

Kazimierz Ajdukiewicz's philosophy of mathematics

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Abstract Ajdukiewicz's account of mathematical theories is presented and analyzed. Theories consist of primary (original) and secondary (derivative) theorems. Theories go through three phases or stages: (a) preaxiomatic and intuitive, (b) axiomatic but intuitive, (c) axiomatic and abstract, whereas the final stage takes two forms: definitional and formal. Each stage is analyzed. The role of the concepts of truth, evidence, consequence, and existence is examined. It is claimed that the second stage is apparent or transitory, whereas the initial and final stages are vital and constitute two salient attitudes to mathematics, focused on truth or consequence respectively. It is also claimed they are attitudes rather than stages, and the crucial difference between them concerns effectiveness. The chief question of philosophy of mathematics turns out to be to determine whether mathematical theories are assertive or hypothetical.

Keywords Ajdukiewicz · Mathematics · Deduction · Theory · Development

Kazimierz Ajdukiewicz was a pioneer of meta-mathematics related to mathematical logic and played a significant role in many pivotal metalogical discoveries. Although a metaphysical account of mathematical objects is hardly to be found in his work, there are undoubtedly valuable, methodologically oriented contributions to the philosophy of mathematics. The contributions turn out to be original, partially prophetic, locally requiring improvement, correction or completion, and didactically brilliant. They amount to a comprehensive account of mathematical (deductive) theories (see Borkowski 1965, 1966; Batóg 1995; Murawski 2014; Woleński 1985, 1989).

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Theories in general

As Roman Murawski pointed out, Ajdukiewicz's account of mathematical theory was strongly influenced by the formalism of David Hilbert, whose lectures Ajdukiewicz attended in 1913 in Göttingen (Murawski 2014: 101–102). Ajdukiewicz followed Hilbert's formalism but he also used terminology tainted by psychologism. He often spoke about “thoughts”—that is propositional contents expressed by sentences. However, Murawski claims that in this case there is no psychologism (Murawski 2014: 106). A significant view of a theory is that it is a set of propositions (statements, formulas) some of which are admitted (affirmed), others are not. Those admitted are called theorems. The set of theorems is a subset of the set of propositions.

Propositions consist of words or terms, some of which may be borrowed from underlying theories, e.g. the term ‘and’ in arithmetic is borrowed from logic. Terms which are not borrowed are known as characteristic terms. At most some characteristic terms are defined by means of other characteristic terms and borrowed terms. They are *derivative*. The undefined characteristic terms are *primitive*.

Theorems are divided into *primary (original)* and *secondary (derivative)* theorems. The distinction is ambiguous, but usually does not cause misunderstandings. In some contexts the distinction is parallel to that of direct and indirect justification. A proposition is justified directly if it is justified by means of no other proposition of the theory in question. Otherwise it is justified indirectly (Ajdukiewicz 1959: 67–68). In other contexts, especially where only deductive theories are involved, some theorems are designated as *primary (original)*. The set of theorems is identical with the set of all consequences of the primary theorems. Hence, all consequences of the primary theorems, and they alone, are necessarily classified as theorems. Theorems which are not primary are called *secondary (derivative)* theorems. In the first context, it is legitimate to say that the theorems are simply primary theorems, and those justified with use of the primary theorems. In the other context, it is legitimate to say that the theorems are simply the primitive theorems and all their consequences.

In his early works, published before the Second World War, Ajdukiewicz differentiates among the sciences depending on the legitimate ways to justify primary theorems. In the deductive sciences no proposition would be considered a primary theorem but as a self-evident truth, i.e. the propositions expressing the semantic content of words (postulates). In the natural (empirical) sciences purely factual (perceptual) theorems are primary also. And in humanities theorems based on the comprehension of texts are also primary (Ajdukiewicz 1938a: 287, 1938b: 133). The distinction into primary and derivative theorems is taken here in the second sense. By contrast, in his last works, Ajdukiewicz divides the sciences according to the legitimate ways to justify secondary theorems. In the deductive sciences deduction is the uniquely legitimate way to derive secondary theorems, whereas in the inductive sciences both deduction and induction are taken as legitimate (Ajdukiewicz 1965b: 178). The first sense of the distinction into primary

and derivative theorems is involved here. In practice, both before and after the war, primary as well as secondary theorems were taken into account.

Ajdukiewicz shares the uncontroversial view that deduction is the only legitimate way to justify secondary statements in the deductive sciences (Ajdukiewicz 1965b: 178–179). Other kinds of justification cannot play any other than a heuristic role. In particular no experience can justify secondary mathematical statements (Ajdukiewicz 1938b: 134–138). No such uncontroversial account of primary mathematical statements could be provided (cf. Murawski 2014: 107–111).

Development of deductive theories

Ajdukiewicz's account of the development of mathematical theories is quite original. In his interwar works Ajdukiewicz speaks of two stages or phases of the theoretical development of mathematics: *preaxiomatic* and *axiomatic* (Ajdukiewicz 1938b: 137). In the post-war period, and in particular in his last, unfinished, and posthumous work, *Logika pragmatyczna* (1965b), the axiomatic stage is subdivided into *axiomatic intuitive* and *axiomatic abstract* stages. In addition, the preaxiomatic stage is often called *preaxiomatic intuitive* (Ajdukiewicz 1965b: 6–7). Ajdukiewicz does not explain what he means, but it may be argued that the idea of the abstract stage is virtually contained in his early works concerning the nature of formal theories (Ajdukiewicz 1921b: 2–3, 9–10). Hence the doctrine remains coherent and stable; however it seems probable that the philosophical significance of the abstract stage was not sufficiently evident until Ajdukiewicz's last years. Ajdukiewicz must have finally realized that the opposition between the abstract and intuitive stage is fundamental; it is even more important than the earlier recognized opposition between the preaxiomatic and the axiomatic stages.

