becomes when we're forced to choose one of (1), (2), which should we prefer?

I think that we should pretty much always prefer to gain explanatory power at the loss of parsimony than go the other way around, at least when all other things are, indeed, equal. Let me sketch a brief argument why.[5] What is the theoretical cost of having more ontological commitments? As far as I can tell, the main cost is that taking on commitments to more kinds of entities runs the risk of falsifying the theory. If we commit ourselves to the existence of something which turns out not to exist, we'll have made an error, and have a false theory on our hands. Such risks are, indeed, theoretical costs, as are any commitments we take on which might wind up false. However, they are *just as costly* as any other such risky commitments we take on by making assertions — they are not *more costly*. So when we decide which theories to adopt, and we weigh the costs of ontological commitments, which come along with the theory, we should weigh these the same way we do any potentially false claims the theory makes.

If this is correct, then when we are in a position to decide whether to adopt an ontologically profligate, but more explanatory theory or one which is more parsimonious and less explanatory, the question comes down to whether we should take on a greater risk of falsehood in the hope of having a more explanatory theory. I think the answer

[5] This argument is, of course, deeply indebted to Lewis [25], though I'll refrain from citing chapter and verse as I go into it. It's worth noting that it involves an appeal to *inference to the best explanation*, and this has been discussed in detail by Lipton, information concerning which can be found in [26]. The notion of a "satisfying explanation" to which I appeal is, perhaps, best understood as an appeal to an explanation being the "loveliest," in his terminology, but I'll leave this appeal somewhat vague here.

here should be a resounding "yes." In general, we are better off taking the liberal attitude of seeking truth than we are taking the more conservative approach of avoiding falsehood; from this perspective, if the only thing we have to lose by taking onboard commitments to further entities that provide us with better explanatory power is the risk of falsehood, we should do so.

3. Ontic TR Frames and Informational F Frames

Let's cash out the sense in which you might understand TR semantics as being more ontic and F semantics as being more informational. As mentioned, this distinction is not hard and fast. There are realist readings offered of some forms of F semantics (as in [24]), and there have certainly been informationally flavored readings of semantics in the TR framework, such as in Dunn's work on program interpretations [17; 14; 15].

Having said this, it does seem to be the case that, as Punčochář [33] notes, there is a tendency for proponents of the TR semantics to defend realist readings and those of the F semantics to defend informational readings. Perhaps, the most clear point of distinction between these approaches concerns the interpretation of disjunction; a bit of discussion of the history here is salient. Urquhart [44] first attempted to give an operational semantics for relevant logics employing a frame with points obeying the truth condition for disjunction common from Kripke semantics for modal and intuitionist logic. Taking $[\![\mathcal{A}]\!]^M$ as the collection of points of a frame satisfying a formula $\mathcal{A}$ in a model M, this truth condition is:

$$[\![\mathcal{A} \vee \mathcal{B}]\!]^M = [\![\mathcal{A}]\!]^M \cup [\![\mathcal{B}]\!]^M.$$

The problem, well known, is that if one attempts to interpret the conditional in terms of a binary operation $\otimes$, as

$$[\![\mathcal{A} \to \mathcal{B}]\!]^M = \{s \colon \forall t (t \in [\![\mathcal{A}]\!]^M \Rightarrow s \otimes t \in [\![\mathcal{B}]\!]^M)\},$$

then one winds up with models for which standard relevant logics are not complete (see Dunn and Restall [18], for more details). In order to resolve the problem, one must adopt a different truth condition either for disjunction or the conditional. The F framework takes the former route, and the TR framework takes the latter.

It's been noted many times over the years (e.g., [23; 13; 22; 24]) that the standard truth condition for disjunction is ill-suited to interpretations of points in the frame as informational, motivating the move made by proponents of the F framework. For instance, suppose we take frame elements to represent information states, such as those available to an agent in the course of a reasoning task. There's no good reason to suppose that whenever such an agent has information supporting a disjunction, they'll have information supporting either disjunct. For instance, Sherlock Holmes may have enough information to know "either Moriarty or Queen Victoria committed the murder" without having information adequate to pin down the identity of the killer. This is one way that an informational reading is especially well suited to the F style semantics.

The kind of situation-theoretic reading often offered for TR semantics can avoid this issue by taking situations themselves *not* to be the sorts of things which agents directly cognize. On this sort of picture, what an agent cognizes is not a situation but rather a proposition (or collection thereof), which *type* situations, but need not *be* situations themselves. Situations are, perhaps, well understood as *inexact truthmakers* which support the truth of propositions — in this case, it is a plausible claim that they support a disjunction just in case they support one of the disjuncts.[6]

Another way this tendency comes up is that the most standard interpretation of the ternary relation, using *channel theory* Barwise [3], as in [35; 42], has a realist flavour. It posits mind-independent links between situations to interpret the ternary relation. The operation of the F semantics, on the other hand, is usually read informationally, as concerning the result of applying an *epistemic* or *informational* action on bits of information, sentences, or theories [44; 41; 28]. I won't go into further detail, but hopefully this suffices to bolster Punčochář's case that there is a tendency for TR semantics to be read ontically and F semantics to be read epistemically/informationally.

Now let's turn to the equivalence of the frameworks.

4. A Sketch of Equivalence Between TR and F

This section is, as the title suggests, just a sketch — a more detailed investigation of these matters is certainly possible. I'll give basic details to provide the reader an indication of how the construction works, going in either direction, and how it naturally proceeds through the three layers on which I'll be focused later. A fuller presentation of a narrower result concerning the logic **R** can be found in [33].

The main aim is to show that the TR and F semantic frameworks are equivalent in the sense that from a model on a frame of one kind, we can construct a model on a frame of the other kind which satisfies just the same formulas. This goes to show that the frameworks capture, in a sense, the same *data*, leaving the question of the choice between them up to other theoretical considerations. In this section, I'll introduce a form of the TR semantics and a form of the F semantics and then show how to construct one from the other in a simple, uniform way.

