

A note on the normative dimension of the St. Petersburg paradox

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1. INTRODUCTION

Daniel Bernoulli (1700-1782) is widely known as the perspicacious solver of a very popular paradox, named after the journal where it was published, the *Commentarii Academiae Scientiarum Imperialis Petropolitanae*. However, in Gerard Jorland's words, «the paradox in the St.Petersburg problem is that there is a paradox» (Jorland, 1987, p.157). In 1713, Nicolas Bernoulli, Daniel's cousin, asked P.R de Montmort to calculate the mathematical expectation of a coin-toss game, in which the payoff function was 2^{n-1} , i.e., A pays B one coin if the result of tossing a coin is heads the first time ($2^{1-1}=2^0$), two coins if this result occurs in the second toss (2^{2-1}), four if in the third (2^{3-1}), and so on. Considering that the corresponding probability amounts to $1/2^n$, the mathematical expectation of the game is given by an infinite geometric series of common ratio 1, that is, a divergent series. For more than two centuries, it was taken to be infinite¹.

In fact, neither Nicolas nor Daniel considered the paradox as a mere probabilistic puzzle: it was rather a contradiction between the presumed expected value of the game and the *common*

sense (le bon sens) arithmetically expressed in the concept of *expectation (expectatio)*. To grasp this latter dimension, let us recall another classical riddle, the Problem of Points (*le problème des partis*)², for which Pascal and Huygens originally coined the concept of mathematical expectation in the mid 1650s. Let us think of a game –e.g. Heads and Tails– to be won by the first player that wins, say, five rounds; suppose that for any reason, the game is interrupted when one of the gamblers has already won three rounds, while the other only one: which would be the *fair distribution* of the stakes, then? According to both Pascal and Huygens, if a gambler wanted to abandon the game, selling to another individual the advantage gained –i.e. his right to receive a certain proportion of the stakes at that stage of the game–, the *just price* would amount to the mathematically expected gain. That would also be the *fair distribution* sought in case of interruption: each player would be given a part of the stakes in accordance with his expectation.

For Nicolas Bernoulli, however, it was quite difficult to admit that a reasonable man (*un homme de bon sens*) would be willing to pay an infinite sum in order to play the St. Petersburg game. To account for this exception an alternative calculation of the just price of a gamble was due and Daniel Bernoulli's merit would have been to articulate a different approach on the basis of his *expected utility functions*. Probably echoing the justification of these advocated by von Neumann and Morgenstern in 1944, Lorraine Daston argued that the key to Daniel's solution lay in transferring the concept of expectation from a *legal* to an *economic* framework (Daston, 1988, p. 71). Instead of calculating just prices, Bernoulli would have modelled a decision-making process in which a psychological variable (the gambler's utility) would account for the hedging of the value of the stake, in such a way as to render the price of the game reasonable³.

The aim of this paper is to provide an alternative interpretation of Daniel Bernoulli's solution of St. Petersburg paradox based on an analysis of the precise conceptual transfers on which

was moulded upon. Following a suggestion by the late Ernest Coumet (Coumet, 1972), I will contend that the concept of *mathematical expectation* was itself constructed on the basis of a transfer which assimilated games and contracts, incorporating thus a juridical standard of fairness (*just prices*). I will argue that this criterion was the source of the contradiction perceived in the St. Petersburg's gamble by both Nicholas and Daniel Bernoulli, so that the latter's solution consisted namely in the replacement of a normative standard mathematically expressed by an alternative one. In this view, expected utility functions would convey a moral intuition on what constituted a judicious decision-making, rather than a positive model of certain types of economic behaviour.

In the next three sections, I will present the three conceptual transfers which constitute the original normative dimension of the concepts of mathematical expectation and expected utility. First, the assimilation of games and contracts operated by certain Schoolmen will be discussed. The second section is devoted to the formal analysis of this projection carried out by Pascal and Huygens to develop the concept of mathematical expectation. Finally, I will focus on the alternative analysis performed by D. Bernoulli to produce his expected utility functions. In order to render the discussion of these transfers more precise, I will draw on a particular theory of metaphor that I borrow from the cognitive linguist George Lakoff (Lakoff & Johnson, 1999). The virtue of this theory is that it deals with metaphors as mappings in quite a simple and intuitive way, and even those who do not share its principles may well accepted without much trouble just for the sake of the analysis.