Preaxiomatic intuitive stage

The initial phase of any mathematical theory is *preaxiomatic (intuitive)*. Since there are no other preaxiomatic phases, it is permissible to drop the word 'intuitive'. However, one should be constantly aware of the vital feature shared with the axiomatic intuitive phase in contrast with the axiomatic abstract phase. Three features are listed, characteristic of mathematical method at the preaxiomatic stage: (a) a formula may be considered a primary theorem if it seems evident (obvious); (b) a formula may be considered a secondary theorem if it is an evident consequent to some previously recognized theorems; (c) a word may appear in any formula only if the word is either itself comprehensible or has been defined by means of the words previously allowed to appear in formulas (Ajdukiewicz 1965b: 181). Roughly speaking, at the initial stage of a theory mathematical statements are formulated in vernacular and may be accepted if and only if they are either evident or consequent to previously accepted ones.

The preaxiomatic phase seems to match the spontaneous, reflexive, instinctive way to practice mathematics, the way everyone experiences at school.

A distinctive feature of the preaxiomatic phase of mathematics is its peculiar *openness*. Ajdukiewicz says literally that neither the set of primary theorems nor the vocabulary is ever closed: “In a deductive science at its intuitive preaxiomatic stage the range of primary theorems, i.e. the theorems to be accepted without a proof, is never closed. A researcher may legitimately appeal at any time to a theorem, which he has neither proved nor has ever previously accepted without a proof, if only he considers the theorem to be generally evident. Nor is the list of terms at his disposal ever closed even though he has not defined them. The only requirement for a word to be used legitimately is that it be commonly understood.”¹ (Ajdukiewicz 1965a, b: 181)

Patency or obviousness is the crucial and deciding criterion at the initial stage of mathematical theory. It is the criterion for primary theorems: “[...] at the preaxiomatic stage any theorems are acceptable without proof which we could expect more or less everyone would recognize as evident.”² (Ajdukiewicz 1938b: 137) It is also the criterion for deriving one theorem from others: “A deductive science appears at the preaxiomatic, say primitive, stage as a system of theorems interrelated by logical connections, especially the consequence relation, where truth is attributed to a theorem based on the theorem being consequent to other true theorems, and assumptions are statements possessing some degree of obviousness. These theorems are neither explicitly stated, nor do they necessarily appear as ultimate assumptions: sometimes they appear as [derivative] theorems. At the initial stage it is sufficient for any theorem to be either evident or related to evident theorems by evident consequence relations.”³ (Ajdukiewicz 1921b: 1) Finally what counts is the criterion of vocabulary: “All concepts are intuitively given, the word that designates it is either comprehensible with no further explanation or is reducible by an array of definitions to directly comprehensible words, to those commonly said to require no explanation. Furthermore, words requiring no explanation as well as concepts needing no analysis are not explicitly listed and do not always necessarily appear.”⁴ (Ajdukiewicz 1921b: 1) It is obvious that self-

¹ ‘W nauce dedukcyjnej, uprawianej w stadium intuicyjnym przedaksjomatycznym, liczba twierdzeń pierwotnych, tj. twierdzeń przyjmowanych bez dowodu, nie jest nigdy zamknięta. Badacz może w każdej chwili odwołać się do jakiegoś twierdzenia, którego ani nie dowiódł, ani bez dowodu już poprzednio nie przyjął, jeśli tylko uważa je za powszechnie oczywiste. Nie jest też nigdy zamknięta lista terminów, którymi się może posługiwać, mimo że ich nie zdefiniował. W każdej chwili wolno mu się posłużyć terminem, którego nie zdefiniował, jeśli tylko uważa, że jest on powszechnie w taki sam sposób rozumiany.’

² ‘[...] na stadium przedaksjomatycznym dopuszczalnymi bez dowodu twierdzeniami są wszelkie twierdzenia, mogące liczyć na to, że dla wszystkich mniej więcej ludzi są oczywiste.’

³ ‘W owym przedaksjomatycznym, nazwijmy je pierwotnym, stadium występuje nauka dedukcyjna jako system twierdzeń stosunkami logicznymi, a nade wszystko stosunkiem wynikania powiązanych, z których każdemu przypisuje się prawdziwość opartą na wynikaniu z innego prawdziwego twierdzenia, zaś jako założenia występują sądy, którym przysługuje jakaś oczywistość. Sądy te nie są ani wyraźnie wymienione, ani też niekoniecznie występują jako założenia ostateczne: czasami bowiem występują jako twierdzenia. W owym pierwotnym stadium wystarczy, jeżeli każdy w nauce występujący sąd jest bądź oczywisty, bądź też oczywistymi związkami wynikania powiązany z sędami oczywistymi.’

⁴ ‘Każde pojęcie jest intuicyjnie dane, wyraz, który je oznacza, bez bliższych wyjaśnień zrozumiały lub też przez szereg definicji sprowadzony do wyrazów zrozumiałych bezpośrednio, o których się zwykło mówić, że ich wyjaśniać nie trzeba. Poza tym owe już nie wymagające określenia wyrazy i te nie

evidence is the universal criterion in the initial phase of mathematics. Hence, it may be of some interest that no account of self-evidence has been provided.

Axiomatic intuitive stage

Whereas openness is distinctive of the preaxiomatic phase, absence of openness indicates the transition to the second, axiomatic intuitive phase. The range of primary theorems as well as words becomes fixed or established in a sense. In the early works only the establishment of the range of theorems is mentioned: “[...] hence, in the axiomatic phase of a deductive science, a few explicitly listed sentences, called axioms, are ultimate premises, accepted without proof and constitute the basis for proofs.”⁵ (Ajdukiewicz 1938b: 139). Fixing terminology (*ustalenia terminologiczne*) appears in the last work:

The transition to the axiomatic phase consists in fixing the range of primary theorems, i.e. those acceptable without proof within the scope of a given science, as well as the range of primary terms, i.e. those which we can have recourse without definition. Once the axiomatic phase has been reached, it is no longer legitimate to accept any evident statement without proof or to use any commonly understood word without definition. It would be legitimate only in case the statement or the word in question appears on a corresponding list.⁶ (Ajdukiewicz 1965a, b: 182)

Ajdukiewicz emphasized that at the axiomatic intuitive stage the concept of primary theorem has been at once tightened and expanded: “[...] the previous rule [relating to primary theorems] has been at once tightened and expanded in the following way. Only some explicitly listed propositions, accepted without proof, are allowed as the ultimate premises of any proof, namely propositions that are true without a shadow of a doubt and that together seem sufficient to derive all the theorems that in a given domain are deducible from self-evident truths. These explicitly listed theorems, hereafter recognized as the ultimate, unproven premises of all proofs, have been

Footnote 4 continued

wymagające analizy pojęcia nie są wyraźnie wymienione i też niekoniecznie zawsze jako takie występują.’

