# MARTIN'S CONJECTURE AND STRONG ERGODICITY

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Abstract. Under the assumption of Martin's Conjecture, we prove a strong ergodicity result for the Turing equivalence relation $\equiv_T$.

## 1. Introduction

Throughout this paper, $\equiv_T$ denotes the *Turing equivalence relation* on $\mathcal{P}(\mathbb{N})$, which is identified with the Cantor space $2^{\mathbb{N}}$ by identifying subsets of $\mathbb{N}$ with their characteristic functions. By Martin [11, 12], if $A \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then either $A$ contains a cone or else $2^{\mathbb{N}} \smallsetminus A$ contains a cone. (Here a subset $A \subseteq 2^{\mathbb{N}}$ is said to be *$\equiv_T$-invariant* iff $A$ is a union of $\equiv_T$-classes.) This easily implies that if $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel function which takes a constant value on each $\equiv_T$-class, then there exists a cone $C$ such that $f \restriction C$ is a constant function. As Friedman [5] points out, this can be regarded as an ergodicity result for $\equiv_T$. In this paper, we shall show that if Martin's Conjecture [9] on degree invariant Borel maps is true, then $\equiv_T$ satisfies a much stronger ergodicity result.

Before we can give a precise statement of our main result, we first need to recall some of the basic notions of the theory of countable Borel equivalence relations. Suppose that $E$, $F$ are countable Borel equivalence relations on the standard Borel spaces $X$, $Y$ respectively. Then the (not necessarily Borel) map $f : X \to Y$ is said to be a *homomorphism* from $E$ to $F$ iff $x \mathrel{E} y$ implies $f(x) \mathrel{F} f(y)$ for all $x, y \in X$. If $f$ is a Borel map and satisfies the stronger condition that $x \mathrel{E} y$ iff $f(x) \mathrel{F} f(y)$ for all $x, y \in X$, then $f$ is said to be a *Borel reduction* and we write $E \leq_B F$. Finally, if there exists a countable-to-one Borel homomorphism $f : X \to Y$ from $E$ to $F$, then we say that $E$ is *weakly Borel reducible* to $F$ and write $E \leq_B^w F$. In this case, we say that $f$ is a *weak Borel reduction* from $E$ to $F$.

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**Definition 1.1.** (a) A countable Borel equivalence relation $E$ is said to be *universal* iff $F \leq_B E$ for every countable Borel equivalence relation $F$.

(b) A countable Borel equivalence relation $E$ is said to be *weakly universal* iff $F \leq_B^w E$ for every countable Borel equivalence relation $F$.

For example, by Dougherty-Jackson-Kechris [3], the orbit equivalence relation $E_\infty$ arising from the shift action of the free group $F_2$ on two generators on $2^{F_2}$ is a universal countable Borel equivalence relation. Of course, if $E$ is a universal countable Borel equivalence relation, then $E$ is weakly universal. It is currently not known whether the converse holds. However, Kechris [18, Corollary 4.9] has pointed out that the Turing equivalence relation $\equiv_T$ is weakly universal; and Dougerty-Kechris [4] have shown that if Martin's Conjecture holds, then $\equiv_T$ is not universal. (The material in Thomas [18, Section 4] is entirely due to Kechris and Miller.) In the final section of this paper, we shall prove the following result.

**Theorem 1.2.** *Assuming Martin's Conjecture, there exist uncountably many weakly universal countable Borel equivalence relations up to Borel bireducibility.*

Theorem 1.2 is a straightforward consequence of the techniques of Thomas [18], together with the following strong ergodicity result for $\equiv_T$. (For the standard measure-theoretic version of strong ergodicity, see Definition 4.1.)

**Definition 1.3.** Let $E$ be a countable Borel equivalence relation on the standard Borel space $X$. Then $\equiv_T$ is said to be *$E$-m-ergodic* iff for every Borel homomorphism $f : 2^{\mathbb{N}} \to X$ from $\equiv_T$ to $E$, there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $f$ maps $C$ into a single $E$-class.

**Theorem 1.4.** *Assuming Martin's Conjecture, if $E$ is any countable Borel equivalence relation, then exactly one of the following conditions holds:*

(a) *$E$ is weakly universal.*

(b) *$\equiv_T$ is $E$-m-ergodic.*

Of course, if $E$ is a weakly universal countable Borel equivalence relation, then $\equiv_T \leq_B^w E$ and hence $\equiv_T$ is not $E$-$m$-ergodic. Thus conditions 1.4(a) and 1.4(b) are mutually exclusive.