2. 1556: GAMES AS CONTRACTS

There is, in fact, an ancient tradition in which a deal to gamble was assimilated to a sort of contract and there were a number of theological discussions of this analogy available before the XVI century⁴. However, as for the immediate sources of the concept of mathematical expectation, I share Coumet's hypothesis (Coumet, 1972, pp. 591-ff) about the Spanish

Dominican Domingo de Soto (1495-1560) being the most likely source of Pascal's discussion. As a matter of fact, Soto's treatise *De Iustitia et Iure* was reprinted about thirty times (including eight in Lyon, two in Antwerp, and seven in Venice) before the end of the 16th century. This inaugurated a series of Scholastic monographs on the topics therein contained (at least twenty-two, taking account only of Spanish authors in a hundred-year lapse after its second edition, from 1567 to 1670). Among Pascal's closest friend there was a prominent jurist, Jean Domat, who could have well informed him of this Scholastic discussion⁵, of which, in any case, textual traces abound in the vocabulary deployed to introduce the concept of mathematical expectation.

The first step in my analysis will focus thus on book IV of *De Iustitia et Iure*⁶, which is devoted to the analysis of commutative justice, specially of property rights (*dominium*). The fifth question considers its transference, and the second article («Numquidnam per ludum dominium transferatur») discusses whether the gains acquired in gambling were legitimately owned or should be rather restored to the original owner. With a view to asserting the former, Soto proceeds by means of an analogy: a game would be a kind of contract voluntarily arranged by the gamblers, entirely similar to those employed to insure seaborne commodities, and therefore perfectly legitimate from a juridical standpoint.

Even though Soto does not provide an explicit statement of the analogy, a metaphorical mapping could be extracted from certain passages⁷. It would read as follows:

Source domain

Target domain

Insurance contract

Game

Partners

Gamblers

Investment (<i>sors</i>)	Stake
Gain / Loss	Payoff
Risk (<i>periculum sortis</i>)	Risk
Expectation (<i>spes</i>)	Expectation

The most relevant aspect of this mapping, insofar as our probabilistic inquiry is concerned, lies in the *inferential structure* mapped onto the target domain (games). For Soto had also discussed the fairness of insurance contracts, that is, whether there was anything usurious about them, and the criteria he established for their assessment applied also, as we will now see, to gambling contracts. According to Soto and the Schoolmen, in a fair insurance contract the partners constitute a kind of society; they are hence obliged to assume certain risks in the enterprise, if the contract is to be distinguished from a loan (risk being the criterion of ownership)⁸. Soto leaves the fixing of the just price of an insurance contract⁹ to the partners involved, except for the following proviso: *the insurance policy* (i.e. the insurer's share of the expected profits) *must be in proportion to the risk assumed* (i.e. the insured capital)¹⁰.

In my view, this proviso constitutes what Lakoff would denominate the *invariance principle* of the metaphor¹¹. According to Lakoff, a metaphorical mapping does not only consist of, say, *horizontal* (term to term) correspondences, but also of *vertical* relations, so that the stakes, the payoff and the risk –in the target domain- are linked through the concept of expectation, as their homologous terms were in the source domain. In other words, in case this proportionality rule between the insurance policy and the risk assumed was cancelled, the metaphor would be partially dismantled.

This invariance principle would be reflected in the very delicate balance existing between positive justice and prudence in the Scholastic tradition. Soto was arguing within a *natural*

law framework, not necessarily coincident with the positive legislation of every single country. Fixing the particular proportion that constituted a just price was a natural right of the partners to be exercised *prudentially*, i.e., without settled rules of universal application¹². In my opinion, this is the source of St.Petersburg paradox as I will try to show later on.

2. 1654: IN SEARCH OF JUST PRICES FOR GAMES

In this section, I will try to show the extent to which this metaphor was still alive among those who formalised the concept of mathematical expectation, namely Blaise Pascal (1623-1662) and Christian Huygens (1629-1695). Two different sorts of traces might be found in their texts. On the one hand, there is lexical evidence of the presence of Soto's conceptual mapping scattered among the texts in which Pascal discusses the Problem of Points¹³, as in the following passage of his *Treatise on the Arithmetical Triangle* (1665):

To understand the rule of distribution, the first thing that should be considered is that the money the gamblers have bet no longer belongs to them, (do not belong to them anymore), since they have renounced ownership or owning it. (to own it). But they have received in exchange the right to expect whatever chance might give them, according to the conditions that were agreed beforehand.

Yet Since it is a voluntary law, they can break it at will. They can therefore quit the game at any stage and renounce whatever they might have expected of chance - inversely to what they did on entering the game- so that they can regain the ownership of something. In this case, the distribution of what should belong to them should be strictly proportional to what they might rightfully expect from chance. It should be thus indifferent for any gambler to take this amount [of the stake] and retire or continue playing. This fair distribution is what we call the *parti*. (Pascal, 1963, p. 57)

Pascal speaks of gambling as Soto had done one hundred years before: «a contract is an action

between two people, from which a certain obligation arises for them both» (Soto, 1968, p. 541), i.e., «a voluntary law», that entitles the gamblers to expect (*attendre*) a certain gain. «It does not matter if the loser receives no compensation (*emolumenti*) whatsoever for his money (*pecunia*); since, as I said before, he put it at risk voluntarily expecting (*cum spe*) to get it back by chance» (Soto, 1968, p. 314). Besides this lexical vicinity, there is also evidence that Soto's proportional rule for the fixation of just prices (the invariance principle of the mapping) organised Pascal's argument concerning the Problem of Points. Let us recall that at issue was the fair distribution of the stake when the game had been interrupted before it ended. According to the passage already cited, for Pascal the gamblers should share it in proportion to the gain they are entitled to obtain at that moment, which is worth the money they would risk to enter the game then. In other words, balancing the risked capital and the expected gain, which was Soto's recommendation for settling just prices in this kind of contracts.