⁵ “[...] w stadium aksjomatycznym zatem jakiejś nauki apriorycznej ostatecznymi przesłankami, przyjętymi bez dowodu, na których wolno się w dowodach opierać, są nieliczne wyraźnie wymienione zdania, zwane aksjomatami.’

⁶ ‘Przejsście w stadium aksjomatyczne polega na tym, że lista twierdzeń pierwotnych, tj. takich, które przyjmujemy nie dowodząc ich już w obrębie danej nauki, jak również lista terminów pierwotnych, tj. takich, którymi wolno nam się posługiwać bez podawania ich definicji, zostaje w pewnej fazie rozwoju tej nauki zamknięta. Po przejściu w stadium aksjomatyczne nie wolno już przyjmować każdego zdania oczywiście bez dowodu i nie wolno bez definicji posługiwać się dowolnym terminem powszechnie zrozumiałym, lecz wolno to czynić tylko wtedy, gdy zdanie to, czy ów termin, znajduje się na odpowiedniej liście.’

named *axioms* of the deductive science in question.⁷ (Ajdukiewicz 1938b: 138) The range of primary theorems has been tightened, for to be a primary theorem it is no longer sufficient that it be evident. It is also required that it be explicitly listed. The range has also been expanded, because to be a primary theorem it is also no longer necessary that it be evident. It is required that it be true beyond a doubt; however, truthfulness may be established deductively: “[...] not every self-evident truth within the scope of a science is its axiom, a certain statement becomes an axiom of the science only if it has been explicitly listed as such. Secondly, not every axiom need be immediately obvious, because the position of an axiom may be held by any proposition which seems true beyond a doubt, as it has been established by deriving the proposition deductively from obvious statements, even though it itself is not obvious.”⁸ (Ajdukiewicz 1938b: 138–139)

The description of the preaxiomatic phase can be summarised in the following way: theorems are formulated in vernacular and may be accepted if and only if they are either evident or are the consequents of previously accepted ones. The epitome of the axiomatic intuitive phase would be as follows: theorems are formulated in some fixed and specific part of the vernacular and may be accepted only if they either appear on some fixed list of evident theorems or are the consequents of those previously accepted.

Underlying theories

The inner hierarchy of theorems, introduced at the axiomatic stage, is accompanied by an outer hierarchy of theories: “Among deductive systems there exists a kind of hierarchy, because some deductive systems are based upon others, which means that they include the axioms of underlying systems among their own axioms. For example, all mathematical deductive systems are based on the deductive system of formal logic. Every mathematical system considers (tacitly for the most part) the axioms of logic to be theorems acceptable without proof, i.e. axioms. Deductive systems of geometry are based on the deductive system of arithmetic (as well as logic, of course) in the above mentioned sense. Hence, the system of formal logic is the ultimate basic deductive system, as it serves all other systems as a basis and is

⁷ “[...] zacieśniono i rozszerzono zarazem dotychczasowy przepis w sposób następujący. Jako ostateczne przesłanki wszelkiego dowodu, przyjmowane bez dowodu, poczęto dopuszczać tylko *pewne* wyraźnie wymienione spośród twierdzeń, co do których prawdziwości nie nasuwała się żadna wątpliwość, i które razem wzięte uważano za wystarczające do tego, aby z nich wydedukować wszystkie twierdzenia, dające się w danej dziedzinie wyprowadzić dedukcyjnie z pewników. Te wyraźnie wskazane twierdzenia, dopuszczalne odtąd jako ostateczne niedowodzone przesłanki wszelkiego dowodu, nazwano *aksjomatami* danej nauki apriorycznej, czyli dedukcyjnej.”

⁸ “[...] nie każdy pewnik należący do zakresu danej nauki będzie jej aksjomatem, lecz tylko wtedy pewnik stanie się aksjomatem danej nauki, jeśli zostanie jako taki wyraźnie wymieniony. Po drugie nie każdy aksjomat musi być pewnikiem, gdyż w roli aksjomatu może także figurować zdanie, które nie jest bezpośrednio oczywiste, lecz którego prawdziwość nie budzi w nas żadnej wątpliwości dzięki temu, że potrafiliśmy się o niej przekonać, wyprowadzając to twierdzenie na drodze dedukcji z sądów oczywistych.”

itself based on no other system.”⁹ (Ajdukiewicz 1938b: 139–140) This view was reinforced as Ajdukiewicz came to consider that the sciences are based on one another (Ajdukiewicz 1965b: 182–184). The hierarchy of theories or even of sciences is such that on higher levels of this hierarchy non-evident axioms are allowed for. It has not often been noticed that the hierarchy of theories was originally described by Thomas Aquinas at the beginning of his *Summa Theologica* (p. I, qu. 1).

Reasons for axiomatization

Ajdukiewicz examined the reasons for passing from the preaxiomatic to the axiomatic intuitive phase in many of his works. But these explanations differ from work to work and include: (a) exploration of the foundations of knowledge, (b) uncovering logical connections among primary theorems, and (c) discovering paradoxes (antinomies).