As we mentioned earlier, Martin's Theorem on the $\equiv_T$-invariant Borel subsets of $2^{\mathbb{N}}$ easily implies that $\equiv_T$ is $\Delta(2^{\mathbb{N}})$-$m$-ergodic, where $\Delta(2^{\mathbb{N}})$ denotes the identity relation on $2^{\mathbb{N}}$. On the other hand, there are currently no nonsmooth countable Borel equivalence relations $E$ for which it has been proved that $\equiv_T$ is $E$-$m$-ergodic. In particular, it is not known whether $\equiv_T$ is $E_0$-$m$-ergodic, where $E_0$ denotes the Vitali equivalence relation on $2^{\mathbb{N}}$.

This paper is organised as follows. In Section 2, we shall first formulate the version of Martin's Conjecture that we will be assuming throughout this paper and then we shall derive some simple but useful consequences. In Section 3, we shall prove Theorem 1.4; and in Section 4, we shall prove Theorem 1.2. In Section 5, we shall present two easy applications of the results of the earlier sections, which answer questions of Boykin-Jackson [2] and Thomas [18], modulo Martin's Conjecture.

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## 2. Martin's Conjecture

Throughout this paper, by Martin's Conjecture, we shall always mean the following special case of the more general conjecture (also known as the 5th Victoria Delfino Problem) formulated by Martin in Kechris-Moschovakis [9].

**Martin's Conjecture** $(MC)$**.** *If $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$, then exactly one of the following conditions holds:*

(i) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $f$ maps $C$ into a single $\equiv_T$-class.*

(ii) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $x \leq_T f(x)$ for all $x \in C$.*

In the remainder of this paper, we shall write $(MC)$ to indicate that the (currently known) proof of a given statement makes use of Martin's Conjecture.

**Theorem 2.1** $(MC)$**.** *If $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$, then exactly one of the following conditions holds:*

(i) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $f$ maps $C$ into a single $\equiv_T$-class.*

(ii) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $f \restriction C$ is a weak Borel reduction from $\equiv_T \restriction C$ to $\equiv_T$. Furthermore, in this case, if $D \subseteq 2^{\mathbb{N}}$ is* any *cone, then $[\, f(D)\,]_{\equiv_T}$ contains a cone.*

*Proof.* Suppose that (i) fails. By Martin's Conjecture, there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $x \leq_T f(x)$ for all $x \in C$. Clearly $f \restriction C$ is countable-to-one and so $f \restriction C$ is a weak Borel reduction. Let $D \subseteq 2^{\mathbb{N}}$ be any cone and let $D_0 = D \cap C$. Since $f \restriction C$ is countable-to-one, it follows that $f(D_0)$ is a Borel subset of $2^{\mathbb{N}}$ and this implies that the $\equiv_T$-saturation $[\, f(D_0)\,]_{\equiv_T}$ is also a Borel subset. By Martin's Theorem [11, 12], since $[\, f(D_0)\,]_{\equiv_T}$ is a $\leq_T$-cofinal $\equiv_T$-invariant Borel subset of $2^{\mathbb{N}}$, it follows that $[\, f(D_0)\,]_{\equiv_T}$ contains a cone. □

Condition 2.1(ii) is reminiscent of the conclusion of the "unique ergodicity argument" first introduced by Adams [1] in the measure-theoretical setting and later exploited by Thomas [16, 17] and Hjorth-Kechris [7]. Of course, the following result is an immediate consequence of Theorem 2.1 and implies that $\equiv_T$ is not countable universal. (Here $\equiv_T \sqcup \equiv_T$ denotes the disjoint union of two copies of the Turing equivalence relation $\equiv_T$.)

**Corollary 2.2** $(MC)$**.** $\equiv_T \,<_B\, (\equiv_T \sqcup \equiv_T)$.

*Observation* 2.3. Suppose that $C = \{x \in 2^{\mathbb{N}} \mid z \leq_T x\}$ is a cone. Then the map $y \mapsto y \oplus z$, where $y \oplus z$ denotes the usual disjoint sum, is a weak Borel reduction from $\equiv_T$ to $\equiv_T \restriction C$ and hence $\equiv_T \restriction C$ is weakly universal.

