Holography and Emergence

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Abstract

In this paper, I discuss one form of the idea that spacetime and gravity might 'emerge' from quantum theory, i.e. via a holographic duality, and in particular via AdS/CFT duality. I begin by giving a survey of the general notion of duality, as well as its connection to emergence. I then review the AdS/CFT duality and proceed to discuss emergence in this context. We will see that it is difficult to find compelling arguments for the emergence of full quantum gravity *from* gauge theory via AdS/CFT, i.e. for the boundary theory's being metaphysically more fundamental than the bulk theory.

Contents

1 Introduction

One of the greatest recent advances in theoretical physics is surely the phenomenon known as the Anti-de-Sitter/Conformal Field Theory (AdS/CFT) 'correspondence' or 'duality', also variously known as the gravity/gauge correspondence, the string/gauge correspondence, the bulk/boundary correspondence, or the gravity/fluid correspondence. AdS/CFT is an instance of a 'holographic' duality that says, roughly, that a string theory or its low-energy limit on AdS space (i.e. the *bulk theory*) is equivalent to a quantum gauge theory on the boundary of AdS space (i.e. the *boundary theory*).

It is striking that contemporary discussion of holographic dualities (and in particular AdS/CFT) is rife with the use of the terms 'emergent' and 'emergence' (see e.g. [23, 15, 9, 25, 3, 17]). These terms have no standard meaning in the physical literature. For instance, some authors use fairly thin notions[1] while others have a more robust meaning in mind.

Another important distinction can be drawn with regard to the 'emergence' of gravity in AdS/CFT.[2] On the one hand, there are those who use the term *perturbatively*, i.e. to suggest that some classical supergravity theory emerges in the appropriate *limit* of a gauge theory. On the other hand, there are those who make a more ambitious claim that draws directly on the belief that AdS/CFT is an exact *duality*, i.e. a full theory of quantum gravity emerges from the gauge theory.[3]

[1] For instance, El Showk and Papadodimas [9] seem to relativize this notion to 'the point of view' of a particular theory and Berenstein [3] emphasizes emergence in the sense that the boundary conformal field theory is a *definition* of the bulk theory.

[2] I thank an anonymous referee for suggesting that I make this distinction more explicit.

[3] It is hard to tease these uses apart in the actual literature, since many authors seem to discuss both perturbative and non-perturbative aspects in the context of emergent quantum gravity (e.g. Section 1.3.2 of [15] and Section 5.3 of [23]). Greene (p. 238 of [11]) seems to be one of the few authors who explicitly makes a strong metaphysical claim about the (non-perturbative)

In this survey article, I shall focus on trying to make sense of the second, more ambitious claim, for two reasons. First, while the perturbative emergence of gravity-like structures from gauge theory is interesting (and dealt with by other papers in this volume), it is possible to discuss some aspects of it without drawing on duality. For instance, the appearance of stringy-structures in the large N limit of gauge theory can be discussed without invoking AdS/CFT (see e.g. [4] for a philosophical treatment of this topic). Thus, one of the novel possibilities introduced by the phenomenon of duality is the possibility of a form of emergence based on the exact equivalence between two theories.[4]

Second, while there is a clear sense of which theory is 'fundamental' in most discussions of perturbative emergence, this is not necessarily so in the context of a duality: to give a concrete example in the context of AdS/CFT, classical gravity can be seen as emerging in the appropriate limit from either the bulk quantum gravity theory, or from the boundary gauge theory. (Of course, it is far easier to define the gauge theory side of the duality than the gravity side, and so we often prioritize the gauge theory for reasons of convenience.[5]) Thus, if one is interested in the 'metaphysical' emergence of a phenomenological theory from a fundamental theory (and not in a pragmatic reading of what is or isn't 'fundamental'), then it makes sense to focus on the more ambitious claim. Furthermore, the outcome of this inquiry may then help one to interpret various forms of perturbative emergence that hold between the dual theories.

Thus, whenever I speak of the 'emergence of gravity' in this paper, I will be referring to the emergence of the full quantum gravity (bulk) theory from gauge theory, except where explicitly stated otherwise.

Broadly speaking, the plan of this paper will be to survey the notions of a duality and emergence *from* duality in the first half, after which I proceed to a survey of AdS/CFT for philosophers, and a discussion of emergence from duality in this context. The outcome of my discussion of the ambitious claim will be deflationary: we seem to have no good reason to believe it.

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emergence of the entire bulk theory from the boundary theory. On the other hand, one should also note that authors always have the perturbative limit in mind when they talk about the emergence of classical gravity or supergravity.

[4]NB: I do not of course mean to deny that a duality – an exact correspondence – can be helpful for constructing, identifying, or understanding various forms of perturbative emergence. However, there nonetheless remains a sense in which duality plays a merely epistemic role in such scenarios.

[5]On the other hand, when we are interested in the strong coupling regime of a gauge theory, where there is very little understanding of how to perform field theory computations, we tend to prioritize the gravity side of the duality. One might say that whichever theory one can do calculations in (i.e. whose weak coupling regime corresponds to the other theory's strong coupling regime) is the one that is more fundamental – but this is merely to turn 'fundamentality' into a shorthand for calculational convenience.

1.1 Prospectus

AdS/CFT is a (conjectured) example of a physical *duality*, viz. (roughly) the statement that two distinct physical theories are in some sense equivalent, or describe the same physical degrees of freedom. Section 2.1 thus begins by discussing the received view that this equivalence is to be cashed out in terms of mutual definability, and how this conflicts with one naive precisification of the semantic view of theories, viz. what I call the *model isomorphism criterion* below. On the other hand, there is a very natural category-theoretic formalism for conceptualizing dualities within mathematics (Section 2.1.1) which captures much of what the naive precisification misses. Section 2.1.2 surveys the rich and complex array of *physical* dualities, of which AdS/CFT is an instance. This is followed by a discussion of metaphysical interpretations of duality (Section 2.1.3), in preparation for our discussion of emergence.

Emergence from duality is the topic of Section 2.2: I seek to contrast this notion with other notions of emergence in the philosophical and physical literature, some of which involve claims related to definitional extension and supervenience. We shall see that emergence from duality is significantly different from these other notions. In particular, I will propose that here the asymmetric relation of emergence is grounded in the claim that one of the dual theories is metaphysically more fundamental than the other.

Section 3 lays out the framework of AdS/CFT and concludes with a discussion of emergence within the context of this particular duality. Another article in this volume emphasizes the string-theoretic aspects of AdS/CFT; in this article I will emphasize a more recent point of view, which has been important for applications of AdS/CFT to condensed matter physics, in particular for modeling non-Fermi liquids and strange metals. Sections 3.1-3.3 provide an introductory survey of Anti de Sitter (AdS) spacetime, conformal field theory (CFT), and the AdS/CFT duality respectively. Section 3.4.1 explains a key aspect of the 'dictionary' between the dual theories, viz. the 'field-operator correspondence', which relates fields on the bulk spacetime to local operators of the boundary QFT. Section 4 then returns to the topic of emergence in light of the details provided by the foregoing sections. Finally, Section 5 provides a summary and conclusion.

2 Duality and emergence

2.1 Duality

The phenomenon of 'duality' is one of the central themes of twenty-first (and late twentieth) century physics. In the broadest terms, it can be characterized as the statement that two physical theories, i.e. the 'dual pairs' related by duality, are in some sense equivalent.[6] In practice, the details of how this equivalence is cashed out will vary from case to case. For instance, in the case of S-duality, there is a *symmetry* between the dual theories, but in other cases – e.g. the AdS/CFT duality that is the subject of this paper – there may be no such symmetry. Nonetheless, a feature that all dualities share is the existence of a *dictionary-like* correspondence:[7]

> (Dictionary) If theory A and theory B are related by duality, then the (terms of the theory A which represent the[8]) observables, processes, and fundamental entities of theory A can be expressed in terms of those of theory B, and vice versa.

As it happens, there is an orthodoxy in the philosophy of science literature about how one should think of scientific theories, viz. the 'semantic view of theories', which holds that a theory is a class of models. Thus, on one natural precisification of the semantic view, what it means for two theories to be equivalent is just for them to have isomorphic models – call this, following Halvorson [13], the *model isomorphism criterion.* But as Halvorson convincingly argues, the model isomorphism criterion is inadequate even for accommodating the most elementary examples of (what we would intuitively think of as) an equivalence between theories of different mathematical objects.

