Probabilistic and Geometric Languages in the Context of the Principle of Least Action

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Abstract

This paper explores the question of the unification of the three basic languages of physics, the geometric language of forces, the geometric language of fields or 4-dimensional space-time, and the probabilistic language of quantum mechanics. I will show that on the one hand, equations in each of these languages may be derived from any form of the Principle of Least Action (PLA). On the other hand, Feynman's 'path integral' method could explain the physical sense of these particular forms of PLA. In conclusion, I will show that the axioms of classical and relativistic mechanics become consequences of Feynman's formulation of quantum mechanics.

Keywords: minimal principles, Hamilton's principle, path integral, interpretation quantum mechanics, probability causality.

1 Introduction

We are used to describing each field of nature within the framework of specific branches of science using special methods or languages. Here, under the terms 'method' or 'language' of a branch of science I mean not only a formal set of definitions, axioms, logical rules and mathematical tools. The terms also mean ontology, including a concept of causality and reality[1].

[1]V. Heisenberg wrote about the language of science as a set of concepts, logic and ontology axioms [1]. T. Kuhn speaks about the languages used in science and included many assumptions about nature [2]

It is well accepted that very different languages coexist in physics. They are based on concepts of forces, fields, streams, stability, space-time geometry, statistics, probability, and other concepts. For instance, the motion of macroscopic objects is described in terms of forces or fields. A language of curved space-time is used for the description of cosmological objects. The geometric representation of objects and deterministic causality unite both these languages. However, philosophical foundations of these languages are different. Until the present, many scientists have considered that the probabilistic language of thermodynamics, especially of non-equilibrium, is a statistical approximation of classical mechanics. After the Copenhagen formulation of quantum mechanics (QM), some physicists have accepted that geometry and deterministic causality are not applicable to micro objects. Some of these physicists, like Max Born, are sure that the wave theory must dispose of the means of translating its results into the language of ordinary objects mechanics [3]. All attempts to reduce quantum probability to statistics or to consider the probabilistic description as incomplete have failed. Some interpretations of the physical meaning of QM have appeared [4–9]. However, it remains unclear how to reconcile the classical laws of nature with the impossibility of the deterministic definition of quantum events in time and space. This study attempts to give a partial answer to this question.

R. Feynman believed that every decent physicist-theorist knows six or seven theories that describe of the same physical facts [10]. It is known that the philosophical foundations of such theories often contradict each other. Scientists do not like this at least for aesthetic reasons. Scientific knowledge equals an awareness of connections [11]. These connections are realized within the language of the scientific community, which is, in turn, connected with the dominant paradigm [2]. Thus, mutual understanding is defined by a common language.

In this paper I will describe how to remove some contradictions between geometrical descriptions in terms of forces, fields and 4-dimensional space-time with probabilistic laws of QM. My solution is based on the variational principle — the Principle of Least Action (PLA). The physical and philosophical meaning of this principle may be disclosed by means of Feynman's formulation of QM using the 'path integral' or 'many paths' method.

It is considered that the axioms of classical mechanics, classical field theory, and general relativity are based on the happy guesses of their creators (though that is not the entire truth). In my approach, these theories are the necessary consequences of QM. The equations of the main fields of physics can be represented as special cases of quantum equations. This is because the geometric description of motion in n-dimensional space can be represented as a convenient mathematical approximation of more fundamental probabilistic descriptions.

The concept of PLA was formulated by P. Maupertuis in 1744. Euler gave it a mathematical form, and J. Lagrange, J. D'Alamber, W. Hamilton, K. Gauss, H. Helmholtz, and others took part in the improvement of PLA. Einstein considered that general relativity could be derived from this 'single variational principle' [12]. Planck named it a more universal law of Nature than law of conservation of energy and momentum, so PLA 'dominates above all reversible phenomena of physics' [13]. Eddington wrote about two great generalizations of science: PLA and the second law of thermodynamics [14]. Moore states that 'this principle lies at the core of much of contemporary theoretical physics' [15]. Attention to PLA has not weakened, especially in connection with quantum physics [16–22]. PLA is used also in cosmology [23–27].

To prove my approach, I show that the basic equations of some physical theories are equivalent to one of PLA's forms, which are all equivalent to each other. I will show that each form of PLA could be represented as a special case of Feynman's 'path integral', based on the concept of quantum probability amplitudes.

