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MATHEMATICS AND FICTION II: ANALOGY

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1. *Introduction*

The object of this paper is to study the analogy, drawn both positively and negatively, between mathematics and fiction. The analogy is more subtle and interesting than fictionalism, which was discussed in part I. Because analogy is not common coin among philosophers, this particular analogy has been discussed or mentioned for the most part just in terms of specific similarities that writers have noticed and thought worth mentioning without much attention's being paid to the larger picture. I intend with this analogy (looking at others' comparisons) to shed a little light on what is going on in mathematics, how one can understand it a bit other than experientially. This intention is philosophical and the way that I am attempting to accomplish it is also philosophical. I shall conclude my attempt to explain how it is possible and even natural for mathematics and fiction to have the analogy they have, taking it for granted, as argued in part I, that they are not to be identified. To this end I shall discuss philosophers' comparisons, mainly those of Hodes, Resnik, Tharp, and Wagner, who are the writers that seem to me to have written most thoughtfully and sufficiently extensively about fiction in making the comparison and of Körner, whose comparison is different. Whether either of these comparisons or the more general analogy is of permanent philosophical interest will have to be decided by philosophers now that they have had a fuller examination. I shall mention some other writers' reference to fiction, but not all; indeed, I am sure that I have not even found all the comparisons that there are.

In order to study the analogy with fiction, I shall depend upon the view of what mathematics is about that I outlined in part I. I begin with a one-paragraph recapitulation of that view. I think mathematics is about relations rather than objects. I disclaim any originality in making this claim; it is often remarked on but then ignored. In part I, I cited indications of this view from Newton, Poincaré, Russell, Mac Lane, Atiyah, Davis, Isaacson. I think that this subject matter accounts for much that is distinctive about mathematics, for its objectivity, for its comparative success as an intellectual discipline evolving over two and a half millennia and inspiring virtually all others, natural science in particular. Seeing mathematics this way is more a matter of a gestalt switch than of coming to acknowledge something that can be argued for. It is more of an observational premise than a philosophically interesting conclusion; that is perhaps why the persons cited above are mostly mathematicians. In part I, I pointed out that physics abstracts physical relations among physical things for study; when the relations are further abstracted so that they are no longer physical they can be the subject matter of mathematics, for instance, set membership, distance, order, one-to-one correspondence. They can be re-applied to physical things or to anything else once they have been thought about on their own, the way to do which is mathematical by postulating (in the thin mathematical sense) abstract objects to have the mathematical relations: elements in sets, points at distances in metric spaces, and so on. Relations among the relations are studied by reifying relations already studied. Just as the interesting way to engage interest in fictional characters is narrative, the interesting way to engage interest in mathematical objects and their relations is deduction. Narrative is interesting because it tells one what has happened to the persons; deduction in mathematics is interesting because it says what are the implications. In both cases, results. In the two cases the corresponding narratives and deductions are what are important; they are about relations more than they are about the relata. Deduction and narration are different, as I emphasized in part I.

The original comparison with fiction was not so much making the ontologically dominated point that is being made by contemporaries but used 'fictional' as a way of saying 'abstract'. Part I concluded that this Pickwickian use of 'fiction' was not an error, however misleading it may seem if taken out of context. This was an older tradition relaunched by Vaihinger [1924], acknowledging his debt to Bentham and earlier thinkers. His main theme is understanding, and he saw that we understand something by seeing how it is the same as and how it is different from (*both* are instructive) other things. Mathematics is not narrative in form, but one may see something about mathematics by thinking of it as if it were narrative. If one is clear that mathematics is not narrative, then this seeing is a fiction in Vaihinger's sense. 'All cognition is the apperception of one thing through another.' (p. 29) When the other is mathematics, this is mathematical modelling, and we now call the mathematics used a mathematical model. This use of the term 'model' is what Harold T. Hodes, in his paper to be studied below [1984], calls the 'ordinary' sense.

> A model is useful or informative because of what Mary Hesse called a 'positive analogy' between it, a well-understood portion or aspect of reality, and that ill-understood portion of reality which it models. Analogies are analogies in certain, and not in all, respects. To fully

> understand a model one must see 'where' the sustaining positive analogy runs out. (p. 126)

To go farther is to turn 'a benign picture into a false theory' (p. 126). As he does not point out, to attack an analogy because it has limitations is the opposite mistake. If the limitations are too severe, then one discards the analogy because the positive analogy is too small, but the negative just needs to be noted. What one needs is enough correspondences (some similarities, some differences) to be interesting and with enough and important enough similarities (the positive analogy) to outweigh the inevitable differences (the negative analogy). Hodes distinguishes carefully between truth *simpliciter* and its mathematical model, truth in a model which, he says, 'provides a transparent and mathematically tractable model ... of the less tractable notion of truth' (p. 131). One can use the notions of truth in a model and truth in a story even without the absolute notion of truth. In part I, I indicated how mathematics and narrative employ similar devices in their successful discussion of relations by postulating things to have those relations.

> For a discussion of relations in the way that people care about, what one needs is to engage the intelligence and imagination (not primarily or necessarily visual) of the reader with entities related by the relations to be discussed. (Thomas [2000], p. 325)

Despite the similarity of some devices used, the subject matters and other devices prevent narrative and mathematics from being identified as in the fictionalism of Hartry Field ([1980], [1989]), which it was a conclusion of part I to reject.

2. *Make-believe*

There is little question that the fundamental idea of pre-modern or Arabic algebra, that is, the solving of equations using letters as unknowns, is one of the outgrowths of the pretend play that I mentioned back in part I, section 3, Psychology. I do not know how the idea was originally hit upon, but the effective way to teach and use it is to pretend that letters are numerals and work with them on that basis, seeing ultimately which numbers fulfil conditions set out in a problem. But what is pretend play doing at the basis of the third great mathematical theory (after arithmetic and geometry)? The same thing it is doing in *reductio ad absurdum*;[1] it is just one of the very effective ways that we think. There are no limits to pretence; children at Auschwitz

[1] This occurrence of make-believe is drawn attention to in Netz [1999], p. 55. Netz calls on make-believe elsewhere in his soberly historical book, pp. 54 ff., 198, 267. Another recent publication (Sherry [1999]), draws attention to the make-believe aspect of ancient Greek mathematics, calling it idealization and also using the Vaihinger expression, to *see* a

had a game called 'going to the gas chamber'.[2] All I can say to anyone that objects is that he (I'm guessing his sex) should think of the august profession of the law and the frequency with which one thing is deemed to be the case when it is not: the deceased is deemed to have sold all property, which shall be included in the estate at current values. It is mainly fictions such as these that led Jeremy Bentham to consider fictions and their importance. The objector is invited to 'deem' that the letters are numerals *etc.* The children that learn to do it, however, can pretend. We have two sorts of term for a single reality, the dignified term and the undignified term, but pretence is make-believe even if it is called deeming (of which there seems to be no nominal form).

Much discussion of fiction seems to depend on a clear presumption that all non-fiction is deadly accurate, which is of course not true. A great deal of theoretical discourse pertains to entities existing within their respective theories and not necessarily existing in the real world, and inexact statements are made about entities that surely exist. The parallel has been drawn attention to by Gideon Rosen in [1990], though the point I use below appears in David Lewis [1983] and M. Devitt [1980] and is discussed in Kent Bach [1987]. Utterances of 'There is a brilliant detective at 221b Baker Street' are, Rosen says, to be taken as 'elliptical renderings' (p. 331) of 'In the Holmes stories, there is a brilliant detective at 221b Baker Street.' Not only is the latter statement true but also one can believe it without any commitment to believing the former. He calls 'In the Holmes stories' or such a 'story prefix' (Lewis calls it an intensional operator) and says that quantification within the scope of a story prefix is not existentially committing. He further claims that the 'such' of the previous sentence includes things other than fiction in the normal sense, citing 'according to Leibniz's monadology' and mentioning scientific theories and metaphysical speculations. And the story prefix is often implicit without its absence's being deceptive. He points out why Field does not take fiction seriously, with its talk of stories and linguistic entities so offensive to a strict nominalist (p. 338). In the mode of historical writing that helps the reader to engage imaginatively with the characters on the world stage, very much the same happens as in fiction. Characters are drawn (though not usually postulated) and traced through a story — interestingly if it is a good story; whether what has been said is judged by critics to be good history too depends upon how true the story is in the usual more or less absolute sense of true. What the story says, including what follows from the story, does not depend upon what happened in the world but on what

specific 'arrangement of pebbles *as* a group of as many even numbers as we please' (p. 634). Sherry does not mention 'make-believe' by name.

[2] I. A. Opie and P. Opie [1969], p. 331, quoted in Kendall L. Walton [1990], p. 12.

was said in the story. The historical question is whether truth in the story matches truth in the world, a question that can be answered only after the former is determined in accordance with the same canons that one uses for fiction. Even histories need a story prefix, as 'in Gibbon's *Decline and Fall*'. Mathematics, however, does not; the modality of mathematics is instantly recognizable by virtually all adults.

The explanation of grown-up make-believe offered by Kendall Walton [1990] starts where we learn how to do it, with children's pretend play, and in particular with the sort of game that involves props. He has a couple of boys pretend that the stumps in a wood are bears, the point of the props being to make the game not entirely subjective. There is objectivity introduced by the presence or absence of a stump in any particular location, independently of the boys' noticing or failing to notice it. It is 'fictional' in the game that a bear is there or not there depending on the presence or absence of a stump.[3] The game of make-believe to which the reader of a novel is invited has the novel as its main prop providing objectivity for the game and allowing the game — unlike a propless daydream — to be discussed intelligibly with another reader. Daydreams are prototypical examples of subjectivity. Walton attributes to this prop-based objectivity one of the affinities between fictionality and truth (p. 42): 'We can be unaware of fictional truths or mistaken about them as easily as we can about those aspects of the real world on which they depend.' (p. 42) While, as I have already said, there is a good deal more to mathematics than make-believe, make-believe is essentially involved in the mathematical thinking that could not be done better by a computer. The objectivity of mathematics has its basis in shared experience of relations (shared in stories to a large extent), but the function of mathematical texts as props representing its objectivity is sufficiently important to have led some in the past to the formalist fantasy[4] that the text was all there was. The relations that mathematics is about are no more in the text than the objects (infinite in several ways) that many non-formalists considered inadequately represented in finite mathematical texts. But the texts do contribute to the objectivity of the mathematics.

Walton asks whether *The Origin of Species* prescribes imaginings, and his conclusion is negative.

[3] Azzouni [2000] puts the props and rules of chess into parallel with objects of fiction in accounting for mathematical objectivity. But he wants truth not fictionality.