There are a number of available variations on the theme of "TR semantics" and "F semantics," and the versions I sketch here are chosen in a way which is partially due to my own, perhaps idiosyncratic, preferences and partially in order to simplify the presentation. I'll deal here with basic forms of the TR and F semantics appropriate for the relevant logic **B** — the correspondence available between frame conditions and further axioms or rules which may be added to **B** to obtain further logics is well known, and we do not need to go into it here. I take the propositional language to be defined, as usual, from a set of atomic formulas $\mathbb{P}$, the logical constant **t**, and the connectives $\neg, \wedge, \vee, \rightarrow$ (of arities 1, 2, 2, 2, respectively). I'll use $\mathcal{L}$ to denote the language.

[6] For related discussion, into which I'll not go further here, see Deigan [11]. As a related point, note that both the TR and F semantic frameworks commonly employ the standard truth condition for conjunction in terms of set intersection. One upshot of this, in the case of the situation-theoretic picture, is that we obtain the validity of the distribution law immediately from the fact that a powerset algebra, with unions and intersections, is a distributive lattice. The justification of the distribution law has been discussed in relevant circles (e.g., in Belnap [5] and Restall [37]), and this raises a potential avenue of objection against reading the situation-theoretic line as realist. I won't go into this question further, but note it as a potential difficulty.

4.1. **Ternary Relation Frames and Models.**

Definition 1. A ternary relation (TR) frame $\mathfrak{F}$ is a tuple $\langle W, N, R, *\rangle$ where $\varnothing \neq N \subseteq W$, $R \subseteq W^3$, and $*\colon W \longrightarrow W$ are such that, given the following definitions:

$$\leq = \{\langle \alpha, \beta\rangle \in W^2 \colon \exists \gamma \in N(R\gamma\alpha\beta)\}$$

$$\mathcal{P}(W)^{\uparrow} = \{X \subseteq W \colon \forall \beta \in W(\exists \alpha \in X(\alpha \leq \beta) \Rightarrow \beta \in X)\},$$

the following constraints are satisfied:

(tr1) $\langle W, \leq\rangle$ is a poset.
(tr2) $N \in \mathcal{P}(W)^{\uparrow}$.
(tr3) If $\alpha' \leq \alpha, \beta' \leq \beta, \gamma \leq \gamma'$ and $R\alpha\beta\gamma$, then $R\alpha'\beta'\gamma'$.
(tr4) If $\alpha \leq \beta$ then $\beta^* \leq \alpha^*$, and furthermore, $\alpha^{**} = \alpha$.

Before defining models on TR frames, let's fix a couple other definitions. First,

Definition 2. Given a set $\Gamma \subseteq \mathcal{P}(W)^{\uparrow}$, we fix the following:

$$[\Gamma) := \{Y \in \mathcal{P}(W)^{\uparrow} \colon \exists X_1, \ldots, X_n \in \Gamma(\bigcap_{j \leq n} X_j \subseteq Y)\}$$

Briefly, $[\Gamma)$ is the least *filter*, on the distributive lattice $\langle \mathcal{P}(W)^{\uparrow}, \cap, \cup\rangle$, containing Γ.[7]

Definition 3. Given $X, Y \in \mathcal{P}(W)^{\uparrow}$, let

$$X \to Y = \{\alpha \colon \forall \gamma(\exists \beta \in X(R\alpha\beta\gamma) \Rightarrow \gamma \in Y)\}$$

$$\neg X = \{\alpha \colon \alpha^* \notin X\}$$

Definition 4. A model M on a TR frame $\mathfrak{F}$ is a function of type $\mathbb{P} \longrightarrow \mathcal{P}(W)^{\uparrow}$, extended to a valuation $[\![\cdot]\!]^M \colon \mathcal{L} \longrightarrow \mathcal{P}(W)^{\uparrow}$ as follows:

(1) $[\![p]\!]^M = M(p)$	(4) $[\![\mathcal{A} \wedge \mathcal{B}]\!]^M = [\![\mathcal{A}]\!]^M \cap [\![\mathcal{B}]\!]^M$
(2) $[\![\mathbf{t}]\!]^M = N$	(5) $[\![\mathcal{A} \vee \mathcal{B}]\!]^M = [\![\mathcal{A}]\!]^M \cup [\![\mathcal{B}]\!]^M$
(3) $[\![\neg\mathcal{A}]\!]^M = \neg[\![\mathcal{A}]\!]^M$	(6) $[\![\mathcal{A} \to \mathcal{B}]\!]^M = [\![\mathcal{A}]\!]^M \to [\![\mathcal{B}]\!]^M$

A formula $\mathcal{A}$ is satisfied by M on $\mathfrak{F}$ just in case $N \subseteq [\![\mathcal{A}]\!]^M$; it is satisfied by $\mathfrak{F}$ in case it is satisfied by any model on $\mathfrak{F}$; it is valid on a class $\mathcal{F}$ of TR frames just in case it is satisfied by each $\mathfrak{F} \in \mathcal{F}$.

4.2. **Fine-Style Frames and Models.** The semantics in this section does not quite follow Fine's original presentation. The most salient point is that I explicitly include an operation, $\sqcap$, which interprets disjunction. Since much of what I say here concerns disjunction, I pull this out explicitly and state some conditions concerning it ((f1), (f5), and (f6)) in order to clarify its behavior. For instance, (f5) is the constraint, noted by Humberstone [23], which enforces distribution in a general setting. Fine does not include a detailed discussion of disjunction, but I render these constraints explicit for the purposes of comparison.