Additional evidence about the presence of our juridical metaphor among these early probability theorists might also be found in the arguments by means of which Christian Huygens (a doctor *in utroque iure*) mathematised the concept of expectation. Let us examine in this respect a few passages extracted from *De Ratiociniis in Ludo Aleae*, a short piece by Huygens after a brief stay in Paris and published in 1657 as an appendix of a compilation of mathematical questions edited by Frans van Schooten¹⁴.

In *De Ratiociniis...*, Huygens addresses the Problem of Points in a systematic way, restating, first of all, his *règle des partis*:

One's Hazard or Expectation to gain any Thing is worth so much, as, if he had it, he could purchase the like Hazard or Expectation in a just and equal Game. (Meusnier, 1992, pp. 7-8)¹⁵

He then proves the following proposition, concerning a two players' game

If I expect either a or b , that I can obtain with equal facility, my expectation is said to be worth $(a+b)/2$. (Meusnier, 1992, p. 9)¹⁶

Note that a and b are the terms of a deed, that is, the winner's and the loser's payoff, respectively. According to the aforementioned *règle*, a gambler's expectation amounts to the price he had paid to enter a similar game, and, as long as both players assume equal risks, they are equally entitled to obtain a or b . For the same reason, there is only one price for them, still unknown: x . «Hic autem ludus justus est»: up to this point, the problem is entirely conceived in Soto's terms, no hint of probabilities yet.

Huygens then proceeds as follows: given that both players have paid (bet) x , the stakes will amount to $2x$. If the loser obtains b , the winner will gain $2x-b$, that is, a . It is easy to see, then, that $2x-b = a$, and hence: $x = (a+b)/2$. If a third gambler wants to replace one of the two at a certain stage of the game, here is a method to calculate *quo pretio me eam ipsi vendere aequum sit*. There also lies a very significant deviation from one of the Schoolmen's central insights, their reluctance to fix a definite value for just prices.

Even if Huygens was still trying to fix a fair price reasoning along the schema originally settled by Soto, it must be noticed that this mathematised concept of expectation replaced a prudential proportion for a well defined arithmetical mean. I.e., instead of leaving to the prudential deliberation of the gamblers the fixation of the proportion between risk and betting they estimate most convenient in each particular case, the mathematical expected gain of a gamble settled its just price once and for all. I contend that it is the generality of this rule what was put at stake by the St. Petersburg's game: Huygens' calculating procedure yielded its *a priori* just price, but this was such that no reasonable gambler would be willing to pay it. In the following section, I will discuss this interpretation of the paradox, while the final one will be devoted to analyse the conceptual foundations of the mathematical solution proposed by Daniel Bernoulli.

3. 1738: THE PARADOX OF JUST PRICES

At the beginning of the 18th century, the technical concept of expectation was still not distinguished from its ordinary usage, as can be inferred from a scholium in *Ars conjectandi*, Huygens' opusculum, which had been reproduced and annotated by Jacob Bernoulli (1654-1705) himself¹⁷. The book was published eight years after Jacob's death by his nephew Nicolas (1687-1759), a doctor *in utroque jure*, and, as we already know, author of our paradox.

In his correspondence with Pierre Rémond de Montmort, Gabriel Cramer and Daniel Bernoulli, additional traces of our juridical metaphor are to be found: according to Nicolas, the paradox affected its very foundations, as stated in a letter addressed to Montmort, dated 20th February 1714:

From all this, I conclude that the just value of a certain expectation is not always the mean that we obtain through dividing by the addition of all possible cases the sum of the products of each expectation by the number of cases that yield it. This goes against our fundamental rule. (*apud J. Bernoulli, 1975, p. 558*)

In other words, there would be an exception for the general rule for the calculation of just prices stated by Huygens. In his doctoral dissertation *De usu artis conjectandi in iure* (1709), Nicolas Bernoulli had inverted Soto's metaphorical projection, applying the mathematical concept of expectation to the analysis of naval insurances¹⁸. In *On the Law of War and Peace* (II, 12, 23) Grotius, whose authority Nicolas claimed to determine the legal status of naval insurance contracts (Bernoulli, 1975, p. 318), argued that such price should be estimated according to the *common estimation*. Now, according to Nicolas, the merchants could *sell their expectations (spei alteri vendere)* on the basis of a *fair and rational (jure et rationaliter)* anticipation of its value (Bernoulli, 1975, p. 318). In other words, contracts could now be analysed as games, so that the mathematical calculation of their prices would conform

to the fair ordinary estimation discussed by both Soto and Grotius¹⁹.