The question of how the semantic view should be developed in order to cope with duality is an interesting one, but it lies beyond the scope of this essay. Here I only wish to add grist to Halvorson's mill by pointing out that the model isomorphism criterion is incompatible with the notion of duality, for duality is precisely an equivalence between two theories that describe (in general) different physical structures, i.e. theories with non-isomorphic models. Indeed, were the models isomorphic, there could be no non-trivial implementation of (Dictionary). Of course, the two different theories may give rise to *derivative* structures which represent the theory, and for which the model isomorphism criterion is indeed an adequate criterion of equivalence. For

[6] In some special cases, e.g. $\mathcal{N} = 4$ Super Yang-Mills in four dimensions, the duality relates two sectors of the same theory.

[7] In addition to the metaphor of a 'dictionary', sometimes the metaphor of 'translation' is also used, e.g. in this passage from [2]: 'For example, in this paper we have calculated the stress-energy tensor of a gauge theory, with dissipative corrections, by rephrasing the problem in the language of gravity, in a regime where the calculation is tractable, then translating the result back into the language of gauge theory.' This way of speaking is fine so long as one understands that it is only meant loosely, and indeed it is easy to see the attraction of thinking of different theories as 'languages', all describing the same reality. On the other hand, if taken literally then it is misleading: the predicates on either side of the duality simply *mean* different things, even if they have the same extension.

[8] I make this additional clarification here to avoid any confusion. In the physics literature, it is standard to use 'observable' to mean 'term of the theory which represents an observable'. *Mutatis mutandis* with other such terms.

instance, in AdS/CFT the (quantum) bulk and boundary theories are supposed to give rise to isomorphic Hilbert spaces, see e.g. p. 90 of [1].

So one road to formalizing the 'equivalence' of duality has been foreclosed. However, in Section 2.1.1 I will review a mathematical notion (i.e. *an equivalence of categories*) that fares much better than the model isomorphism criterion at capturing the (Dictionary) aspect of duality. Despite the existence of flexible and subtle mathematical tools, we should remember that physical dualities are rich and complex and often escape rigorous formalization in all but the simplest cases.[9] Section 2.1.2 surveys some of these physical dualities, and Section 2.1.3 discusses the metaphysical interpretations that can be given to duality at this level of generality.

In Section 2.2, I will propose that the question of 'which of the dual pairs is more metaphysically fundamental' is crucial for spelling out what emergence amounts to in the context of duality, and in particular, in the context of AdS/CFT. I also sketch how emergence in this sense fits into the wider landscape of discussions of emergence in philosophy and physics.

2.1.1 In mathematics (category theory)

Let us begin by considering the (Dictionary) aspect of duality, i.e. the mutual definability of dual theories. One class of examples where such phenomena can be formalized comes from category theory, where the relevant notion of 'equivalence' is called an *equivalence of categories*.[10]

More precisely, let $\mathcal{A}$ and $\mathcal{B}$ be two distinct categories, which we can think of as 'theories of different mathematical objects, along with their structure-preserving morphisms'. We then say that these categories are *equivalent* just in case there exist functors

$$F : \mathcal{A} \longrightarrow \mathcal{B} \qquad G : \mathcal{B} \longrightarrow \mathcal{A} \tag{1}$$

that obey the following natural isomorphisms (i.e. isomorphisms in the category of functors):

$$\varepsilon : FG \longrightarrow 1_{\mathcal{B}} \qquad \eta : GF \longrightarrow 1_{\mathcal{A}} \tag{2}$$

[9]However, recent work on extended functorial QFT may hold some promise for formalizing relatively better understood examples of holographic duality.

[10]A terminological caveat: in the context of category theory, the term 'duality' is usually reserved for a specific kind of equivalence involving a contravariant functor, which reverses the direction of morphisms from one object to another. However, in this paper I will be using 'duality' in the more general physicist's sense.

Evidently, (1) provides a mapping between the objects and morphisms of $\mathcal{A}$ and $\mathcal{B}$ – thus one can say that any 'sentence' (i.e. concatenation of morphisms) in $\mathcal{A}$ can be mapped to a 'sentence' in $\mathcal{B}$ and vice versa.

For instance, there is an equivalence of categories between the category of sets **Set**, on the one hand, and the category of complete atomic Boolean algebras **CBA**, on the other hand. The relevant functor from **Set** to **CBA** sends a set X to its power-set, which is in turn isomorphic to a complete Boolean algebra with the set X of atoms. Clearly, a set is not isomorphic to its power-set; thus the failure of the model isomorphism criterion despite the manifest equivalence of **Set** and **CBA**.

However, note that we have not quite done away with 'isomorphism' in this definition of equivalence: rather, we have lifted the notion to a higher category, viz. the category of categories **Cat**, and proceeded to *weaken* the notion of isomorphism in **Cat** by 'relaxing' equalities to isomorphisms in the equation (2).

2.1.2 In physics

So much for pure mathematics. We now proceed to the realm of physics, where things are in general woolier and more complex, and so often resist formalization. The following list of dualities is of course far from exhaustive, and is merely meant to emphasize that AdS/CFT is but one of large number of phenomena – all exhibiting (Dictionary) – that fall under the rubric of duality.[11]

- Kramiers-Wannier duality: This is an equivalence between an Ising model at low temperature to another Ising model at high temperature. For instance, the free energy in the low temperature theory is related to the free energy in the high temperature theory.

- S-duality: This is an equivalence (indeed a symmetry) between a quantum field theory at weak coupling to a quantum field theory at strong coupling. For instance, S-duality says that an $\mathcal{N} = 4$ Super-Yang-Mills theory with gauge group G and coupling τ is equivalent to an $\mathcal{N} = 4$ Super-Yang-Mills theory with gauge group ${}^LG$ (the Langlands dual group to G) and coupling $-1/\tau$.

- T-duality: This is an equivalence between a string theory compactified on a circle of radius R with a string theory compactified on a circle of radius $1/R$.

[11] One should also remember that many of these phenomena are related conceptually, and even mathematically. For instance, Kramers-Wannier duality can be seen as an inspiration of sorts for S-duality: the former relates theories at low-temperatures to theories at high temperatures, whereas the latter relates theories at weak-coupling to theories at strong-coupling. And S-duality can be reduced to T-duality in certain contexts (see e.g. [14]).

- Holographic duality: This is an equivalence of a gravitational theory on the bulk (i.e. interior) region of some spacetime with a theory on the boundary region of that spacetime.

The case of interest to us, viz. holographic duality, is remarkable in that it was motivated by general considerations long before specific examples, e.g. AdS/CFT, were discovered.[12] Consider that the No-Hair Theorem tells us that a stationary black hole is characterized only by its mass, charge, and angular momentum. Since the matter that collapses into a black hole can have an arbitrarily large entropy, whereas its final, stationary state has no entropy, it would thus seem that this process violates the second law of thermodynamics. On the other hand, the Area Theorem (which says that the area of a black hole event horizon never decreases in time) suggests that this violation may only be apparent: perhaps an increase in the area of the horizon compensates for the loss in matter entropy. Indeed, this led Bekenstein to conjecture that the entropy S_{BH} of a black hole is proportional to its horizon area A. Later work by Hawking on black hole radiation suggested that

$$S_{BH} = \frac{A}{4}. \tag{3}$$

This was followed by 'entropy bound' arguments, which e.g. show that the entropy of a matter system which collapses to form a spherically symmetric black hole is bounded by the area of the smallest sphere that encloses the system.

Recall that the entropy of a system is a sort of measure of its physical degrees of freedom. Thus the above considerations suggest that the number of degrees of freedom in the bulk of some region is in some sense equivalent to the number of degrees of freedom on the boundary of that region – this statement is nothing but holographic duality in its crudest, most general, form. As we shall see below, AdS/CFT goes far beyond this by giving a precise account of the theories that live on the bulk and its boundary respectively, and how one can construct a 'dictionary' that relates the two. The claim that their degrees of freedom match up can be then be verified by starting from this more fundamental picture.[13]

2.1.3 Metaphysical interpretations

We have just discussed duality in mathematics and physics. One might then go on to ask how one should interpret physical duality from a *metaphysical* point of view. For instance, one obvious question is whether

[12]See Bousso's excellent and comprehensive review paper [5] for a detailed review of what follows, as well as references to the original work of Hawking, Bekenstein, et al.

[13]See e.g. the naive computation on pp. 9-10 of [19].

dual theories describe the same reality, albeit in very different ways, or if one of the dual pairs is metaphysically more fundamental than the other. And what are the principled reasons for the various verdicts given? More systematically, the metaphysical options are:

1. The dual theories are merely different descriptions of the same reality.
2. The dual theories describe different realities, and neither is more fundamental.
3. One of the dual theories is more fundamental than the other.

