How to construct Remainder Sets for Paraconsistent Revisions: Preliminary Report

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Abstract

Revision operation is the consistent expansion of a theory by a new belief-representing sentence. We consider that in a paraconsistent setting this desideratum can be accomplished in at least three distinct ways: the output of a revision operation should be either non-trivial or non-contradictory (in general or relative to the new belief). In this paper those distinctions will be explored in the constructive level by showing how the *remainder sets* could be refined, capturing the key concepts of paraconsistency in a dynamical scenario. These are preliminaries results of a wider project on Paraconsistent Belief Change conduced by the authors.

Introduction

In a working group at BRAON'17 (Third Madeira Workshop on Belief Revision, Argumentation, Ontologies, and Norms), the very definition of revision was discussed in the context of an inconsistent-tolerant setting: given the logical possibility of contradictory but non-trivial belief sets (a direct consequence of considering an underlying paraconsistent logic), some authors propose that the revision could be understood as a plain expansion (cf. for instance (Priest 2001; Girard and Tanaka 2016)). The questions adduced by the referred working group were: could it still be rationally and even logically possible for the reasoner to demand from a revision operator a non-contradictory output in a paraconsistent scenario? If so, which definition of revision should be considered? Is it really necessary to equate revision with a plain expansion?

Paraconsistent logics are based on the study of contradictory yet non-trivial theories, exposing a clear distinction between *triviality* and *contradictoriness*. As we understand it, the classical desideratum of consistency, in a paraconsistent setting, splits itself into two distinct ones: non-triviality and non-contradiction. More: since contradictions are distinct, this last concept can be considered with respect to a specific belief-representing sentence (namely, the input). In this paper the relation between those will be constructively explored. We suggest new constructions for remainder sets that fulfill the above desiderata and also circumvent some issues advanced by the literature, as the failure of *extensionality* in general.

On AGM

AGM-style belief revision describes an idealized agent, with a potentially infinite set of belief-representing sentences closed under logical consequence. To express the closure, we are going to use the consequence operator Cn: for a given underlying logic $\mathbf{L}$, $K\vdash_{\mathbf{L}}\alpha$ if and only if $\alpha \in Cn(K)$. Hence the criterion that K is closed under logical consequence can be formally expressed by:

$$K = Cn(K)$$

The agent's dynamics is given by operations that describe the change from one belief set to another. These operations are:

Expansion. An expansion occurs when new information is simply added to the set of the beliefs of an agent. As a result of an expansion, the belief set can become inconsistent. The outcome of an expansion of a belief set K by a sentence α will be denoted by $K + \alpha$.

Contraction. A contraction occurs when information is removed from the set of beliefs of an agent. The result of a contraction of K by a sentence α will be denoted by $K - \alpha$.

Revision: A revision occurs when new information is added to the agent's belief set. When performing a revision some beliefs can be removed in order to ensure consistency. Contrary to expansion, revision preserves consistency (unless the new information is itself inconsistent). The result of a revision of a belief set K by a sentence α will be denoted by $K * \alpha$.

Formally we have the following:

Definition 1. *The expansion of K by α ($K + \alpha$) is given by*

$$K + \alpha = Cn(K \cup \{\alpha\})$$

The explicit construction for contraction adopted is the *partial meet contraction*, constructed as follows (the results of this section are from (Alchourrón, Gärdenfors, and Makinson 1985)):

1. Choose some maximal subsets of K (with respect the inclusion) that do not entail α.
2. Take the intersection of such sets.

The *remainder* of K and α is the set of all maximal subsets of K that do not entail α. Formally the definition is the following:

Definition 2 (Remainder)**.** *The set of all the maximal subsets of K that do not entail α is called the* remainder set *of K by α and is denoted by $K \perp \alpha$, that is, $K' \in K \perp \alpha$ iff:*

(i) $K' \subseteq K$.

(ii) $\alpha \notin Cn(K')$.

(iii) *If $K' \subset K'' \subseteq K$ then $\alpha \in Cn(K'')$.*

Typically $K \perp \alpha$ may contain more than one maximal subset. The main idea constructing a contraction function is to apply a *selection function* γ which intuitively selects the sets in $K \perp \alpha$ containing the beliefs that the agent holds in higher regard (those beliefs that are more entrenched).

Definition 3 (selection function)**.** *A selection function for K is a function γ such that, for every α:*

1. *$\gamma(K \perp \alpha) \subseteq K \perp \alpha$ if $K \perp \alpha \neq \emptyset$.*
2. *$\gamma(K \perp \alpha) = \{K\}$ otherwise.*

The *partial meet contraction* is the intersection of the sets of $K \perp \alpha$ selected by γ.