Definition 5. A Fine-style (F) frame $\mathfrak{G}$ is a tuple $\langle S, S_P, \sqsubseteq, \otimes, -, @\rangle$, where $\varnothing \neq S_P \subseteq S$, $\sqsubseteq \subseteq S^2$, $\otimes \colon S^2 \longrightarrow S$, $- \colon S_P \longrightarrow S_P$, and $@ \in S$ so that the following constraints are satisfied:

[7] For further information on lattices and related topics, the reader may consult Dunn and Hardegree [16].

(f1) $\langle S, \sqsubseteq \rangle$ is a meet (written $\sqcap$) semi-lattice.
(f2) If $s \sqsubseteq s'$ and $t \sqsubseteq t'$ then $s \otimes t \sqsubseteq s' \otimes t'$. Also, $@ \otimes s = s$, for any $s \in S$.
(f3) $-$ is an order-inverting involution: so $s \sqsubseteq t \Rightarrow -t \sqsubseteq -s$ and $--s = s$.
(f4) If $x, y \in S$, $a \in S_P$, and $x \otimes y \sqsubseteq a$, then there are $x', y' \in S_P$ s.t. $x \sqsubseteq x'$, $y \sqsubseteq y'$, $x' \otimes y \sqsubseteq a$, and $x \otimes y' \sqsubseteq a$.
(f5) If $s \sqcap t \sqsubseteq u$, then there are $s', t' \in S_P$ s.t. $s \sqsubseteq s', t \sqsubseteq t'$, and $s' \sqcap t' \sqsubseteq u$.
(f6) If $s \sqcap t \not\sqsubseteq u$, then there are $s', t' \in S_P$ s.t. $s \sqsubseteq s'$, $t \sqsubseteq t'$, and $s' \sqcap t' \not\sqsubseteq u$.

Definition 6. Given $X, Y \in \mathcal{P}(S)^{\uparrow}$, let

$$\begin{aligned} X \rightsquigarrow Y &= \{s \in S \colon \forall t \in S(t \in X \Rightarrow s \otimes t \in Y)\} \\ X \sqcup Y &= \{s \in S \colon \exists t, u \in S(t \sqcap u \sqsubseteq s \ \&\ t \in X \ \&\ u \in Y)\} \\ {\sim} X &= \{s \in S \colon \forall t \in S_P(s \sqsubseteq t \Rightarrow -t \notin X)\} \end{aligned}$$

Definition 7. A model on an F frame $\mathfrak{G}$ is a function L of type $\mathbb{P} \longrightarrow \mathcal{P}(S)^{\uparrow}$ required to satisfy the constraint, for any $p \in \mathbb{P}$,

$$\forall t \in S_P(s \sqsubseteq t \Rightarrow t \in L(p)) \Rightarrow s \in L(p)$$

L is extended to a full valuation $|\cdot|^L \colon \mathcal{L} \longrightarrow \mathcal{P}(S)^{\uparrow}$ required to satisfy the following clauses:

(1) $|p|^L = L(p)$ (2) $|\neg\mathcal{A}|^L = {\sim}|\mathcal{A}|^L$
(3) $|\mathcal{A} \wedge \mathcal{B}|^L = |\mathcal{A}|^L \cap |\mathcal{B}|^L$ (4) $|\mathcal{A} \vee \mathcal{B}|^L = |\mathcal{A}|^L \sqcup |\mathcal{B}|^L$
(5) $|\mathcal{A} \to \mathcal{B}|^L = |\mathcal{A}|^L \rightsquigarrow |\mathcal{B}|^L$ (6) $\forall t \in S_P(s \sqsubseteq t \Rightarrow t \in |\mathcal{A}|^L) \Rightarrow s \in |\mathcal{A}|^L$

A formula A is satisfied on L just in case $@ \in |\mathcal{A}|^L$. Satisfaction on a frame and validity w.r.t. a class of frames are defined as for TR models.

4.3. **From TR to F.** Now let's show that from an arbitrary TR frame, we can construct an F frame such that for any model on the original frame, we can obtain a model on the new frame which satisfies the same formulas as the original model.

Definition 8. Given a TR frame $\mathfrak{F}$, let its F-mate $\mathfrak{F}^F = \langle S^F, S^F_P, \sqsubseteq^F, @^F, \otimes^F, -^F \rangle$ be defined as follows:

(i) $S^F = \{\Gamma \subseteq \mathcal{P}(W)^{\uparrow} \colon \forall Y \in \mathcal{P}(W)^{\uparrow}(\exists_{i \leq n} X_i \in \mathcal{P}(W)^{\uparrow}(\bigcap_{i \leq n} X_i \in \Gamma \Rightarrow Y \in \Gamma))\}$
(ii) $S^F_P = \{\Gamma \in S^F \colon \forall X, Y \in \mathcal{P}(W)^{\uparrow}(X \cup Y \in \Gamma \Rightarrow X \in \Gamma \text{ or } Y \in \Gamma)\}$
(iii) $\sqsubseteq^F = \subseteq$
(iv) $@^F = [\{N\})$
(v) $\Gamma \otimes^F \Delta = \{Y \in \mathcal{P}(W)^{\uparrow} \colon \exists X \in \Delta(X \to Y \in \Gamma)\}$
(vi) $-^F\Gamma = \{X \in \mathcal{P}(W)^{\uparrow} \colon \neg X \notin \Gamma\}$, for $\Gamma \in S^F_P$.

The idea of the construction is that we take the set of filters, w.r.t. $\langle \mathcal{P}(W)^{\uparrow}, \cap, \cup \rangle$, S^F, the set of prime filters thereon, S^F_P, an order $\sqsubseteq^F$, a logical point $@^F$, and a pair of operations appropriate for the interpretation of implication and negation on Fine-style frames, $\otimes^F$ and $-^F$. So we have, essentially, constructed a set of *non-prime* points out of (up)sets of prime points and defined operations appropriate to interpret the logical vocabulary, all in accordance with the structure of F frames.