If the first option is true, then a robust form of emergence does not even get off the ground. The second option is touched on briefly in Section 3.5, but it lies off our main path of inquiry; it is really the third option on which the claim of emergence hinges, as I shall explain in the next subsection. Thus, most of Section 3.5 is devoted to exploring the extent to which one is justified in saying that in AdS/CFT, the boundary theory is metaphysically more fundamental than the bulk theory.

2.2 Emergence from duality

There is no standard use of the term 'emergence' in philosophy, let alone in popular discourse; thus it is important to begin by clarifying terms. I shall use the term 'top theory' to mean the emergent theory and the term 'bottom theory' to mean the theory from which the top theory emerges. Despite the manifold uses of emergence, most authors take it to have at least the following focal meaning:

> (Focal): Emergence is an asymmetric relation between a top theory X and a bottom theory Y such that X displays novel features with respect to Y.

The vagueness about this 'relation' between the top theory and the bottom theory is deliberate, as the relevant relation will vary with the context, and even after fixing a context, authors will disagree about what the appropriate relation is! So for instance, some claim – while others deny – that X is a definitional extension of Y or is supervenient on Y. Yet others endorse some kind of part-whole relation between the objects of Y and the objects of X.

Fortunately, there is no need for us to enter such controversies, as we are concerned solely with the emergence of one theory from another, where the two theories are thought to be 'equivalent' in the sense that they are related by a duality. I shall call this *Emergence from Duality* (ED). However, it will be useful to contrast ED with two cases that are much discussed in the literature in order to (i) emphasize how very different ED is from these cases, and (ii) explain why the standard tools of definitional extension and supervenience are largely irrelevant for characterizing ED.

The first contrast case with ED is emergence in a sense that is incompatible with definitional extension.[14] (Recall that X is a definitional extension of Y just in case Y, when suitably augmented with a set of definitions, contains X as a subtheory.) This 'logical emergence' holds that in addition to (Focal), the top theory X is emergent just in case it cannot be deduced from a complete knowledge of the bottom theory Y and the laws that govern it. But it is clear that logical emergence cannot make sense of ED: since dual theories are mutually definable, each member of a dual pair is a definitional extension of the other.

We can also contrast ED with the sort of emergence that is exhibited in the emergence of thermodynamics from statistical mechanics; for instance in the topic of phase transitions. In the latter case, novel features emerge in the top theory via the taking of a limit (the thermodynamic limit) and coarse-graining (the renormalization group). Furthermore, some (e.g. Butterfield and Bouatta [7]) would argue that the top theory is a definitional extension of the bottom theory (thus the slogan 'emergence is compatible with reduction' in [6]). Again, it is clear that such emergent phenomena differ greatly from ED: there is no limit-taking or coarse-graining going on in ED, since dual theories are supposed to contain the same amount of information.

Since we will ultimately be discussing emergent quantum gravity in AdS/CFT, it is worth comparing the dialectic at this stage to a recent discussion of emergent spacetime in Loop Quantum Gravity (LQG) by Wuthrich (in Chapter 9 of [27]). Wuthrich argues that definitional extension and supervenience are inappropriate reductive relations for understanding emergent spacetime in LQG, and the same is true of ED for the above reasons. However, Wuthrich also plausibly makes the case that emergent spacetime in LQG can be understood by means of a combination of ordered approximating and limiting procedures. This is where it is crucial to disambiguate two sorts of emergence that can be discussed within the context of AdS/CFT. The emergence of a full theory of quantum gravity from gauge theory, which is our topic of interest, does not turn on a limit and is thus different from emergence in LQG and other approaches. However, and as we shall see later, there is a sense in which classical supergravity emerges from gauge theory in the large N and large t'Hooft parameter (which I later call λ) limit. This phenomenon is indeed similar to Wuthrich's claim about emergent space-time in LQG, since it involves both a judicious choice of limit-taking and approximations (i.e. one needs to justify certain perturbative solutions).

[14] For an expression of this view, see the section in [21] on epistemological emergence, and also [10].

How then to understand ED? In order to make progress on this question, it may help to pose a second question, viz. why is emergence discussed only with respect to some, and not other forms of duality? So for instance in the $\mathcal{N} = 2$ case of S-duality, no one speaks of the instantons of one theory emerging from the monopoles of its dual theory or vice versa; whereas in the case of AdS/CFT, the bulk theory and its features are sometimes said to emerge from the boundary theory. A possible first answer to this question is the 'novelty' criterion in (Focal): since S-duality is a relation (indeed a symmetry) between two QFTs, one might argue that neither of the relata are sufficiently novel relative to the other in order to merit the apellation 'emergent'. But this is at any rate clearly insufficient. To begin with, novelty is a somewhat subjective notion; but more importantly, emergence is an asymmetric relation whereas novelty is a symmetric one. That is to say, even if gravity and QFT are novel relative to each other, one still needs to explain why the former is emergent from the latter but not conversely.

More plausibly, one one can try to ground the asymmetry of ED in the thought that the bottom theory is metaphysically more fundamental than the emergent, top theory. Indeed, it is fairly common in physics to use 'emergence' to indicate that the bottom theory describes the fundamental degrees of freedom, whereas the top theory describes the phenomenological degrees of freedom.This is thus the route that we shall follow in our quest to make sense of, and evaluate, emergence in AdS/CFT.[15]

We have arrived at the following idea: in order to establish that a theory is emergent from its dual, one must first argue that its dual is more fundamental. In the next section, I shall discuss two lines of thought that might move one towards this conclusion. The first thought (Phenom) is that if one of the dual theories provides what we would think of as a complete explanation of physical phenomena that we can in principle detect and predict, then its dual must describe some sort of veiled, underlying reality, and thus be more fundamental. The second thought (Explanation) is that if one theory – but not its dual – has the resources to explain a fundamental concept such as entropy, then that theory should be considered to be more fundamental. However, the proper explanation and evaluation of both ideas requires a more detailed picture of AdS/CFT. In the next section, I give such a picture, and then return to (Phenom) and (Explanation) in its final subsection.

[15]To avoid confusion: note that there are other ontological views of emergence on which the top theory has novel, irreducible, metaphysical properties – in which case the bottom theory is arguably not fundamental. These will not be relevant for our topic, but see e.g. Merricks [20] who considers a view of this sort with respect to the causal powers of objects.

3 Emergence in AdS/CFT

In order to address the issue of emergent spacetime and gravity and AdS/CFT, we will first need to review AdS spacetime (Section 3.1) and conformal field theory (Section 3.2) separately, before proceeding to sketch the duality between them (Section 3.3). We will see that AdS/CFT duality can be motivated in two different contexts, viz. a string-theoretic context (Maldacena's original motivation) and a context that only relies on gauge field theory. One of the most vivid illustrations of the AdS/CFT dictionary is given by the 'field-operator correspondence', which I review in Section 3.4.1. I also discuss an application of this dictionary to the hole argument in Section 3.4.2. With these details in the foreground, we revisit in Section 3.5 the question of emergence, and in particular (Phenom) and (Explanation) as mentioned earlier.

Before proceeding a quick disclaimer is necessary: the AdS/CFT literature is vast. So in a short survey such as this, it is impossible to do justice to most of it – or even the focal case which below I call Classic AdS/CFT!. In particular, technical discussions of supersymmetry, the large N limit, various generalizations, applications to condensed matter physics, confinement, QCD and the quark-gluon plasma, etc. have all been omitted. Fortunately, there are many good review articles (e.g. [1, 16, 8, 24, 22]) from which to learn this material and many of the original papers are very readable; (indeed my sketch of the material below closely follows various aspects of the magisterial [1] and [19]).

3.1 Review of AdS space

In the focal case of AdS/CFT duality, the bulk theory is a theory of gravity (and other fields) whose dynamical spacetime is asymptotically Anti-de Sitter (AdS). Thus it behoves us to understand some basic facts about $(p+2)$-dimensional AdS space, denoted AdS_{p+2}. In order to aid the reader's visual imagination, I will only discuss the case where $p = 1$, viz. AdS_3; the extension to the general AdS_{p+2} will be obvious, so the reader should have no trouble in Section 3.3 modifying various facts to the case of AdS_5.

It will first be convenient to review the notion of compactification: roughly, this is a mathematical operation through which a non-compact space is turned into a space all of whose points are a finite distance away from every other point. So for instance, $\mathbb{R}^n$ can be compactified into the sphere S^n by adding a point 'at infinity', and $(n+1)$-dimensional hyperbolic space H^{n+1} – the simplest space with negative curvature – can be compactified into the $(n+1)$-dimensional disk D_{n+1} via a conformal rescaling $g' = \Omega^2 g$ of the metric, where $\Omega > 0$. Note that the boundary of compactified hyperbolic space is compactified Euclidean space in one less dimension. Similarly, we shall see that the boundary of compactified AdS spacetime is compactified

Minkowski spacetime in one less dimension.