Definition 4 (partial meet contraction)**.** *Let K be a belief set, and γ a selection function for K. The partial meet contraction on K that is generated by γ is the operation $-_\gamma$ such that for all sentences α:*

$$K -_\gamma \alpha = \bigcap \gamma(K \perp \alpha).$$

By the Levi identity, revision $K * \alpha$ is defined as a prior contraction by $\neg\alpha$ followed by a expansion by α. As it can be easily understood, the prior contraction assures the consistency of the result.

The partial meet revision (the construction for revision defined over the partial meet contraction) is defined as follows.

Definition 5 (partial meet revision)**.** *Let K be a belief set, and γ a selection function for K. The partial meet revision on K that is generated by γ is the operation $*_\gamma$ such that for all sentences α:*

$$K *_\gamma \alpha = \left(\bigcap \gamma(K \perp \neg\alpha)\right) + \alpha$$

An operation $*$ on K is a partial meet revision if and only if there is a selection function γ for K such that for all sentences $\alpha : K * \alpha = K *_\gamma \alpha$.

Partial meet revision is axiomatically characterized as follows:

Observation 6. *The operator $*$ is an operator of partial meet revision for a belief set K if and only if it satisfies the following postulates:*

(K$*$1) $K * \alpha = Cn(K * \alpha)$. **(Closure)**

(K$*$2) $\alpha \in K * \alpha$. **(Success)**

(K$*$3) $K * \alpha \subseteq K + \alpha$. **(Inclusion)**

(K$*$4) *If $K + \alpha$ is consistent, then $K * \alpha = K + \alpha$.* **(Vacuity)**

(K$*$5) *If α is consistent, then $K * \alpha$ is consistent* **(Consistency)**

(K$*$6) *If $Cn(\alpha) = Cn(\beta)$, then $K * \alpha = K * \beta$.* **(Extensionality)**

On Paraconsistent Belief Revision

Some approachs on Paraconsistent Belief can be found, for instance, in (Restall and Slaney 1995), (Chopra and Parikh 1999), (Tamminga 2001), (Priest 2001), (Mares 2002), (Girard and Tanaka 2016) and (Testa, Coniglio, and Ribeiro 2017). A brief overview on some of these inquiries can be found in (Fermé and Hansson 2018). The main objective of this work is to refine some results of the so-called AGMp system, following directly the original AGM model (with suitable adjustments), advanced in (Testa, Coniglio, and Ribeiro 2017). This system is designed over a class of paraconsistent logics called **LFI**s to be further introduced.

Paraconsistent Logics and LFIs

The Logics of Formal Inconsistency (**LFIs**), advanced by (Carnielli and Marcos 2002) and further developed mainly in (Carnielli, Coniglio, and Marcos 2007) are a family of paraconsistent logics that encompasses most of paraconsistent systems with a supraclassical character, where it is possible to re-encode classical reasoning within it (cf. (Carnielli and Coniglio 2016) by a comprehensive textbook).

Roughly speaking, withing LFIs it is possible to express the notions of inconsistency and consistency inside the object language. The sentential unary connective $\circ$ of *formal consistency* is the more frequently used, where the sentence $\circ\alpha$ is intended to formally express the meaning that 'α is consistent'. As a consequence, contradiction does not generate triviality in general, unless the sentence involved is consistent. In formal terms, for any logic $\mathbf{L}$ that is a LFI, denoted by a consequence operator $\vdash_{\mathbf{L}}$, the following does not hold:

Observation 7. *Explosion principle*

$$\alpha, \neg\alpha \vdash_{\mathbf{L}} \beta,$$

but a distinct form of it is always the case:

Observation 8. *Gentle explosion principle*

$$\alpha, \neg\alpha, \circ\alpha \vdash_{\mathbf{L}} \beta.$$

The distinctions given by the LFIs not only separates the notion of contradiction from deductive triviality (like every paraconsistent logic), but also contradiction from inconsistency as well non-contradiction from consistency (in the object language level). So there is a clear distinction between contradictions that can be accepted from those that cannot. The idea to be captured is that no matter the nature of the contradictions a reasoner is willing to accept, there still are contradictions that cannot be accepted at all.