Firstly, precision may be lacking in the preaxiomatic phase, though it may be considered inevitable. According to Ajdukiewicz, the inaccuracy is such as to raise questions concerning the actual foundations of theories. Ajdukiewicz appears tacitly to regard the axiomatic phase as somehow superior to the preaxiomatic one. It could be even said that deductive sciences feature an inherent drive for axiomatization. Ajdukiewicz speaks literally of satisfaction:

The above described [preaxiomatic] phase cannot satisfy science. The alleged absolute comprehensibility of primitive symbols often turns out to be illusory, and consequently the theory turns out to be ambiguous. Similarly, the apparent obviousness of primary assumptions becomes relative and subjective. Hence, science must aim to establish ultimate assumptions and concepts, as any vagueness relative to them may become destructive. Ultimate assumptions and concepts must necessarily be explicitly listed, and they must be actually ultimate, which means that every theorem of the science must be reducible to them.¹⁰ (Ajdukiewicz 1921b: 1–2)

⁹ ‘Wśród systemów dedukcyjnych istnieje pewna hierarchia, niektóre bowiem systemy dedukcyjne opierają się na innych w tym sensie, że aksjomaty tego innego systemu zaliczają między swoje własne aksjomaty. I tak np. wszystkie systemy dedukcyjne matematyczne opierają się na systemie dedukcyjnym logiki formalnej. Każdy bowiem system matematyczny zalicza (zwykle milcząco) aksjomaty logiki do twierzeń przyjmowanych bez dowodu, tj. do aksjomatów. Zwykle systemy dedukcyjne geometrii opierają się we wspomnianym wyżej sensie na systemie dedukcyjnym arytmetyki (i oczywiście także logiki). Najbardziej podstawowym systemem dedukcyjnym jest więc system logiki formalnej, służy on bowiem wszystkim innym systemom dedukcyjnym za podstawę, a sam żadnego innego systemu dedukcyjnego nie zakłada.’

¹⁰ ‘Wyżej scharakteryzowane stadium nie może nauki zadowolić. Rzekomo bezwzględna zrozumiałość symboli pierwotnych okazuje się niejednokrotnie złudna, a co za tym idzie, teoria staje się wieloznaczna. Podobnie w związku z objawem powyższym oczywistość założeń pierwotnych staje się względna i subiektywna. Musi tedy nauka dążyć do ustalenia założeń i pojęć podstawowych, bo niejasność w ich zakresie może stać się zgubna. Założenia i pojęcia podstawowe muszą być koniecznie wyraźnie wymienione i być w istocie założeniami podstawowymi, tj. do nich musi być każde twierdzenie występujące w nauce sprowadzone.’

Secondly, the discovery of logical connections among the self-evident propositions, i.e. primary theorems of mathematics, may also incite the search for axiomatization:

When at the preaxiomatic stage consequences were deductively derived from self-evident propositions, it turned out among other things that some of the latter are deductively derivable from others. [...] The discovery that some self-evident statements are deductively derivable from others brought to mind the idea to change the previous rule concerning which theorems should be recognized as acceptable without a proof as the basis for all proofs.¹¹ (Ajdukiewicz 1938b: 138)

Thirdly, Ajdukiewicz points to paradoxes as a reason for the sweeping reform of mathematics:

It has happened repeatedly that apparently obviously false conclusions seem to follow obviously from obvious premises. An example from the history of Greek science is the proof that there exist segments incommensurate with one another. [...] It subverted Greek mathematicians' trust in self-evidence and induced one of them, Euclid, to reconstruct the method of geometry, namely to give it the form of an axiomatic system, hence to pass from the preaxiomatic to the axiomatic phase, though still intuitively.¹² (Ajdukiewicz 1965a, b: 182)

Paradoxes, especially those called antinomies that consist in proofs of contradictory theorems, seem commonly to be considered as the chief reason to turn to the axiomatic phase (Batóg 1977: 5).

The peculiarity of the second stage

Four observations seem to be vital. Firstly, the concept of self-evidence remains present at the axiomatic intuitive stage. The primary theorems are not simply any theorems at all, they are evident theorems. Ajdukiewicz claims that they may actually not be evident, though they must be established as unshakeable in one way or another. Similarly, the listed primitive terms comprise only comprehensible terms.

Secondly, the intention seems to be that the theorems at those two stages be exactly of the same kind. For all the theorems consequent to the axioms should also

¹¹ 'Gdy w stadium przedaksjomatycznym poczęto z pewników na drodze dedukcyjnej wyprowadzać wnioski, stwierdzono m. i., że niektóre pewniki dają się wyprowadzić z innych na drodze dedukcyjnej. [...] Zwrócenie uwagi na to, że niektóre pewniki dają się na drodze dedukcji wyprowadzić z innych, nasunęło myśl, aby zmienić dotychczasowy przepis odnoszący się do tego, jakie sądy wolno nam przyjmować bez dowodu jako ostateczne przesłanki wszelkiego dowodu.'

¹² 'Niejednokrotnie bowiem się zdarzało, że z oczywistych przesłanek w sposób również oczywisty zdawały się wynikać wnioski, które wydawały się oczywiście fałszywe. W historii nauki greckiej przykładem tego był dowód istnienia odcinków niewspółmiernych. [...] Podważyło to ufność matematyków greckich do oczywistości i skłoniło jednego z nich, mianowicie Euklidesa, do przebudowy metodologicznej struktury geometrii, mianowicie do nadania jej postaci systemu aksjomatycznego, a więc do przejścia ze stadium przedaksjomatycznego w stadium aksjomatyczne, choć wciąż jeszcze intuicyjne.'

be consequent to the self-evident statements, since the axioms are consequent to them. On the other hand, the axioms should be sufficient to deduce all the theorems of preaxiomatic mathematics. Thus at the axiomatic intuitive stage the sole intention is to put order into preaxiomatic mathematics in order, to tidy it up. However, the intention is clearly unfeasible. The deep reasons for this will become clear soon. At the moment it is sufficient to notice that one of the chief reasons to enter the axiomatic phase is to avoid antinomies. And antinomies are clearly deducible from self-evident statements. Thus it seems that at the axiomatic intuitive stage a theory at the preaxiomatic stage is actually tightened. Mathematics at its second stage turns out to be simply a fragment of preaxiomatic mathematics, selected or singled out for one reason or another. The *tightness* is the essential difference at the axiomatic stage in contrast with the preaxiomatic stage's openness. On the other hand, if preaxiomatic mathematics is to be tightened, a question can arise whether there are good reasons to expand it in any way. For instance, one could ask why the range of the axioms is to be limited to the consequences of self-evident propositions? Perhaps the truthfulness of an axiom could be established in any way, including external ways. Why not? These and similar questions lead to the final, axiomatic abstract phase of mathematical theory, and enlighten the profound nature of the axiomatic intuitive stage.