[4] This reductionist stance is still current. See Dales [1998]. Reductionisms can always be maintained by those prepared to ignore what they ignore. Wang ([1986], p. 19) cites Gödel's saying more than once 'How strange (is it) that the positivists (and empiricists) do philosophy by cutting off parts of their brain (in excluding conceptual knowledge)?'

> In writing his book Darwin no doubt intended to get readers to believe certain things. But there is no understanding to the effect that readers are to believe whatever the book says just because it says it. (p. 70)

What Walton means here is that Darwin is not making it fictional that what he says is as he says it. That is perfectly true, but on the other hand readers do have to imagine what he writes in order even to consider it judiciously before believing it on the basis of the evidence that he adduces. Walton uses Darwin as an example of scientific, historical, and mathematical writing as a contrast to fiction. Hayden White [1981b] on the other hand, writing about history suggests that the

> value attached to narrativity in the representation of real events arises out of a desire to have real events display the coherence, integrity, fullness, and closure of an image of life that is and can only be imaginary. The notion that sequences of real events possess the formal attributes of the stories we tell about imaginary events could only have its origin in wishes, daydreams, reveries. (p. 23)

So imaginings are required outside fiction but are not in Walton's sense 'prescribed', prescribed as make-believe, as he makes clearer by adding that 'we cannot conclude that it prescribes imaginings, even if believing involves imagining' (p. 71). If Darwin had been writing a mathematics book, imaginings would be even more important but again would not in Walton's sense be prescribed. For Walton, 'considering or entertaining propositions falls short of imagining them' (p. 71); he is probably right that what we need to do, both in reading fiction and doing mathematics is more than just entertaining the material. Hao Wang, in [1974], doubts that one is likely to get good scientific results just pretending rather than having the kind of commitment characteristic of Gödel (p. 324). On the other hand, van Fraassen pointed out in his review [1975] of Putnam's *Philosophy of Logic* that Abraham Robinson and Paul Cohen are counterexamples.[5] I call it imaginative engagement; it need not involve ontological commitment but it is more than just taking up a propositional attitude (*cf.* Currie [1990], p. 21). 'The secret life of Walter Mitty' is not a proposition. Walton does say that *Gulliver's Travels* warrants 'A war was fought over how to break eggs,' despite giving us no reason to think any such war was ever actually fought. But Swift's world is implicit in such a statement. The intensional operator signalling a story is applicable

[5] In his [1986], Wang acknowledges the exception of Robinson, specifically attributing to him the view that we need only 'pretend that infinite sets exist' (p. 198). The practical usefulness of belief rather than pretence is a purely psychological matter neither implying nor implied by anything about the extra-mental. Dare I suggest a confusion between what is being discussed (quantified over) and what is in the world, the model and the modelled?

here; one can rightly say that 'in Darwin's *Origin of Species* ...', where it is not fictional that ..., but ... *is* what the book says. Creationists would read Darwin as they would read Swift; Walton acknowledges that for such a reader Darwin's book is a representation like a novel. Because *Gulliver's Travels* is a work of fiction, the story prefix can more readily be omitted; if one omits it from biology the reader thinks the work's world is the real world and not *merely* the work's world. Gregory Currie ([1990], p. 1) thinks that 'whether, or in what proportion to be instructed or delighted' by a work depends upon knowing whether it is non-fiction or fiction. I think that part of the point of scientific writing is to convince internally that the work is non-fiction — as Swift convinces us internally that his book is fiction although it is cast in the form of a memoir. Whether I am right for scientific writing is debatable, but a mathematical work *must* be convincing or it does not qualify as mathematics. It does not do this by failing to prescribe imaginings, for example, geometrical constructions in Euclid's *Elements* and since. What we cannot do we *must* imagine.[6]

Perhaps I need to note the distinction between objectivity (which is based on common experience of relations as in science and for which the evidence is intersubjectivity) and verisimilitude, which Popper claimed was the aim of science. Since we do not claim truth for our mathematical premises, we can hardly claim truth for our conclusions, but verisimilitude ought not to elude us; what we are saying is not meant to be false and is always subject to disproof by (mathematical) counterexample. Objectivity is hardly a sufficient condition for verisimilitude, since, as Walton has pointed out, fictional make-believe can be objective.[7] He did not need to point out that it could lack verisimilitude nor that it could have it, as White claims history aims to have. The objectivity-subjectivity division cuts across the fiction-non-fiction division. Mathematics, like fiction, can have verisimilitude without needing to have 'a reality independent of itself to answer to', which Walton rightly considers to be what makes ordinary discourse or thought true ([1990], p. 102). In both cases verisimilitude is a feature of the whole not dependent upon the truth value of individual sentences. Woods [1974] uses bet-sensitivity in the absence of truth to indicate that there are right and wrong answers to questions in fictional contexts, *e.g.*, Sherlock Holmes's street address. I need hardly add that subjective impressions are formed — even needed — on both sides of the non-fiction-fiction divide. Currie mentions an imaginary

[6] *Cf.* the eating of mud pies in Gareth Evans [1982], p. 356.

[7] W. W. Tait [2001] traces the view that the objectivity of mathematics 'concerns, not primarily the existence of objects, but the objectivity of mathematical discourse' to G. Cantor ('Über unendliche, lineare Punktmannigfaltigkeiten, 5', *Mathematische Annalen* 21 (1883), 545–586).

example of a historical fiction that happens by chance to be precisely true ([1990], p. 9); its verisimilitude is perfect without its failing to be fictional. Using the story of Oedipus as an example, C. S. Lewis wrote

> ... we have just had set before the imagination something that has always baffled the intellect; we have seen how destiny and freewill can be combined, even how freewill can be the modus operandi of destiny. The story does what no theorem can quite do. It may not be 'like real life' in the superficial sense, but it sets before us an image of what reality may well be like at some more central region. ([1982], p. 39, quoted in Mary Warnock [1994], p. 97)

Like the objects of mathematics, the characters of fiction do not need to be real for the relations manifested in the mathematics and the fiction to be appropriately applicable to relations in the world. Reading fiction, like knowing mathematics, prepares our imaginations for such application, which for many is the main point of knowing mathematics. Walton calls the analogous why-bother question the 'chief aesthetic question about fiction' (p. 241). With objectivity and verisimilitude assured, interest can shift to questions of centrality, importance, style, and other subjective values. It needs to be emphasized that even so logical a structure in mathematics as a proof has a dimension second to the logical one in which the significance of the entities participating in the proof are revealed; this is analogous to the way a plot, as well as having its chronological function, makes 'significant wholes out of scattered events'.[8]

Having indicated some similarities between mathematics and fiction, let me elaborate a little the gross distinction between the aims of mathematics and fiction.[9] In fiction, an author postulates for the imagination of a reader, the reading progression is temporal both for the reader and for the characters, and inference on the part of the reader is needed but incidental to the imagining of what is going on. The main difference for history is only at the postulation stage; the rest works as in fiction — or perhaps one should say the same for fiction as for veridical reportage. Different readers imagine

[8] Ricoeur [1981]. He credits the idea expressed to Louis O. Mink at p. 174.

[9] I say nothing elsewhere about aesthetics, where there is another marked contrast. Writing stories is hard work; the fun is in reading them. Reading mathematics, on the other hand, is hard work; creating it is where the fun lies. Poincaré expresses this commonly held attitude in writing of his way of reading mathematics, 'I find it more convenient to do proofs over than to examine thoroughly those of the author. My proofs are generally far poorer, but they have for me the advantage that they are mine.' (Letter of Poincaré to G. Mittag-Leffler, 1889 2 5, quoted in Philippe Nabonnand, 'The Poincaré–Mittag-Leffler Relationship', *The Mathematical Intelligencer* 21 (1999), No. 2, 58–64).

differently both internally (one assumes) and in the ways they would verbalize their imaginings. If the story is a myth, legend, or history, both author and reader already know the broad outline of the story's milieu. The view of mathematics I propose is intricately different if utterly distinct. It is exceptional for the author and reader not already to know the broad outline of the mathematical area under discussion; the comparable fiction is more like a myth than a freestanding story like *Robinson Crusoe*. In a piece of mathematics, an author postulates (or takes over standard posits) for the imagination of a reader, the reading progression is temporal for the reader but logical for the matter discussed, imagining what is going on is necessary on the part of the reader but subserves the understanding of the logical relations being expounded. The relation between imagination and inference is opposite in the two cases; imagination for the sake of inference for mathematics, inference for the sake of imagining in fiction. Different readers may imagine mathematics differently internally, but not usually in how they would verbalize their conclusions, partly because, unlike the imaginings that are the aim of a reader of narrative, they have been provided in verbal form by the author as proved conclusions. This is what narrative does not do directly; only the bald facts are so delivered: Claudius, Gertrude, Hamlet, Laertes, Ophelia, and Polonius are all dead. In history, time has passed; all conclusions are provisional and arbitrary. The conclusion is not the point of a narrative, even of a murder mystery. C. S. Lewis argues that the point of 'Jack and the Beanstalk' is the fear of the monstrous, something that cannot even be said, only communicated — in this instance as fear of giants.[10] On history, Hayden White attributes to Louis O. Mink the view that

> narrative has the power to teach what it means to be *moral* beings (rather than machines endowed with consciousness) more or less capable and shrewd enough to carry out our intentions as we conceive them. ([1981a] p. 253)

Searle, agreeing with Walton as quoted above, gives a reason why fiction matters.

> Part of the answer would have to do with the crucial role, usually underestimated, that imagination plays in human life, and the equally crucial role that shared products of the imagination play in human social life.
>
> . . . Almost any important work of fiction conveys a 'message' or 'messages' which are conveyed *by* the text but are not *in* the text. Only in such children's stories as contain the concluding 'and the moral of the story is . . . ' or in tiresomely didactic authors such as Tolstoy do we get an explicit representation of the serious speech

[10] [1982] cited in Mary Warnock [1994], p. 88.

> acts which it is the point (or the main point) of the fictional text to convey. ([1979], p. 74)

He seems to think that one can put the point into words. A thorough consideration of what is accomplished by stories would leave the traditional province of epistemology and enter the more nebulous realm of significance, not to mention the creation of meaning (*cf.* Ricoeur's second dimension of narrative referred to in the previous paragraph). Even in these realms there are correspondences with mathematics, where significance is not something amenable to proof but emerges in proofs. The law is different from both history and fiction — and certainly also from mathematics, but the verdict at a trial is more like the verification of a theorem than the moral or end of a story. Different persons can get different benefits from knowing and even believing a story without necessarily accepting or even knowing all logical consequences that might be drawn by others. I can easily agree with both Resnik and his fellow structuralist Stewart Shapiro, who, while less concerned with fiction than Resnik, writes that 'there is nothing fictional about the ordinary language of arithmetic' ([1997], p. 125).