The paradox in St. Petersburg's gamble would arise in the contradiction between the value calculated in accordance with Huygens arithmetical rule and such ordinary estimation. The risk of losing and the hope of gaining had "no proportion to the event", and therefore could not be considered fair. To correct this, Nicolas would have liked to solve the paradox in strictly juridical terms, as revealed in his response to his cousin, dated 5th April 1732, after receiving Daniel's *Specimen* (1738)²⁰

It is not a question of measuring either the utility or the pleasure we obtain from a sum of money we might gain. Neither is it a question of looking for an equivalence between any of them, but rather to discern the amount of money that one gambler is obliged to pay to another, according to justice or fairness, for the advantage that he hands him in this particular game of chance -or in any other sort of game-, so that the game could be considered equitable. [...] (*apud* Bernoulli, 1975, p. 566)

For his part, Daniel seems to have been perfectly aware of his cousin's concerns, despite his holding quite a different view. The opening paragraph of his *Specimen...*, referred to Nicolas' *regle fondamentale*, reads as follows:

Proper examination of the numerous demonstrations of this proposition that have come forth indicates that they all rest upon one hypothesis: *since there is no reason to assume that of two persons encountering identical risks, either should expect to have his desires more closely fulfilled, the risks anticipated by each must be deemed equal in value*²¹. No characteristic of the persons themselves ought to be taken into consideration; only those matters should be weighed carefully that pertain to the terms of the risk [*ad conditiones sortis*]. The relevant finding might then be made by the highest judges [*iudices supremi*] established by public authority. (Bernoulli, 1954, pp. 23-24)

Besides, once he had introduced the concept of *emolumentum* and its mathematical development (§§3-12), Daniel announces his purpose of building up «a complete theory», «as has been done with the traditional analysis» (*hypothesi communi*). He gives no hint of what that *traditional theory* is, other than an alternative account of some «significant points among those which at first glance occurred to me»: i.e., *games of chance* (§§ 13-14) and *naval insurances* (§§ 15-16: let us recall Soto); then, in the final paragraphs of the *Specimen...*(§§ 17-19), he presents his solution of the St. Petersburg paradox.

It is clear that both cousins were moving along the same metaphorical paths: games were still treated as contracts, and the St. Petersburg paradox was perceived as a glaring exception to the mathematical rule that enabled the jurist to solve the Problem of Points. The question is whether Daniel intended to replace an old juridical theory by an economic approach –as defended by Lorraine Daston–, or rather to shift the grounds of the discussing reintroducing prudential considerations. The only way to assess it is through a detailed examination of the argumentation displayed in his 1738 paper, since there is no other parallel text available. Our analysis will discuss first Bernoulli’s conceptual turn regarding the juridical view of mathematical expectation, and how it is later reflected in his concept of *emolumentum*.

4. WHAT WAS DANIEL BERNOULLI TRYING TO ACCOMPLISH?

But really there is no need of judgement [*iudicia*] but of deliberation [*consilia*], i.e., rules [*regulae*] would be set up whereby anyone could estimate his prospects from any risky undertaking in light of one’s specific financial circumstances. (Bernoulli, 1954, p. 24)

«Non iudicia sed consilia». It seems to me that an accurate reading of these two terms, *iudicium* and *consilium*, would be extremely helpful in order to understand Daniel’s point. Both terms admit an informal reading, but their opposition suggests a technical one, since in classical Roman Catholic theology (that of Aquinas and Soto) *judgement* and *deliberation*

received a separate treatment. According to Aristotle [e.g., *EN*, III 3, 1112a18-ff] and Aquinas [*S.Th.*, I-II, q.14], *deliberation* (*boúleusis* in Greek, *consilium* in Latin) consisted of the assessment of the different means available to attain an intended end, whenever no fixed criterion could account for the variety of contingent circumstances involved in its achievement. It was a matter of *prudence* and, as such, related to *morals*, while, in turn, *judgement* concerned *politics*, it had to do with law²².

Note, in addition, that Bernoulli equated *concilia* with *regulae*, which were no foreign notions among the Schoolmen. As a result of a moral process (described in *S.Th.*, I-II, qq.6-21), rules were, indeed, set up to assess the proper attainment of our ends²³. According to Bernoulli, a rule would be, for instance, the proposition that states how to calculate expected values, or his own proposition concerning the estimation of expected utilities (*emolumenta media*). Rules, in Bernoulli's view, are closely connected with *consilia*, where the Latin term *consilium* combines both *deliberation*, as translated above, and *advice*, as employed by Bernoulli in the following passage:

Another rule which may prove useful can be derived from our theory. This is the rule that it is *advisable* to divide goods which are exposed to some danger into several portions rather than to risk them all together (Bernoulli, 1954, p. 30; italics added).