Definition 9. *In order to avoid ambiguity, the following notation is useful (for any logic $\mathbf{L}$ represented by Cn):*

a. *A set A is contradictory if and only if there is a sentence β such that $\{\beta, \neg\beta\} \subseteq Cn(A)$.*

b. *A set A is contradictory with respect to α (or is α-contradictory) if and only if, for some β equivalent to α (that is, for α and β such that $Cn(\alpha) = Cn(\beta)$), $\{\beta, \neg\beta\} \subseteq Cn(A)$.*

c. *A set A is trivial if and only $\perp \in Cn(A)$.*

d. *A sentence α is contradictory if and only if the set $\{\alpha\}$ is contradictory.*

e. *A sentence α is trivial if and only if the set $\{\alpha\}$ is trivial.*

Definition 10. *The most basic* **LFI** *in the family considered is the propositional logic* **mbC**. *The language $\mathbb{L}$ of* **mbC** *is generated by the connectives $\wedge, \vee, \rightarrow, \neg, \circ$.*

Definition 11 (**mbC** (Carnielli and Marcos 2002))**.** *The logic* **mbC** *is defined over the language $\mathbb{L}$ by means of a Hilbert system as follows:*

Axioms:

(A1) $\alpha \rightarrow (\beta \rightarrow \alpha)$
(A2) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow (\beta \rightarrow \delta)) \rightarrow (\alpha \rightarrow \delta))$
(A3) $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$
(A4) $(\alpha \wedge \beta) \rightarrow \alpha$
(A5) $(\alpha \wedge \beta) \rightarrow \beta$
(A6) $\alpha \rightarrow (\alpha \vee \beta)$
(A7) $\beta \rightarrow (\alpha \vee \beta)$
(A8) $(\alpha \rightarrow \delta) \rightarrow ((\beta \rightarrow \delta) \rightarrow ((\alpha \vee \beta) \rightarrow \delta))$
(A9) $\alpha \vee (\alpha \rightarrow \beta)$
(A10) $\alpha \vee \neg\alpha$
(bc1) $\circ\alpha \rightarrow (\alpha \rightarrow (\neg\alpha \rightarrow \beta))$

Inference Rule:

(Modus Ponens) $\alpha, \alpha \rightarrow \beta \vdash \beta$

It is worth of noticing that (A1)-(A9) plus *Modus Ponens* constitutes an axiomatization for the classical positive logic $\mathbf{CPL}^{+}$, so that **mbC** can be understood as a extension of it, adding few constraints on negation and formal consistency by axioms (A10) and (bc1). Further constraints can be given by the axioms of **mbC**'s extensions, for instance: (ciw) $\circ\alpha \vee (\alpha \wedge \neg\alpha)$, (ci) $\neg\circ\alpha \rightarrow (\alpha \wedge \neg\alpha)$, (cl) $\neg(\alpha \wedge \neg\alpha) \rightarrow \circ\alpha$, (cf) $\neg\neg\alpha \rightarrow \alpha$, (ce) $\alpha \rightarrow \neg\neg\alpha$ and (cc) $\circ\circ\alpha$. A detailed taxonomy on **LFI**s can be found on the references.

Regarding implication, recall that deduction holds for *any* propositional logic where (A1) and (A2) can be derived when MP is the unique inference rule.

Observation 12 (deduction meta-theorem (Carnielli, Coniglio, and Marcos 2007))**.** *The* **mbC** *calculus satisfies the following:*
$\Gamma, \alpha \vdash_{mbC} \beta$ *iff* $\Gamma \vdash_{mbC} \alpha \rightarrow \beta$

Regarding paraconsistent negation, note that $\mathbf{CPL}^{+}$plus $\alpha \vee \neg\alpha$ is too weak (as expected). This axiom reflects that the truth-value of α partially determines the truth-value of $\neg\alpha$: if α is false, then $\neg\alpha$ must be true; but if α is true, $\neg\alpha$ can be either true or false. The only axiom that deals with the formal consistency in **mbC** is $\circ\alpha \rightarrow (\alpha \rightarrow (\neg\alpha \rightarrow \beta))$: similarly, if both α and $\neg\alpha$ are true, $\circ\alpha$ must be false.