Conformal compactification is useful for two reasons. First, it allows us to represent the causal structure of a spacetime in a perspicuous fashion. Second, the conformal compactification of a spacetime X allows us to define the concept of a spacetime's being 'asymptotically X': this means that after conformal compactification its boundary structure is the same as X's.

In order to illustrate how this works, let us consider 2-dimensional Minkowski spacetime $\mathbb{R}^{1,1}$ whose metric is $ds^2 = -dt^2 + dx^2$. Via a conformal rescaling that results in the new metric $ds^2 = (4\cos^2 u_+ \cos^2 u_-)^{-1}(-d\tau^2 + d\theta^2)$, where $u_\pm = (\tau \pm \theta)/2$ and $|u_\pm| < \pi/2$, this spacetime can be conformally compactified into the following rectangle:

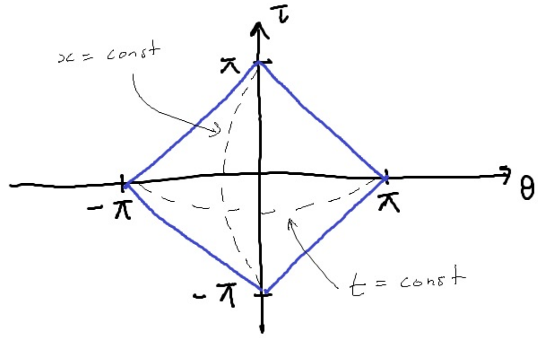

Figure 1: Compactified Minkowsi space

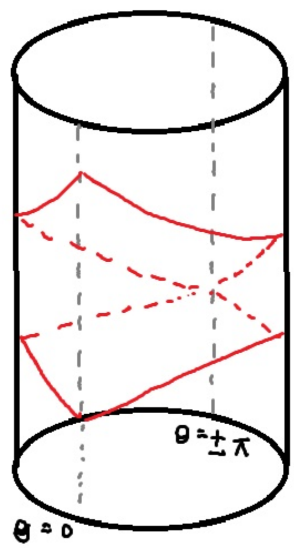

Figure 2: Compactified Minkowski space embedded in a cylinder

The above rectangle can in fact be embedded in the cylinder $\mathbb{R}\times S^1$ (by identifying $\theta = -\pi$ and $\theta = \pi$) and the metric analytically continued to the whole cylinder so that it is the maximal extension of compactified Minkowski spacetime. Furthermore, the conformal group $SO(2,2)$ of $\mathbb{R}^{1,1}$ acts on the compactified spacetime.

We now proceed to describe AdS_3 space and how it can be conformally compactified. AdS space is a maximally symmetric space with constant negative curvature, and its simplest description is as a hyperboloid embedded in the flat Lorentzian space, in this case $\mathbb{R}^{2,2}$.[16] More explicitly, the hyperboloid is the locus of

$$-X_0^2 + X_1^2 + X_2^2 - X_3^2 = -R^2 \tag{4}$$

in the ambient space $\mathbb{R}^{2,2}$, whose metric is

$$ds^2 = -dX_0^2 + dX_1^2 + dX_2^2 - dX_3^2. \tag{5}$$

It is immediately evident from the form of (4) and (5) that the hyperboloid respects the $SO(2,2)$ symmetries of the ambient space. Indeed it is also evident that AdS_3 space is homogeneous, i.e. the $SO(2,2)$ action takes any point to any other point. The below figure shows the hyperboloid as parameterized by $(X_0, X_3, r = \sqrt{X_1^2 + X_2^2})$, i.e. as the locus of $X_0^2 + X_3^2 = R^2 + r^2$. Since the range of r (from 0 to ∞) is represented twice, note that each point of the hyperboloid represents a semicircle whose ends are identified with those of the semicircle on its reflection in the X_0-X_3 plane. The identification folds together the two halfs of the hyperboloid to form a 'Torpedo' cigar-shape, and so we see that the the manifold does not have a boundary at $r = 0$.

[16]Note that although $\mathbb{R}^{2,2}$ has two time-like directions, AdS_3 really only contains one time-like direction, as can be readily seen by proving that two orthogonal time-like do not exist at any point on the embedded hyperboloid.

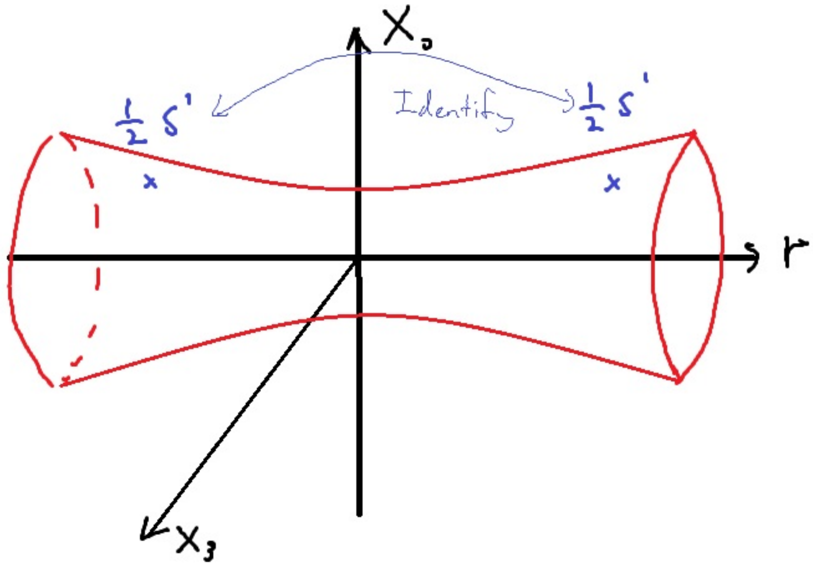

Figure 3: AdS_3 hyperboloid with semi-circles identified

We can now choose the following global coordinates for AdS_3, i.e. $X_0 = R \cosh \rho \cos \tau$, $X_3 = R \cosh \rho \sin \tau$, and the spacelike directions $X_i = R \sinh \rho \; \Omega_i$, where $i = 1, 2$ and $\sum_i \Omega_i^2 = 1$. These cover the hyperboloid once for the range $0 \leq \rho$ and $0 \leq \tau < 2\pi$, and allow us to rewrite the metric of AdS_3 in the form $ds^2 = R^2(-\cosh^2 \rho \; d\tau^2 + d\rho^2 + \sinh^2 \rho \; d\Omega^2)$. Unfortunately, this description of AdS space is problmatic from the physical point of view, as it has closed timelike curves in the τ direction and so is not 'causal'; however, the problem is easily remedied by passing to the universal covering space, i.e. by unwrapping the τ coordinate so it extends from $-\infty$ to ∞ without any identifications. It is this unwrapped space that is usually referred to as AdS in discussions of AdS/CFT duality.

Just as in the case of hyperbolic space, we can compactify AdS_3 by introducing new coordinates and conformally rescaling the metric. In particular, one finds that the compactified metric takes the particularly simple form of $ds^2 = -d\tau^2 + d\theta^2 + \sin^2 \theta \; d\Omega^2$ on $\mathbb{R} \times$ (the two-dimensional disk D^2). To be explicit: at each spacelike hypersurface of constant $\tau \in \mathbb{R}$, D^2 is parameterized by the S^1 metric $d\Omega^2$ and $0 \leq \theta < \pi/2$, where the boundary S^1 lies at $\theta = \pi/2$. It is thus evident that the boundary of compactified AdS_3 is just (the maximal extension of) compactified Minkowski space, i.e. the cylinder $\mathbb{R} \times S^1$. This also lends a sense to a spacetime that is asymptotically AdS_3, i.e. it is a spacetime which can be compactified into a region with the same cylindrical boundary structure.

To what extent can particles probe AdS space? Although the boundary of AdS is an infinite distance away from the center, it turns out that a massless particle (travelling along a null geodesic) can reach the boundary in finite time. Thus, the Cauchy problem for massless particles is ill-posed until one specifies appropriate boundary conditions; for instance, once possible choice has the massless particles reflecting off the boundary and returning to the original position, as in the diagram below.[17] Massive particles, on the other hand, do not reach the boundary and have an oscillatory solution around $\rho = 0$, as shown in the figure below.