This construction proceeds by a two step process. We start from the TR frame, and then we consider the space of *propositions* thereon, given by $\mathcal{P}(W)^{\uparrow}$, and it is out of this space that we define our desired F frame. Intuitively speaking, we construct propositions out of elements of W, and from there we construct elements of the desired F frame. Let us verify that $\mathfrak{F}^F$ is, indeed, an F frame when $\mathfrak{F}$ is a TR frame.

Fact 9. *If $\mathfrak{F}$ is a TR frame, then $\mathfrak{F}^F$ is an F frame.*

Proof. It suffices to ensure that $\mathfrak{F}^F$ verifies conditions (f1)–(f6). For (f1), just note that $\sqsubseteq^F = \subseteq$ does have a meet, namely, $\cap$, and so $\langle S^F, \sqsubseteq^F\rangle$ is, indeed, a meet semi-lattice.

For (f2) we have two things to check. First, suppose $\Gamma \sqsubseteq^F \Delta$ and $\Sigma \sqsubseteq^F \Theta$ are the case, and furthermore, that $X \in \Gamma \otimes^F \Sigma$. Therefore, there is a $Y \in \Sigma$ such that $Y \to X \in \Gamma$, and so $Y \in \Theta$ and $Y \to X \in \Delta$, and so $X \in \Delta \otimes^F \Theta$. Since X was arbitrary, this suffices to prove that $\Gamma \otimes^F \Sigma \sqsubseteq^F \Delta \otimes^F \Theta$. Next, we want to show that for any $\Gamma \in S^F$, $@^F \otimes^F \Gamma = \Gamma$. First, if $X \in @^F \otimes^F \Gamma$, then there is a $Y \in \Gamma$ s.t. $Y \to X \in @^F$. But $Y \to X \in @^F$ holds iff $N \subseteq Y \to X$ holds in F and so $Y \subseteq X$ holds there, and so if $Y \in \Gamma$ then $X \in \Gamma$, since $\Gamma \in S^F$. For the converse, if $X \in \Gamma$ then, since $X \subseteq X$ always holds, we have $X \to X \in @^F$, and so $X \in @^F \otimes^F \Gamma$, as desired.

For (f3), we again have two things to prove. First, suppose that $\Gamma \subseteq \Delta$ and that $X \in -^F\Delta$. Thus, $\neg X \notin \Delta$ and so $\neg X \notin \Gamma$, and so $X \in -^F\Gamma$, as desired. Note, further, that $X \in -^F -^F \Gamma$ holds iff $\neg\neg X \in \Gamma$ iff $X \in \Gamma$.

For (f4), suppose that we have $\Gamma, \Delta \in S^F$ and $\Theta \in S^F_P$ s.t. $\Gamma \otimes^F \Delta \subseteq \Theta$. To obtain a $\Delta' \supseteq \Delta$ such that $\Gamma \otimes^F \Delta' \subseteq \Theta$, consider the pair

$$\langle \Delta,\ \Delta^- = \{X \in \mathcal{P}(W)^{\uparrow} \colon \exists Y \notin \Theta (X \to Y \in \Gamma)\}\rangle.$$

Now by definition Δ is a filter on $\langle \mathcal{P}(W)^{\uparrow}, \cap, \cup\rangle$, and it is fairly easy to verify that Δ^- is an ideal.[8] Furthermore, we can show that $\Delta \cap \Delta^- = \varnothing$. In fact, there are no $X_1, \ldots, X_m \in \Delta$ and $Y_1, \ldots, Y_n \in \Delta^-$ s.t. $\bigcap_{1\le i\le m} X_i \subseteq \bigcup_{1\le j\le n} Y_j$. With this fact, since $\langle \mathcal{P}(W)^{\uparrow}, \cap, \cup\rangle$ is a distributive lattice, we can employ [16, Corollary 13.4.6] to infer that there is a $\Delta' \in S^F_P$ s.t. $\Delta' \supseteq \Delta$ and $\Delta' \cap \Delta^- \neq \varnothing$.[9] From this, we can infer that $\Gamma \otimes^F \Delta' \subseteq \Theta$, as desired.

Let's show the key fact, assuming, for contradiction, that for each Y_j there is a $Z_j \notin \Theta$ such that $Y_j \to Z_j \in \Gamma$, and so $\bigcap_{1\le i\le m} X_i \to \bigcup_{1\le j\le n} Z_j \in \Gamma$ and so $\bigcup_{1\le j\le n} Z_j \in \Theta$, contradicting the assumption that $\Theta \in S^F_P$.[10]

The argument needed to obtain a $\Gamma' \supseteq \Gamma$ s.t. $\Gamma' \in S_P$ and $\Gamma' \otimes \Delta \subseteq \Theta$ is similar, so elided. Also, the proofs of (f5) and (f6) are straightforward and, for reasons of space, are left to the reader. $\triangleleft$

This suffices to show that the F-mate of a TR frame is an F frame, as desired. It remains to show how, given a model M on a TR frame $\mathfrak{F}$, to obtain a model on $\mathfrak{F}^F$ which will satisfy the same formulas. For this, we adapt the definition of a *canonical valuation*, given in Bimbó and Dunn [6, p. 23].

[8] The key facts are: $(X \to Y) \cap (Z \to U) \subseteq (X \cup Z) \to (Y \cup U)$, and if $X \subseteq Y$ then $Y \to Z \subseteq X \to Z$.

[9] Note, this step is analogous to the use of the Pair Extension lemma in completeness proofs for relevant logics w.r.t. their TR frame semantics, for instance in [2, §48.3].

[10] This relies on the fact that $(X \to Y) \cap (Z \to U) \subseteq (X \cap Z) \to (Y \cup U)$.