Definition 13 (Valuations for **mbC** (Carnielli and Coniglio 2016))**.** *A function $v : \mathbb{L} \rightarrow \{0, 1\}$ is a* valuation for **mbC** *if it satisfies the following clauses:*

$(v\wedge)$ $v(\alpha \wedge \beta) = 1 \Leftrightarrow v(\alpha) = 1$ *and* $v(\beta) = 1$. **(Conjunction)**

$(v\vee)$ $v(\alpha \vee \beta) = 1 \Leftrightarrow v(\alpha) = 1$ *or* $v(\beta) = 1$ **(Disjunction)**

$(v \rightarrow)$ $v(\alpha \rightarrow \beta) = 1 \Leftrightarrow v(\alpha) = 0$ *or* $v(\beta) = 1$ **(Implication)**

$(v\neg)$ $v(\neg\alpha) = 0 \Rightarrow v(\alpha) = 1$ **(Paraconsistent/Weak negation)**

$(v\circ)$ $v(\circ\alpha) = 1 \Rightarrow v(\alpha) = 0$ *or* $v(\neg\alpha) = 0$ **(Formal Consistency)**

The semantical consequence relation associated to valuations for **mbC** is defined as expected: $X \models_{\mathbf{mbC}} \alpha$ iff, for every **mbC**-valuation v, if $v(\beta) = 1$ for every $\beta \in X$ then $v(\alpha) = 1$. The following result is well-known:

Observation 14 (Adequacy of **mbC** w.r.t. bivaluations (Carnielli and Coniglio 2016))**.** *For every set of formulas $X \cup \{\alpha\}$: $X \vdash_{\mathbf{mbC}} \alpha$ if and only if $X \models_{\mathbf{mbC}} \alpha$.*

Remark 15. *Despite of the fact that we are considering in this presentation the logic* **mbC** *and extensions, it is worth noticing that the constructions here depends on more general restrictions, so that they can encompass a wider class of logics.*

Remark 16 (derived bottom particle and strong negation)**.** *The* falsum *(or bottom) is defined in* **mbC** *by means of the formula $\bot_\beta =_{def} \beta \wedge \neg\beta \wedge \circ\beta$, for any formula β. From this, the classical (or strong) negation is defined in* **mbC** *by $\sim_\beta\alpha =_{def} (\alpha \rightarrow \bot_\beta)$. Since $\bot_\beta$ and $\bot_{\beta'}$ are interderivable in* **mbC**, *for any β and β', then $\sim_\beta\alpha$ and $\sim_{\beta'}\alpha$ are also interderivable. Hence, the strong negation of α will be denoted simply by $\sim\alpha$. The same applies to $\bot$.*

The following propositions may prove useful for assessment of further results (they can be easily checked by valuations of Definition 13).

Proposition 17 (some properties of **mbC**)**.** *The following hold:*

i. $\bot \vdash \alpha$
ii. $\sim \alpha \vdash \neg\alpha$ *and so* $\vdash \sim \alpha \rightarrow \neg\alpha$
iii. $\circ\alpha \wedge \neg\alpha \vdash \sim \alpha$, *but* $\sim \alpha \nvdash \circ\alpha \wedge \neg\alpha$
iv. $\neg\neg\alpha \nvdash \alpha$

Remark 18. *As usual, $\alpha \leftrightarrow \beta$ is an abbreviation for $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$.*

Proposition 19. *The following hold in* **mbC**:

$\alpha \leftrightarrow \beta \nvdash \neg\alpha \leftrightarrow \neg\beta$

Since a classical negation $\sim$ can be defined in **mbC**, that logic can be understood as an expansion of the classical propositional logic **CPL** by adding a paraconsistent negation $\neg$ and a consistency operator $\circ$ satisfying certain axioms.

In formal terms, consider **CPL** defined over the language $\mathbb{L}_0$ generated by the connectives $\wedge, \vee, \rightarrow, \neg$ (observe that in $\mathbb{L}_0$ $\neg$ represents the classical negation instead of the paraconsistent negation of **mbC**). If $Y \subseteq \mathbb{L}_0$ then $\circ(Y) = \{\circ\alpha : \alpha \in Y\}$. Then, the following result can be obtained:

Observation 20 (Derivability Adjustment Theorem (Carnielli, Coniglio, and Marcos 2007))**.** *Let $X \cup \{\alpha\}$ be a set of formulas in $\mathbb{L}_0$. Then $X \vdash_{\mathbf{CPL}} \alpha$ if and only if $\circ(Y), X \vdash_{\mathbf{mbC}} \alpha$ for some $Y \subseteq \mathbb{L}_0$.*

Remark 21. *From now on, let us assume a* **LFI**, *namely* **L**$=\langle\mathbb{L}, Cn_{\mathbf{L}}\rangle$, *such that* **L** *is an extension of* **mbC**. *Since the context is clear, we will omit the subscript, and simply denote the closure by Cn.*

The AGMp system

Let us assume a non-trivial state K such that $K = Cn(K)$.