3. *General Comparison*

Various authors have used fictional devices in writing about mathematics, leaving aside Turing machines. Two books have used the device of fictional characters instead of fictional mathematical objects. Philip Kitcher, in explaining his view of mathematics as being ultimately about the world and depending upon humans' activity in the world — from which they idealize[11] the beginning of their mathematics, had occasion to describe idealization of human activity.

> Arithmetic owes its truth not to the actual operations of actual human agents, but to the ideal operations performed by ideal agents. I construe arithmetic as an *idealizing theory:* the relation between arithmetic and the actual operations of human agents parallels that between the laws of ideal gases and the actual gases which exist in our world. We may personify the idealization, by thinking of arithmetic as describing the constructive output of an ideal subject, whose status as an ideal subject resides in her freedom from certain accidental limitations imposed on us. (p. 109)

[11] He says that this makes statements of 'arithmetic, like statements of ideal gas theory, turn out to be vacuously true' ([1984], p. 117, n. 18). This interpretation makes arithmetic fiction in Field's sense, which is not an adequate view of either the arithmetic or the fiction, as the analogy of ideal gas theory confirms.

While the expression of this view has some connection with intuitionism and constructivism, as Kitcher notes, he does not import those methodological or epistemological views. What we can do in mathematics can be put in terms of 'what powers should be given to the ideal constructive subject' (p. 110). It is not clear why this subject is thought of as personal rather than as an automaton. Kitcher *is* clear that this is just a manner of speaking.

> In regarding mathematics as an idealizing theory of our actual operations, I shall sometimes talk about the ideal operations of an ideal subject. That is not to suppose that there *is* a mysterious being with superhuman powers. (p. 110)

It just lets him tell the story and avoid the two opposite errors that he wishes to steer between, mathematics as 'the investigation of the consequences of arbitrary stipulations' (p. 160) and as description of 'Platonic objects, structures, operations' that are in the world (p. 161). The aim seems to be much the same as mine of idealizing relations among ordinary things in the world.

The other author to use much the same device — but more — is Brian Rotman. If mathematics can be constructed, then it can be deconstructed ([1993], pp. 34 f.). This book seems to be the first such serious study of mathematics. Rotman points out that in mathematics, imperatives are addressed to a third party, the *agent*, who is neither the author nor the reader but executes whatever arithmetic or geometric feats (p. 73) are required. The agent is a more developed version of Kitcher's ideal subject, 'a wholly mechanical and formal proxy for the Subject' (p. 76), a fictional (because capable of infinite actions) computer. In an earlier work he draws a very specific comparison between proofs and narratives, expressed in what I assume to be metaphorical form:

> Presented with a new proof or argument, the first question the mathematician . . . is likely to raise concerns 'motivation': he will in his attempt to understand the argument — that is, follow and be convinced by it — seek *the idea behind the proof.* He will ask for the story that is being told, the narrative through which the thought experiment or argument is organized. [It may be unknown or obvious.] . . . Nevertheless a leading principle is always present — acknowledged or not — and attempts to read proofs in the absence of their underlying narratives are unlikely to result in the experience of felt necessity, persuasion, and conviction that proofs are intended to produce, and without which they fail to *be* proofs. (p. 18 of [2000])

Roberto Torretti, in a paper read at a meeting before the publication of Field's [1980], claims as unique Bunge's combination of 'staunch realist in the philosophy of physics' ([1981], p. 400) and fictionalist in philosophy of mathematics, 'though Henri Poincaré may have been groping after it' (p. 400). He mentions, only to reject them as too far 'removed from the reality of mathematics' (p. 407), Wittgenstein's pronouncements in favor of fictionalism in the *Bemerkungen über die Grundlagen der Mathematik*. The paper attempts a taxonomy of fictionalism somewhat too soon for it to be effective; one category includes physical idealization. But it also contains a discussion of application in a fictionalist context that touches upon issues much more often ignored. He assumes physical realism and says that mathematics is applied by the theorems' antecedents' acquiring 'referents of which they, and hence also the respective consequents, are true' (p. 408). An odd metaphor, acquiring. Applying mathematics is a human intellectual process, not something that happens to things in the physical world. Moreover, it takes place by our selectively applying a whole model, as Quine has pointed out, to a whole physical (let us say physical for the sake of argument despite mathematics' being applied to much else) system, not just a specific theorem. To take the classic example of application, Euclidean geometry to mechanics, one is using an infinite space, which Newton thought justified but we do not (part of the negative analogy). This means that not one of the theorems of Euclidean geometry exactly applies in physical space, but that fact does not deter us nor invalidate the merely approximate conclusions drawn. Torretti claims that 'the statements of applied mathematics must be regarded as fully interpreted, at least in so far as they are concerned with reality' (pp. 408 f.). Presumably he requires that physical space be infinitely divisible, something that is true of Euclidean space (more negative analogy). Detailed examination of Torretti's examples would take too long here.[12]

From these three examples of authors that have made some use of fiction, we turn to the most limited form of the comparison, the merely ontological point, typically made for its epistemological significance, that there is no more ontological commitment to objects of mathematics than to characters of fiction. This point, which has now been made from time to time for over thirty years is in reaction to the analytic view that we can refer to and reason about only what is in some sense real. Gareth Evans is responsible for a forthright expression of the idea, which he attributes to Frege, 'that there can [not] be a way of thinking about something unless there is something to be thought about in that way' ([1982], p. 22). This is not the place to carry on the argument against that axiom, as Avrum Stoll [1998] has recently called it on account of its lacking 'arguments, reasons, or evidence'. I simply mention that it has been widely ignored particularly by analytic philosophers in the science-fiction examples of which many are so fond and refer to the growing literature: Stephan Körner ([1966] and [1967]), John Woods ([1969] and

[12] He deserves some credit for discussing in the philosophy-of-mathematics literature the application difficulties that are discussed in philosophy of science but are usually ignored in discussion of mathematics. Recent exceptions are Otavio Bueno [1997] and Mark Steiner [1998].

[1974]), Peter van Inwagen [1977], Richard Routley [1980], Terence Parsons [1980], Albert Menne [1982], Ed Zalta [1983], Howard Margolis [1987], Tharp ([1989] and [1991]), Charles Crittenden [1991], Linsky and Zalta [1995], Mark Balaguer ([1996] and [1998]), Eddy Zemach [1998], Sarah Hoffman [1999], Mark Sainsbury ([1998] and [1999]), Ken Akiba [2000]. The problem addressed in much of this literature is that of reference to what is *known* not to exist, a problem much worse than that faced in discussions of mathematics, where one is normally happy to assume the existence of what one discusses, at least for the sake of argument. Only in the worst case is one doing so for the sake of a *reductio* showing that the entities assumed do *not* exist, *e.g.*, the integers $m, n,$ such that m/n is in lowest terms and equals the square root of two. That anything should depend upon the sincerity of such an assumption I find amusing. A similar medieval prejudice broke down by the sixteenth century.[13]

The earliest of the merely ontological comparisons to literary fiction I know of is Stephan Körner's in his contribution to Lakatos's 1966 conference, where he mentioned the world of Dickens (his novels, not his period) and said that 'mathematical theories carry no heavier ontological burdens than do works of literary imagination' ([1967], p. 137).[14]

A reason not to air this matter fully here is that it is a matter of empty singular terms. Fiction is only used as an example of our facility with such, sometimes with little care's being given to the fiction. (Zalta's treatment of stories, for example, completely eliminates the narrative element in favour of what others call 'the world of the story', and Zemach's remarks on pretence are strikingly inadequate.) Mathematics may or may not be another example; nothing much is said about mathematics with such comparisons except to reveal the ontological commitments of their authors.

The quotation at the beginning of section 8 Mathematics not Fiction in part I from Leslie Tharp [1989] (repeated here) would be typical if it were not elaborated as we shall see in section 6.

> The comparison [of mathematics to fiction] is not intended in any pejorative sense whatsoever. Rather, we wish to focus attention on the technical fact that myth and other fiction frequently operate with meaningful everyday concepts, but without objects. In fiction one has all along been using ordinary logical forms and inferences in contexts where no objects are referred to. ([1989], p. 167)

[13] So Rivka Feldhay [2000], pp. 56–63.

[14] What he means is elaborated somewhat in his [1966]. The comparison is now commonplace; Brown [1999] calls Hilbert's attribution of 'existence' to whatever is consistent 'innocuous; it's a kind of fictional existence' (p. 100).

His point is that the logical forms of fiction are taken over from everyday conversation, and fiction is full of inferences as daily life is full of inferences, despite fiction's reference to objects outside of daily life. This illustrates that such inferences are possible, have been done virtually forever, and can be done in mathematics as well. And they are; even if mathematical objects do exist, their existence is not relevant to our production of mathematics any more than the population of what Routley/Sylvan called 'Meinong's jungle' need to be seen to be believed or at least discussed. Shakespeare never met Hamlet. As Tharp puts it, our talk of mathematical objects 'is only a manner of speaking about the concepts' ([1989], p. 167) — with which we denote them. Since I do not wish to pass judgement on the existence question, I shall use the more neutral 'denote' when I do not know whether what is denoted is real or not.

On the one hand we have the locution 'Let X be a separable Hilbert space', and on the other hand we have 'Once upon a time there were four little rabbits.' (*Peter Rabbit*) John Searle writes of non-deceptive pretending or 'imitating the making of an assertion' ([1979], p. 65). The meaning of the assertion is not changed thereby ([1979], p. 66); 'telling stories really is a separate language game' ([1979], p. 67). Searle's description of theatrical acting is interesting because of being done in terms of speech acts. When he says that an actor pretends (not deceptively) to be the character, all can agree; when he says that the actor also pretends to perform the character's speech acts, one is tempted to disagree, but, recalling that the speech act is the reality behind / below / in the speech, one has to agree that, while the speech is *really* delivered, the speech act accomplished thereby (think of a death sentence) is *not* thereby accomplished. Searle has the playwright writing a recipe for the pretence of the actors, 'rather than engaging in a form of pretense itself' (p. 69, an indication that his description of a novelist's engaging in pretence was wrong, being inconsistent with the playwright's entirely parallel actions). I think that on plays Searle has it about right: 'a play as performed is not a pretended *representation* of a state of affairs but the pretended state of affairs itself, the actors pretend *to be* the characters' (p. 69). A novelist in parallel is giving directions to a reader for imagining states of affairs; perhaps novel writing is parasitic on play writing (certainly the timing is right). The overall analogy suggests then that proof *writing* might be parasitic on oral showing of the truth of assertions, as it undoubtedly is, two cases of coming to say what was previously only shown, there being no reason why such a boundary should be immovable.