2 Four methods to describe motion of body and their philosophical foundations

There are four methods to predict the flight path of a body thrown angularly to the horizon (see [28, 29]).

The first method. Newtonian theory says that the body has inertia and is attracted by the Earth with a certain force. The forces of inertia and gravity depend on the body's weight. The actual movement at each moment is a sum of movements caused by these two forces. According to Newton's idea, the body has a mysterious internal 'tendency' of moving straight with constant speed. If the body 'feels' the effect of external forces, it is accelerated. It is assumed that the force effects are felt at a distance (not locally) and depend on the height of the body from the center of the Earth. If body's initial position and vector of velocity are known, we can write an equation to calculate all points of its trajectory. As a result, the actual trajectory is defined as the sum of two virtual trajectories — the horizontal and the vertical.

The second method. If we do not like the 'mystical' effect at a distance, we can describe the same body's flight in terms of field theory. The field is a collection of numbers at each point of space. These numbers, called 'potentials' vary from one point of space to another, and if we put the body at any point of space, we find the force acting on the body in the direction in which the potential decreases most rapidly. In other words, this force is proportional to the speed of the potential decrease or the vector of force is the antigradient of potential energy. The actual body's trajectory is determined by the force at each point in space.

It could seem that the body 'probes' space for virtual trajectories around itself and rushes along trajectories where the potential of the gravitational field is minimum. The faster the potential decreases, the faster the body rushes (proponents of this idea usually say that the force acting on the body is greater). This field formulation allows us to predict the body's flight, if we know what happens in the present moment at each point around it. The clause 'at the present moment' is important, because such virtual 'probes' of space do not take any real time. Unfortunately, without metaphors we cannot explain how the body 'learns' the value of the potential at the neighbouring points.

The third method. Another method of predicting a body's flight path is very different from the first or the second methods, in a philosophical sense. It is not necessary to know what is happening at a close moment in time or at the neighbouring points of space: we only need to know the body's initial and final positions in space and time. The Principle of Least Action (PLA) states that the actual body's trajectory from one point to another in the same time is the one, from all possible movements, for which a functional called 'the action' is minimum or stationary. Hamilton's form of PLA[2] says that along the real body's trajectory the difference between its average kinetic and potential energies reaches a minimum in comparison with all possible trajectories. Differential equations of the body's motion in the gravitational field (Euler-Lagrange equations) could be derived from PLA [31]. Each virtual path of the body corresponds to a certain amount of the action, but only that path is valid for which the action is minimum. Only this way is observed as real and it exactly coincides with the results of the two previous ways. Now we do not need to think about any forces. We also do not need for a fictitious inertial force, because in the absence of a potential field, the body's way with the least action is a straight line with constant speed[3].

[2]This principle arise from the optical-mechanical analogy with Fermat's principle of, by which the light moves along the path, which takes less time. Schrödinger in his Nobel speech showed that only in terms of the wave method of observation do Hamilton's and Fermat's principles open their true value [30].

[3]PLA has an advantage over the principle of conservation of energy and variational principles of mechanics (D'Alamber's, virtual displacements, Gauss', etc.). In one equation, PLA gives the relation between the values of space, time and potential [32].

The fourth method. In general relativity theory, there are no attractive forces and potential fields of gravity. Instead, common geometric space-time is curved under the influence of the Earth's mass. The body moves inertially along a world line (called a geodesic) in a space-time between initial and final events. The form of the geodesic is calculated by an equation for 4-dimensional space-time. For Earth's conditions the result of calculations coincide with results of the previous three methods and the form of the geodesic is accurately described by PLA of classical mechanics [33]. For a free body, the actual world line between two events is the one, from all possible world lines, for which a value of the body's proper time is maximum or stationary. This line is the geodesic. This principle is called Principle of Maximum Proper Time[4]. For weak gravitational fields and low velocities, it is reduced to PLA in Hamilton's form (see the third method).