Partial meet AGMp revisions In (Testa, Coniglio, and Ribeiro 2017) it is shown that a paraconsistent revision $K*\alpha$ can be defined by Levi identity as in classical AGM, that is, by a prior contraction by $\neg\alpha$ followed by a expansion by α (Definition 5). It is worth of noticing that one of the focus of that paper was showing the possibility of defining *external revision* for paraconsistent closed theories, in the sense of *reverse Levi identity* as defined by Hansson for Belief Bases. For our intends and purposes, this construction will not be taken into consideration – nevertheless, it should be noted that the results here advanced applies when taking into consideration the proper features of that operation.

In terms of postulates, the AGMp internal revision is characterized as the classical operation, but without the *extensionality* postulate, and changing *consistency* by *non-contradiction*. It should be noticed that in (Testa, Coniglio, and Ribeiro 2017) *vacuity* was replaced by *relevance*, since it was proven that both postulates are equivalent in standard, supraclassical and deductive logics). So the following holds:

Observation 22. *(Testa, Coniglio, and Ribeiro 2017) The operator $*$ is an operator of AGMp partial meet internal revision for a belief set K if and only if it satisfies closure, success, inclusion, vacuity and the following:*

(K$*$5') *if α is non-contradictory, then $K * \alpha$ is not contradictory* **(non-contradiction)**

The postulate of *non-contradiction* above is an adaptation of the classical postulate of *consistency*. That's exactly the necessity of still demanding a non-contradictory output for revision operation that will be further discussed. Furthermore, AGMp presupposes that K itself is also non-contradictory – in fact, in order to keep generality (in the sense of taking into account a contradictory belief set as an input), it could be said that (K$*$5') should specifically guarantees that $K * \alpha$ in not α-contradictory.

Extensionality lost The weakness of a paraconsistent negation has the advantages of allowing contradictions. Nevertheless, this same property come with the cost of loosing extensionality in general. By definition 5 and the negative results of propositions 19 and 17(iv), it is easy to check that, given the paraconsistent negation properties, partial meet paraconsistent revision is not extensional.

In order to restore a suitable form of extensionality, some assumptions on the underlying logic should be made, as proposed by (Testa, Coniglio, and Ribeiro 2017). We advance a refinement in the constructions in order to preserve that postulate in weaker LFIs (and other paraconsistent logics).

Non-contradiction *vs.*Triviality It is clear that AGMp assumes that the output of a revision should still be non-contradictory (with respect to the input). Despite that fact, a non-trivial revision was suggested in that paper, defined by the Levi identity applied to the strong negation.

Definition 23.

$$K *_\gamma \alpha = \big(\bigcap \gamma(K \perp \sim \alpha)\big) + \alpha$$

By proposition 17 iii. and definition of *remainder* it is easy to check that this construction assures that the output is not trivial, by retracting $\neg\alpha$ or $\circ\alpha$, as long α itself is non-trivial.

Some Refinements on the Paraconsistent Framework

In order to restore extensionality in the paraconsistent setting, as well to better capture the distinction between *non-contradictoriness* and *triviality*, a new definition of reminder set is advanced. It will be shown that, in classical Belief Revision, this construction is equivalent to the classical reminder set. Furthermore, in the paraconsistent setting, this construction defines a revision where the output is demanded to be non-trivial (denoted by $\breve{*}$), and suitable modifications on it defines revisions in which the output is non-contradictory with respect to the input (denoted by $\bar{*}$), and non-contradictory in general (denoted by $\bar{\bar{*}}$) – those revisions, when the underlying logic is CPL, are proven to be equivalent with the classical one. Those features, as we understand, captures the results of the *Derivability Adjustment Theorem*, advanced in the proposition 20.

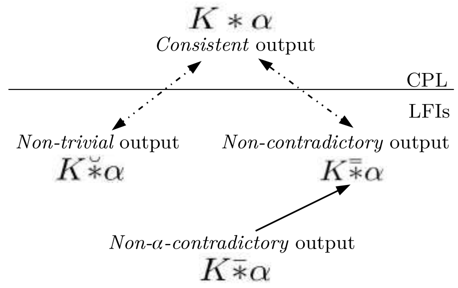

Figure 1: Relation between the revisions

Remainder sets: new constructions to revisions

Recall the definition of *remainder set*: for defining classical revision $K * \alpha$, we want to expand by α the intersection of some maximal subset of K that does not entail $\neg\alpha$ – in logical terms, a remainder for revision is designed to be a "$\neg\alpha$-saturated" subset of K.