One does not pretend to refer to a real Sherlock Holmes; one really refers to the fictional Sherlock Holmes (Searle [1979], p. 72), 'a non-entity who is a somebody' (Woods [1974], p. 29). (A CBC radio news report on 1999 8 2 said that the Abbey National Building Society at Holmes's fictional address receives up to forty letters to him each week.) Searle regards the distinctions between 'serious' and 'fictional' discourse and between both of them and 'serious discourse about fiction' as helping 'us to solve some of the traditional puzzles about the ontology of a work of fiction' (p. 70). In serious discourse about the world, Sherlock Holmes does not exist and references to him are empty. In fictional discourse, if Sherlock Holmes is a character in a story, then Holmes is said to exist for what I have called in the mathematical context 'the sake of argument', and references to him are not empty. In serious discourse about a Holmes story, Holmes exists as above and references to him are not empty but are context sensitive; a sentence taken out of context and thereby changed to being an ordinary serious sentence would turn into an error in just the way modular-arithmetic calculations become errors when they are seen out of context by a non-mathematician and mistaken for ordinary arithmetic.[15] Searle then asks how it is 'possible for an author to "create" fictional characters out of thin air' (p. 71). Fortunately this question need not be answered for the analogy to be informative (*cf.* Peter Caws and his co-intentional objects).

4. *Postulation — Hodes*

Hodes [1984] takes a view of fictional objects, which he regards as a 'natural, harmless — at least when one is not doing philosophy — and even helpful' (p. 125) model for mathematical objects, that involves 'pretending to posit' (p. 126) them. He calls this 'accepting the mathematical-object picture', calling it 'Wittgensteinese' (p. 126).[16] One is, sure enough, pretending that the rules of the game are true of these objects, but to do this surely one is *really* positing them, not pretending to do so. It seems to me that postulation is pretending that things are real for the sake of argument (the conversational sense) or affirming that they are real (the philosophical sense). What would it mean to pretend to pretend or pretend to affirm that they are real? I think that Hodes must be taken to mean that to postulate conversationally pretends to postulate philosophically; that makes sense although it does not seem right. The action, as ours, has to be independent of whether the posits themselves

[15] Gareth Evans agrees; 'serious discussion of "what went on in the novel" or "what went on in the play" also involves pretence' ([1982], p. 364).

[16] Hodes makes quite clear in his [1990] that the mathematical-object theory, to transform 'this picture into a theory of the alethic underpinnings of mathematical discourse' is a 'natural error; but an error nonetheless' (p. 254). In this paper he replaces the name for his view used in 1984, 'coding-fictionalism' with the undescriptive 'alternative theory', 'some closed singular terms, including those that are properly mathematical, do a sort of semantic work that is not designation.' (p. 237)

are real in whatever sense, since the positing is one's own activity not the object of one's activity. In fiction, the characters are postulated in a less explicit way, but just as clearly and in the merely conversational sense. No one is deceived by its opening sentence into thinking that *Peter Rabbit* is a biography or an animal-breeding tract. 'These are the "things" we are going to talk about now' is precisely what the opening sentence of a story or the *dramatis personae* of a play announces. There is no pretence on the part of the writer, but there is an invitation to the reader/spectator to pretend, to imagine the action of the story as the writer has presumably imagined it.[17] This is the mathematical/narrative way of discussing relations interestingly. Nominalizing relations in order to refer to them with 'definite referring expressions' (as Searle would call them ([1969], p. 26)) so restricts what one can say about them as to produce syntactic paralysis on the part of a writer and terminal boredom on the part of a reader. (Of course we use singular and plural definite and indefinite referring expressions for the objects we postulate to talk about them.) This is what philosophy typically does with relations. One can think of collinearity and non-collinearity all day to no purpose, but as soon as one thinks of three points that *are* collinear one has order to think of or that *are not* collinear one has a triangle to be thinking about. Each is an interesting relation but only if something *has* it. Kant comes close to seeing this (though for objects themselves) when he says that philosophy 'must always consider the universal *in abstracto*' whereas mathematics 'can consider the universal *in concreto*' because of the way it operates (A 735/B 763). Kant regarded intuition as much more important than has been fashionable since Poincaré; but just because one no longer appeals to or depends upon intuition is not to say that we get on without it. Frege made clear the importance for his view of relations at *Grundlagen* §70[18] in spite of having to define them, incomplete as they are unless completed by objects.

What does one gain by this postulation? According to Hodes

[17] This is not the place to deal with the issue of who is pretending. It seems to me that a storyteller, known to be such by the audience, is not independently pretending but is actually non-deceptively telling a story, and that the audience are listening to a story. The pretence is of the reality of the persons or places in the story or of the events described if it is they that are not real. This is the shared pretence between storyteller and audience, an important social practice. Independent pretence on the part of the audience is cynicism, as in listening to the virtues of a used car, and on the part of the storyteller (what Lewis says is the case in [1983] and takes back in a postscript on account of Walton's arguments) is usually lying, sometimes only irony (Paul Grice [1989], p. 54). Evans [1982] on p. 30 discusses Frege's being muddled about this in *Posthumous Writings* at p. 130 and takes up what I call the lying position on p. 353, clarifying that he distinguishes storytelling from lying in n. 31 on p. 359.

[18] P. 82 of Austin's translation; *cf.* Michael Dummett [1991], p. 39, and Frege's *Begriffsschrift* §9, quoted there.

> Higher-order logic is notationally messy and logically complex. For purposes of everyday life, and even for advanced research in pure number theory, there is no need to express arithmetic propositions in a notation that exhibits the higher-order nature of the thoughts involved. Such a 'coding device' loses nothing (except philosophical confusion) and gains much.
>
> ... Philosophical rigor does not require that we abandon these advantages to first-order mathematical discourse, but only that we see it right. (pp. 144 f.)

If one asks which of these, ordinary arithmetic or higher-order logic, is actually mathematics, only prejudice will lead to the logic. It is the arithmetic that mathematicians do, whatever may be the 'nature' of what they are doing or — more obscurely — 'the thoughts involved'. In his later [1990] he is clear that it is the lower-level 'encoding of higher-order logics' (p. 254) that is mathematics.

Hodes draws out a bit the negative analogy between storytelling and mathematics, observing that a mathematician is constrained by logic (third-order logic, he says) in a way that a storyteller, 'who does not make genuine assertions or even express propositions with truth values (for the most part)' (p. 145) is not. A mathematician does not

> pretend to make assertions, he makes primary assertions indirectly (in Searle's terminology[19]), by pretending to make secondary assertions 'about' fictions.
>
> Pressing this disanalogy further, it might be urged that the critic or literary historian does (or should) express propositions with truth values; but the propositions expressed will have their truth values contingently, depending on the whims of those who created the fictions under discussion. Mathematical assertions, on the other hand, are not dependent on the whims of the ur-mathematician. (pp. 145 f.)

Hodes does not find this disanalogy persuasive. He attacks the 'whim' notion, saying that Shakespeare could not have invented a radically different plot involving the very character of Hamlet, 'a prince who comes to believe that his uncle has murdered his father'. Another attack would be that the exercise of whim is historical by the time a theatregoer sees *Hamlet*; one can only write a new play, not change the old play and make it be Shakespeare's *Hamlet*. Still another is that the mathematician has some room for exercise of whim too; logic is not so constraining that it decides, for instance, how many dimensions a geometry has.

[19] 'Indirect Speech Acts' in Searle [1969]. I disagree with his interpretation of Searle's term.

In his [1990] Hodes rejects the identification of mathematical and fictional objects, despite his calling the former 'second-rate' and 'not among "the furniture of the universe" ' (p. 235). But he notes that 'mathematical discourse involves genuine assertion.' (p. 255)

> But metafictional discourse is different [I take it from fictional discourse]: attributions of fictional content (e.g., 'Hamlet was Danish', or even 'Hamlet existed'), construed as if prefixed with 'According to Shakespeare's *Hamlet*', can have truth-values; some singular terms in them (e.g., 'Hamlet' in the above example) contribute to determining that truth-value, but do so without designating anything. (p. 255)

5. *Postulation — Resnik*

In the chapter 'Positing Mathematical Objects' of his book [1997], to which I have already referred, Michael Resnik considers fiction because his account of the postulation of mathematical objects looks so much *to him* like the creation of fictional characters. On account of its being a discussion by someone that does not want to claim any similarity with fiction, it is worth looking at in some detail. 'The basic idea is that humans brought mathematical objects into their ken by positing them.' (p. 175) These are mathematical objects that exist timelessly regardless of having been posited; 'we do not create mathematical objects by postulating them' (p. 188). What he means by 'posit' them is 'to introduce discourse about them and to affirm their existence' (p. 185), as I quoted piecemeal in part I. He discusses three problems among others, how 'mathematicians came to believe in new types of mathematical objects' (p. 176) compatibly with realism, how mathematics is distinguished from fiction, and 'how in positing mathematical objects we manage to refer to them' (p. 175).

It is good to see that there is some common basis for the discussion:

> Positing mathematical objects involves nothing more mysterious than the ability to write novels, invent myths, or theorize about unobservable influences on the observable world.' (pp. 184 f.)

Mathematical posits, like Resnik's physical examples, the Ether and phlogiston, can be tentative at first, as he points out; being tentative is more important on the physical or philosophical meaning of posit than on the conversational.

> It takes very little to justify this sort of tentative positing. The only major worry is that it be a waste of time. (p. 185)

Protomathematical[20] work usually guarantees that the time will not be wasted, for one usually knows useful results before abstracting the mathematical relations from real or lower-level mathematical ones. Resnik points this out but not that it is the relations among the lower-level objects that the abstraction is needed to discuss. What we are really thin on is justification for the step from tentative philosophical postulation (which might as well be conversational) to philosophical postulating 'decisively' (p. 185). I take it to be still unjustified in the case of mathematical objects and an empirical question in the case of a physical posit. He writes:

> Furthermore [on account of protomathematical considerations, ancient mathematicians] had reason to believe that the new theoretical framework would allow them to simplify, unify, and extend the mathematical principles they had already developed, tested, and applied. (p. 185)

He mentions that this step of abstraction is what Kitcher would call a 'rational interpractice transition' ([1984], pp. 225 f.) and blames on Kitcher's approach

> the worry that while it certainly was scientifically rational for mathematicians to *introduce* and *promote* the new practice, it may not have been rational for them to *believe* in the new objects. This is because they may have had ample evidence for the utility of their new theories but little evidence for their truth. ([1997], p. 186)

Note that using the conversational meaning for posit dissolves the worry. Evidence for the dependability of mathematical theories accumulated quickly, but there has never been any acknowledged evidence that their objects exist any more than ghosts, the third of Resnik's physical examples. There are isolated reports of intuitings.