The point is that rules would be not only a result of individual deliberation, but also a means of advice, i.e. criteria of intersubjective assessment: after applying his rule to the assessment of naval insurances, Bernoulli concludes:

A man would act *unadvisedly* if he were to offer to sponsor this insurance for six hundred rubles when he himself possesses less than 29878 rubles. However, he would be *well advised* to do so if he possesses more than that amount. (Bernoulli, 1954, p. 30; italics added)

To advise was also a part of the scholastic doctrine of *consilium*, once again in accordance with Aristotle²⁴. In any case, even if Bernoulli were somehow familiar with those philosophical doctrines, it would be obviously absurd to make him a Schoolman. If he was actually attempting to set up a theory, those allusions could simply indicate that he wished to place it in the traditional domains of prudence, which do not coincide with those of the Law.

However, Bernoulli's fundamental concern was not speculative: his «new theory on the measurement of risk» was entirely aimed at solving the St. Petersburg paradox. He was not only introducing the basics of a doctrine concerning deliberative rules, perhaps an ethical one: *he stated a deliberative rule (non iudicia sed consilia) aimed at improving our actual behaviour under uncertainty.*

The substitution of a bounded payoff function for the mere expression of the monetary gain to make the series representing expectation converge also had to be justified in non-mathematical terms. Therefore, Bernoulli argued -against his cousin- that engagement in risky enterprises was not primarily motivated by the justice of its price: it was an individual decision, based on *prudential* criteria. Consequently, the mathematical concept of expectation could be prudentially restated.

The normative justification of this prudential criterion would be its coincidence with the ordinary practice of price estimation, defended, as we have seen, by the theorists of naval insurances. “All our propositions harmonize perfectly with experience”, argued Bernoulli (1954, p. 31), since “a person who is fairly judicious by natural instinct might have realised and spontaneously applied much of what I have here explained” (*ibid.*). Therefore, he proceeds to restate in his own terms “the procedure customarily employed by merchants in the insurance of commodities transported by sea” (Bernoulli, 1954, p. 29). This is precisely the sort of justification for deliberative criteria to choose among uncertain prospects that the Aristotelian tradition could provide: the imitation of the deliberative patterns of those who we

consider wise decision-makers (Vega, 1998)²⁵.

The Bernoullian notion of *emolumentum* comes to the fore now, because, in the Scholastic tradition, *common utility* was the key concept for the prudential estimation of natural prices²⁶.

If Pascal and Huygens had transformed that intersubjective criterion into an objective algorithm, Bernoulli converted it into an individual rule of behavior. All of them retained its normative dimension: Pascal spoke of the Law; Bernoulli, of deliberation.

That moral overtone can be clearly perceived in a well-known letter addressed by Gabriel Cramer to Nicolas Bernoulli, dated 21 May 1728 (*apud* Bernoulli, 1975, pp. 560-61]. Trained in philosophy, and being himself a part-time lecturer in the same discipline, Cramer had named *Esperance Morale* what Bernoulli knew as *emolumentum medium*, and so, *morally speaking (moralement parlant)*, discussed the *moral value of wealth (Valeur Morale des Richesses)*, the moral value of goods (*Valeur Morale des Biens*). Bernoulli was plainly sympathetic to Cramer's approach, to the extent of reproducing a substantial part of his letter in his *Specimen...*: their respective theories were «so similar», that it seemed to him «miraculous» that they had «independently reached such close agreement on this sort of subject» (Bernoulli, 1954, p. 33).

Cramer's rule, entirely assumed by Bernoulli (*quod meum quoque est*), stated that:

Sensible men should estimate money proportionally to the use they can make of it.

(*apud* Bernoulli, 1975, 560).

Cramer also equated the moral expectation of a certain gain with *the pleasure (le plaisir)* one hopes to derive therefrom –a correlative amount of *pain (chagrin)* would be expected in case of loss. The psychological connotation of these two terms (also employed by Nicolas Bernoulli in his aforementioned letter to his cousin) may have suggested a parallel reading of Bernoulli's *Specimen...*, in which *emolumentum* was to be translated as *utility*, being an

antecedent of the marginalist view (e.g., Kauder, 1965; Schlee, 1992).