3 Feynman's formulation of quantum mechanics

In 1942, Richard Feynman [34] used the same ideas as C. Huygens and A. Fresnel, which had formerly inspired E. Schrödinger to the wave equation, and proposed a new formulation of QM. He replaced the classical concept of a particle's motion along a 'single' and unique path by a representation of motion along an infinite set of conceivable paths, which are mathematically described by a functional integral over them. The particle moves simultaneously along all possible paths, each of these associated with a wave of probability or quantum amplitude. The quantum amplitudes of all paths are extinguished at the final point, so that the maximum probability corresponds to the actual path, for which the variation of some functional is zero. Feynman called this functional 'the action' by analogy with classical mechanics, and connected it with the quantum phase of waves of probability [35]. Every possible path of the particle is associated with the phase, and the amplitudes near the actual path are nearly in one phase. Thus, they reinforce each other and generate significant effects, observable as 'real'. Other paths exist too (they are called virtual or imaginable), but they are not observed or, more precisely, their probability to be observed are very small. It could be called as 'probabilistic existence'. The probability of observing them is given by the square of the modulus of the amplitude (wave function). This formulation of QM is mathematically equivalent to Heisenberg's matrix method and to Schrödinger's wave equation [36].

[4]Using the Principle of Maximal Aging, we can study stars and black holes without the tensors and field equations of general relativity (see [23, 24]).

4 Classical body and quantum mechanics

We now need to consider how QM relates to the flight of the classical body. The classical laws are deterministic, they accurately predict a body's behaviour, and, it would seem, they are not connected with the probability of the microcosm. Nevertheless, Feynman concluded that QM is more primary than classical mechanics and general relativity, as far as fundamental laws of physics can be expressed in the form of PLA [37]. Even the relationship between symmetry and the laws of conservation, as articulated in Noether's theorem is based on PLA, which follows from the laws of quantum mechanics [38].

According to Feynman, a classical body, as well as a photon or an electron moves simultaneously along all possible paths or world lines between initial and final events. As the phase of quantum amplitude is very high, a set of world lines, making a significant contribution to the probability of the body's detection, reduces to a narrow bundle. In the limit it contracts to the single world line predicted by the Hamilton's classical form of PLA [17]. It is the same in the third way, above.

What Newtonian physics treats as cause and effect (the force producing acceleration), the quantum 'many paths' view treats as a balance of changes in phase produced by changes in kinetic and potential energy [19]. So classical mechanics and field theory become short-wave approximations of QM, and the action is given the meaning of the phase of quantum amplitude. It is no longer necessary to use the concept of forces acting on the body. It is enough that the body simultaneously 'passes' along all possible paths from one point to another and 'selects' the path, for which the action is minimum [39]. Perhaps the term 'selects' is superfluous in this case, because the classical trajectory is selected not by the body, but by the rule of addition of the quantum amplitudes' phases for the possible paths.

According to E. Taylor's figurative expression [17], a stone moving with nonrelativistic speed in the region of a small space-time curvature obeys nature's command: *Follow the path of least action*! The stone moving with any possible speed in curved space-time, obeys nature's command: *Follow the path of maximum aging (or maximum proper time)*! The electron obeys nature's command: *Explore all paths*! [17]. Taylor proposes a scheme where PLA, on the one hand, is a limiting case of the Principle of Maximal Aging, and on the other hand, a limiting case of Feynman's principle 'Explore all paths'. In other words, Newtonian mechanics becomes a limiting case and an approximation of general relativity and QM at the same time. I suggest extending Taylor's scheme using his metaphors, my additions are indicated by dashed lines (Fig. 1).