**Corollary 2.4** $(MC)$**.** *If $A \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then $\equiv_T \restriction A$ is weakly universal iff $A$ contains a cone.*

*Proof.* If $\equiv_T \restriction A$ is weakly universal, then there exists a weak Borel reduction $f : 2^{\mathbb{N}} \to A$ from $\equiv_T$ to $\equiv_T \restriction A$. By Theorem 2.1, it follows that $[\, f(2^{\mathbb{N}})\,]_{\equiv_T}$ contains a cone. □

On the other hand, if $\equiv_T$ is countable universal, then $(\equiv_T \sqcup \equiv_T) \leq_B \equiv_T$ and this easily implies that there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $\equiv_T \restriction (\, 2^{\mathbb{N}} \smallsetminus C\,)$ is also countable universal. Consequently, it would be very interesting to obtain lower bounds on the Borel complexity of $\equiv_T \restriction (\, 2^{\mathbb{N}} \smallsetminus C\,)$ for cones $C \subseteq 2^{\mathbb{N}}$. In Section 5, we shall prove that there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $\equiv_T \restriction (\, 2^{\mathbb{N}} \smallsetminus C\,)$ is *not* essentially free.

*Remark* 2.5. In [6], answering a question of Thomas [18], Hjorth proved that the universal countable Borel equivalence relation $E_\infty$ was not Borel bireducible with a smooth disjoint union of essentially free countable Borel equivalence relations. This can also be seen as follows. Suppose that $E = \bigsqcup_{z \in 2^{\mathbb{N}}} E_z$ is the smooth disjoint union of the essentially free countable Borel equivalence relations $\{E_z \mid z \in 2^{\mathbb{N}}\}$ and that $f : 2^{\mathbb{N}} \to \bigsqcup_{z \in 2^{\mathbb{N}}} X_z$ is a Borel reduction from $\equiv_T$ to $E$. Then there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $f(C) \subseteq X_z$ for some fixed $z \in 2^{\mathbb{N}}$; and since $\equiv_T \restriction C$ is weakly universal, it follows that $E_z$ is also weakly universal. But this contradicts Thomas [18, Corollary 4.8], which says that weakly universal countable Borel equivalence relations are not essentially free.

## 3. The Proof of Theorem 1.4

Theorem 1.4 is a straightforward consequence of Theorem 2.1, together with the following characterization of weak Borel reducibility.

**Theorem 3.1** (Kechris-Miller)**.** *If $E$, $F$ are countable Borel equivalence relations on the uncountable standard Borel spaces $X$, $Y$ respectively, then the following conditions are equivalent:*

(i) $E \leq_B^w F$.

(ii) *There exists a countable Borel equivalence relation $S \subseteq F$ on $Y$ such that $S \sim_B E$.*

*Proof.* It is clear that condition (ii) implies condition (i), since if $f : X \to Y$ is a Borel reduction from $E$ to $S$, then $f$ is a weak Borel reduction from $E$ to $F$. The more interesting converse direction is an immediate consequence of Theorem 4.4 and Proposition 4.10 of Thomas [18]. (As we mentioned earlier, the material in Thomas [18, Section 4] is due Kechris-Miller.) □

*Proof of Theorem 1.4.* As we pointed out earlier, it is clear that conditions (a) and (b) are mutually exclusive. Suppose that $f : 2^{\mathbb{N}} \to X$ witnesses the failure of condition (b). Since $\equiv_T$ is weakly universal, it follows that $E \leq_B^w \equiv_T$. Hence, applying Theorem 3.1, we can suppose that $X = 2^{\mathbb{N}}$ and that $E \subseteq \equiv_T$. Then $f$ is also a Borel homomorphism from $\equiv_T$ to $\equiv_T$. By assumption, $f$ does not map any cone into a single $E$-class and this easily implies that $f$ does not any cone into a single $\equiv_T$-class. Hence, by Theorem 2.1, there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $f \restriction C$ is countable-to-one. Since $\equiv_T \restriction C$ is weakly universal and $(\equiv_T \restriction C) \leq_B^w E$, it follows that $E$ is also weakly universal. □

The following corollary appears to be consistent with Kechris's Conjecture that $\equiv_T$ is countable universal and even with Hjorth's Conjecture that *every* weakly universal countable Borel equivalence relation is countable universal. (I should perhaps point out that Hjorth denies having ever made this conjecture.)

