

# Resolving Insolubilia: Internal Inconsistency and the Reform of Naive Set Comprehension—An Addendum

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A further reformulation of Naive Set Comprehension related to that proposed in “Resolving Insolubilia: Internal Inconsistency and the Reform of Naive Set Comprehension” (2012) is possible in which contradiction is averted not by excluding sets such as the Russell Set but rather by treating sentences resulting from instantiation of such sets as the Russell Set in their own descriptions as invalid. So the set of all sets that are not members of themselves in this further revision is a valid set but the claim that that set is or is not a member of itself is not validly expressible. Such an approach to set comprehension results in a set ontology co-extensive with that permitted by the Naive Set Comprehension Principle itself. This approach (that may be called Revised Set Comprehension II) has as strong a claim to consistency as that formulated in “Resolving Insolubilia: Internal Inconsistency and the Reform of Naive Set Comprehension.”

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In “Resolving Insolubilia: Internal Inconsistency and the Reform of Naive Set Comprehension” (2012), it was shown that the set theoretical paradoxes are occasioned not by self-reference, vicious circles, or a set’s being “oversize” but rather by the descriptions of sets implicated in the paradoxes containing a descriptive matrix of the form “ $\neg(x \in y)$ .” The most notable, indeed the archetype, is to be found in Russell’s Paradox occasioned by its description “ $\neg(x \in x)$ .” All the set theoretical paradoxes with the exception of the Burali-Forti Paradox were shown to be traceable to such a matrix.

It was further shown that a set comprehension system based on the Naive Set Comprehension system (but excluding putative sets whose descriptions contain the matrix “ $\neg(x \in y)$ ”) must be consistent because once the Russell matrix is excluded from set descriptions every permissible set description becomes a linguistic truth of naming and thus can be considered as an analytical truth. So the set of green things is merely a collective expression for all things that are green, but the description of the Russell Set leads to contradiction because of the underlying structure of its biconditional description and when the individual instantiated in the description is the Russell set itself.

Unlike the Null Set (its standard description is the collection of all those individuals which are not identical to themselves), the Russell set is non-empty but an instantiation of the Russell set into its own description directly leads to a contradiction. The Russell Paradox also serves as a proof that unrestricted “Naive

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Set Comprehension is both inconsistent and false in a bivalent system. Although a set theory lacking a set comprehension principle is conceivable (set descriptions giving rise to contradiction might conceivably be excluded on an ad hoc basis), such an approach risks paralyzing uncertainty to the orderly functioning of reasoning about sets.

Traditional approaches to the set theoretical paradoxes are generally in two categories:

- Limiting the recognition of sets in set comprehension so as to prevent the paradoxes from arising such as limiting set comprehension to the cumulative hierarchy as in Zermelo-Frankel Set Theory;
- Limiting sentence making and/or instantiation so as to prohibit the making of certain sentences asserting membership or non-membership of certain sets. Examples include Whitehead and Russell's Type Theory, von Neumann-Bernays-Gödel Set Theory, and Quine's stratification of sets in his systems.

All such existing approaches have the drawback of a relatively impoverished set ontology limiting the expressive usefulness of the set concept. In addition, there is the philosophical problem caused by the non-logical bases of the prohibitions. A practical concern in many of these systems is that potentially useful sets such as the Universal Set that are not productive of contradiction or other problems are excluded from comprehension.

The approach proposed in "Resolving Insolubilia: Internal Inconsistency and the Reform of Naive Set Comprehension" is in the first category; Russell's Set and similar sets are excluded by that approach so the fatal instantiations can never arise. This approach offers a very large set ontology and it was claimed in that paper that it offered the largest possible short of inconsistency.

On reflection, it is suggested that an even larger ontology is possible, indeed one co-extensive with that of Naive Set Comprehension itself because available comprehension is governed by the Naive Comprehension Principle. Rather than excluding sets such as the Russell Set from set comprehension, it appears feasible to merely exclude certain sentences arising from instantiations in the problematic set descriptions. On this approach, it thus becomes possible to talk about the Set of All Sets that are not members of themselves. The set of all abstract concepts, for example is not a member of the Russell Set. But the set of all non-aligned countries is a member of the Russell Set in that it conforms to the Russell Set's membership description. Under this alternate system, it is not validly expressible to state that the Russell Set is or is not a member of itself. Under this approach every predicate describes a set, empty or non-empty; intensional concepts correspond with extensional ones.

Under Reformed Set Comprehension as originally proposed the naive comprehension principle is valid except where the following schema ("the schema") is applicable:

$$(\forall x) (\exists y) (x \in y \Leftrightarrow P(x)) \Rightarrow (\exists u) (u \in y \Leftrightarrow \neg(u \in y))$$

Proof of u's existence is to be judged by inference solely from the putative set description via the standard rules of inference and general logical principles. Hence, the existence of a relevant entity u must be inferred from the set description alone and P(x) must be  $\neg(x \in y)$  or at least contain it in some form such as a conjunct.<sup>1</sup> An example is where there are multiple quantifiers in the description where it can be inferred that there is one instance where all the variables take the same identity.

The proposed alternate system (which is called Reformed Set Comprehension II) mandates that no instantiation for u can be validly made in set descriptions conforming to the schema above if one could otherwise infer:

$$u \in y \Leftrightarrow \neg(u \in y)$$

Such an instantiation does not therefore produce a valid sentence in Reformed Set Comprehension II.

It is clear a contradiction cannot arise under either version of Reformed Set Comprehension if either the set descriptions conforming to the schema are excluded or sentences arising from the prohibited instantiations are excluded from the relevant systems. In both cases, it is the instantiations that generate the inconsistency not the existence of the questionable sets per se. The claims for the consistency of the first and this new version of Reformed Set Comprehension are made on the same analytical foundation. The second has therefore strong claims to parity of reasoning with the original version.

The advantages of both versions are therefore overwhelming: consistency and untrammelled set ontologies although this latest version has the greatest expressive power short of contradiction.

## Notes

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1. Or its equivalent: any proposition containing other propositional connectives can of course be reformulated to one containing conjunction and negation alone.

## Works Cited

Thompson, Neil. "Resolving Insolubilia: Internal Inconsistency and the Reform of Naïve Set Theory." *Philosophy Study* 2 (2012): 417-31.