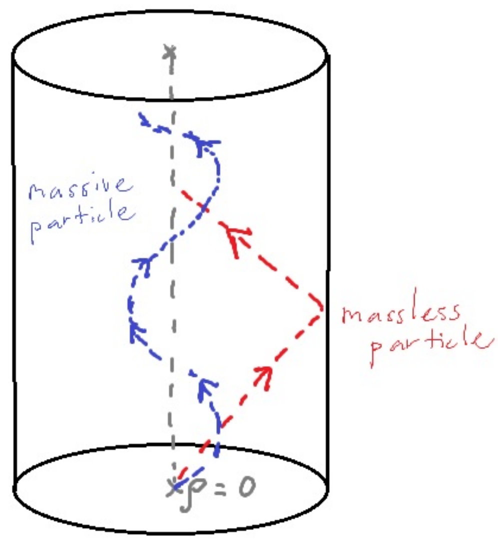

Figure 4: massless and massive particles in AdS spacetime

Although the global coordinates are conceptually transparent and cover all of AdS, for calculations in AdS/CFT it is often advantageous to use a coordinate system called the *Poincare patch* that covers only one half of the hyperboloid. In this coordinate system, we single out X^2 from the X^i, thus breaking the $SO(2)$ symmetry of the circle S^1 that is parameterized by X_1 and X_2. More precisely, we set $X_\mu = \frac{R}{z} x^\mu$ (where

[17]This non-trivial relationship between the bulk and the boundary is an intimation of the much more radical relationship described by AdS/CFT.

$\mu = 0, 1$ and $z > 0$), $X_3 + X_2 = \frac{R}{z}$, and $-X_3 + X_2 = v$.[18] The locus equation (4) thus becomes

$$\frac{R}{z}v - \frac{R^2}{z^2}x_\mu x^\mu = -R^2, \tag{6}$$

where we are contracting the μ indices with the metric $\eta_{\mu\nu} = (-+)$. Using it to solve for v, we can then convert the AdS_3 metric from

$$ds^2 = d(\frac{L}{z})dv - X_0^2 + X_1^2 \tag{7}$$

to the Poincare patch form

$$ds^2 = \frac{R^2}{z^2}(dz^2 + dx^\mu dx_\mu). \tag{8}$$

Notice that z can be thought of as a 'radial coordinate' in AdS space that parameterizes a continuous family of $\mathbb{R}^{1,1}$ Minkowski spaces, the largest of which lies at the boundary $z = 0$ (the constant R is typically called the AdS radius). Furthermore, the Poincare AdS metric (8) solves Einstein's equations with negative cosmological constant, i.e. $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \Lambda g_{\mu\nu}$. Indeed by using the equations of motion, one can easily show that for AdS_3 the cosmological constant Λ is $-6/(2R^2)$.

The sort of bulk theory implicated in the focal case of the AdS/CFT correspondence is in fact a supergravity theory, which requires an understanding of the AdS supergroup and Killing spinors. Unfortunately, I do not have the space to review these notions here (the interested reader is referred to pp. 47-54 of [1]), although the above will suffice to provide a rudimentary understanding of the correspondence. In particular, it is sufficient background to work out the duality between a scalar field on AdS space and a QFT on its boundary via the state-operator correspondence, as discussed in Section 3.4.1 below.

3.2 Review of Conformal Field Theory

We now turn to the notion of a conformal field theory (CFT), which is the sort of QFT implicated on the boundary side of the focal case of AdS/CFT duality. Unlike more familiar Poincare-invariant QFTs, CFTs are invariant under the conformal group (of Minkowski space), which is the smallest group containing both the Poincare group and the inversions $x^\mu \mapsto -x^\mu/x^2$. In particular, conformal symmetry includes a scale invariance symmetry that links physics at different length scales, implying that (unlike more familiar QFTs) CFTs do not have an S-matrix. Furthermore, the larger symmetry places very strong constraints on the correlation functions, e.g. conformal invariance essentially determines the two-point function for scalar primaries. Other special features of CFTs include a one-to-one correspondence between local operators

[18] Note that were we to choose $z < 0$ then the chart would cover the other half of the hyperboloid.

and states in the radial quantization, the effectiveness of operator product expansion and other algebraic techniques, and the existence of rigorous mathematical tools for constructing such theories in 2 dimensions. CFTs are an important object of study in their own right; however, they are also important for understanding more familiar QFTs, which (typically) have a renormalization group flow from a scale-invariant fixed point in the UV to a scale-invariant fixed point in the IR.

The conformal group is the set of transformations that leaves the spacetime metric invariant up to an overall (in general position-dependent) rescaling, i.e. $g_{\mu\nu} \mapsto \Omega^2(x) g_{\mu\nu}$. Such transformations preserve angles although they obviously distort distances in general. Note that if the metric is dynamical, one can interpret a conformal transformation as a (metric-preserving) diffeomorphism $x \mapsto x'$, $g_{\mu\nu} \mapsto g'_{\mu\nu}$ followed by a Weyl transformation $x \mapsto x'$ which does not preserve the metric.

The conformal group of $\mathbb{R}^{p,q}$ can be divided into translations, Lorentz transformations, scalings, and special conformal transformations, whose infinitesimal generators are respectively written as $P_\mu = -i\partial_\mu$, $M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$, $D = ix^\mu \partial^\mu$ and $C_\mu = -i(x_j x^j \partial_\mu - 2x_\mu x_j \partial^j)$, where μ, ν run over all coordinates, whereas j only runs over the spatial coordinates. It is easy to verify that they form a Lie algebra, whose commutation relations tell us how the generators transform under Lorentz transformations: P^μ and C^μ are vectors, D is a scalar, and $M_{\mu\nu}$ is a rank-2 tensor. There are also three commutation relations that will be particularly important for us, and so we write them out explicitly:

$$[D, P_\mu] = iP_\mu, \tag{9}$$

$$[D, C_\mu] = -iC_\mu, \tag{10}$$

$$[C_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - M_{\mu\nu}). \tag{11}$$

Similarly to the $\mathfrak{su}(2)$ case, these commutation relations tell us that P^μ and C^μ are raising and lowering operators respectively for the eigenvectors of D, which is a 'diagonal' operator. Note too that by making judicious identifications between generators of the conformal group of $\mathbb{R}^{p,q}$ and generators of $SO(p+1, q+1)$ it is immmediately evident that these groups are isomorphic; thus our earlier claim that the conformal group of $\mathbb{R}^{1,1}$ is $SO(2, 2)$.

Just as in more familiar cases (e.g. classifying the irreducible representations of the Poincare group), the representations of the conformal group are classified by the relevant Casimirs, which here correspond to spin and the eigenvalues of D.[19] For our purposes, we are thus interested in those representations containing

[19] Note that unlike the case of the Poincare group, $P_\mu P^\mu$ is not a Casimir of the conformal group, because the scaling operator D does not commute with the Hamiltonian H.

fields (or states) which are eigenfunctions of D. These have eigenvalues $-i\triangle$, where $\triangle$ is called the *scaling dimension* of the field. So for instance, under a scaling, a spinless field transforms as $\phi(x) \mapsto \phi(x') = \lambda^{\triangle}\phi(0)$, the infinitesimal form of which is $[D, \phi(0)] = -i\triangle\phi(0)$. Since the spectrum of any unitary field theory should be bounded from below, each representation has a field of lowest dimension (called a *primary operator*) that gets annihilated by the lowering operator C_μ. We can thus build up the entire spectrum of the theory by listing all the primary operators and hitting these repeatedly with the raising operator P^μ. Two points from this excursus into representation theory are relevant to AdS/CFT: first, knowledge of the Hilbert space of the CFT is necessary for checking that the bulk and boundary theories have isomorphic Hilbert spaces; and second, the scaling dimension $\triangle$ plays an important role in the duality dictionary – it is related to the mass of a field in the bulk theory.

The sort of CFT that is relevant to the focal case of AdS/CFT is in fact an $\mathcal{N} = 4$ *super*-conformal $SU(N)$ gauge theory, to match the super-AdS space alluded to above; again here supersymmetry lies beyond the scope of our discussion. Another significant omission from this review is a discussion of the large N (of $SU(N)$) limit of gauge theory, which is crucial for establishing results at our present stage of knowledge about AdS/CFT, and for understanding how perturbative gauge theory can approximate perturbative string theory.

3.3 Review of AdS/CFT duality

The term AdS/CFT pays tribute to what has historically been the focal case of this duality, viz.

> (Classic AdS/CFT) Four-dimensional $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills gauge theory is equivalent to type IIB string theory with $AdS_5 \times S^5$ boundary conditions.