**Corollary 3.2** ($MC$)**.** *Let $E$, $F$ be countable Borel equivalence relations on the standard Borel spaces $X$, $Y$ respectively. Suppose that $E$ is weakly universal and that $F$ is not weakly universal. If $f : X \to Y$ is a Borel homomorphism from $E$ to $F$, then there exists a Borel subset $Z \subseteq X$ such that:*

(i) *$E \restriction Z$ is weakly universal; and*
(ii) *$f$ maps $Z$ into a single $F$-class.*

*Proof.* Let $g : 2^{\mathbb{N}} \to X$ be a weak Borel reduction from $\equiv_T$ to $E$. Then $h = f \circ g$ is a Borel homomorphism from $\equiv_T$ to $F$. By Theorem 1.4, $\equiv_T$ is $F$-$m$-ergodic and hence there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $h$ maps $C$ to a single $F$-class. Since $g$ is countable-to-one, it follows that $Z = g(C)$ is a Borel subset of $X$; and since $(\equiv_T \restriction C) \leq_B^w (E \restriction Z)$, it follows that $E \restriction Z$ is weakly universal. Thus $Z$ satisfies our requirements. □

## 4. The Proof of Theorem 1.2

Before we begin the proof of Theorem 1.2, we first need to recall the standard measure-theoretical version of strong ergodicity.

**Definition 4.1.** Suppose that $E$, $F$ are countable Borel equivalence relations on the standard Borel spaces $X$, $Y$ and that $\mu$ is an $E$-invariant probability measure on $X$. Then $E$ is said to be *$F$-ergodic* iff for every Borel homomorphism $f : X \to Y$ from $E$ to $F$, there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that $f$ maps $Z$ into a single $F$-class.

*Remark* 4.2. More generally, if $E$ is $F$-ergodic and $f : X \to Y$ is a $\mu$-measurable homomorphism from $E$ to $F$, then there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that $f$ maps $Z$ into a single $F$-class. To see this, recall that there exists a Borel map $g : X \to Y$ such that $g(x) = f(x)$ for $\mu$-a.e. $x$. It is easily checked that

$$W = \{x \in X \mid g(\,[x]_E\,) \text{ is } \textit{not} \text{ contained in a single } F\text{-class } \}$$

is an $E$-invariant Borel subset of $X$ with $\mu(W) = 0$. Hence, after adjusting the values of $g$ on $W$, we can suppose that $g$ is a Borel homomorphism from $E$ to $F$. The result follows easily.

We shall make use of the following result, which was proved in Thomas [18].

**Theorem 4.3.** *There exists a Borel family $\mathcal{G} = \{G_\alpha \mid \alpha \in 2^{\mathbb{N}}\}$ of finitely generated groups, each with underlying set $\mathbb{N}$, such that the following conditions are satisfied:*

(a) *$G_\alpha$ has an infinite normal subgroup $N_\alpha$ such that $N_\alpha \cong SL_3(\mathbb{Z})$.*
(b) *$G_\alpha$ has no nontrivial finite normal subgroups.*
(c) *If $\alpha \neq \beta$, then $G_\beta$ does not embed into $G_\alpha$.*

For each $\alpha \in 2^{\mathbb{N}}$, consider the shift action of $G_\alpha$ on $2^{G_\alpha} = 2^{\mathbb{N}}$. Then the usual product probability measure $\mu$ on $2^{\mathbb{N}}$ is $G_\alpha$-invariant and the *free part* of the action

$$X_\alpha = \{x \in 2^{\mathbb{N}} \mid g \cdot x \neq x \text{ for all } 1 \neq g \in G_\alpha\}$$

has $\mu$-measure 1. Let $E_\alpha$ be the corresponding orbit equivalence relation on $X_\alpha$. Then the following result is an easy consequence of Popa's Cocycle Superrigidity Theorem [14]. (For example, see Thomas [18, Section 5].)

**Theorem 4.4.** *If $\alpha \neq \beta$, then $E_\beta$ is $E_\alpha$-ergodic with respect to $\mu$.*

Clearly $(\equiv_T \times E_\alpha)$ is weakly universal for each $\alpha \in 2^{\mathbb{N}}$. Hence Theorem 1.2 is an immediate consequence of the following result. (In an earlier version of the proof of Theorem 4.5, I assumed $\mathbf{\Sigma}^1_1$-Determinacy in order to obtain the measurability of $\mathbf{\Sigma}^1_2$ sets. I am grateful to Alexander Kechris for providing the following elegant method for eliminating the hypothesis of $\mathbf{\Sigma}^1_1$-Determinacy.)