Second, the distinction from fiction. There is no question that positing in the conversational sense is what is done to create fictional persons, places, and things in much fiction. One cannot generalize strictly about fiction as I shall quote Tharp's writing in the next section. Since the construction of stories is constitutive of human thinking in a rather low mode[21] and mathematics is a main subject matter of a higher mode, it seems likely that in the course of intellectual evolution stories came before mathematics. Fiction and mathematics both require pretending. As Resnik says, his speculations about how the latter was begun

[20] *Sc.* pre-axiomatic, informal. This is a term I have previously misattributed. It was first used, so far as I know, by Philip Kitcher [1984], p. 117.

[21] This is a term of Margaret Donaldson's [1993] discussed in part I.

> do presuppose that before the ancients ever posited mathematical objects they had already developed the ability to communicate in written languages, to use pictures, diagrams, and words to represent things that are absent or merely imagined, to speculate, and, finally, to hypothesize and theorize about new kinds of entities. (p. 182)

His point in mentioning this presupposition is that 'none of these abilities involve supernatural processes', which is how he characterizes Frege's 'grasping thoughts' (p. 182). The positing itself requires no new skill, as pointed out in the quotation from his pp. 184 f. near the beginning of this discussion. After the considerations involving 'ghosts, the Ether, and phlogiston', he goes on.

> Positing mathematical objects probably produced significant changes in ancient mathematical practice and hastened the arrival of the mathematical method as we now know it. For the nature of the new objects meant that reasoning from postulates governing them would play a much more authoritative role than perceptual verifications. (pp. 185 f.)

Now, as he mentioned on p. 180, positing can be implicit or explicit. Numbers were surely posited implicitly, since no postulates from ancient times have come down to us or are referred to. The extension of 'number' was even enlarged gradually to include one and much later zero. Nevertheless, they were reasoned about as objects, their relations being the whole concern with them. Geometry, as we know, was more explicitly organized if not in an existential way. But I must object to the phrase 'the nature of the new objects' because their nature was not determined, discussed, or used. Indeed it is still widely disputed whether they have a nature or whether, if any, it is relevant to mathematics, not least by structuralists like Resnik. The arithmetic postulates were implicit, amounting to arithmetic, and the geometric, while some were explicit, referred only to relations among the objects, not to their existence or nature. This is an immediate and large difference from fiction, where persons, places, times, and things are posited *as* persons, places, times and things as well as being in certain specific relations that form the focus of interest. Their natures as persons, places, times and things are implicit in the story and bring unbidden an enormous amount of presupposition that is lacking in mathematical postulation where ideally (an ideal not easily or quickly achieved) only the focus-of-interest relations are available for reasoning. Hence the so-called 'incompleteness of mathematical objects' (p. 194), something that is less true of fictional objects and something Resnik flags as needing explanation rather than abolition. The matter of nature, though it may have been introduced by a mere slip of the pen, is of some importance. The matter of incompleteness is analogous in mathematics and fiction (and, Woods [1969] claimed,[22] to historical characters like Julius Caesar).

The presuppositions of fiction (corresponding to the axioms of a piece of mathematics) are implicit, more like those of protomathematics. There are certainly not the same constraints as on mathematics. Mathematical posits are more like provincial try-outs, subject to modification if they do not work out the way that was intended, 'hypotheses that we are prepared to modify or withdraw in the face of evidence that they are inconsistent, have unwanted models, fail to yield the consequences we seek, or poorly fit our broader mathematical and scientific programmes' (p. 189). The specific disanalogies Resnik mentions between mathematics and fiction do not turn on syntax or reference but chiefly their operating in distinct modalities with 'different standards of accessibility, clarity, precision, rigour, coherence, and thoroughness (p. 189)' and different rôles in intellectual life, only the former having 'an (apparently) ineliminable place in science (p. 189)'. One of the disanalogies between mathematical and scientific objects that Resnik thinks some might find troubling, is physicists' attempting to detect new posits. But no one tries to find fictional characters either. So 'positing does not fictionalize mathematics', agreed; 'or detract from our justification in recognizing mathematical objects or truths about them' (p. 190). Recognizing mathematical objects? With the exception of Maddy's small finite sets — a mere claim — this is just not done. With 'truths about them', Resnik is introducing the last of his questions.

In considering how we use mathematical language to refer to the independently existing mathematical objects that Resnik says we refer to, he is mainly concerned with anti-realists' bridling 'at the idea that *anyone* can hold beliefs *about* mathematical objects on the grounds that, if they exist, they cannot stand in the causal relations necessary to establishing and preserving reference' (p. 187). Since he is concerned to distinguish mathematics *from* fiction, he does not rely on the analogy *to* fiction to reverse the causal relations to the we-to-them direction that grounds both fictional and mathematical denotation by stipulation. He rightly rejects the demand for they-to-us causal relations and attempts to answer two questions:

[22] Incompleteness is a Meinongian idea, used extensively in *Über Möglichkeit und Wahrscheinlichkeit* (1915) and, according to Peter Simons [1999], also by R. Ingarden, *Das literarische Kunstwerk* (1931), but there are hints of it in Lear's Aristotle (Part I, section 5, The Distant Past). It has been discussed by Terence Parsons [1980] and Charles Parsons [1982] and taken up by Akiba [2000]. Mathematics is constrained to use only the relations attributed to its objects (*as if* they were incomplete) even if they are real and hence not incomplete.

> *The Genesis Question*: How did we come to use a certain term to refer to a given object?
> and
> *The Criterial Question*: When does a given term refer to a given object? (p. 191)

At first I thought that my reservation about these questions was that mathematical objects are not given in the way that physical objects are given — 'just there', like Mount Everest. But a closer look revealed that the sense of 'given' is the normal mathematical sense, *specific* ('not independently given' (p. 188)). I have chosen the word specific to point to what I now think needs consideration. Since we do not meet a mathematical object but only encounter it in literature, the referential damage, as it were, is already done before we assign a term with which to refer to it. The object has been *specified* and thereby denoted even if not referred to (with reference's demand for existence). How? 'Let (a, b, c) be the point of intersection of the three planes p_1, p_2, p_3, in E^3.' The reference of (a, b, c) is in a sense fixed, but the reference, in the mathematical sense, is already in the denotative phrase 'the point of intersection of the three planes p_1, p_2, p_3'. All that is necessary to refer to a mathematical object is to denote it. This is usually done with a descriptive phrase involving relations. As usual with denotation, it does not matter whether the denoted object exists. Not wanting to write about the empty set, one usually determines that what one has defined exists, but this is in the mathematical sense of existence, not metaphysical. If the planes p_1, p_2, p_3, intersect in no point, then that can be routinely determined, and that is the only thing that will have been learned about the denoted point *making use of the denotation*. This exemplifies what I mean by saying that the problem of mathematical reference precedes the genesis question. Resnik agrees (p. 192) that 'in positing we often describe first'. Fictional characters too need only be specified for us to be able to refer to them; there is a clear analogy here.

Another difficulty with reference to mathematical objects that needs to be acknowledged is that they are not specified exactly and so can be referred to no more exactly. Whatever satisfies the specification is denoted, but it is left a whatever (I return to this point at the end of the paper.). In the above example, a standard occurrence is that (a, b, c) turns out to be a whole line of points. It is also possible for (a, b, c) to be the polynomial $ax^2 + bx + c$ and just to be *calculated* as the solution of linear equations that bear the planar interpretation. Like fiction, mathematical sentences do not deal with what analytic philosophers would call fully determinate thoughts; hence Quine's insistence on application of mathematics. This does not mean that they are not viable freestanding thoughts. Application is, however, when mathematical terms are used to refer fully in the everyday sense of reference. A theory of mathematical reference ought not do too much.

Aside from the above considerable difficulty for any straightforward theory of reference, there seems to be a large one that Resnik has ignored. He relativizes reference to each natural language and argues for disquotational answers to his criterial question:

> (Sing) For any x, the singular term 't' refers to x just in case $x = t$, where t is a schematic letter standing for English singular terms. [... and]
> (Pred1) For any x, the predicate F refers to x just in case x is F, where F is a schematic letter standing for one-place English predicates. (p. 193)

The main trouble that I see with this is that it is not relativized to theory. As a result, mathematical ideograms like '$+$' and '0', which 'have different meanings in Boolean Algebra and Number Theory' are 'ruled out' (p. 193). Just as it is an analogue of 'truth in the story' that one aims at in mathematics, so we use 'reference in the theory' not any sort of absolute reference. Even within a single theory there is a difficulty in referring to objects if one thinks of them as having an identity of their own prior to their being denoted. We have no way to tell whether we have identified them correctly or incorrectly. The existence of the integers $\mathbb{Z}$ does not guarantee that '1' refers to 1 rather than to -1. It would not matter nor could we know if all our talk of the integers were the sign-reverse of their 'actual' selves — if actual selves made any sense so that our sign-reversed arithmetic was 'actually' frequently wrong. Would it matter if Rosencrantz and Gildenstern were impersonating each other? Would it matter if the actors were impersonating each other?

Independently of the degree of success of Resnik's theory of reference, its discussion illustrates the uselessness of adding affirmation to the conversational postulation needed to conduct mathematical discourse and tell stories.

6. *Inference — Tharp*

Tharp gave some consideration to how fiction works and was somewhat true to his intention to compare despite slipping quickly into identification (discussed in part I). We can charitably take his identifications to be metaphorical.

> Fiction undeniably involves a great number of stipulations, but we claim that the most significant difference between fiction and mathematics is that the latter is based on a few peculiarly sharp concepts. ([1989], p. 168)

The sharpness of mathematical concepts is what allows inference chains of arbitrary length to be depended upon. A lack of initial vagueness prevents vagueness, as with rumour, from engulfing all with the passage of time, a point made particularly with respect to ancient Greek mathematics by Netz [1999].[23] In his posthumous sequel (Part II [1991]), Tharp repeats his point about precision at greater length. One thing that we do do in mathematics that is 'distinctive' ([1991], p. 183) of it is to rule out troublesome borderline cases by the way we frame our concept definitions. Those concepts are normally 'extrapolations of ordinary concepts', an idea of Bernays that he quotes from Bernays in [1989], p. 171. The sharpness of mathematical concepts derives, in my opinion, from their being specified entirely in terms of the relations, also precise, in which their supposed instantiations stand and not on borderlines. Tharp draws attention to the pervasiveness of implicit definition (his example is of course relational) in mathematics.