Thirdly, the procedure of deriving one theorem from another is not tightened. It remains purely intuitive. Describing the preaxiomatic phase, Ajdukiewicz referred to (a) primary theorems, (b) derivative theorems, and (c) the lexicon. However, describing the axiomatic intuitive phase he tightens only (a) primary theorems and (c) lexicon and does not explain why (b) derivative theorems are to be passed over. One possible solution is that Ajdukiewicz had in mind the history of mathematics. Another solution has to do with the boundary between the content of a theory and its underlying logic. In the axiomatic intuitive stage the content of a theory is established, whereas the underlying logic remains untouched. Actually, the above sketched solutions are not mutually exclusive. For it is simply the case that mathematical logic was not axiomatized for hundreds of years after Euclid had first axiomatized mathematical theory.

Fourthly, the term 'axiom' appears which has not been used to describe the preaxiomatic phase. It follows that the word 'axiom' is not a synonym for the word 'primary theorem'. Ajdukiewicz claims that neither are all axioms self-evident truths nor are all self-evident truths axioms. It should be also added that all axioms are primary theorems, but not conversely. Primary theorems are referred to as axioms only if they are explicitly listed. Although it is not perfectly clear what such listing should exactly mean or what a list in question should look like, the distinction will shortly turn out to be vital.

Axiomatic abstract stage

The two first phases share a certain intuitive character. The meaning of the terms to be used is based upon the vernacular and the theorems are to be certain:

It is distinctive for the intuitive phase of deductive sciences that primitive terms, i.e. the terms requiring no definition, are to be understood in their existing meaning and that primary theorems or axioms are to be evident when understood in that very way, i.e. to be evident for everyone without proof. The primary difference between the intuitive and the abstract practice of deductive sciences lies therein that in the second stage we abstract from the existing meaning of specific primitive terms; their meaning is yet to be established.¹³ (Ajdukiewicz 1965a, b: 188)

At the final i.e. axiomatic and abstract stage, neither the existing meaning of the terms to appear in statements nor any external criteria for accepting or rejecting propositions are involved. It seems reasonable to claim that at the abstract stage any arbitrary propositions may be listed as axioms and theorems are exactly the consequences of the axioms.

There are two versions of the axiomatic abstract method of mathematics. In the first version, the specific terms play a role analogous to that of unknowns in mathematical equations, whereas axioms are actually semantic postulates. In the other version, statements are actually schemata. Let us call the first version *definitional* and the second *formal*. In earlier works one or other version appears, whereas in the posthumous *Logika Pragmatyczna* (1965b) they are clearly distinguished.

The definitional method

As just mentioned, in the first, definitional version of the axiomatic abstract method, axioms take the position of semantic postulates. This means that any existing meaning they have is to be totally ignored. Apart from any existing meaning the postulates are considered true by convention. Thus the characteristic terms, appearing in the postulates, are to be understood in any arbitrary way, provided the meaning makes the postulates true: “By retaining the meaning of loanwords borrowed from the sciences on which we depend, we decide that characteristic terms of the science in question name such objects that meet conditions put on them by axioms, apart from what the terms designated in their existing meaning and whether they had any meaning whatsoever.”¹⁴ (Ajdukiewicz 1965a, b: 188) An interwar text contains a similar account of formalized theory (Ajdukiewicz 1938a:

¹³ ‘Stadium intuicyjne nauk dedukcyjnych tym się charakteryzuje, że terminy pierwotne, tj. terminy, których się używa bez podawania ich definicji, bierze się w znaczeniu zastanym i od twierdzeń pierwotnych czy od aksjomatów żąda się, aby były przy zastanym znaczeniu zawartych w nich terminów oczywiste, tj. by były dla każdego przekonujące bez podawania dowodu. Zasadnicza różnica między intuicyjnym sposobem uprawiania nauk dedukcyjnych a sposobem występującym w stadium abstrakcyjnym polega na tym, że się w tym drugim stadium abstrahuje od zastanego znaczenia swoistych terminów pierwotnych, że się ich znaczenie dopiero konstryuuje.’

¹⁴ ‘Zachowując mianowicie znaczenie terminów zapożyczonych z nauk, na których się opieramy, postanawiamy co do terminów swoistych danej nauki, że te terminy mają być nazwami takich tworów, które czynią zadość warunkom, jakie na nie nakładają aksjomaty, nie zważając na to, czego nazwami te terminy były przy ich dotychczasowym (zastanym) znaczeniu i czy w ogóle miały dotychczas jakies znaczenie.’

294–295). That understanding of the abstract stage is related to the account by J. D. Gergonne (Kneale and Kneale 1962: 385).