Non-trivial remainder We define the *remainder* of K with respect to α as the set of all maximal subsets of K that, when expanded by α, are non-trivial (that is, do not entail $\perp$). This modification goes in the line of the one presented by (Delgrande 2008), advanced for Horn clause contraction function but here understood in the general context of an operation in logics without negation, as suggested by (Ribeiro 2012). We do, of course, have a negation – but given some weak properties of it (like the loss of extensionality as advanced before) the idea is to design the remainder not relaying on that.[1]

[1]There are some authors that discuss what are the necessary and sufficient conditions for a negation to be, effectively, a negation, taking the paraconsistent one as an example. This analysis

Formally the definition is the following:

Definition 24 (non-trivial remainder)**.** *Let K be a belief set, and let α be a formula. A set $X \in K \Downarrow \alpha$ if and only if:*

(i) $X \subseteq K$.
(ii) $X \cup \{\alpha\} \nvdash \bot$.
(iii) *If $X \subset X' \subseteq K$ then $X' \cup \{\alpha\} \vdash \bot$.*

$K \Downarrow \alpha$ is the non-trivial remainder of K with respect to α.

Remark 25. *It is clear that if $X \in K \Downarrow \alpha$, then $\{\neg\alpha, \circ\alpha\} \nsubseteq Cn(X)$, and that $X \cup \{\alpha\} \nvdash \beta \wedge \neg\beta \wedge \circ\beta$ for all β.*

Non-trivial revision A selection function for K is a γ defined as above. The *partial meet non-trivial revision* is, also, the intersection of the sets chosen by the selection function expanded by α.

Definition 26. *Let K be a belief set, and γ a selection function for K. The partial meet non-trivial revision on K that is generated by γ is the operation $\check{*}_\gamma$ such that for all sentences α:*

$$K \check{*}_\gamma \alpha = \bigcap \gamma(K \Downarrow \alpha) + \alpha.$$

An operation $\check{*}$ is a partial meet non-trivial revision if and only if there is a selection function γ for K such that for all sentences $\alpha : K\check{*}\alpha = K\check{*}_\gamma\alpha$

It should be noticed that this operation does not explicitly use the construction of a contraction operator, as it is classically done by AGM – where revision is defined by Levi identity, alluding the contraction by the negated formula (cf. def. 5). Instead, the sentences to be retracted in order to accommodate the new belief-representing sentence are chosen directly by the revision's construction. The same happens with the further revisions to be presented.

Of course contraction could still be defined by a Harper-like identity, but that's not our focus on this paper.

Non-contradictory remainder with respect to the input A less permissive remainder can be defined – relative to contradictions. In a nutshell, it is designed to retract $\neg\beta$ from K, for all β equivalent to the new belief-representing sentence α – re-encoding the characteristics of the classical one, but now relative to a paraconsistent setting (endowed with a weak negation).

Definition 27 (non-contradictory remainder with respect to the input)**.** *Let K be a belief set, and let α be a formula. A set $X \in K{\downarrow_\alpha}\alpha$ if and only if:*

(i) $X \subseteq K$.
(ii) *For all $\beta \equiv \alpha$, $X \cup \{\alpha\} \nvdash \beta \wedge \neg\beta$.*
(iii) *If $X \subset X' \subseteq K$ then there exists a sentence $\beta \equiv \alpha$ such that $X' \cup \{\alpha\} \vdash \beta \wedge \neg\beta$*

$K{\downarrow_\alpha}\alpha$ is the non-α-contradictory remainder of K with respect to α.

Remark 28. *It is clear that if $X \in K{\downarrow_\alpha}\alpha$, then $\{\neg\alpha\} \nsubseteq Cn(X)$. More: $X \nvdash \neg\beta$ for all $\beta \equiv \alpha$.*

is of interest to Belief Revision, since much of the properties carried out by the systems tacitly assumes the properties of a classical negation.