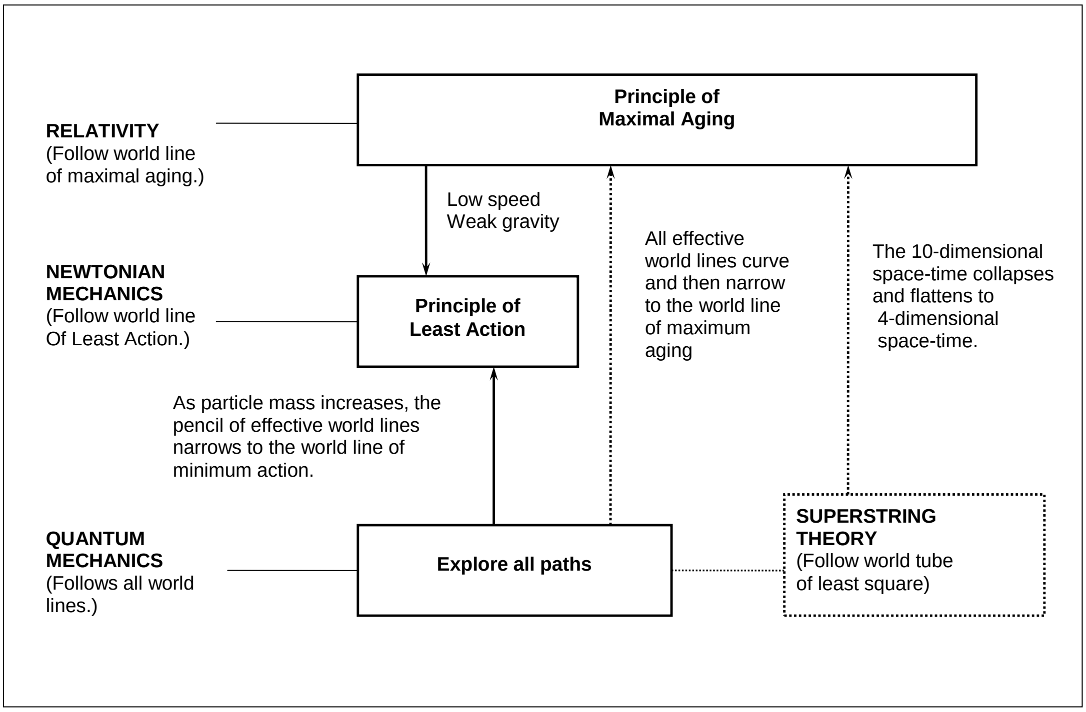

Figure 1: Story line showing the principle of least action sandwiched between relativity and quantum mechanics [17]. In addition relativity becomes a limiting case and an approximation of quantum mechanics.

Firstly, the Principle of Maximum Aging could be considered as the limiting case of Feynman's principle, 'Explore all paths', for strong gravity. Particle mass increases and under the influence of massive objects the pencil of effective world lines for this particle curves. After adding the phases of all their amplitudes, they reduce to the one world line of maximum aging. Secondly, we could apply nature's command, 'Explore all paths', for the space-time of any dimensions and curvature. It is known that the Principle of Maximum Aging is applied only for the smooth 4-dimensional space-time. Using the formalism of QM for such space-time, an infinite and even negative probability inevitably arises. One of the mathematical solutions of this problem is offered by a theory of superstrings, which requires extra dimensions instead [40]. According to this theory, in each point of 4-dimensional space-time, there are six extra collapsed dimensions. If superstring theory is true (there is no evidence of this), we can assume that the Principle of Maximum Aging is also an approximation of the quantum method of 'path integrals'. When the scale increases, the n-dimensional space-time collapses and flattens into a 4-dimensional one[5]. All possible paths of micro-objects are stable only in 4-dimensional space-time, and therefore, all possible paths in 10-dimensional space-time reduce to the paths in 4-dimensional space-time due to interference. Then not only classical mechanics, but also general relativity becomes a limiting case and approximation of QM.

5 Discussion

In the study, we have attempted to combine the languages of physics by means of PLA. The philosophical status of PLA has always been unclear. One of the reasons is the phantom of final cause. The explanation of PLA by the simplicity and perfection of nature, understudied by teleology did not coordinate with any scientific paradigms. Gradually, PLA turned into a pure heuristic rule. Opponents of philosophical interpretations of PLA were J. D'Alamber, J. Lagrange, K. Jacobi, A. Einstein, I. Prigogine, and others. Mach found that variational principles of mechanics are no more than other mathematical formulations of Newtonian laws and that they do not contain anything new. However, Mach added that modern mathematics did not provide any other method to formulate a covariant and at the same time a compatible system of field equations [42].

M. Born wrote that Einstein's law of gravity, which includes Newtonian laws as the limiting case, could also be derived from PLA. Following Mach, Born emphasized that extreme descriptions talk not about properties of nature but about our aspiration for economy of thinking [43]. According to other opinions, PLA not only has methodological meaning, but expresses the unity and interconnection of symmetry and conservation principles and causality [44]. PLA summarizes not only a physical causality, but also the regularity, necessity, probability and connection of states [45]. The law of conservation of energy, as well as other laws of conservation can be derived from the action and variational principles [46, 47]. It is believed that QM in Feynman's form appears as the generalization of classical mechanics [48] and that the application of 'path integrals' provides a clear and elegant language with which to describe the transition from classical to quantum physics [49].