> We believe that implicit definitions are absolutely basic to mathematics and that the use of such definitions has interesting implications concerning the codification of mathematical concepts. That is, one cannot assume that mathematical concepts are codified in some standard independent way and that one then looks at the codifications and infers propositions about the concepts. ... in mathematics we believe that even so basic an axiom as, "Every number has a successor", is, for the most part, implicit definition, a stipulation of one of the things the concepts of number theory are to permit. ... we set down the axiom and let it (partially) define the concepts implicitly. ([1991], p. 189)

Inferences in narrative are made, as he says (of fiction, but the point is general), 'more or less unconsciously' ([1989], p. 169). One cannot read a story and make any sense of it without drawing inferences; doing so is an essential part of the creative act of reading.[24] It is part of the work of interpretation that is part of the pleasure of reading a story; no story tells it

> [23]The empirical world is recalcitrant, it does not yield to logic, and this is because it behaves by degrees, by fine shades, by multiple dimensions. Shading into each other, the chains of the relations operating in the real world break down after a number of steps. ... Mathematical objects are different. Or are they just assumed to be different? (p. 197)
>
> At some stage, some Greeks ... decided ... to demand that in discussions of relations of area and the like, the make-believe of ideal transitivity should be entertained. Here is finally the make-believe, the abstraction truly required by Greek mathematics. (p. 198)

I have already referred to von Freytag-Löringhoff along the same lines in part I (his p. 30).

[24] A reader constructs a possible *fabula* along lines hinted at by Morton [1996] and described in more detail by Umberto Eco in terms of what he calls *inferential walks* ([1979], Chapter 8).

all. In Part II, he says of fiction that it 'presents many complications and difficulties',

> In practice, such liberties may be taken with the concepts that it is difficult to state any interesting generalizations which are not refutable by some counterexample from actual literature. ([1991], p. 180)

This is true, of course, and, if there were some exception, then a piece of actual literature could be created specifically to supply a counterexample. For an analogy, however, one does not need ironclad generalizations as one would to show, for instance, that mathematics is a species of fiction, the view into which Tharp was clearly tempted to lapse at the beginning of Part I and clearly gave up in Part II. It is quite adequate for our purposes to think of classical literature up to, for instance, Shakespeare, or just Greek myths and children's stories; we are not concerned with subtleties but with broad patterns. Tharp points out again that inference is part of narrative (Walton calls it close to indispensable ([1990], p. 143)) and adds that it *has* been since before the invention of mathematics (he too assumes stories come first).

> But surely the human race used the ordinary logical forms and inferences in myth with great success for countless millen[n]ia before the invention of set theory, or of any other systematic mathematics. This alone casts doubt on the necessity of assuming that one is, in some sense, speaking of real objects. ([1989], p. 183)

Gareth Evans (in [1982], basing his discussion on Walton's papers before [1990]) offers what he calls 'a recursive principle' for inference in fiction.

> If $A_1 \ldots A_n$ is a set of make-believe truths, and the counterfactual 'If $A_1 \ldots A_n$ were true, then B would be true' is true, and there is no set of make-believe truths $A'_1 \ldots A'_n$ such that the counterfactual 'If $A'_1 \ldots A'_n$ were true, then B would not be true' is true, then B is make-believedly true. (p. 354)

His discussion as a whole makes it clear that fiction is not an unproblematic analogue.

> By formulating the rules of the game of make-believe explicitly in terms of couterfactuals, I do not intend to suggest that the capacity to understand the counterfactual idiom is a more primitive capacity than the capacity to engage in games of make-believe. In fact I think that they are the *same* capacity. (Using counterfactuals is engaging in a purely cognitive pretence; though it might sometimes be better to speak of cognitive acts within the scope of a supposition rather than a pretence.) (p. 355)

On the whole he prefers use of his notion of scope to direct use of the verb 'pretend' because it is clearer what is supposed/pretended (nn. 33, 37 of chapter 10).

Tharp does more than just mention modality, but does not 'think it really appeals to other worlds and their possible inhabitants. Rather, I think one is saying that the concept itself does not rule out some object satisfying it.' ([1989], p. 183) He elaborates.

> We have claimed that the modal propositions of arithmetic are primarily about concepts, and are about ordinary objects and activities in the indirect sense that the concepts may be applied to ordinary objects arising from ordinary activities, such as an actually constructed inscription. In particular, existential assertions such as 'there is a number ...' may go far beyond anything humanly feasible. The discomfort with modal treatments of mathematics is reminiscent of the everyday interchange of 'can' and 'may'. One sometimes says 'Herr Schmidt can drive 150 kph on the Autobahn' when he actually cannot (because, say, his Volkswagen won't go that fast). Obviously, what one means is that he *may,* that is, the relevant rules permit such speed. We interpret the mathematical modalities in such a 'may' sense: one may construct an inscription with 99^{99} strokes — the concepts undeniably permit it. ([1989], p. 187)

This permissive interpretation of mathematical modality may not be unique to Tharp. In 'What is Mathematical Truth?', Putnam ([1975], pp. 60–78) wrote,

> What [a mathematician] asserts is that certain things are *possible* and certain things are *impossible* — in a strong and uniquely mathematical sense of 'possible' and 'impossible'. In short, mathematics is essentially *modal* rather than existential ... ([1975], p. 70)

In his introduction to Tharp [1989], Charles Chihara [1989] notes that 'it might be argued' that to use the mathematical modality to interpret mathematics is illegitimate because it presupposes mathematics. Like Tharp, I see no convincing reason for accepting what Chihara calls the crucial premise that the notion of possibility he needed in his semantics was a specifically mathematical one. This notion belongs to Putnam (the critic Chihara quotes), not Tharp.

Part of the point of the comparison with fiction is to point out that our general intellectual and imaginative capacity is in play; one peculiar to mathematics need not be hypothesized. Mathematics describes relations that we can, without leading ourselves into inconsistency — a traditionally necessary[25] condition — imagine (not visually) with sufficient clarity and distinctness (the classical description of sharpness) to be able to draw consequences with dependability. One of the functions attributed to fiction is its enlarging effect on the imagination, showing us, as it does, the temporal if not logical consequences of behaviours in potentially real — imaginable — circumstances. Among their other virtues, Shakespeare's great tragedies are studies in procrastination, treason, jealousy, and vanity, to nominalize relational circumstances displayed. My view, that what both simpler fiction (throughout) and mathematics (at bottom) are about is relations of *kinds* that hold among ordinary everyday persons and things, ties the meanings of terms closely to ordinary meanings. It is an observation of some importance that fiction in particular typically does not invent new relations to hold among characters; even science fiction, which at least in its early years invented new things, did not invent new relations. It is more or less necessary for the understanding of fiction that words be used in their usual senses; often lacking apparent reference, there is no other way for them to be understood. In his defence of realist semantics for mathematics, despite not having a realist ontology, van Fraassen pointed out in his review [1975] of Putnam's [1971] that the fictionalist response is not a Goodman-Quine reconstruction of mathematics.

> But the Fictionalist seems to proceed differently. As far as ferreting out the inferential structure is concerned, we may imagine him saying, the realist semantics is successful — why provide another one? But mathematics remains just as practically useful and intellectually interesting if we stop thinking of it as true. The realist has clearly described the picture that bewitches us — the picture that guides inference. Let us just add that this is a matter of make-believe. We speak 'about' mathematical entities as if we are speaking about real things — it matters not at all whether they *are* real. ([1975], p. 742)

One needs to note that this is not saying that they are not or cannot be real, it is equally useful whether we know or do not know.

What Tharp means about the kind of inference he intends, referring (in his limited sense) to concepts, he exemplifies with the example that all whales are mammals. Because the concept being a whale includes the concept being a mammal whether there are any whales left or not, the statement is intended to indicate that one need not be referring to past, present, or future whales. He could have referred to mythical beasts to make his point (one needs to keep in mind that the document quoted was not prepared for publication, just left on the author's death). The relation to which he wanted to draw

[25] See, however, Chris Mortensen [1995], and review by J. P. Van Bendegem, *Philosophia Mathematica* (3) 7 (1999), 202–212.

attention is conceptual; 'our knowledge of the general modal truth comes from certain self-knowledge, knowledge as to how we intend to use these concepts to classify certain objects.' ([1991], p. 188)

Anyone comparing mathematics with narrative that may be fiction or not indifferently must, as Tharp does, reject the reference-based treatment of truth and deduction given by Benacerraf in 'Mathematical Truth' [1983] because in it Benacerraf sets aside the notion of *meaning* in favour of that of *reference* ([1991], pp. 180 f.). The point Tharp is making in the following example is partly anti-ontological, but mainly epistemological; that is at any rate why I am quoting it: to make clearer the kind of reasoning, dependent on meaning, that he attributes to mathematics.

> An architect can describe quite precisely a skyscraper so tall it would crumble under its own weight if one attempted to built [*sic*] it. In fact he can describe quite precisely an infinitely tall skyscraper, by using a clause analogous to $(x)(Ey)Sxy$. He can then reason and prove theorems about the skyscraper ('There are infinitely many washrooms'). All materials which go into the description seem quite finite, and it is far more plausible to look to *that description,* than to look to infinite abstract skyscrapers, or to other infinite structures, for an explanation of the resulting theorem. Even if there were some such structures external to us — whatever that means — we would still need to describe the particular structure under consideration. That is, it is highly implausible that we have any way of directly attaching ourselves to any such entity, so we must pick out the one we wish to discuss, distinguishing it from all of the many slightly different mathematical structures by some appropriate description. Thus we are appealing to our powers of description or our powers of meaning. We might as well stop with an appeal to those powers and not postulate unhelpful entities.
>
> ...Only certain concepts will give the sharpness wanted in mathematics, so our choices of concepts are not completely arbitrary [not to mention other reasons]. And once we have picked the initial concepts, we have no control over the consequences. There is some sort of objectivity and great definiteness both in the original limitations of choice, and in the uncontrollable consequences. I am not sure how much explanation can be given of this kind of objectivity ...([1989], p. 192)

The postulation Tharp has in mind is the philosophical kind. He reiterated his point in Part II, where he wrote, 'We believe that mathematics is possible just because one can, in certain cases, reason quite clearly about what is permitted, be it feasible or not.' ([1991], p. 185)

7. *Inference — Wagner*

Steven Wagner in 'Arithmetical Fiction' [1982], considered arithmetic and fiction, accepting Benacerraf's argument that numbers cannot be sets or anything else, 'number words *as standardly meant* lack reference' (p. 255), and he seems horrified by the idea of leaving 'the question of existence open' (p. 255). Wagner pursues the analogy far enough to remark that it 'does capture the intuitive distinction between arithmetical knowledge and error' (p. 264), falsity in the story. He points out that the story view gives a satisfactory way of talking and thinking about numbers, noting that it has apparent similarities to both logicism and formalism,[26] while being plainly different from both. As I have remarked, mathematics used to be thought to be more like the history it is written like than the perhaps-fiction that Wagner portrays it as. I think that this change accounts for the word 'unintentionally' in Wagner's claims as a 'fictionalist':

> For the fictionalist, identifications of 2 with sets — that is, assertions that 2 is some set — are like continuations of a story, the original 'story' being an unintentionally fictional concept of number which we may elaborate in various ways. Any such identification is mathematically correct if it is consistent with the original concept, and to offer several is analogous to spinning different tales of Red Riding Hood's adult life on separate occasions. If we keep our stories apart, no inconsistency arises. (p. 256)[27]

Wagner's story view gives an account of such freedom as one has in spinning the mathematical kind of story. His

> [f]ictionalism, however, respects apparent linguistic fact. Since it is only a thesis about the *existence* of our supposed objects of discourse, it lets sentences of arithmetic mean just what they seem to mean. (p. 256)

The freedom is not complete. The non-existence of a licence number for my car (since I have no car) does not license a host of complex *and informative* sentences about that number. Obviously neither Wagner nor anyone else considering fiction is resorting to one's freedom to make arbitrary statements about the contents of the empty set. 'Nothing will come of nothing' is true mathematically as well as elsewhere. The story view is informative *because*

[26] To the extent that mathematics needs to transfer truth — as when applied — logicism is right. Surely this is the important relation of mathematics, as of reason in general, to truth. To the extent that mathematics depends upon the objectivity of written arguments, formalism is right.