A more accurate interpretation of *emolumentum* could be obtained, in my opinion, if we locate it within the juridical and economic framework discussed above. Let me examine, firstly, Bernoulli's presentation of the concept: it is introduced in §§3-5, while §§6-11 provide a mathematical account of its estimation. The starting point of his argument is the following example: suppose that a poor man is given a lottery ticket that will yield with equal probability either nothing or twenty thousand ducats: would he be wrong if he sold it for nine thousand ducats? And what would happen, in case he was rich? In order to obtain a rule (*regula*) that could account for both cases, instead of mathematical expected values, Bernoulli states:

The determination of the *value* of an item must not be based on its price, but rather on the *utility* it yields. The price of an item is dependent only on the thing itself and is equal for everyone; the utility, however, is dependent on the particular circumstances of the person making the estimate. (Bernoulli, 1954, 24)²⁷

Bernoulli does not define *emolumentum* in his paper, nor does he provide a single synonym for it, except for the aforementioned comments on Cramer. Therefore, it would not be absurd to suppose that most of his contemporaries –at least, the readers of scientific journals- would have understood what he meant solely by virtue of their linguistic training.

The most probable origin of *emolumentum* is the verb *emolo* (to grind out): thus *emolumentum* would be *the payment to a miller for the grinding of, say, corn* (e.g., Ernout & Meillet, 1939, s.v.). Latin synonyms were *lucrum*, *quaestus*, *compendium*, *fructus* or *redditus* (e.g., Lewis, 1993, s.v.), and so it is mainly translated as *advantage* or *benefit*, whereas in medieval Latin it was read as *effect*, *progress* or *achievement*²⁸. That was also the primary sense it acquired in French²⁹, whereas no German term was apparently derived therefrom. It is also sometimes possible, when translating from classical Latin, to translate *emolumentum* as

utility, though it usually happens that the latter may be exchangeable with any of the aforesaid terms, as there is no sharp contrast between them.

On the other hand, Bernoulli almost always employs *emolumentum* with reference to the concept of *lucrum* (gain), since that was the chief mathematical statement of the *Specimen...*:

Ita vero valde probabile est *lucrum quodvis semper emolumentum afferre summae bonorum reciproce proportional* (Bernoulli, 1982, p. 224)

Now it is highly probable that *any increase in wealth, no matter how insignificant, will always result in an increase in utility which is inversely proportionate to the quantity of goods already possessed.* (Bernoulli, 1954, p. 25)

In §§ 3-12, for instance, both terms are implicitly or explicitly linked thirteen times, while *emolumentum* only appears unconnected with *lucrum* three more times. *Lucrum* is related to *emolumentum* by verbs such as *afferre*, *accipere*, *venire* or *evenire*.

Given that these two words were often synonymous, it may have been a bit strange for a Latin reader to find them together. In such a huge corpus as the *Patrologia Latina* is, *emolumentum* and *lucrum* very rarely appear next to each other, and when they do, they are most often exchangeable, or at least, they both present an analogous denotation.³⁰

Emolumentum also had a special connotation when used in legal contexts, to mean the *financial advantage* derived from (or inherent to) a certain wealth³¹. Let me quote Soto again:

It does not matter if the loser receives no compensation whatsoever for his money since, as I said before, he put it at risk voluntarily expecting to get it back by chance.
(Soto, 1968, p. 314)³²

To sum up, in my view, the most adequate translation of Bernoulli's *emolumenta* would be certainly utility, provided that it is not opposed to *gain*, as if the former meant a kind of subjective (say, psychological) magnitude, and the latter its objective reference (say, cash).

Emolumentum was usually employed regarding somebody's interests, as it consisted in the output of his pursuit. Hence, *emolumenta* were, right from the beginning, *useful* or *pleasant* things, no matter whether *material* or *immaterial*, and not the quality of being useful or pleasant itself. As I see it, that translation would be entirely consistent with the theological framework discussed above, since in the Aristotelian tradition utility and pleasure were concomitant with the attainment of an intended end, and could not be dissociated therefrom [EN, 1172a20-ff].

5 CONCLUDING REMARKS

Though the analysis of the origins of the expected utility functions is far from closed, I think that there is substantial evidence for what I consider the main conclusion of this piece: they were conceived namely as normative models of individual rationality, on the basis of which prudent decisions could be made. This does not imply that they were void of empirical content (Bernoulli, 1954, p. 31), quite the opposite: a prudential procedure is grounded on past experiences, particularly those of whom we consider wise decision makers. In the end, this interpretation of expected utility functions is not as deviant as it might seem to be at first: it is precisely the one advocated by Leonard J. Savage (against von Neumann & Morgenstern) when he built these functions into his *Foundations of Statistics* as a normative standard of probabilistic rationality –(Savage, 1972, p. 57). The underlying motivation of this alternative was precisely to account for the exceptions (Allais Paradox, in Savage's case) that inevitably appear when we try to generalize expected utility into a scientific model of economic behaviour. In this respect, the discussion of the origins of the concept of mathematical expectation carried out in this paper suggests that the history of probabilistic expectation as a tool for the analysis of decision making processes should account first for the transformation of their underlying normative schemas into positive models of decision making processes.