Following L. Euler, J. Lagrange, and W. Hamilton, the creators of QM borrowed an optical-mechanical analogy from geometrical optics (Fermat's and Huygens's prin-

[5]It is assumed that the string moves in space along the world-sheet or world-tube. To calculate the trajectory of its movement, we should minimize the analog path's length – area of tube [41].

ciples) and mechanics – Hamilton's principle [50]. Hilbert and Einstein used the same analogy when they wrote their equations of general relativity [51]. Perhaps methodological convenience is not the sole reason for this analogy. Possibly, the usage of the same analogy in descriptions of two different languages points at their common essence.

General relativity is able to unite Newtonian mechanics and the field theory of object motions in 4-dimensional space-time with any curvature. However, this language is not applied to Planck's scale. Only one language successfully works on all three levels: the language of PLA. At the level of 4-dimensional space-time, Einstein's equations are equivalent to the Principle of Maximum Aging for free particles and to PLA for gravitational fields. At the classical level, Newtonian equations and field equations are equivalent to PLA for free bodies and fields of different types [52]. At the quantum level, the field's equations are equivalent to the 'path integrals' method. The last method, in its turn, explains why PLA works at all levels above. So just this method of Feynman could answer why quantum, classical and relativistic objects obey the same principles.

Physics is an interesting science. The same observed result could be obtained in at least four languages with close mathematical precision. Each of them is based on different logical and philosophical grounds. So which one of them is correct? I think, that this is not the right question and that every language is correct and useful in its own field of nature. We should formulate another goal – to find a method of combining classical and relativistic geometric language with the probabilistic language of QM. I assume that it could be the language based on PLA and Feynman's formulation of QM. It has several advantages, as described below:

1. The basic physical theories can be represented as approximations and limiting cases of this language. We need no concept of 'the force', replacing it with changing phases of quantum amplitudes.

2. This language accounts for the transition from probabilistic to deterministic causality. It is enough to connect the minimal principles with the concept of probability.

3. Results predicted by this language correspond to observations of micro-, macro- and mega- objects, for any speeds and for any dimension's space.

4. This language is based on the simple set of concepts; it has simple and universal mathematical tools - the calculus of variations.

Of course, there are some difficulties with this language. The 'path integral' method has some problems in quantum field theory. It is unexplained why PLA in each field of physics has such very different forms. We do not know what the similarities between all forms of the action are. For classical mechanics, the action is the difference between average kinetic and average potential energy. For general relativity, the action is the proper time. For QM, the action is the probability amplitude. There are other questions as well. Why is the action always extremal? Why is any form of the action invariant concerning transformations of space-time? How is the action connected with energy, space and time?

However, the main problem is philosophical. Our common sense protests against the explanations of the essence of phenomena proposed above. If PLA is not simply a convenient method, and 'path integrals' is not the only useful metaphor, as most physicists think, then how is it possible that everyday objects locate simultaneously at different points of space-time? The scientists say that it happens virtually, but they do not explain what that means. Does it happen, really or not really? The most radical idea of the language based on PLA and 'path integrals' is that not only quantum particles, but also any classical objects 'explore' all possible paths. Because of interference of their possible paths, classical objects are found in the state or at the path corresponding to the minimal action. What does the term 'explore' really mean when applied to inanimate matter? Feynman did not point at any philosophical sense of his method, considering it only a convenient formalism and pointing out its shortcomings [53]. The answer is to accept the logic of QM in Feynman's formulation for explaining classical objects' behavior; we should revise our views on reality and causality. Following V. Heisenberg, V. Fock [54], D. Bohm [5] and K. Popper [55], we should go back to Aristotle's idea about existence as development from possibility into reality, and should recognize classical determinism as the limiting case of probabilistic (not statistical) causality.

6 Conclusion

The overall results show that the method of 'path integrals', created by R. Feynman for QM, is able to justify and explain the physical sense of some forms of PLA. For this it is enough to replace the classical representation of objects' motion along a single and unique trajectory by simultaneous motions along an infinite set of possible trajectories or world lines. These motions are described by Feynman's integral over all trajectories.

PLA of classical mechanics can be derived as an approximation from QM laws for scales much larger than Planck's. At the same time, PLA of classical mechanics is an approximation of general relativity for low speeds and weak gravity. In addition, I assumed that equations of general relativity could be considered approximations of the laws of QM when the intricate multidimensional space-time is collapsing into smooth 4-dimensional space-time. Then the axioms of classical and relativistic mechanics become necessary consequences of QM. As a result, equations of the main fields of physics could be represented as special cases of equations of QM.

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