[27] *Cf.* Zalta [1983], p. 152.

one does not need to invent new semantics to deal with stories. Standard semantics is built at least as much on stories as on non-narrative prose — more, Turner ([1996], discussed in part I) would say. Wagner does want his story view to depart as little as possible from the linguistic, epistemological, and mathematical constraints set out in Benacerraf's [1983].

The linguistic and mathematical constraints the story view observes with the substitution of 'truth-in-F, where F is a particular fiction' (p. 259) for ordinary truth 'to appease intuition' (p. 259), within which interpretation 'our usual distribution of truth values is retained' (p. 260). And since ordinary truth has been abandoned by Bourbaki and most mathematicians[28] for a long time, there is no loss in this, despite its formalistic appearance. Wagner draws a distinction between arithmetic and '(ordinary) fiction in respect of being unasserted, asserted only within tacit fictional operators, or asserted with the understanding that we are talking about an unreal world — whichever view of fiction one might prefer' (p. 259). Claiming that 'Such views would run counter to common sense about our straightforwardly assertorical use of arithmetic', he insists on keeping to 'an entirely standard view of our discourse' and 'denying the truth of obvious, elementary theorems' (p. 259). In saying this Wagner is just catching up with mathematicians' shunning of a metaphysical basis for mathematics along the lines of what Frege attempted for arithmetic (without objecting to the mathematical studies now called 'foundations'). In a contribution to the discussion that he explicitly labels as 'not brought out clearly enough' by past fictionalists, he says that the untruth of arithmetic, which has for millennia been a paradigm of truth, 'make[s] vivid how our standards of evidence may be systematically wrong in an entire domain of discourse' (p. 260).

> Such cases might be described as ones in which we depart from the policy of settling questions ultimately in the light of our overall theory. Without hesitation we appraise a variety of statements about Hamlet as being possibly, probably, or unquestionably true, yet we reconsider when, stepping back, we observe that according to the comprehensive body of doctrine we accept as the literal truth, Shakespeare's text is no standard of truth in the same sense. Similarly, the fictionalist locates the intuitive certainty of arithmetic in our ingrained practice of deciding arithmetical questions by reference to a concept of number which is itself accepted uncritically. If our total theory gives us no reason to believe that we are conceiving something real (and I claim that the success of applied arithmetic

[28] Saunders Mac Lane's [1986] is a knowledgeable and articulate indication of current attitudes.

> is not a reason), then we have no assurance that our arithmetical 'truths' actually hold. (p. 260)

('Literal' properly contrasts with 'figurative'; the proper contrast here is less standardly denominated, but serious/fictional would be better.) There can be no doubt that the intuitive certainty of arithmetic, which predates all critical thinking, is ingrained practice. Fortunately it is one that no one discourages. It is good to see someone opposing the unargued-for Quine-Putnam indispensability thesis.

The epistemological constraint Wagner phrases thus: 'The assignment of meanings should not conflict with apparent facts about knowledge.' (p. 257) The modest (if not negligible) loss in truth is intended for 'big epistemological gain' by the fictionalist who 'supposedly', instead of facing the notorious and 'perhaps insoluble problem of knowledge about a Platonic realm', 'does not even allow problems about knowledge to get started' (pp. 260 f.). Wagner (who is identifying himself less clearly with the fictionalist here) does not agree that the trade-off is so simple. He claims that the 'alleged problem about Platonic objects of knowledge is our lack of causal interaction with them' (p. 261), but he accepts Mark Steiner's objections to this allegation. 'Causal' is not of the essence here; the only relations I have seen alleged are intuition and postulation; in conversational postulation we are the cause and the mathematical objects are the effect. It is at this point that it is important to note that Wagner is not a nominalist. He has no problem with abstract objects; he just thinks mathematical ones are useless. Accordingly, he can point out that we do not know how it is that we know about many other abstract objects, 'features, ways, propositions, possibilities, states, properties' (p. 261), for example, but yet we do know about them. For him, the real trouble with non-fictional views of number, the set-theoretical definitions of number in particular, is that they differ. If we have $2 = \{\{\emptyset\}\}$ and $2 = \{\emptyset, \{\emptyset\}\}$, then we also have $\{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$, which is plainly unacceptable. As Wagner claims, 'The identity of the uniquely right set-theoretical construction of the number series (which may be the null construction) required by [the standard non-fiction view] *looks* unknowable.' (p. 262) This is not, he points out, unknowability in any deep matter.

> The natural numbers and the small (hereditarily) finite sets are paradigms of conceptual transparency. What we have, in short, are elementary questions about elementary things. *Their* unknowability is anomalous — and *anomalously* unknowable truths violate the epistemological constraint. (p. 262)

Wagner does more than any previous writer I have noticed to try to understand what he means in comparing mathematics to fiction. For his purposes, he is prepared to be less abstract than I (part I, n. 42),

> to identify a piece of fiction with a set of statements (roughly, the text), the consequences of which are, roughly, the truths-in-this-fiction. More accurately, consequences of the text plus certain other propositions, among them various beliefs of the author's, should be added, and other consequences, such as some the author could not imagine, might be dropped; but while adjustments of this sort would help us to get the notion of fictional truth just right, we may ignore them here. They would not save the law of bivalence for fictional truth: many questions about the events in a story would still lack answers. (p. 263)

He gives examples and allows for further developments of fictional stories, noting that 'it is senseless to ask which of these is *really* the *correct* development' (p. 263). He then moves specifically to arithmetic.

> What usually passes for arithmetical truth is (mostly) untrue on the fictionalist's picture, but it is true-about-numbers ('N-true') much as 'Ophelia drowned' is true-in-*Hamlet*. That is, the so-called truths of arithmetic are determined by a certain concept of number which, although really fictional, is nonetheless our point of reference for settling questions about number. It seems that what we regard as the truth in arithmetic, and what is therefore really N-true, is what follows from the description of the numbers as forming an ω-sequence. . . .
>
> What is inconsistent with the same description is N-false and is ordinarily taken really to be false. (pp. 263 f.)

(Notice that Wagner's claim earlier in his paper that he was working with an 'analogy between arithmetic and fiction' that is 'only suggestive' (p. 255) seems to have slipped. This is not surprising in him or in others; one of the gains of having a good analogue is to be able to talk of the target area in terms of the analogue, in this case, the 'story view'. This is what Lakoff and Johnson call metaphor; it need not be the identification error.) Wagner gives examples, the point of which is epistemological. The 'N-truths are, if untrue, not truly knowable, but fictionalism does capture the intuitive distinction between arithmetical knowledge and error' (p. 264), as I quoted above. And

> Whatever definition of knowledge we hold will generate an appropriate notion of N-knowledge in a fairly obvious way. We can, for example, define p to be N-known if p is believed, N-true, and arrived at (or arrivable-at) in a way that reliably yields N-truths. . . .
> Fictionalism, in short, having reinterpreted arithmetical truth, disturbs our views on knowledge no further. We N-know what we seem to know, and when we think there is no way to answer a question, even in theory, we are right. (p. 264)

He gives an example of what he means; trying to figure out whether $2 = \{\{\emptyset\}\}$ '*feels* like trying to divine the number of hairs on Hamlet's head, and a fictionalist offers parallel accounts of these cases.' (p. 264)

For Wagner, mathematical truth is not just truth in a story but truth in the contextually relevant story. One does not answer questions about Cordelia looking in *Hamlet*, and one needs also to answer arithmetical questions with the standard arithmetical axioms. And 'context-dependence of fictional truth ..., *for a fictionalist*, legitimizes multiple constructions of number' (p. 264) preserving consistency; same myth, different story. We can have $2 = \{\{\emptyset\}\}$ and $2 = \{\emptyset, \{\emptyset\}\}$ in different stories; the transitivity problem cannot arise.

A fictionalist view of mathematics has to view it as a large myth, so large as to be capable of supporting all of the pairwise inconsistent stories that are told within it.

8. *Unanswerable Questions*

As Steven Wagner admitted above, there are questions one cannot answer. I think that this is where ontological and other external questions should be put.

Richard E. Grandy, in his paper 'Shadows of Remembered Ancestors: Mathematics as the Epitome of Storytelling' [1996], takes the time to consider fiction a bit, mentioning right away unanswerable questions.

> One of the many abilities of humans that may set us apart from other species is the ability to tell stories. ...
> One of the special features of our avowedly fictional stories is the incompleteness of the properties of the characters.
> ... there seems to be no fact of the matter whether Sherlock Holmes weighted more than 14 stone. ...
> In our imagination we can construct stories with fewer and fewer definitive characteristics. We can ... imagine a process which produces a new result at each step with no special defining characteristics except that the new object is indeed new. We can imagine an infinite sequence of objects each distinct from the other. And the story need include *no* other properties. (p. 171)

He goes right on, giving the impression that the ability discussed previously is connected to the ability discussed next, 'We have a faculty to imagine structures ...' (p. 171). We are able to describe these structures in such a way that others are able to imagine structures that are 'isomorphic' so far as we can tell by discussion with the other imaginers. He focusses on relations connected with operations, as does Kitcher, rather than relations in general. Mathematical structures

> are typically suggested by generalization and abstraction from physical operations, as in the case of the natural numbers, or by generalization from previous mathematical constructions, as when we move from formal finitary languages (themselves suggested by natural languages) to infinitary languages. (p. 171)

As he says, the position he is sketching requires one remarkable capacity, that of conveying '(mostly) consistent mathematical stories about the imagined structures and to understand the stories well enough that we (to some degree, mathematicians to a greater degree) tell matching stories' (p. 171). With the usual philosopher's ontological preoccupation, he points out the contrast with platonism's requiring two 'miracles', as he calls the existence of the platonic objects and of our 'faculty like perception that gives us access' (p. 172). The platonic miracles are widely denied; the story capacity is undeniable. He counts Maddy's perceptual view of mathematics (sets, anyway) and Resnik's structural view as having affinities to what he wants to say, though he thinks that both 'run into a wall when they attempt to make the transition from "perception of finite structure" to infinite structure' (p. 172).