Definition 10. Given a model M on a TR frame $\mathfrak{F}$, we fix $|\cdot|^M : \mathcal{L} \longrightarrow \mathcal{P}(S^F)^{\uparrow}$ by setting $|\mathcal{A}|^M = \{\Gamma \in S^F : [\![\mathcal{A}]\!]^M \in \Gamma\}$.

Fact 11. *Given a model M on a TR frame $\mathfrak{F}$, the model $|\cdot|^M$ on $\mathfrak{F}^F$ has the following properties:*

(1) $|\mathcal{A}|^M \in \mathcal{P}(S^F)^{\uparrow}$, *for every* $\mathcal{A} \in \mathcal{L}$ (2) $|\boldsymbol{t}|^M = \{\Gamma \in S^F : @ \subseteq \Gamma\}$

(3) $|\mathcal{A} \wedge \mathcal{B}|^M = |\mathcal{A}|^M \cap |\mathcal{B}|^M$ (4) $|\mathcal{A} \vee \mathcal{B}|^M = |\mathcal{A}|^M \sqcup |\mathcal{B}|^M$

(5) $|\mathcal{A} \to \mathcal{B}|^M = |\mathcal{A}|^M \rightsquigarrow |\mathcal{B}|^M$ (6) $|\neg\mathcal{A}|^M = \sim |\mathcal{A}|^M$

(7) $\forall \Delta \in S^F (\forall \Gamma \in S^F_P (\Delta \subseteq \Gamma \Rightarrow \Gamma \in |\mathcal{A}|^M) \Rightarrow \Delta \in |\mathcal{A}|^M)$

(8) *For any* $\mathcal{A} \in \mathcal{L}$, $N \subseteq [\![\mathcal{A}]\!]^M$ *iff* $@^F \in |\mathcal{A}|^M$.

Proof. (1) is immediate from the definition. The others we can prove by induction on the complexity of formulas. For (2), the only atomic case, we can show:

$$|\mathbf{t}|^M = \{\Gamma \in S^F : [\![\mathbf{t}]\!]^M = N \in \Gamma\} = \{\Gamma \in S^F : [\{N\}) = @ \subseteq \Gamma\}$$

(3) is immediate, and left to the reader. For (4), note that the right-to-left direction is immediate from the fact that $[\![\mathcal{C}]\!]^M \subseteq [\![\mathcal{A} \vee \mathcal{B}]\!]^M$ holds for $\mathcal{C} \in \{\mathcal{A}, \mathcal{B}\}$, so let's consider the converse. Note that if $[\![\mathcal{A} \vee \mathcal{B}]\!]^M = [\![\mathcal{A}]\!]^M \cup [\![\mathcal{B}]\!]^M \in \Gamma$, then it's immediate that $[\![\mathcal{C}]\!]^M \in [\{[\![\mathcal{C}]\!]^M\})$ holds for $\mathcal{C} \in \{\mathcal{A}, \mathcal{B}\}$ and $[\{[\![\mathcal{A}]\!]^M\}) \cap [\{[\![\mathcal{B}]\!]^M\}) \subseteq \Gamma$, which suffices to show that $\Gamma \in |\mathcal{A} \vee \mathcal{B}|^M$, as desired. For (5) and (6), the standard kind of arguments given in completeness proofs (for instance, those in Restall [36]) suffice, and verifying these are left to the reader.

For (7), we proceed by contraposition. Suppose that $\Delta \in S^F \cap \overline{|\mathcal{A}|^M}$, so that $[\![\mathcal{A}]\!]^M \notin \Delta$. We want to show that there is a $\Gamma \in S^F_P$ s.t. $\Delta \subseteq \Gamma$ and $[\![\mathcal{A}]\!]^M \notin \Gamma$. For this, however, it suffices to employ Dunn and Hardegree [16, Corollary 13.4.6], fixing $\{X \in \mathcal{P}(W)^{\uparrow} : X \subseteq [\![\mathcal{A}]\!]^M\}$, noting that this is an ideal which doesn't overlap Δ, and thus we can obtain a prime filter Γ on $\langle \mathcal{P}(W)^{\uparrow}, N, \cap, \cup, \to, \neg \rangle$ s.t. $\Gamma \supseteq \Delta$ and $\Gamma \cap \{X : X \subseteq [\![\mathcal{A}]\!]^M\} = \varnothing$, so that $[\![\mathcal{A}]\!]^M \notin \Gamma$ as desired.

For (8), it suffices to note that:

$$@ \in |\mathcal{A}|^M \iff [\![\mathcal{A}]\!]^M \in @ = [\{N\}) \iff N \subseteq [\![\mathcal{A}]\!]^M \qquad \triangleleft$$

Points (1)–(7) guarantee that $|\cdot|^M$ is well-defined, giving rise to a model on $\mathfrak{F}^F$. Point (8) gives the desired property, that the formulas satisfied by $|\cdot|^M$ on $\mathfrak{F}^F$ are just those satisfied by M on $\mathfrak{F}$. So we can state:

Theorem 12. *Given any TR frame $\mathfrak{F}$, and model M thereon, we can construct an F-frame $\mathfrak{F}^F$ and a model satisfying just those formulas satisfied by M.*

This gives us one half of our puzzle; that any formulas satisfiable on a TR frame are satisfiable on some F frame.

4.4. **From F to TR.** This direction is quite similar, and is, in any case, better understood. In Fine's original paper, especially, the part reproduced in Anderson et al. [2, §51.5], he considered in some detail the relationship between his frames and the TR frames presented by Sylvan, Meyer, and their collaborators Routley et al. [40]. The method I'll employ is a bit different from his, but shares some similarities.