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¹ A classical account of St.Petersburg paradox is Todhunter (1949). For a mathematical overview of its development, see Jorland (1987); its conceptual implications are discussed in Daston (1980) and (1988), pp. 168-ff; Shoesmith (1983) and Jallais & Pradier (1997).

² For a discussion, see Daston (1980), although my own view is much more indebted to Coumet (1972), inspired himself by a game-theoretician —G.-Th. Guilbaud (1952). I have also benefited from Meusnier (1996).

³ Cf. von Neumann & Morgenstern (1953), p. 28. The gap between Bernoulli and von Neumann is partially (but superbly) bridged in Pradier (1998).

⁴ For instance, the Franciscan origins of this analogy are explored in Ceccarelli (1999). See also Franklin (1991) and (2002, pp. 258-88). A discussion of the prehistory of the Problem of Points can be found in Meusnier (2004).

⁵ For a discussion of Pascal from a political standpoint, see Ferreyrolles (1996). An account of Pascal's philosophy from the standpoint of games is Thirouin (1991). For a formal reconstruction of his wager, cf. Garcia de la Sienna (2002).

⁶ For a brief account from an economic perspective, see Gómez Camacho (1998a), pp. 59-ff. A good introduction to *De Iustitia et Iure* is Ramos-Lissón (1976).

⁷ See Soto (1968), pp. 312-317. The statement of the metaphor is as follows [from now on, I will quote the 1556 text without any correction of errata, etc.]: «Primum ex forma non vitiatur iure nature: nam praeterqua quam posset uterque ludentium gratis donare, illa non est simplex donatio: sed quaedam pactio, Do ut des. Nempe mea pecunia periculo expono, ut tu vicissim exponas tua. Et tanti aestimat periculo unius, quanti alterius» [p. 314]. Games became then an analogue of naval insurance contracts: «Et (de qua lib.6 dicturi sumus) contractus assecurationis maritimarum mercium uno Doctorum consensu tanquam licitus habetur: in quo

tamen plurimum dominatur humano loquendi more fortuna» [*loc.cit.*]. On the different types of contracts, cf. pp. 541-546; on naval insurance contracts, cf. pp. 579-581. I leave out a careful examination of the textual evidence of the metaphor, which is certainly impossible to carry out in a few pages.

⁸ Cf. Soto (1968), p. 579: «Pro quacunq̄ue re quae pretio aestimabilis sit, potest quisq̄ue mercedem recipere: in tuto autem rem constituere, quae periculis est exposita, pretio aestimabile: ergo quisq̄ue potest illud pretio redimere, atq̄ue adeo qui periculum illud subiit, idem recipere pretium. Atq̄ue id ex eo potissime quod uterq̄ue se periculo summittit: videlicet tam dominus navis solvendi pretium, si salva res sit, quam alter solvendi merces si pereant.»

⁹ Cf. Soto (1968), p. 580: «Sunt quidem qui stupidum illum censeant, qui spe lucrando centum aut mille, periculum subeat navis institoris valentis forsā viginti aut triginta milia. His autem respondemus, de pretio quidem nihil nos decertare potest quidem esse iustum, vel iniustum. Ipsi sibi contrahentes viderint.»

¹⁰ Cf. Soto (1968), p. 577 : «Ad secundum concedimus posse quempia decem millia, verbi gratia, ducatorum in societatem conferre eo pacto ut no subeat periculum nisi sex aut octo milium. Attamen tunc non potest ex lucro recipere nisi illam praecise partem quae illi pecuniae respondet, cuius ipse periculum subit: nam reliqua censetur sociiis mutuare quippe qui suo periculo illa suscipiunt: atq̄ue adeo pars illa lucri quae periculo illius provenit illorum est qui eide sese submiterun discrimini.»

¹¹ For an application of invariance principles to mathematical reasoning, see Lakoff & Nuñez (1997).

¹² Cf. *S.Th.* I-II, q.94 and Soto (1968), pp. 28-ff and 546-ff: «Pretium vero quod non est lege positum, non indivisibile est, sed latitudinem habet iustitiae: cuius unu extremum dicitur rigidum, alteru vero piu sed medium, moderatum. Ut quae resiuste venditur decem, iuste quoque vendit tum undecim, tum etiam novem. Atq̄ue ratio huius est quod prudentia humana,

qua per supradictorum considerationes de pretio existimatur, nequit puctim attingere metam: sed arbitramento quodam» (p. 547). See Gómez Camacho (1998b) for a discussion.

¹³ In fact, Pascal explicitly stated that the discussion of the Problem of Points concerned the Law (*Ius*). Cf. Pascal (1963), p. 102: «Novissima autem ac penitus intentatae materia tractatio, scilicet de compositione aleae in ludis ipsi subjectis, quod gallico nostro idiomate dicitur *faire les partis des jeux*, ubi anceps fortuna aequitate rationis ita reprimetur ut utriusque lusorum quod jure competit exacte semper assignetur.»