Non-contradictory revision with respect to the input

Definition 29. *Let K be a belief set, and γ a selection function for K. The partial meet non-contradictory revision with respect to α on K that is generated by γ is the operation $\bar{*}_\gamma$ such that for all sentences α:*

$$K \bar{*}_\gamma \alpha = \bigcap \gamma(K{\downarrow_\alpha}\alpha) + \alpha.$$

An operation $\bar{*}$ is a partial meet non-contradictory revision with respect to α if and only if there is a selection function γ for K such that for all sentences $\alpha : K\bar{*}\alpha = K\bar{*}_\gamma\alpha$

Non-contradictory remainder

Definition 30 (non-contradictory remainder)**.** *Let K be a belief set, and let α be a formula. A set $X \in K \downarrow \alpha$ if and only if:*

(i) $X \subseteq K$.
(ii) *For all $\beta \in \mathbb{L}$, $X \cup \{\alpha\} \nvdash \beta \wedge \neg\beta$.*
(iii) *If $X \subset X' \subseteq K$ then there exists a sentence $\beta \in \mathbb{L}$ such that $X' \cup \{\alpha\} \vdash \beta \wedge \neg\beta$*

$K \downarrow \alpha$ is the non-contradictory remainder of K with respect to α.

Remark 31. *It is clear that if $X \in K{\downarrow}\alpha$, then $\{\neg\alpha\} \nsubseteq Cn(X)$. More: $X \nvdash \beta \wedge \neg\beta$ for all $\beta \in \mathbb{L}$.*

Non-contradictory revision

Definition 32. *Let K be a belief set, and γ a selection function for K. The partial meet non-contradictory revision on K that is generated by γ is the operation $\bar{\bar{*}}_\gamma$ such that for all sentences α:*

$$K \bar{\bar{*}}_\gamma \alpha = \bigcap \gamma(K \downarrow \alpha) + \alpha.$$

An operation $\bar{\bar{*}}$ is a partial meet non-contradictory revision if and only if there is a selection function γ for K such that for all sentences $\alpha : K\bar{\bar{*}}\,\alpha = K\bar{\bar{*}}_\gamma\alpha$

Relation between the remainders

Proposition 33. *The following identities hold:*

i. $K \Downarrow \alpha = K\bot(\alpha \to \bot) = K\bot \sim \alpha$
ii. $K\bot\alpha = K \Downarrow (\alpha \to \bot) = K \Downarrow\sim \alpha$

33.i. is an expected result, given definition 23. As it can be perceived, 33.ii. is an intermediate result for further defining contraction via non-trivial remainder.

Proposition 34. *In general, the remainder sets $K\bot\neg\alpha$, $K\Downarrow \alpha$, $K\downarrow \alpha$ and $K\downarrow_\alpha \alpha$ are different from each other.*

This is a predictable feature, since the concepts of triviality and contradiction are distinct in a paraconsistent setting and, moreover, contradictions are distinct to each other.

However, given the fact that in **CPL** all contradictions are alike, and equivalent to $\bot$, it is easy to check the following:

Proposition 35 (The classical collapse of consistency, non-triviality, non-contradictoriness and non-α-contradictoriness)**.** *When the underlying logic is* **CPL***:*

$$K\bot\neg\alpha = K \Downarrow \alpha = K \downarrow \alpha = K \downarrow_\alpha \alpha$$

From construction to postulates

In this section we will present each one of the paraconsistent revision functions through a set of postulates that determine the behavior of each one of these functions – establishing conditions or constrains that they must satisfy, as it is classically done. Through the postulates, the refinement made in the constructive level in order to capture the distinction between non-contradictoriness and triviality can be highlighted. In the paraconsistent setting, the consistency desideratum (classicaly captured by the *consistency* postulate) adduce three distinct new postulates: *non-triviality*, *non-α-contradiction* and *non-contradiction*, capturing respectively the intuition that the revision output should be non-trivial, non-contradictory relative to the new information to be incorporated or non-contradictory in general.

Another important feature of the new constructions advanced in this paper is that the revision of a belief set by logical equivalent sentences produces the same output in general – captured by the postulate of *extensionality*. Recall that this property was not valid in general in paraconsistent systems, as aforementioned.

Non-trivial partial meet revision

Proposition 36. *If $\breve{*}$ is an operator of non-trivial partial meet revision for a belief set K, then it satisfies the following postulates:*

(K$\breve{*}$1) $K\breve{*}\alpha = Cn(K\breve{*}\alpha)$. **(Closure)**

(K$\breve{*}$2) $\alpha \in K\breve{*}\alpha$. **(Success)**

(K$\breve{*}$3) $K\breve{*}\alpha \subseteq K + \alpha$. **(Inclusion)**

(K$\breve{*}$4) *If $K + \alpha$ is non-trivial, then $K\breve{*}\alpha = K + \alpha$.* **(Vacuity)**

(K$\breve{*}$5) *If α in non-trivial, then $K\breve{*}\alpha$ is non-trivial.* **(Non-triviality)**

(K$\breve{*}$6) *If $Cn(\alpha) = Cn(\beta)$, then $K\breve{*}\alpha = K\breve{*}\beta$.* **(Extensionality)**

Example 37. *Let $K = Cn(\{\neg\alpha, \gamma, \neg\gamma\})$. It is clear that K is non-trivial, since $\circ\gamma \notin K$.*
It can be easily checked that $\neg\alpha \in K\breve{*}\alpha$, since $\circ\alpha \notin K$. Furthermore, this operation does not retract γ nor $\neg\gamma$ from K.