The formal method

In the other, formal, version of the axiomatic abstract method, no meaning whatsoever is attributed to characteristic terms of the science in question, be they existing or conventional:

Practicing deductive sciences at the axiomatic abstract stage [in the formal way] we ignore the existing meaning of characteristic primitive terms and prejudge nothing about their meaning. The characteristic primitive terms are handled like variables, which have no meaning other than that fixed by syntactic categories. Hence, axioms as well as theorems of a science at the abstract stage are no longer statements, i.e. they cannot be considered true or false. They become rather propositional schemata, because the characteristic primitive terms to appear in those schemata are essentially free variables, not bound by any quantifier.¹⁵ (Ajdukiewicz 1965a, b: 191)

A similar account of formalized theories appear in another interwar work:

[...] a range of symbols is mentioned; those symbols appear in axioms and based on the axioms a range of “theorems” is “proved”. These symbols have no attributed meaning. However, why are they called symbols rather than strokes or ornaments? After all, to mean something seems to be essential for any symbol. Primitive symbols of deductive sciences are undoubtedly not symbols in the sense that vernacular words are. [...] And yet, they differ from common strokes and ornaments as well. [...] Both symbols of deductive sciences and pieces in chess are symbols in the sense that they are interrelated in one way or another. For symbols of deductive sciences these interrelations are axioms and other theorems. [...] Hence, symbols of deductive sciences are called symbols not because they supposedly “mean” or “denote” anything, but because they play a specific “role”, because they are related to one another in precisely determinate ways’ (Ajdukiewicz 1921b: 2–3)¹⁶

¹⁵ ‘Uprawiając nauki dedukcyjne w stadium aksjomatyczno-abstrakcyjnym, abstrahujemy od zastanego znaczenia ich swoistych terminów pierwotnych i nie przesądzamy niczego o ich znaczeniu. Swoiste terminy pierwotne traktujemy więc tak, jak traktujemy symbole pierwiastkowe, których znaczenie (poza ich kategorią semantyczną) jest zupełnie nieokreślone. W ten sposób zarówno aksjomaty, jak i twierdzenia nauki dedukcyjnej uprawianej w sposób abstrakcyjny przestają być zdaniem, o których można orzec, że są prawdziwe lub że są fałszywe, a stają się schematami zdaniowymi, albowiem występujące w nich swoiste terminy pierwotne są co do swej istoty symbolami zmiennymi i to nie związanymi żadnym kwantyfikatorem.’

¹⁶ ‘[...] wymieniamy szereg symboli; symbole te wchodzi w skład aksjomatów, na ich zaś podstawie «udowadniamy» szereg «twierdzeń». Symbolom tym nie przypisujemy żadnego znaczenia. Dlaczego je jednak nazywamy symbolami, a nie kreskami lub ornamentami? Wszakże—zdawałoby się—istotą symbolu jest, że symbol coś znaczący. Niewątpliwie symbole pierwotne nauk dedukcyjnych nie są w tym znaczeniu symbolami, jak np. wyrazy mowy potocznej. [...] Mimo to jednak różnią się od zwykłych kreszek lub ornamentów, [...] Zarówno symbole nauk dedukcyjnych jak i figury w szachach są o tyle

Ajdukiewicz's account of formalized theories is the standard, contemporary metalogical one. Symbols whose use in formulas is legitimate are simply listed and syntactic rules are structural and purely formal. Similarly, axioms are listed. In comparison to the axiomatic intuitive stage, the procedure to derive secondary theorems is regulated and controlled by the standard concept of proof. Proof is defined inductively, by means of modus ponens and rules of definitional replacement only. Roughly speaking, a proof of a formula A in a given theory T is any such finite sequence of formulas A_1, A_2, \dots, A_n , such that A_n is identical to A and, for any i from 1 to n , it is the case that A_i is either an axiom of logic, or a characteristic axiom of the theory T , or is derivable from earlier formulas in the sequence by means of modus ponens or some definitional replacement. The set of axioms is considered to be closed under substitution. The set of characteristic theorems being empty, the theory is a system of logic. Hence, the intuitive consequence has been tightened at this stage. Metalogical features of a theory are understood in the standard way as well (Ajdukiewicz 1965b: 192–217, 1966).

As Ajdukiewicz's account of formalized theories is standard there is no need to describe it in detail. It is worth noting instead that Ajdukiewicz was a pioneer of such an account that he had first merely sketched in the early dissertation (1921a) and later developed in book-length form in a posthumously published work (1965b). As Ludwik Borkowski and Tadeusz Batóg affirm, at the beginning of Ajdukiewicz's research, i.e. from 1913 till the late 1920s, formal logic was at the forefront of Ajdukiewicz's publications. In particular, Ajdukiewicz pioneered meta-mathematical (metalogical) investigations (Ajdukiewicz 1966: 10; Borkowski 1965: 11; Batóg 1995: 53; Murawski 2014: 102). His post-doctoral dissertation *Z Metodologii nauk dedukcyjnych* (Ajdukiewicz 1921a) appeared post-dated in 1920 and was translated from Polish into English by Jerzy Giedymin under the title *From the methodology of the deductive sciences* (Ajdukiewicz 1966). It consists of three essays: "The logical concept of proof," "On proofs of consistency of axioms" and "On the notion of existence in deductive sciences." Some parts of it appeared also separately (e.g. Ajdukiewicz 1921b). The dissertation is actually the first Polish contribution to meta-mathematics, based upon mathematical logic, and has played a significant role in many meta-mathematical discoveries, whereas mature meta-mathematics was not established earlier than in the mid-thirties (Borkowski 1965; Batóg 1995; Murawski 2014).

Existence of mathematical objects

Among standard meta-mathematical contributions there appears Ajdukiewicz's slightly strange account of the concept of existence in mathematical theories. It applies to 'absolutely pure deductive theories', where it is only in a metaphorical,

Footnote 16 continued

symbolami, o ile występują w takich a takich związkach. Związkami tymi dla symboli nauk dedukcyjnych są aksjomaty i twierdzenia. [...] Są tedy symbole nauk dedukcyjnych symbolami nie dlatego, jakoby «coś znaczyły» albo «coś oznaczały», lecz dlatego, że mają określoną «rolę», dlatego, że występują w ściśle określonych związkach.'

i.e. improper sense that we may speak of 'truth' (Ajdukiewicz 1966: 40). Although the account is hardly satisfactory, it seems worth mentioning. The work is quite early and, despite some doubts, delivers an interesting example of an early version of ontological commitment, the metalogical theory of definition or something similar.