However, it is important to emphasize that the term is something of a misnomer, as it is also used to refer to dualities whose bulk geometries are not AdS, and whose boundary field theories are not conformal. Here are some of the ways in which Classic AdS/CFT generalizes: first, it turns out that a duality still holds for non-conformal field theories obtained either by perturbing a conformal field theory, or by considering a stack of Dp-branes for $p \neq 3$. Second, if we change the geometry of the boundary side by considering AdS times some Einstein space, we find a duality with quiver gauge theories. Third, the assumption of supersymmetry can be dropped or weakened by modifying the gauge theory's Hamiltonian, leading to yet other instances of the duality (but beware: breaking supersymmetries greatly reduces the stability of the theory). At any rate, despite these generalizations, it will convenient to limit ourselves to Classic AdS/CFT for the purposes of this paper.

Some circumstantial evidence for the Classic AdS/CFT can be gathered by examining the symmetries on both the bulk (AdS) side and the boundary (CFT) side and checking that they match up. So, for instance, on the bulk side one sees that the geometric symmetries of $AdS_5 \times S^5$ are $SO(4,2)$ acting on AdS_5 and $SO(6)$ acting on the 5-sphere S^5. On the other hand, the bulk theory, i.e. four dimensional $\mathcal{N} = 4$ $SU(N)$ Super-Yang-Mills theory has an $SO(4,2)$ conformal symmetry and an $SO(6)$ symmetry that rotates its scalars. Furthermore, one can also show that there are 32 supersymmetries on both sides: these arise as Killing spinors on the bulk side and from the super-conformal algebra on the boundary side. Of course, this might be dismissed as a coincidence without a more direct argument for the duality: in Section 3.3.1 I review a perspective on such an argument that is motivated by string theory, and then one that is purely based on gauge theory in Section 3.3.2.

3.3.1 The stringy context

The original argument for AdS/CFT duality, which I now sketch, was given by Maldacena in [18] and made essential use of type IIB string theory in a flat $\mathbb{R}^{9,1}$ spacetime. It provides compelling evidence for the conjecture, but not an actual proof, because only *perturbative* string theory is used. To begin with, one considers a stack of N parallel D3-branes close to each other which couple to gravity (i.e. distort the metric) with strength $\lambda \sim N g_s$, where g_s is the dimensionless string coupling.[20] The duality is then motivated by contemplating two rather different descriptions of this setup: first, the weak coupling regime when $\lambda << 1$ and second, the strong coupling regime when $\lambda >> 1$.

When $\lambda << 1$, i.e. when the spacetime is nearly flat, then one obtains open strings (ending on the brane) describing the excitations of the brane, and closed strings in $\mathbb{R}^{9,1}$ describing the excitations of empty space. However, in a certain low-energy limit, these two systems decouple, and the brane system is described by an effective four-dimensional $U(N) \cong SU(N) \times U(1)$ Super-Yang-Mills theory. Indeed, the $U(1)$ factor also decouples and one is left with precisely the gauge theory of the boundary side of the AdS/CFT duality.

On the other hand, when $\lambda >> 1$ the gravitational back-reaction of the brane becomes important, and the metric describes a black-hole-like object called an *extremal black 3-brane*. Just like a black hole, this object has a horizon, and its near-horizon geometry is $AdS_5 \times S^5$. Again, in the same low-energy limit as

[20] When viewed from the gauge theory point of view, this λ is also called the t'Hooft coupling, and is often expressed as $\lambda \sim g_{YM}^2 N$, where g_{YM} is the Yang-Mills coupling and N is from $SU(N)$.

before, we obtain two decoupled systems, viz. type IIB string theory in $AdS_5 \times S^5$ near the horizon, and closed strings in $\mathbb{R}^{9,1}$ in the asymptotically flat region.

We have thus arrived at two different descriptions of the low-energy physics of type IIB string theory, one at large λ and the other at small λ. Since both descriptions contain a common factor, viz. the decoupled closed strings in $\mathbb{R}^{9,1}$, one natural move is to subtract out this factor, and conjecture that the full gauge theory description is equivalent to the full gravity (string theory) description. Of course, it is possible to have weaker forms of the conjecture: for instance, one might conjecture that the equivalence is *only* valid for large λ.[21] However, the strong form of the conjecture discussed in this paper is the most interesting, and the most often discussed, form of AdS/CFT.

One can also obtain a *classical* (on the bulk side) version of the duality by (i) taking the $N \to \infty$ limit of the $SU(N)$ gauge theory, which suppresses quantum corrections since $g_s \sim 1/N$; and (ii) taking the the $\lambda \to \infty$ strong coupling limit, which reduces the string size $l_s \sim \lambda^{-1/4}$. When there are no quantum corrections and the string size is much smaller than the AdS radius R, then we have a classical supergravity theory on the bulk which is dual to an $N, \lambda \to \infty$ quantum gauge theory on the boundary. For instance, this derived duality is often used to obtain qualitative results about QCD, which is in many respects similar to the boundary gauge theory, e.g. they both exhibit confinement and thermal phase transitions (but there are also significant differences, e.g. QCD is asymptotically free but the bulk gauge theory is not).

What sorts of tests can be performed in order to verify the AdS/CFT duality? Very briefly (see Section 3.2 of [1] for a detailed account), the duality is hard to check perturbatively, since one needs weak coupling to do perturbative calculations, and the weak coupling limit of the boundary theory is the strong coupling limit of the bulk theory, and vice versa. However, some properties of the theories (e.g. some special correlation functions, the spectrum of chiral operators, the moduli space of the theory, etc) do not depend on λ and so tests can be carried out. For instance, one can show that there is a 1-1 correspondence between supergravity particles on $AdS_5 \times S_5$ and the chiral primary operators of the dual boundary CFT.[22]

3.3.2 The field theory context

Although the string theory context has played an important historical role in motivating AdS/CFT, there have also been attempts to motivate the duality directly from the perspective of field theory, e.g. [19, 15]. A

[21]See p. 60 of [1] for a discussion of the different forms of the conjecture.

[22]This then extends to the identification of the spectrum (Fock space) of supergravity particles on AdS with the CFT spectrum generated by chiral primary fields.

starting point for this endeavor is reflection on the Witten-Weinberg theorem, which says that a QFT with a Poincare covariant conserved stress-energy tensor $T_{\mu\nu}$ cannot contain massless momentum-carrying particles of spin greater than one. In particular, this appears to forbid gravitons (i.e. massless spin 2 particles) that are made out gauge degrees of freedom. So how to construct a gravitational theory from a gauge theory, let alone prove an equivalence between them?

An answer suggests itself when one considers that the Witten-Weinberg theorem tacitly assumes that the graviton must live in the same spacetime as the field theory. If one believes in the holographic principle on independent grounds (i.e. black hole thermodynamics) then it seems that one has reason to challenge this assumption; indeed if our gauge theory is 4-dimensional, we should be looking at gravitons on a 5-dimensional spacetime whose boundary is the spacetime of the gauge theory.

But where can this extra spacetime dimension come from? That is to say, how can we identify a local scale in the gauge theory that gives rise to a local spacetime dimension? Fortunately, such a scale exists in a gauge theory in the guise of the renormalization group (RG) flow, whose coupling parameters are local in energy/length scale. (Furthermore, one would expect the gauge theory to be strongly coupled if it is to describe classical gravity, because we know from perturbative calculations that a weakly coupled gauge theory does not resemble gravity.) If we identify the energy scale in the gauge theory with the radial coordinate z, then we can see that the AdS/CFT duality is in some sense a geometrization of the renormalization group, with the UV lying at the boundary $z = 0$ and the IR lying at the $z \to \infty$ limit.

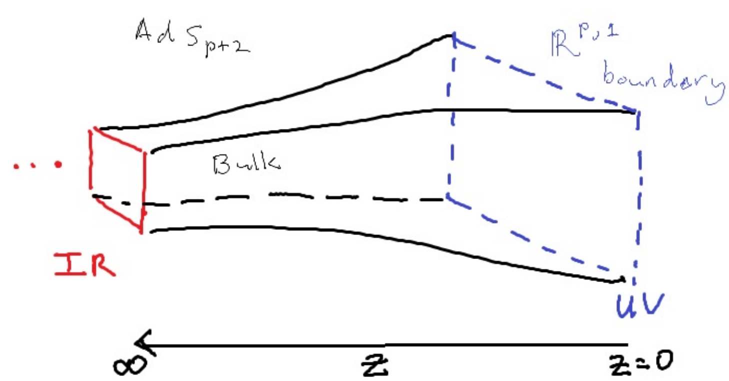

Figure 5: The holographic RG flow

We have arrived at the idea that the gravitational dual of a four-dimensional gauge theory should live in a five-dimensional spacetime whose extra coordinate z can be thought of as a energy scale. If we assume that the couplings do not change with energy-scale, then from dimensional analysis and the requirement of Poincare invariance, it is easy to see that the most general metric consistent with this idea has the form:

$$ds^2 = \frac{R^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2), \qquad (\mu = 0, 1, 2, 3). \tag{12}$$

But this is nothing other than the Poincare patch metric of AdS space that we saw earlier (8). Furthermore, if we are interested in a theory of gravity – i.e. a theory with a dynamical metric – then we should really be looking at spacetimes that are asymptotically AdS.