Definition 13. Given the an F frame $\mathfrak{G}$, we construct $\mathfrak{G}^{TR} = \langle W^{TR}, N^{TR}, R^{TR}, *^{TR}\rangle$, $\mathfrak{G}$'s TR-mate, as follows:

$$W^{TR} = \big\{\Gamma \subseteq \mathcal{P}(S)^{\uparrow} \colon \forall Y \in \mathcal{P}(S)^{\uparrow}(\exists_{i\leq n} X_i \in \Gamma(\bigcap_{i\leq n} X_i \subseteq Y \Rightarrow Y \in \Gamma)) \;\&$$
$$\forall X, Y \in \mathcal{P}(S)^{\uparrow}(X \sqcup Y \in \Gamma \Rightarrow (X \in \Gamma \text{ or } Y \in \Gamma))\big\}$$
$$N^{TR} = \{\Gamma \in W^{TR} \colon @ \in \Gamma\}$$
$$R^{TR} = \big\{\langle \Gamma, \Delta, \Theta\rangle \in (W^{TR})^3 \colon \forall X, Y \in \mathcal{P}(S)^{\uparrow}((X \rightsquigarrow Y \in \Gamma \;\&\; X \in \Delta) \Rightarrow Y \in \Theta)\big\}$$
$$\Gamma^{*^{TR}} = \{X \in \mathcal{P}(S)^{\uparrow} \colon {\sim} X \notin \Gamma\}$$

Fact 14. *If $\mathfrak{G}$ is an F frame then $\mathfrak{G}^{TR}$ is a TR frame.*

Proof. It suffices to prove that (tr1)–(tr4) hold of $\mathfrak{G}^{TR}$.

For (tr1), it suffices to show that the defined $\leq^{TR}$ is, in fact, just $\subseteq$, i.e., that $\exists \Gamma \in N^{TR}(R^{TR}\Gamma\Delta\Theta) \iff \Delta \subseteq \Theta$. For the left-to-right, suppose that $\exists \Gamma \in N^{TR}(R^{TR}\Gamma\Delta\Theta)$ and $X \in \Delta$. If $\Gamma \in N^{TR}$, then $@ \in \Gamma$ and since $@ \subseteq X \rightsquigarrow X$, we have that $X \rightsquigarrow X \in \Gamma$, and thus $X \in \Theta$. Since X was arbitrary, this suffices to show that $\Delta \subseteq \Theta$, as desired. For the converse, suppose that $\Delta \subseteq \Theta$; in fact, since for any $X \in \Delta$ and any $\Gamma \in N^{TR}$ we have $X \rightsquigarrow X \in \Gamma$, we have that $R^{TR}\Gamma\Delta\Theta$, which suffices to show the result (given that $N^{TR} \neq \varnothing$, verification of which fact we leave to the reader).

For the remainder, we'll take the order concerned just to be $\subseteq$ without further comment. For (tr2), we want to show that if $\Gamma \in N^{TR}$ and $\Gamma \subseteq \Delta$ then $\Delta \in N^{TR}$. This is immediate from the definition of N^{TR}.

The arguments needed for (tr3) and (tr4) are quite similar to arguments standardly given to show that the *canonical frame* of a logic is a TR frame, and the reader may consult [2, §48.3] or [40, Ch. 4] for details of this style of argument. ◁

Definition 15. Given the TR-mate $\mathfrak{G}^{TR}$ of an F frame $\mathfrak{G}$ and a model L on $\mathfrak{G}$, let $[\![\mathcal{A}]\!]^L = \{\Gamma \in W^{TR} \colon |\mathcal{A}|^L \in \Gamma\}$.

Now, once again, we just have to verify that the resulting model satisfies the required properties.

Fact 16. *Given a model L on a F frame $\mathfrak{G}$, the evaluation $[\![\cdot]\!]^L$ on $\mathfrak{G}^{TR}$ has the following properties:*

(1) *If $\Gamma \in [\![\mathcal{A}]\!]^L$ and $\Gamma \subseteq \Delta$, then $\Delta \in [\![\mathcal{A}]\!]^L$.*
(2) $[\![\boldsymbol{t}]\!]^L = N^{TR}$
(3) $[\![\mathcal{A} \wedge \mathcal{B}]\!]^L = [\![\mathcal{A}]\!]^L \cap [\![\mathcal{B}]\!]^L$
(4) $[\![\mathcal{A} \vee \mathcal{B}]\!]^L = [\![\mathcal{A}]\!]^L \cup [\![\mathcal{B}]\!]^L$
(5) $[\![\mathcal{A} \to \mathcal{B}]\!]^L = \{\Gamma \in W^{TR} \colon \forall \Delta, \Theta((R^{TR}\Gamma\Delta\Theta \;\&\; \Delta \in [\![\mathcal{A}]\!]^L) \Rightarrow \Theta \in [\![\mathcal{B}]\!]^L)\}$
(6) $[\![\neg\mathcal{A}]\!]^L = \{\Gamma \in W^{TR} \colon \Gamma^{*^{TR}} \notin [\![\mathcal{A}]\!]^L\}$
(7) *For any $\mathcal{A} \in \mathcal{L}$, $N^{TR} \subseteq [\![\mathcal{A}]\!]^L$ iff $@ \in |\mathcal{A}|^L$.*

Proof. The reader is encouraged to check [6] for details of proving completeness for relational frames for distributive multi-gaggles. The details there suffice here, as can be noted by the fact that the complex algebra of an F frame will be a multi-gaggle. The only part of verifying this that is not standard involves verifying that distribution obtains, and the argument style, using (f5), can be found in [23]. ◁

From this we can infer the key fact, which is:

Theorem 17. *From any F frame $\mathfrak{G}$ we can obtain a TR frame $\mathfrak{G}^{TR}$ such that, for any model L on $\mathfrak{G}$ there is a model $[\![\cdot]\!]^L$ on $\mathfrak{G}^{TR}$ satisfying exactly the same formulas as L.*

I've only dealt here with frames appropriate for the basic logic **B**, but there is a well-known correspondence theory for accommodating stronger logics, and it seems likely that these results allow for the above to be generalized to frames appropriate for a wider range of logics (as can be done in the case of **R**, as shown in [33]). For my purposes, the basic form I've given here is enough to make my point, so I'll leave it at that and get back to the philosophical work.