**Theorem 4.5** (*MC*)**.** *If $\alpha \neq \beta$, then $(\equiv_T \times E_\beta) \not\leq_B (\equiv_T \times E_\alpha)$.*

*Proof.* We shall first prove Theorem 4.5 under the additional assumption that the universe $V$ also satisfies $MA + 2^{\aleph_0} > \aleph_1$. Notice that Theorem 4.4 implies that $E_\alpha$ is *not* weakly universal. Suppose that

$$f : 2^{\mathbb{N}} \times X_\beta \to 2^{\mathbb{N}} \times X_\alpha$$

is a Borel reduction from $(\equiv_T \times E_\beta)$ to $(\equiv_T \times E_\alpha)$ and let $\lambda$, $\rho$ be the Borel functions defined by

$$f(r, x) = (\lambda(r, x), \rho(r, x)).$$

For each $x \in X_\beta$, let $\rho_x : 2^{\mathbb{N}} \to X_\alpha$ be the map defined by $\rho_x(r) = \rho(r,x)$. Then $\rho_x$ is a Borel homomorphism from $\equiv_T$ to $E_\alpha$. Since $E_\alpha$ is not weakly universal, Theorem 1.4 implies that there exists a cone $C_x \subseteq 2^{\mathbb{N}}$ such that $\rho_x$ maps $C_x$ to a single $E_\alpha$-class; say, $\mathbf{d}_x$. If $y E_\beta x$ and $r \in C_x$, then $\rho_y(r) E_\alpha \rho_x(r)$ and so $\rho_y(r) \in \mathbf{d}_x$. Hence if $y\, E_\beta\, x$, then $\mathbf{d}_y = \mathbf{d}_x$. Let $R \subseteq X_\beta \times X_\alpha$ be the $\mathbf{\Sigma}^1_2$ subset defined by

$$(x,z) \in R \quad \text{iff} \quad (\,\exists s\,)\,(\,\forall r\,)\,(\, s \leq_T r \text{ implies } \rho(r,x)\; E_\alpha\; z\,).$$

Applying Kondô's Theorem [10], let $h : X_\beta \to X_\alpha$ be a $\mathbf{\Sigma}^1_2$ uniformizing function for $R$. If $U \subseteq X_\alpha$ is an open set, then

$$h^{-1}(U) = \{\, x \in X_\beta \mid (\,\exists y\,)\,(\, y \in U \text{ and } h(x) = y\,\}$$

is a $\mathbf{\Sigma}^1_2$ set. By Martin-Solovay [13], since $MA + 2^{\aleph_0} > \aleph_1$ holds, every $\mathbf{\Sigma}^1_2$ set is $\mu$-measurable and hence $h$ is $\mu$-measurable. Clearly $h(x) \in \mathbf{d}_x$ for all $x \in X_\beta$ and it follows that $h$ is a $\mu$-measurable homomorphism from $E_\beta$ to $E_\alpha$. Since $E_\beta$ is $E_\alpha$-ergodic, there exists a Borel subset $X_0 \subseteq X_\beta$ with $\mu(X_0) = 1$ such that $h$ maps $X_0$ into a single $E_\alpha$-class; say, $\mathbf{c}$.

For each $x \in X_0$, let $\lambda_x : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the map defined by $\lambda_x(r) = \lambda(r,x)$. Then $\lambda_x$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$. If $r$, $s \in C_x$, then $\rho(r,x)$, $\rho(s,x) \in \mathbf{c}$ and it follows that

$$r \equiv_T s \quad \text{iff} \quad \lambda_x(r) \equiv_T \lambda_x(s).$$

Thus $\lambda_x$ induces a Borel reduction from $\equiv_T \restriction C_x$ to $\equiv_T$. Hence, by Theorem 2.1, it follows that $[\,\operatorname{ran}\lambda_x \restriction C_x\,]_{\equiv_T}$ contains a cone $D_x$. In particular, choosing $x$, $y \in X_0$ with $[\,x\,]_{E_\beta} \neq [\,y\,]_{E_\beta}$, there exist $r \in C_x$ and $s \in C_y$ such that $\lambda_x(r) \equiv_T \lambda_y(s)$. But this means that $f(r,x)\ (\equiv_T \times E_\alpha\,)\ f(s,y)$, which is a contradiction.