From here, we could pursue a chain of ideas that would lead us to conjecture the Classic AdS/CFT duality that I outlined earlier. (To repeat: the exact AdS/CFT correspondence has not been *proven*, but it has passed every single non-trivial test that physicists have set for it.) For instance, by adding maximal supersymmetry to a four-dimensional gauge theory, one is led to consider its 10-dimensional type IIB supergravity dual, and thus to consider a bulk spacetime that is asymptotically $AdS_5 \times S^5$ instead of just AdS.[23] However, I would instead like to pursue the more general idea of a dictionary between a gravitational theory (or even field theory) on AdS space and a gauge theory on the boundary of AdS.

3.4 The Dictionary

Earlier in Section 2.1, we discussed the general idea of a dictionary-like relation between mathematical and physical theories. In AdS/CFT duality, a large part of this dictionary is captured by what has come to be known as the 'field-operator correspondence' which relates the basic observables (i.e. fields) of the bulk theory to the basic observables (i.e. local operators) of the boundary gauge theory.

3.4.1 The field-operator correspondence

I now review the prescription given in [12, 26] for matching the fields that live in the AdS spacetime bulk to local operators in the boundary field theory. First recall that a QFT has a path-integral Z_{QFT}, which can be perturbed by product of a source J and operator $\mathcal{O}$ to obtain the generating function

$$Z_{QFT}[J] = \int \mathcal{D}X \exp(-S_{QFT}[X] + \int J(x)\mathcal{O}(x)), \tag{13}$$

where S_{QFT} is the classical action of the QFT. The observables of a QFT are correlation functions of products of local operators, e.g. $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)\rangle$, which are typically computed by taking source derivatives

[23] See p. 4 of [15] for more details.

of the generating function as follows:

$$\langle \prod_n \mathcal{O}_n(x_n) \rangle = \prod_n \frac{\delta}{\delta J_n(x_n)} \ln Z \mid_{J=0} . \tag{14}$$

According to Gubser, Klebanov, Polyakov, and Witten (GKPW), there is a sense in which AdS/CFT duality gives us a new way of computing the partition function of the boundary gauge theory. The thought is as follows: UV perturbations to the classical action of the boundary QFT should correspond to perturbations near the boundary $z \to 0$ of AdS spacetime and so the perturbation should be encoded in the boundary value of one of the bulk fields. Indeed, give some bulk field ϕ, one might hope to use its boundary value ϕ_0 as the source term (earlier called J) of the perturbation. This idea is schematically expressed in the GKPW formula:

$$Z_{QFT}[\phi_0] = Z_{QG}[`\phi \to \phi_0' \text{on the boundary}] \underset{\text{large } N}{\sim} e^{-S_{bulk}}|_{\text{extremum} \quad \phi \to \phi_0} . \tag{15}$$

The GKPW formula says that, in principle, if we knew what the partition function $Z_{QG}[\phi]$ of quantum gravity was, we could relate it to $Z_{QFT}[\phi_0]$ when $\phi \to \phi_0$. However, at present constructing Z_{QG} is out of reach, and we should instead try to relate the bulk and boundary theories in some appropriate limit. A sensible limit to try is $N \to \infty$, where the bulk theory classicalizes and becomes tractable: in this limit, we should be able to use the classical saddle point (i.e. the extremum of the action) of the bulk theory to compute the correlation functions of the boundary theory. In fact, one does not even have to consider a gravitational bulk theory in order to apply the GKPW formula, as a simple example will illustrate.

Let us put aside gravity for a moment and consider a scalar field ϕ in the bulk, propagating on a fixed AdS background. Its action is $S_{bulk} \sim \int d^{d+1}x\sqrt{g}[g^{AB}\partial_A\phi\partial_B + M^2\phi^2]$. For small fluctuations, the equation of motion turns out to just be $(-\Box + m^2)\phi = 0$ and we can solve it using the Fourier-decomposed ansatz $\phi = \exp(ik_\mu x^\mu)f)k(z)$, which yields Bessel functions. For instance, if we plug in $f_k = z^\triangle$ and then examine the solution near the boundary ($z \to 0$), we find the relation $\triangle(\triangle - d) = m^2R^2$. So far, $\triangle$ is just an exponent in the ansatz, but the amazing thing is that we can use the GKPW formula to compute $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim |x|^{-2\triangle}$, thus showing that $\triangle$ can really be interpreted as the scaling dimension $\triangle$ of the boundary CFT. Also, it is straightforward to use the GKPW formula to interpret the solution's leading order fall-off ϕ_0 as a source term for Z_{QFT}. In other words, by taking the derivative of S_{bulk} with respect to ϕ_0, we can compute the expectation value $\langle\theta\rangle$ of the operator θ for which ϕ_0 acts as a source. This simple example shows us that the GKPW formula provides the following dictionary between the bulk theory (on the left) and boundary theory (on the right):

- Scalar field $\phi \longleftrightarrow$ Operator θ
- Scalar field mass $m \longleftrightarrow$ Scaling dimension $\triangle$.

Actually applying the GKPW formula to more complicated scenarios is a subtle matter that lies beyond the scope of this paper, but we can still benefit from its general moral, viz. that it provides a dictionary coupling fields in the bulk (acting as sources near the boundary) to local operators in the boundary. In other words, if we know what sort of source term an operator couples to, then we also know the field to which it is related. Thus, since the stress-energy tensor $T_{\mu\nu}$ of a QFT couples to the metric $g_{\mu\nu}|_{boundary}$ at the boundary, we have the dictionary entry

- Metric field $g_{\mu\nu} \longleftrightarrow T_{\mu\nu}$.

And since gauge fields A_μ couple to global currents J^μ in the form $A_\mu J^\mu$, we also have

- Gauge field A_μ in the bulk theory $\longleftrightarrow$ Global current J^μ in the boundary theory.

Supposing that we have a comprehensive dictionary between the bulk and boundary theories, we can immediately apply it in various ways. The first and most practical use of the dictionary is to perform calculations in one theory that are hard to perform in the other theory (using the bulk theory to obtain qualitative results about confinement, the quark-gluon plasma, and strange metals falls into this category). Second, one might try to use the dictionary as a new definition of gauge QFTs – one that does not inherently turn on a Lagrangian formulation. But the dictionary also has more speculative applications: for instance, if one is wondering whether diffeomorphism symmetry is a physical symmetry of gravity (on pain of underdetermination, cf. the hole argument) or if it is merely a redundancy of description, one only has to look it up in bulk side of the dictionary and see that it does not correspond to anything on the boundary – thus it is a redundancy, albeit an emergent redundancy via the duality. In a similar spirit, the AdS/CFT dictionary has also been invoked to resolve the issue of whether black hole 'information loss' is consistent with the laws of quantum physics – since the evaporation of information in the bulk has a perfectly consistent unitary-evolution description on the boundary, it has been argued that the inconsistency must only be apparent.

4 Emergence in AdS/CFT?

We are now in a position to consider whether the bulk theory emerges (in the sense of ED in Section 2.2) from the boundary theory in AdS/CFT; that is to say, whether the boundary theory is more fundamental than the bulk. As we have seen above, if this is true then AdS/CFT will be a theory of emergent quantum gravity, in our sense. It will also be a theory of emergent (classical or semi-classical) gravity by means of limiting procedures, i.e. in a sense that has not been our main focus in this paper. In both contexts, there will be emergent physical objects such as strings, fields (the metric and other fields), approximate spacetime, and even emergent redundancies such as diffeomorphism symmetry.

The first question to discuss is whether the dual theories describe two different realities, or are instead different descriptions of one reality – as we discussed earlier, the latter precludes the possibility of one theory being more metaphysically fundamental than the other. Following on from the discussion in Section 3.3.1 and Section 3.3.2, the answer to this question is going to turn on background context that one chooses in order to interpret AdS/CFT duality. For instance, one can adopt a string-theoretic context and view the duality as an emergent, derived relation between different limits of the more fundamental string theory that describes reality, in which case neither is more fundamental. On the other hand, one might think that AdS/CFT is a much more general phenomenon which can be motivated entirely within a gauge field theory context and is independent of the string-theoretic argument in Section 3.3.1. One might even go so far as to think that it is a generic sort of relation between quantum gravity theories and gauge field theories. Whether an interpretation at this level of generality is true turns on future work (suffice to say that there are some obstacles to this program, e.g. the difficulty of constructing de Sitter versions of AdS/CFT duality); for the purposes of this discussion I will merely consider known examples of the correspondence within a field theory context, in which case the question of fundamentality once again becomes a live issue.