In Ajdukiewicz's work, two existing theses are analysed and rejected: that mathematical existence is tantamount to consistence, attributed to Henri Poincaré (Ajdukiewicz 1966: 34–37), and that mathematical existence is tantamount to real existence, attributed to Bertrand Russell (Ajdukiewicz 1966: 37–39). According to Ajdukiewicz, mathematical existence is relative to a deductive theory, objects exist or not in a theory. According to Ajdukiewicz, there are three necessary conditions mathematical objects must fulfil to exist: (a) the domain condition, (b) the first consistency condition, and (c) the second consistency condition. Together they form a necessary and sufficient condition of mathematical existence, i.e. an object x exists in a theory T if and only if x meets the conditions (a), (b), and (c) within the scope of the theory T . Furthermore an object x is possible in a theory T if and only if x meets the conditions (a) and (b) within the scope of the theory T (Ajdukiewicz 1966: 44–45).

The domain condition specifies that an object existing in a theory must belong to the domain of the theory (Ajdukiewicz 1966: 40). It seems to be intuitively clear that the domain is similar to the universe of discourse: "[...] we do not attribute e.g. arithmetical existence to the sun, the lamps, the chair, etc." (Ajdukiewicz 1966: 42)

Ajdukiewicz struggled with an accurate formulation of the condition as early as 1920, when neither the salient meta-mathematical concepts nor theorems were known as yet. He claims: "Every system of axioms determines what is usually called the domain of a theory. We usually say that the domain of the theory is the set of [all] objects satisfying the axioms. However, difficulties involved in the concept of truth or satisfaction cannot be disregarded. [...] If, on the other hand, we want to apply this definition to theses of deductive theories, we shall have serious difficulty finding the reality to which the correlates of a priori sentences belong." (Ajdukiewicz 1966: 40) In this account a formula $A(x)$ is satisfied by an object a if and only if the formula $A(a)$ is a consequent of the definition of the object a , whereas the sentence $A(a)$ is true if and only if there exists an object a satisfying the formula $A(x)$. For example, a quadrangular circle satisfies the formula ' x is a circle', but the sentence 'a quadrangular circle is a circle' is not true (Ajdukiewicz 1966: 40).

Ajdukiewicz seems to claim that an object a belongs to the domain of a given theory if and only if the definition of a entails all axioms of the theory. And this is hardly easy to understand. Consider an example: "Let us consider, e.g. the axioms of arithmetic of natural numbers and let us define a as follows: " a is a natural number and a multiplied by 3 equals 2". This definition implies that the object a is an element of the domain of our theory, i.e. that it is a natural number, [...]" (Ajdukiewicz 1966: 43). In what sense does the quoted definition entail all the axioms of arithmetic?

Were other of Ajdukiewicz's claims taken into consideration, it seems probable that the point of the domain condition is that objects should be nameable or even

definable in a given theory (Ajdukiewicz 1966: 35–36). Hence, “satisfying the axioms” in this context is not to be understood in the standard way connected to Tarski-like formal semantics. The concept of satisfying should be understood with respect to the constant terms, appearing in the axioms, rather than to variables. So, to satisfy the axioms seems to mean simply to be described by the axioms, to belong to the model. Perhaps the domain condition is also, in a way, an early version of the translatability condition of the mature theory of definition.

The first consistency condition seems quite clear: an object a meets the condition means that, if a theory T is consistent, then so too must be the theory T enriched with the definition of a (Ajdukiewicz 1966: 43). That matches perfectly the consistency condition of the modern theory of definition.

An object a is possible in a given theory T if and only if a meets both conditions, that of the domain and the first one of consistency. It is definable in the sense of contemporary textbooks.

A possible object a meets the second consistency condition if and only if “it does not restrict the domain of possible objects”, i.e. the definition of a does not entail “any sentences inconsistent with definitions of other possible objects” (Ajdukiewicz 1966: 44). For example, “[...] all categorically formulated axioms of pan-geometry are entailed by the definition of the Euclidean straight line; the latter is also a consistent object. It satisfies, therefore, the first and second requirement with respect to pan-geometry. If on this basis existence were attributed to it in pan-geometry, we would have to attribute existence to the Riemannian straight line on the same basis; in consequence we would have to admit that there exists an inconsistent system [...]” (Ajdukiewicz 1966: 43).

To modernize the second consistency condition one might assume that an object a , possible in a given consistent theory T , meets the condition in question if and only if the object a is possible in every consistent extension T^* of the theory T . However, if Ajdukiewicz did want to say this, then the definition of the object a must have been entailed in a sense by the axioms of the theory T (it would then be more difficult to accept the original version of the domain condition). More accurately, a theorem that exactly one object is identical with a must have been provable in the theory T . This could complete the modern theory of definition. Would this perhaps be the Ajdukiewiczian idea of mathematical existence?

Foundations of primary theorems

The problem of justification of the primary theorems belongs to philosophy of logic as well as mathematics, and even to epistemology. It has been already largely debated (cf. Murawski 2014; Woleński 1985, 1989 and others). In Ajdukiewiczian philosophy of mathematics the question of the foundations of primary theorems is urgent at two intuitive stages. It turns out to be a complex, difficult philosophical problem with no simple solution. At the final stage the question evaporates.

At the initial stages Ajdukiewicz distinguished two versions of the problem of the basis for mathematical primary theorems: one psychological, i.e. the question of sources, the other epistemological, i.e. the question of justification (Ajdukiewicz

1965b: 184–185). The distinction is parallel to that of the general philosophical question of sources of knowledge (Ajdukiewicz 2004: 30–32). The psychological question refers to the origin of the self-evident character of mathematical primary theorems or self-evident truths. There are two principal kinds of answer: either these theorems are verified by sensory experience exceedingly often or the inclination to affirm them is a part of the innate constitution of the human mind (Ajdukiewicz 1965b: 184–185). Ajdukiewicz focuses on the method of justification of primary theorems, and discusses three basic solutions. Primary theorems are either (a) analytic, or (b) synthetic a posteriori (i.e. empirical), or (c) synthetic a priori. Ajdukiewicz found none of these solutions to be satisfactory (Ajdukiewicz 1965b: 185–188). Again, the debate is quite analogous to the general epistemological one (Ajdukiewicz 2004: 32–37).