Finally we shall explain how to eliminate the addtional assumption that $V$ satisfies $MA{+}2^{\aleph_0} > \aleph_1$. First note that $MC$ is equivalent to the following $\Pi^1_2$ statement.

$(MC')$ If $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$, then either:

(a) for all $x \in 2^{\mathbb{N}}$, there exists $x \leq_T y$ such that $f(y) <_T y$; or

(b) for all $x \in 2^{\mathbb{N}}$, there exists $x \leq_T y$ such that $y \leq_T f(y)$.

To see this this, suppose that $MC'$ holds and let $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be a Borel homomorphism from $\equiv_T$ to $\equiv_T$. If (a) holds, then $A = \{\, y \in 2^{\mathbb{N}} \mid f(y) <_T y\,\}$ is a $\leq_T$-cofinal $\equiv_T$-invariant Borel subset of $2^{\mathbb{N}}$; and hence, by Martin's Theorem, $A$ contains a cone $C$. Applying Slamen-Steel [15], it follows that there exists a cone

$D \subseteq C$ such that $f$ maps $D$ into a single $\equiv_T$-class. Similarly, if (b) holds, then there exists a cone $C$ such that $y \leq_T f(y)$ for all $y \in C$. (Of course, it is clear that $MC$ implies $MC'$.)

Fix some $\alpha \neq \beta$ and let $V^{\mathbb{P}}$ be a generic extension which satisfies $MA + 2^{\aleph_0} > \aleph_1$. Then by the Shoenfield Absoluteness Theorem [8, Theorem 25.20], it follows that $V^{\mathbb{P}}$ also satisfies $MC$. Furthermore, it is clear that conditions 4.3(a), (b) and (c) are absolute and so $E_\beta$ remains $E_\alpha$-ergodic in $V^{\mathbb{P}}$. Hence, by our earlier argument,

$$V^{\mathbb{P}} \vDash (\equiv_T \times E_\beta) \not\leq_B (\equiv_T \times E_\alpha).$$

By the Shoenfield Absoluteness Theorem, since this is a $\mathbf{\Pi}^1_2$ property of $\alpha$ and $\beta$, it follows that

$$V \vDash (\equiv_T \times E_\beta) \not\leq_B (\equiv_T \times E_\alpha).$$

□

## 5. Some applications

In this final section, we shall present two easy applications of the results of the earlier sections, which answer questions of Boykin-Jackson [2] and Thomas [18], modulo Martin's Conjecture. Throughout this section, if $c, d \in \mathbb{N}^{\mathbb{N}}$, then $c \leq^* d$ iff $c(n) \leq d(n)$ for all but finitely many $n \in \mathbb{N}$; and $c =^* d$ iff both $c \leq^* d$ and $d \leq^* c$. Similarly, we shall write $c < d$ iff $c(n) < d(n)$ for all $n \in \mathbb{N}$.

It is well-known that the countable Borel equivalence relation $=^*$ is Borel bireducible with the Vitali equivalence relation $E_0$. In particular, $=^*$ is not weakly universal.

**Definition 5.1** (Boykin-Jackson [2])**.** Let $E$ be a Borel equivalence relation on the standard Borel space $X$. Then $E$ is said to be *Borel-Bounded* iff for every Borel map $\varphi : X \to \mathbb{N}^{\mathbb{N}}$, there exists a Borel homomorphism $g : X \to \mathbb{N}^{\mathbb{N}}$ from $E$ to $=^*$ such that $\varphi(x) \leq^* g(x)$ for all $x \in X$

In [2], Boykin-Jackson proved that every hyperfinite countable Borel equivalence relation is Borel-Bounded and asked whether the converse was true. On the other hand, they also pointed out that there are currently no examples of countable Borel equivalence relations which are known *not* to be Borel-Bounded.

**Theorem 5.2** ($MC$)**.** *The Turing equivalence relation $\equiv_T$ is not Borel-Bounded.*

*Proof.* Identifying each $r \in 2^{\mathbb{N}}$ with the corresponding subset of $\mathbb{N}$, let $\varphi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the Borel map such that:

- $\varphi(r)$ is the strictly increasing enumeration of $r \cap 2\mathbb{N}$, if $r \cap 2\mathbb{N}$ is infinite;
- $\varphi(r)$ is the zero function, otherwise.