Even if one detaches AdS/CFT from the context of a more fundamental background theory, one can still press the thought that although the dual theories describe different realities, these realities are metaphysically on a par. That is to say, there exist two parallel universes whose physics is mutually definable and somehow coordinated (via e.g. the field-operator correspondence). But what explains this mysterious coordination? There seems to be some theoretical pressure to either try to unify the universes through some background context (which could bring us back to the string-theoretic argument of Section 3.3.1) or to pursue a reductive strategy and claim that one theory is more fundamental than the other. We now consider the latter strategy, which leads to talk of emergence.

Let us consider how a prominent proponent of string theory makes the move from a holographic duality to claims of metaphysical priority. *Vide* Greene in his 'The Hidden Reality' [11]:

> '...reality – not its mere shadow – may take place on a distant boundary surface, while everything we witness in the three common spatial dimensions is a projection of that faraway unfolding.'
> '...the *holographic principle* envisions that all we experience may be fully and equivalently described as the comings and goings that take place at a thin and remote locus. It says that if we could understand the laws that govern physics at that distant surface, and the way phenomena there link to experience here, we would grasp all there is to know about reality. A version of Plato's shadow world – a parallel but thoroughly unfamiliar encapsulation of everyday phenomena – would *be* reality.' (p. 238 of [11])

At first blush, it seems that Greene is inferring that the boundary theory is more fundamental than the bulk theory from the fact that there is a duality between them, i.e. a non-sequitur. But a more careful reading is possible which places emphasis on the phrase '...thoroughly unfamiliar encapsulation of *everyday phenomena*' (my emphasis). This reading takes seriously the idea (Phenom) that in our world of 'everyday phenomena' (meaning of course 'phenomena' from the perspective of physics, not our senses), everything we might expect to find in nature is described by a quantum gravity theory (type IIB string theory, to fix ideas) and various effective theories derived from it at lower energies. But underlying this phenomenal world is a veiled reality – what we thought of as quantum gravity in our phenomenal $(d+1)$-dimensional world is really (in a metaphysically loaded sense) just a QFT in d dimensions.

This line of thought bears a *loose* resemblance to Plato's Allegory of the Cave, as Greene remarks. However, we can note one irony and one disanalogy with Plato's Cave. First the irony: in Plato's allegory, the shadows on the wall of the cave (i.e. the objects of our experience) are mere*projections* of higher-dimensional reality, and as we have just seen, Greene wants to liken the boundary of AdS to this reality. But insofar as we can make physical sense of the notion of *projection* in AdS/CFT, surely the analogy runs the other way round – it is the boundary that plays the role of the wall of the cave, onto which higher-dimensional objects are projected. Next, and more importantly, the disanalogy: by construction, Plato's allegory is such that the real three-dimensional objects contain more information than their two-dimensional projections on the wall of the cave. This is strongly disanlogous to the AdS/CFT scenario, where the bulk and the boundary are supposed to capture the same amount of physical information!

Perhaps the analogy might be restored if one already had reason to think that the d-dimensional world was fundamental, but arguably one has no independent reason to think this in the context of AdS/CFT. And at any rate, why should the fact that we are embedded in and experience $(d+1)$ physics mean that it is less fundamental than its d-dimensional dual theory?

One possible answer to this question is to take the view of fundamentality suggested by Maldacena in [17], viz. that the constituents of a composite object are more fundamental than the composite itself (this is of course not uncontroversial in metaphysics, depending on the sort of composite one is discussing!). If so, then one might argue that the fact that basic degrees of freedom (i.e. gravitons, or strings representing gravitons) in the bulk theory are dual to composite (or 'bound') states in the boundary theory shows that the boundary theory is more fundamental.

But this strategy is susceptible to several pressing worries. First and more generally, the issue of what counts as a basic degree of freedom is itself a perturbative and interest-relative notion. Second, even if one grants that constituents are more fundamental than their composites, this will not suffice to show that the boundary theory is more fundamental than the bulk theory, because the notion of a composite is here defined *within* the boundary theory. Such an argument thus seems to rest on a damaging circularity. Of course, the idea that constitution can serve as a guide to fundamentality (albeit not *metaphysical* fundamentality) might nonetheless play an interesting role in understanding perturbative notions of emergence associated with AdS/CFT. However, I leave this to future work and turn to a second line of thought in favor of the boundary theory's fundamentality.

The second line of thought, viz. (Explanation), is largely inspired by considerations about how to explain black hole thermodynamics. It runs as follows: in ordinary thermal physics, thermal properties – in particular entropy – can be explained by the physics of the microscopic constituents of matter. On the other hand, it is difficult to find any such explanation in $(d+1)$ dimensions for black hole thermodynamics. Somewhat remarkably, when we view a $(d+1)$-dimensional black hole in the language of its d-dimensional boundary gauge theory, we find precisely a hot gas of gauge bosons, scalars, and fermions, and this explains why black holes display thermodynamic properties. We are then supposed to conclude that the boundary theory is more fundamental than the bulk theory. Unfortunately, this argument sketch is also far from conclusive. First, there is no guarantee that such an explanation will not eventually be forthcoming from the perspective of the $(d+1)$-dimensional bulk theory. Second, one can again press the point that, strictly speaking, what the microscopic description in the boundary theory explains is the thermodynamic properties of the gas in the boundary theory; and only indirectly – via duality – the thermodynamic properties of the black hole. Thus, one cannot use the existence of a microscopic description to argue that the boundary theory is more fundamental.

5 Conclusion and summary

In this paper, I have surveyed both duality in general and AdS/CFT in particular, as well as a notion of emergence that draws essentially upon the exact correspondence between theories that is provided by a duality. The key morals of the general discussion are as follows:

1. Duality is an equivalence between two theories, in the sense that they are mutually definable. However, these theories will not in general be isomorphic. An interesting project for future work is thus to show how the semantic view of theories can accommodate duality phenomena.

2. If one is interested in a robust sense of emergence that draws essentially on duality, then plausibly, one is led to the idea that one of the dual theories is metaphysically more fundamental than the other. On the other hand, and in contrast with some other uses of 'emergence', the top and bottom theories are mutually definable, and so definitional extension (or even supervenience) cannot serve as the asymmetric relation in which emergence is grounded. Emergence from duality also differs from other instance of emergence (e.g. perturbative emergence within AdS/CFT, or thermodynamics as emergent from statistical mechanics) in that the emergent theory is not derived by means of a limit or an approximation.

3. Not all dualities are said to involve emergence, but only those in which the dual theories are sufficiently different from each other, and in which one side of the duality is thought to be more fundamental. We should thus inquire into the reasons (if any) for this asymmetry in fundamentality.

More specifically, the consideration of AdS/CFT brings a rich and complex mathematical apparatus to bear on the topic of emergence from duality. I want to urge two morals for the topics of 'emergent quantum gravity' and 'emergent spacetime'.

First, unlike many other theories which yield some notion of emergent spacetime, AdS/CFT does not in the first instance concern the emergence of spacetime *within* a theory, but rather the correspondence between an theory of quantum gravity, on the one hand, and a gauge theory, on the other. One can then ask if one of these *exact* theories emerges from the other – that has been the central topic of this paper. One can also go on to ask whether any *approximate* theories emerge as limits of either theory. Either way, I want to stress that any emergent gravitational theory will not simply be a theory of spacetime (whether in a quantum or classical form) but will merely be one element of the emergent theory, along with fields, strings, and other objects.

Second, it would seem that we have no good reason to think of the gravitational side of the duality as metaphysically emergent from the gauge theory side, or vice versa. The above arguments have addressed emergence from duality (ED) in AdS/CFT, but the case for the metaphysical priority of one side over the other seems no more promising even in the purely perturbative case, for the symmetry between the theories remains: interesting phenomena in one theory can in principle be described as a limit of the other theory, and vice versa. Thus the most promising metaphysical understanding of AdS/CFT duality is perhaps the one that we briefly considered above (but which was irrelevant to ED): the duality between gravity and gauge can itself be embedded within some larger theory, from which both theories can be seen as emergent – and equivalent – representations of some limit.

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