¹⁴ The Dutch text, with a French translation and a thorough introduction can be found in Huygens (1920), pp. 3-169. Coumet (1979) discusses Huygens' stay in Paris, and his relation with Pascal. Jacob Bernoulli reproduced the Latin text in the first part of his *Ars Conjectandi*, which I read in N. Meusnier's edition.

¹⁵ «Nimirum, in aleae ludo tanti aestimandam esse cujusque sortem seu expectationem ad aliquid obtinendum, quantum si habeat, possit denuo ad similem sortem sive expectationem pervenire, aequa conditione certans». The English translation is owed to John Arbuthnot — quoted in Daston (1980), p. 236.

¹⁶ «Si a vel b expectem, quorum utrumvis aequae facile mihi obtingere possit, expectatio mea dicenda est valere $(a+b)/2$.»

¹⁷ «Ex dictis colligi potest, vocabulum *Expectationis* non sumi hic sensu vulgari, quo communiter expectare vel sperare dicimur quod omnium optimum est, licet nobis pejus accidere possit [...]» [In Meusnier (1992), p. 11]

¹⁸ Nicolas followed Pufendorf's 1672 *De iure naturae et gentium libri octo* (V, 9, 7) in his discussion of games understood as a kind of contracts in which «uterque rem suam aequali periculo exponit», in plain accordance here with Soto: cf. Bernoulli (1975), p. 319.

¹⁹ As for Soto cf. his (1968), p. 580, cited above n. 9. Cf. Pufendorf *De iure naturae et gentium libri octo* (V, 9, 7).

²⁰ Daniel Bernoulli wrote his seminal essay for the 1730-31 volume of the *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, finally published in 1738.

²¹ The Latin text is: «Quod cum nulla sit ratio, cur expectanti plus tribui debeat uni quam alteri, unicuique aequae sint adiudicandae partes» [Bernoulli (1982), p. 223].

²² «Consulere non est propius actus legis, sed potest pertinere etiam ad personam privatam, cuius non est condere legem» (*S.Th.*, I-II, q. 92, a.2, ad 2). Cf. also Soto (1968), p. 21. On the other hand, «Iudicium, quod importat definitionem iusti, pertinet ad iustitiam secundum quid est principaliori modo in praesidente» (*S.Th.*, II-II, q.60, a.1, ad 4). Cf. also Soto (1968), p. 226.

²³ Cf. *S.Th.*, I-II, q.21; q.63, a.2

²⁴ Cf. *EN* 1143b9-ff and *S.Th.* II-II, q.51, a.1

²⁵ Acting against these rules constitute a clear case of unadvised decision-making. On the 4th of July 1731, Daniel wrote to Nicolas: “If only The Bernoullis, who lost so much when the Müllers got bankrupt, paid attention to the very principles that I establish actually, they would probably not have lost as much.” (*apud* Bernoulli 1975, pp. 566-7). And Bernoulli’s hypothesis about the shape of expected utility functions was precisely contested on these prudential terms by Condorcet: the wise merchants apply a different criterion to make their decisions. I owe all these remarks to Pierre-Charles Pradier.

²⁶ Cf. Soto (1968), pp. 546-49 and Molina (1981), pp. 167-ff.

²⁷ Cf. Bernoulli (1982), p 223: «Valor non est aestimandus ex pretio rei, sed ex emolumento, quod unusquisque inde capessit. Pretium ex re ipsa aestimatur omnibusque idem est, emolumentum ex conditione persona.»

²⁸ *E.g.*, Niermeyer (1984), *s. v.*; Habel (1931), *s. v.* The meaning of *emolumentum* remained more or less stable in Latin up to the times of Bernoulli: at least, it was not even included by

Du Cange in his *Glossarium Mediae et Infimae Latinitatis*, first published in 1678 —
Niermeyer (1954), *s.v.*.

²⁹ *Emolument* is found in 18th century French meaning «avantage, profit»: cf. Bloch &
Wartburg (1989), *s.v.*

³⁰ Searching the digital edition of J. P. Migne's *Patrologia Latina* for joint occurrences of
emolumentum and *lucrum*, using proximity operators, within a five-word distance —the
average distance between both terms in Bernoulli (1738). I have only found thirteen, six of
which were clearly synonymous; the phrase *emolumentum lucri* only occurs once.

³¹ A complete summary of legal usage of *emolumentum* in O.Gradenwitz *et al.* (1987).

³² «Neque obstat quod in ludo ille qui perdit nihil recipit *emolumentum* pro sua *pecunia*. Nam, ut
diximus, illud sua sponte dedit pro spe si sibi aleae caderet.» [italics added]