5. Layered Semantics

As per §2.1, a realist account provides for a more explanatorily satisfying picture, and the equivalence results of §4 indicate how it is that, starting from this basis, we can recapture the working of the information-based semantics of the F approach in a more satisfying way using the TR semantics.[11] In any case, regardless of which way one proceeds to do the grounding, the equivalence provides a way of capturing both in one framework with some nice results.

The three-layer picture can be represented as follows — the arrows on the left side indicate explanatory priority (the arrows go from from the thing-grounded to the thing-grounding), and those on the right side order of the "defined in terms of" relation:

$$\begin{array}{ccc}
\text{Information States} & & \Gamma \in S^F \\
\downarrow & & \downarrow \\
\text{Propositions} & & X \in \mathcal{P}(W)^{\uparrow} \\
\downarrow & & \downarrow \\
\text{Situations} & & \alpha \in W
\end{array}$$

As indicated, situations provide the ground of the truth of propositions, and elements of S^F represent the states of information to which agents can find themselves having access. As before, there are good reasons that these should not be required to be prime, as they are not. One can have an information state which includes/supports a disjunctive proposition without supporting either disjunct. Situations, however, understood as inexact truthmakers are prime.

On the base level, we have objects, situations — particularly, something like the *abstract* situations of [4] — to which we have an existential commitment. We take them to be real things, and take propositions to be constructed out of these in systematic ways. Propositions, or representations thereof, are then the constituents of information states, to which agents have cognitive access. For instance, it is by taking in visual information that an agent learns what information the witnessed situation conveys, and they are then in a position to perform various cognitive tasks with that information. Part of the story here is that we don't directly perceive situations, nor do

[11] Assuming, of course, that the version of the F semantics involved is read in a *non-realist* and the TR semantics in a *realist* way.

we express situations directly by our various linguistic/cognitive actions. Rather, what we perceive/express/cognize are propositions and collections thereof into information states. On this line, when we open our eyes and perceive the world around us, what we perceive is not the world directly, but information carried by the world — what we perceive is a fact, not an object.

One place where the distinction becomes most salient is that many situations will be typed by a proposition. This captures the intuitive idea that our available information *underdetermines* the state of the world (the situation) we have information about. When I look at my office, and out the window, there is a great deal of information I get, but the actual world situation I, the office, and the window inhabit supports a great deal more information than that which I obtain by perception. For instance, I may see a drawer, and have a vague idea of what is in it, but may not have access to the more precise information supported by the situation of my office, which specifies precisely what is in the drawer. It is this underdetermination which explains why our information has certain imperfections, such as not being prime.

While in need of further precisification this story provides a skeleton for how a reasonably natural theory of meaning could be constructed on this sort of layered picture, and this in a way which accommodates the nice features of both the ontic and the epistemic/informational readings.

6. Mares–Goldblatt Quantifiers in Layered Semantics

One nice feature of the three-layered semantic picture is that we have three places where we can locate meanings. I've suggested that entailment facts should be understood to be grounded in the world. Having said that, however, we can locate the meanings of other expressions in other places, namely in the proposition or information state layer. One natural kind of expression which would seem to have its meaning most naturally in one of these higher layers may be certain modals which concern the interactions between agents and their available information.

The example I want to consider is the Mares–Goldblatt (MG) [32] interpretation of quantifiers, which I'll suggest most naturally lives at the propositional layer.[12] This provides an interesting contrast with the standard, Quinean, picture of the quantifiers wherein their meanings are to be found in the world, and the arrangements of properties over objects. The picture I'll sketch is similar to Mares' [31] proposed interpretation of the MG semantics, though it differs from his in some respects. Let me begin by recapping the basic elements of the MG semantics, building on the basis of the TR framework.

6.1. **MG Quantifiers in TR Semantics.**

First we extend the basic propositional language (implicit up until now) by a denumerable collection of variables $Var = \{x_n\}_{n\in\omega}$, and the quantifiers $\forall, \exists$. A language signature consists of a set of name constants *Con* and a collection *Pred* of predicate letters of varying arities: the letter c will function as a metavariable over *Con* and P^n over *Pred*, having arity n.

[12] The original form of this semantics was given for quantified extensions of **R**, but it has recently been expanded to include a range of weaker logics in Ferenz [19]; Tedder and Ferenz [43].

Definition 18. An MG frame is a tuple $\langle W, N, R, *, Prop, D, PropFun\rangle$, where $F = \langle W, N, R, *\rangle$ is a TR frame, $Prop \subseteq \mathcal{P}(W)^{\uparrow}$, $D \neq \varnothing$, and $PropFun \subseteq \{\varphi \colon \varphi \colon D^{\omega} \longrightarrow Prop\}$. We stipulate a range of constraints on these things. To that end, given $f \in D^{\omega}$ (called a "variable assignment"), if $f' \in D^{\omega}$ is such that for any $m \neq n$, $fm = f'm$, then f' is an x_n-*variant* of f, written $f' \sim_{x_n} f$.