Then it is clear that for each function $h \in \mathbb{N}^{\mathbb{N}}$, the $\equiv_T$-invariant Borel set

$$S_h = \{ r \in 2^{\mathbb{N}} \mid (\exists s \in 2^{\mathbb{N}})\,( s \equiv_T r \text{ and } h < \varphi(s))\,\}$$

contains a cone. Now suppose that $g : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $=^*$ such that $\varphi(r) \leq^* g(r)$ for all $r \in 2^{\mathbb{N}}$. Applying Theorem 1.4, it follows that there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $g$ maps $C$ into a single $=^*$-class; say, $[h]_{=^*}$. But this means that $S_h \cap C = \emptyset$, which is a contradiction. □

**Corollary 5.3** $(MC)$**.** *If $E$ is a weakly universal countable Borel equivalence relation, then $E$ is not Borel-Bounded.*

*Proof.* If $E$ is weakly universal, then $\equiv_T \leq^w_B E$. Hence, by Theorem 3.1, there exists a countable Borel equivalence relation $S \subseteq E$ such that $S \sim_B \equiv_T$. Applying Boykin-Jackson[2, Lemmas 10 and 11], it follows that $E$ is not Borel-Bounded. □

The proof of Theorem 5.2 makes use of the $E_0$-$m$-ergodicity of the Turing equivalence relation $\equiv_T$. Unfortunately, this argument cannot be carried out within the usual measure-theoretic setting. To see this, suppose that $(X, \mu)$ is a standard Borel probability space and that $\theta : X \to \mathbb{N}^{\mathbb{N}}$ is a Borel map. Then the Borel-Cantelli Lemma implies that there exists a function $h \in \mathbb{N}^{\mathbb{N}}$ such that

$$\mu(\{ x \in X \mid \theta(x) \leq^* h \}) = 1.$$

This simple observation has the following striking consequence.

**Theorem 5.4** $(MC)$**.** *Let $E$ be a countable Borel equivalence relation on the standard Borel space $X$ and let $\mu$ be a (not necessarily $E$-invariant) probability measure on $X$. Then there exists a Borel subset $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \restriction Y$ is* not *weakly universal.*

*Proof.* Let $\varphi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the Borel map defined in the proof of Theorem 5.2. Then, by an easy application of the Feldman-Moore Theorem, there exists a Borel map $\psi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for all $r,\ s \in 2^{\mathbb{N}}$, if $s \equiv_T r$, then $\varphi(s) \leq^* \psi(r)$. Let

$f : X \to 2^{\mathbb{N}}$ be a weak Borel reduction from $E$ to $\equiv_T$ and let $\theta : X \to \mathbb{N}^{\mathbb{N}}$ be the Borel map defined by $\theta = \psi \circ f$. Then there exists a function $h \in \mathbb{N}^{\mathbb{N}}$ such that the Borel subset

$$Y = \{ x \in X \mid \theta(x) \leq^* h \}$$

satisfies $\mu(Y) = 1$. Let $Z = [ f(Y) ]_{\equiv_T}$. Then for each $r \in Z$, we have that $\varphi(s) \leq^* h$ for all $s \equiv_T r$. As in the proof of Theorem 5.2, this implies that $2^{\mathbb{N}} \smallsetminus Z$ contains a cone. Applying Corollary 2.4, if follows that $\equiv_T \restriction Z$ is not weakly universal. Since $(E \restriction Y) \leq^w_B (\equiv_T \restriction Z)$, it follows that $E \restriction Y$ is not weakly universal. $\square$

In particular, assuming Martin's Conjecture, the complexity of a weakly universal countable Borel equivalence relation always concentrates on a null set. This answers Thomas [18, Question 3.22].

*Remark* 5.5. In [6], Hjorth proved that there exists a countable Borel equivalence relation $E$ on a standard Borel space $X$ with an invariant probability measure $\mu$ such that $E \restriction Y$ is not essentially free whenever $Y \subseteq X$ is a Borel subset with $\mu(Y) = 1$. Arguing as in the proof of Theorem 5.4, it follows that there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $\equiv_T \restriction ( 2^{\mathbb{N}} \smallsetminus C )$ is not essentially free.

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