With unanswerable questions we return to the comparisons with fiction that are less interesting. Of a merely methodological Platonist, Chihara [1973] says,

> His position would then be analogous to a standard position regarding works of fiction — one in which the sentence 'Hamlet's nose was $4\frac{1}{2}$ inches long' is regarded as neither true nor false. To adopt such a position, even regarding fictional characters, is not completely uncontroversial, as can be seen from a glance at the recent philosophical literature (*cf.* J. Woods [1969]). But it does appear to be a reasonable option. (p. 64)

This is only one standard position regarding fictional characters, but it has something to recommend it. Chihara's example is, in the literary context, less consequential than the continuum hypothesis in the mathematical context, but the principle is the same even if the example less important. At much the same time Hao Wang was remarking on such indeterminacy. In a novel

> there is much that is left open so that alternative continuations are permissible and not all questions are answerable even in principle. It may simply be indeterminate whether, for example, the hero will remarry or whether his height is five feet, nine inches. The contrast is sometimes characterized as between being true of an object and being true to a concept. A statement about an object is either true or false, but a statement about a concept in this special sense need not be true or false. At places Wittgenstein seems to suggest that many statements about numbers are of this latter type. People have

> also suggested tying up realism with the acceptance of the law of the excluded middle. ([1974], p. 390)

Perhaps objectism rather than realism. Absolute geometry in which the notion of parallelism is not defined has no fact of the matter respecting anything to do with parallelism. One can take the same attitude to the continuum hypothesis, and some do. In that connection Solomon Feferman wrote on the foundations-of-mathematics list using the story analogy,

> In each story, we can go a long way on very little in the way of characters and plot. But then we come to the places that the story leaves undetermined. We do feel in mathematics that the stories, if that's what they are, are less arbitrary than works of fiction. That's because the kinds of objects these are supposed to be about are refined to have a minimum few characteristics, and then one has significantly fewer options as to what to tell about them. [1997]

I think like Wagner that the fiction comparison goes some way toward dissolving both of the famous Benacerraf problems, that of what mathematical objects can be and how we can find a middle path between easy epistemology for unbelievable objects or implausible epistemology for constructed objects. We have the access necessary for a grounding of mathematical knowledge because it is knowledge of idealized relations that hold between ordinary things and then relations that are built on reifications of those relations, and the mathematical objects that it is so hard to become individually acquainted with are like fictional characters, made up to carry the relations that are what it is all really about. Within the textual limits, we have perfect access to fictional characters; some are much better known than almost all real persons. On the one hand, mathematics is more like history than fiction in being so highly constrained in what it can legitimately say. On the other hand, it is more like fable than the driest of history on account of aiming at a kind of truth that is universal not contingent as history is. A fable, the moral of which is not widely regarded as a truth, is a failure as a fable.

9. *Conclusion*

We have seen that the analogy between mathematics and fiction can help to understand some aspects of mathematics in terms of simple fiction. That is motivation enough for me. One of the common motivations for the comparison, in Tharp for example, is to show that mathematical reference and argument are possible even though (he thinks) there is no more ontological commitment to mathematical objects than to fictional characters. The point is well made, but I think it is wiser to set aside the question of the extra-theoretical existence of mathematical objects because, even if the question were meaningful,[29] the answer is beyond us and does not matter. Adam Morton makes this point by saying that there might be 'a world which arithmetic, by a sort of miracle, happened to describe' ([1996], p. 226). As Bourbaki outline elegantly in the English translation of their *Set Theory*, mathematics studies mathematical questions, which concern (but do not consist solely of) sound deduction from assumptions. It is no longer assumed that the assumptions are true in any important philosophical sense nor that the conclusions are either. This is not truth by convention, as Putnam dismissively suggests in his [1971]; it dispenses explicitly with the notion of truth. Mac Lane agrees that the truth of axioms is not what is wanted (but rather features like fruitfulness) nor of theorems (where one wants validity).

Against this some reasons are put forward. Mathematical language seems to refer, and a uniform semantics with the rest of language would be good to have (Benacerraf [1983]), but fictional language also seems to refer. Since we talk the same way about real and imaginary subjects, I claim that semantics has to be ontology-neutral. Mathematics is thought to be discovered rather than invented, but of course we discover the consequences of what we or others have previously invented. The properties of physical inventions are likewise discovered. Mathematical objects are thought by some to help explain applicability, but I think the opposite, as I have explained. Thinking of the Platonic world can be a stimulus to the formulation of new axioms; but on the other hand so can a desire to do mathematics rather than metaphysics. The objects of mathematics are thought by some to contribute to its objectivity, but their lack of influence on our thinking about them makes this wishful thinking in my opinion; and I am not alone. Jody Azzouni ([1994], [1997], [2000]) makes something of the lack of epistemic rôle for mathematical objects. No one is going to discover a mathematical telescope to allow us to check the truth of our axioms by comparison with the objects specified.

In my desire not to do accidental metaphysics, I follow in what Charles Chihara correctly observed to be the mathematical tradition. In his [1973], he wrote that 'most working mathematicians ... would shy away from all ontological questions regarding the actual existence of sets' (p. 62). He suggests that

[29] Even Frege in *Grundlagen*, though not in *Grundgesetze*, held and acted upon the view that it is not, according to Michael Dummett ([1991], pp. 192–199). W. W. Tait writes, 'As a mathematical statement, the assertion that numbers exist is a triviality. What does it mean to regard it as a statement outside of mathematics?' [1986] with a long footnote containing the sentence, 'I think that Carnap [1956] is right that 'external questions' questions of existence have no prima-facie sense.' One of the few arguments from authority that has weight is that something makes no sense.
The earlier Carnap discussed the acceptance of mathematical axioms, as did Wittgenstein, as decisions about language (J. Alberto Coffa [1991], p. 322). I find it puzzling why they did not regard these as decisions about the posited subject matter *of* the language.

> mathematicians construct their systems *as if* they were describing existing objects, *as if* there are such things as sets and numbers, and that he [*sic*] reasons accordingly. Whether such abstract objects exist, he can say, is irrelevant to the question of whether the mathematical theories are intelligible. It is enough that such objects can be conceived. To distinguish Platonists of the latter sort from the ontological Platonists [of whom Chihara's chief example is Gödel], I shall use the term 'mythological Platonist' (pp. 62 f.)

Michael Resnik calls this position *methodological platonism* [1980] and Stewart Shapiro [1997] *working realism*.

Quine ([1969], p. 45) points to what turns out to be a verbal danger to one easy way of expressing my view of mathematical entities. If one says that natural numbers are whatever exhibits arithmetic behaviour, then on the obvious interpretation one runs into Benacerraf's problem of the multiple models of the numbers. If the pronoun 'whatever' is filled out with objects, real or mathematical, then one is dealing with an application or model of the natural numbers respectively, not the numbers themselves. My view of the numbers keeps them at the pronominal level of the unfilled-out 'whatever'. I do not regard this as unreasonable, and can cite Quine himself in support; 'Pronouns are the basic media of reference; nouns might better have been named propronouns.' ([1961], p. 13) Russell's theory of descriptions [1905] uses a *variable* in the same way, 'essentially and wholly undetermined' ([1956], p. 42).[30] Philip Kitcher points out that even if

> there *are* abstract instantiations of mathematical structure, they are no more of interest to mathematics than any other instantiation. We are equally concerned with all the instantiations, and equally unconcerned about any of them. More exactly, we are interested in the structure they share, and it is misleading to formulate the contents of mathematics by identifying one instantiation, even an 'abstract' instantiation, as privileged. ([1984], p. 106)

Azzouni considers problems of reference-fixing a lot in his book ([1994], pp. 21–26), and points out (p. 31) that it is not a *mathematical* worry. Mathematically speaking, it does not matter; this tells us something. This may be the clearest way of saying why I find argument about the existence of these pronouns unfulfilling and why my sympathies lie with those that don't quite see what it would mean for unfilled-out whatevers to exist physically, abstractly, or socio-culturally.

For a variety of reasons, I have not examined the whole literature on fiction and mathematics. I have not even found it all. On the subject of finding it,

[30] Paulos [1998] draws attention to the similarity between pronouns and variables.

I have been perplexed at the degree to which writers on the subject have ignored what others have written on it; separate worlds are not all fictional. For the sake of anyone wanting to look into it, I list the other sources mentioned by Grandy: Vico [1710], Lakatos [1976] on problem evolution, von Glasersfeld [1984], and Resnik [1993]. And others known to me, Liston [1993] and Hersh [1997] and earlier publications, which have left some readers (Torretti [1981], for example) thinking him a fictionalist, a claim he denies.

There are also recurrent comparisons of poetry and mathematics from Scott Buchanan's *Poetry and Mathematics*[31] (1929, second edition, 1962); to Philippe Séguin, whose *Von Unendlichen zur Struktur* (Frankfurt am Main: Verlag Peter D. Lang, 1996; MR 98h:00021) has the subtitle 'Modernity in the poetry and mathematics of Edgar Allan Poe and Georg Cantor'.

That mathematics and fiction, while utterly distinct, have some similarities I have no doubt. Whether these similarities constitute an analogy of sufficient general interest, sufficiently engaging and enlightening, to help those that do not know much about mathematics to gain a better appreciation of it is an empirical question that I intend to put to the test; I am hopeful. Whether they are of sufficient philosophical interest to merit the further consideration of philosophers is also, I suppose, an empirical question that I am in effect putting to the test by writing this paper. I am less hopeful, but I am doing what I can to advance the conversation.[32] The comparisons canvassed in this paper do not suggest that narrative would be useful as a model for mathematics in the Vaihinger/Hesse sense discussed above. In this conclusion I seem to be agreeing with Harold Hodes [1990] that the analogy is not only limited (all are) but too limited.

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[31] Buchanan is more concerned with significance (see note 5) than structure in noting that 'poetry and mathematics are two very successful attempts to deal with ideas' (p. 43). For him deduction is mathematically less important than insight (pp. 62 f.) and narrative is a poetic option (p. 64), but an important one, creating undying (if not eternal) 'objects'. He explores the analogy, negative (p. 135) and positive, from the literary end.

[32] *Cf.* Wang [1974], p. 257.

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