The constraints, taking the definitions of $\rightarrow, \neg$ as operations on $\mathcal{P}(W)^{\uparrow}$ from Definition 6, are:

(MG1) There is a $\varphi_N \in PropFun$ s.t. for all $f \in D^{\omega}$, $\varphi_N f = N$.
(MG2) If $\varphi \in PropFun$, then there is a $\neg\varphi \in PropFun$ s.t. $(\neg\varphi)f = \neg(\varphi f)$.
(MG3) If $\varphi, \psi \in PropFun$, then there is a $\varphi \otimes \psi \in PropFun$ s.t. $(\varphi \otimes \psi)f = \varphi f \otimes \psi f$ for each $\otimes \in \{\cap, \cup, \rightarrow\}$.
(MG4) If $\varphi \in PropFun, n \in \omega$, then there is a $\forall_n \varphi \in PropFun$ s.t.

$$(\forall_n \varphi)f = \bigcup\{X \in Prop \colon X \subseteq \bigcap_{f' \sim_{x_n} f} \varphi f'\}.$$

(MG5) If $\varphi \in PropFun, n \in \omega$, then there is a $\exists_n \varphi \in PropFun$ s.t.

$$(\exists_n \varphi)f = \bigcap\{X \in Prop \colon \bigcup_{f' \sim_{x_n} f} \varphi f' \subseteq X\}.$$

A model M on a MG frame is a multi-type function: it is of types $Con \longrightarrow D$ and $Pred^n \longrightarrow D^n$ (where $Pred^n \subseteq Pred$ is the set of n-ary predicate letters), and we define the combination of M with $f \in D^{\omega}$ as follows, for any $\tau \in Con \cup Var$:

$$M_f(\tau) = \begin{cases} fn & \text{if } \tau = x_n \in Var; \\ M(\tau) & \text{if } \tau \in Con. \end{cases}$$

We define $[\![\cdot]\!]^M$ assigning formulas to elements of *PropFun* inductively as follows (note that $([\![\mathcal{A}]\!]^M)f$, often written $[\![\mathcal{A}]\!]^M_f$, takes a value in *Prop*):

(1) $[\![P^n(\tau_1, \ldots, \tau_n)]\!]^M_f = M(P^n)(M_f(\tau_1), \ldots, M_f(\tau_n))$
(2) $[\![\neg\mathcal{A}]\!]^M_f = \neg([\![\mathcal{A}]\!]^M_f)$
(3) $[\![\mathcal{A} \wedge \mathcal{B}]\!]^M_f = [\![\mathcal{A}]\!]^M_f \cap [\![\mathcal{B}]\!]^M_f$
(4) $[\![\mathcal{A} \vee \mathcal{B}]\!]^M_f = [\![\mathcal{A}]\!]^M_f \cup [\![\mathcal{B}]\!]^M_f$
(5) $[\![\mathcal{A} \rightarrow \mathcal{B}]\!]^M_f = [\![\mathcal{A}]\!]^M_f \rightarrow [\![\mathcal{B}]\!]^M_f$
(6) $[\![\forall x_n \mathcal{A}]\!]^M_f = (\forall_n [\![\mathcal{A}]\!]^M)f$
(7) $[\![\exists x_n \mathcal{A}]\!]^M_f = (\exists_n [\![\mathcal{A}]\!]^M)f$

A formula $\mathcal{A}$ is satisfied by the pair M, f just in case $N \subseteq [\![\mathcal{A}]\!]^M_f$. $\mathcal{A}$ is satisfied by M just in case it it satisfied by M, f for any $f \in D^{\omega}$. $\mathcal{A}$ is satisfied by an MG frame if satisfied by every model on the frame, and it is valid w.r.t. a class of MG frames if satisfied by every frame in the class.

The key innovation in this semantic framework concerns, naturally, the quantifiers. In particular, it is the introduction of the clauses (MG4) and (MG5). Note that, unlike in the standard, Tarskian framework, these are not interpreted just as generalized intersections/unions of "instances." Rather, these are mediated by elements of *Prop* — we don't just consider, when evaluating a quantified claim at a world α, whether all/some instance of the quantified formula holds at α, or even at worlds related to α. Rather, we consider the state of information from α, that is, how α fits into the structure of propositions; in effect, we consider what the information supported by α commits one to. It's by working with *Prop* like this explicitly that Mares and Goldblatt are able to avoid the problems, discovered by Fine [21], with employing the standard Tarskian truth condition. So the interpretation of the quantifiers concerns not just a frame, but this *combined* with a particular complex algebra over that frame — that is to say, it is a form of *general frame semantics*. However this difference isn't just interesting for technical purposes, but also for philosophical purposes.

In particular, by working with this larger structure of information, the MG interpretation of the quantifiers seems to open itself up to readings of these objects other than the traditional reading made famous by Quine [34]. For example, the truth condition for the existential quantifier can be spelled out as

$$\alpha \in [\![\exists x_n \mathcal{A}]\!]_f^M \iff \forall X \in Prop(\alpha \in X \Rightarrow \forall \beta(\exists f' \sim_{x_n} f(\beta \in [\![\mathcal{A}]\!]_{f'}^M) \Rightarrow \beta \in X)).$$

That is, any proposition X which α supports contains any situation β which supports at least one instance of $\mathcal{A}$. That is, the information supported by α must be supported by a situation which supports at least one instance. We are concerned not with an existential commitment at the world of evaluation, but rather with a situation-independent *informational* commitment. In order to be so committed, one *does not need* to be committed to the existence of an $\mathcal{A}$ in any *particular* situation, but rather just be committed to infer the information supported by α in any situation, which does support the existence of an $\mathcal{A}$. To use the preferred terminology of Sylvan [38], we might call this a *particular* quantifier, which simply tracks the commitments associated with commitment to *a particular* one satisfying the formula.

The important thing for my purposes is that the three-layer semantic framework provides the grist both for a realist interpretation of the propositional vocabulary, and an informational interpretation of the quantifiers, in one setting. That this is a strength of the account is, of course, the kind of point Punčochář [33] noted, but it's an advantage we retain even when we are more picky about the grounding of the framework.

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