Remark 38. *By the very definition of vacuity, it is clear that in weaker paraconsistent logics where there is no primitive or defined formal consistency operator (or, equivalently, where there is no strong negation nor bottom particle), a non-trivial revision is a plain expansion.*

Non-contradictory partial meet revision with respect to the input

Proposition 39. *If $\bar{*}$ is an operator of non-contradictory (with respect to the input) partial meet revision for a belief set K, then it satisfies the following postulates:*

(K$\bar{*}$1) $K\bar{*}\alpha = Cn(K\bar{*}\alpha)$ **(Closure)**

(K$\bar{*}$2) $\alpha \in K\bar{*}\alpha$ **(Success)**

(K$\bar{*}$3) $K\bar{*}\alpha \subseteq K + \alpha$ **(Inclusion)**

(K$\bar{*}$4) *If $K + \alpha$ is non-α-contradictory, then $K\bar{*}\alpha = K + \alpha$* **(Vacuity)**

(K$\bar{*}$5) *If α is non-contradictory, $K\bar{*}\alpha$ is non-α-contradictory* **(Non-α-contradiction)**

(K$\bar{*}$6) *If $Cn(\alpha) = Cn(\beta)$, then $K\bar{*}\alpha = K\bar{*}\beta$* **(Extensionality)**

Example 40. *Let K be the same of Example 37. It can be easily checked that $\neg\alpha \notin K\bar{*}\alpha$, but γ and $\neg\gamma$ are still in K.*

Non-contradictory partial meet revision

Proposition 41. *If $\bar{\bar{*}}$ is an operator of non-contradictory partial meet revision for a belief set K, then it satisfies the following postulates:*

(K$\bar{\bar{*}}$1) $K\bar{\bar{*}}\alpha = Cn(K\bar{\bar{*}}\alpha)$ **(Closure)**

(K$\bar{\bar{*}}$2) $\alpha \in K\bar{\bar{*}}\alpha$ **(Success)**

(K$\bar{\bar{*}}$3) $K\bar{\bar{*}}\alpha \subseteq K + \alpha$ **(Inclusion)**

(K$\bar{\bar{*}}$4) *If $K + \alpha$ is non-contradictory, then $K\bar{\bar{*}}\alpha = K + \alpha$* **(Vacuity)**

(K$\bar{\bar{*}}$5) *If α is non-contradictory, $K\bar{\bar{*}}\alpha$ is non-contradictory* **(Non-contradiction)**

(K$\bar{\bar{*}}$6) *If $Cn(\alpha) = Cn(\beta)$, then $K\bar{\bar{*}}\alpha = K\bar{\bar{*}}\beta$* **(Extensionality)**

Example 42. *Let K be the same of Example 37. It can be easily checked that $\neg\alpha \notin K\bar{\bar{*}}\alpha$. Furthermore, this operation also retracts γ or $\neg\gamma$ from K.*

Final Remarks and future works

In a classical setting, ensuring that the negation of the formula to be incorporated is not in the output is necessary and sufficient condition to keep the output non-trivial and forcibly non-contradictory. In paraconsistent reasoning, however, this condition is not necessary in order to ensure non-triviality (since contradictions do not entail triviality in general) nor sufficient in order to ensure non-contradictoriness (since negation is non-extensional in the sense that logically equivalent formulas do not have equivalent negated formulas in general). That asymmetry gives rise to at least three distinct paraconsistent revisions, entailed by more fine-tuned remainders.

Regarding the questions posed at the introduction of this paper, it is worth of noticing that assuming that paraconsistent revision is equivalent with a plain expansion presupposes that (i) consistency is necessarily equivalent to non-triviality in a paraconsistent setting and, furthermore, (ii) that all paraconsistent logics do not endow a bottom particle (primitive or defined). Both assumptions, as we've shown, are not true.

Recovering the extensionality in general is the first step for defining transitively relational partial meet paraconsistent revisions: by considering transitively relational selection functions γ in the remainder sets and, accordingly, by taking into account the supplementary postulates as originally advanced by classical AGM (providing the respective representation theorems).

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