Ajdukiewicz also considers possible justifications of primary theorems by means of (a) direct observation, (b) intuition, or (c) convention, showing that none of them is fully satisfactory. Ajdukiewicz came to think that mathematical systems are hypothetical (neutral) rather than assertoric, i.e. primary theorems depend neither on justification nor assertion (Murawski 2014: 111). This leads, again, to the third stage of mathematical theories' development.

At the axiomatic abstract stage the question of the foundations of primary theorems becomes irrelevant, for the primary theorems at this stage are not supposed to have any foundations whatsoever. On the contrary, they are supposed to be purely arbitrary (Ajdukiewicz 1965b: 187–188). It is worth noting that at no other than the final stage is a deductive science actually purely deductive. At the abstract stage deduction is truly the only method of justification, whereas at both initial stages at least primary theorems are justified, though not deductively.

Reasons for formalization

Although reasons for the change from the preaxiomatic to the axiomatic intuitive phase have been discussed (“Reasons for axiomatization” section), Ajdukiewicz provides no analogical account of the other change, that to the abstract phase. Nevertheless, it is possible to reconstruct an account.

Firstly, the natural continuation of the same development that brought about the axiomatic intuitive stage leads to the abstract stage: (a) the foundations of deductive sciences are further examined; (b) theories are further tightened. Once axioms are listed the question of other lists of axioms becomes crucial. It simply must be asked, sooner or later, what a theory would be like were false axioms listed. Having listed axioms and terms with derivation procedures remaining intuitive requires further control.

Secondly and consequently, the axiomatic but intuitive phase of deductive theories simply fails: (c) the philosophical question of justification of primary theorems remains unanswered, (d) above all the antinomies have not been successfully avoided at this stage. On the contrary, even at the heart of mathematics, in analysis, serious paradoxes have been revealed.

Thirdly, the development of mathematics and logic shows that the intuitive stage is insufficient: (e) the plurality of geometries, and (f) the appearance of mathematical logic calls for reform (Bocheński 1956: 15–17).

All in all, the axiomatic intuitive stage seems to be a merely transitory one. And Ajdukiewicz himself once speaks about going beyond it (Ajdukiewicz 1965b: 188).

Phases or attitudes

When speaking of stages or phases, Ajdukiewicz seriously means a kind of development. He means that practically every mathematical theory goes through the three stages, at least in a mathematician's soul, even if he admits that this image involves an idealization (Ajdukiewicz 1921b: 1). As has already been mentioned, he also speaks of overcoming phases (Ajdukiewicz 1965b: 188). On the other hand, the second stage seems to be somehow transitory (“Reasons for formalization” section). A question then arises whether these stages of mathematical theories are not something like general methodological attitudes towards mathematics or versions of the method rather than evolutionary phases. Perhaps the stages' sequence might have been an accidental effect of the method's development. There seem to be a serious number of formalized abstract theories without the slightest hint of an intuitive stage. On the other hand, intuitive mathematics seems unavoidable and is actually present in metatheory (metalanguage), in many kinds of application, etc.

Truth and consequence

To solve the problem just raised it seems necessary to ask what the nature is of Ajdukiewicz's stages from a contemporary point of view. The actual nature and depth of the difference from stage to stage requires examination. It is possible to have the same theory constructed or described by means of two or more different methods, e.g. the classical propositional calculus constructed by means of matrices, tableaux or natural deduction is in fact, as regards the content, one theory.

When speaking of changes in the stages of mathematical theories Ajdukiewicz uses such terms as ‘open’ and ‘closed’, e.g. the ranges of primitive terms and primary theorems remains open at the initial stage, but are closed at the axiomatic stage. Ajdukiewicz never explains what being open or closed are supposed to mean. He claims that axioms are to be ‘explicitly listed’ (Ajdukiewicz 1938b: 139) or ‘find themselves on a proper list’ (Ajdukiewicz 1965b: 182). Those phrases are certainly metaphorical. If the axioms had needed to be literally listed, no infinite axiomatics would have been possible, and as early as 1920 Ajdukiewicz is analyzing infinitely axiomatized theories (Ajdukiewicz 1966). He is perfectly aware of the existence of such infinitely axiomatized theories as Zermelo's set theory or Peano's arithmetic (Ajdukiewicz 1965a, b: 183).

My hypothesis is that that the fundamental problem of the nature of mathematics that Ajdukiewicz tacitly confronted in his account is the problem of effectiveness.

The final stage of mathematical theory is in fact the stage of a positively implemented semi-recursive or semi-decidable theory. The set of formulas, the set of axioms, and the procedure of derivation are recursive (recursively decidable), whereas the set of theorems is at least positively semi-recursive (Boolos et al. 2003: 73, 80). This is why the intuitive stage is inevitable: the concept of truth is not recursive or semi-recursive, although the concept of consequence is, as Tarski's Theorem and Gödel's First Incompleteness Theorem state (Boolos et al. 2003: 223–225). Hence, the profound boundary in mathematical method is that of at least a positively semi-recursive effectiveness. This is why the axiomatic but intuitive stage was destined to fail.

If the above formulated hypothesis is correct, there are two mutually irreducible attitudes towards the method of mathematics: one focused on the truth and the other on consequence. At the intuitive stage one asks such questions as 'how many parallel lines there are to a given line?'. At the abstract stage one asks such questions as 'what follows from the Riemannian or Euclidean theory?'

The intuitive and abstract versions of mathematics correspond perfectly to another account Ajdukiewicz provides, namely the concepts of assertive and hypothetical deductive systems. A deductive system is assertive if and only if its primary theorems are asserted, otherwise the system is hypothetical. Logical values of theorems of hypothetical systems are irrelevant, it is the consequence interrelations which are essential (Ajdukiewicz 1960a, 1965b: 192). If the above formulated hypothesis is correct, the chief question of Ajdukiewiczian philosophy of mathematics is, therefore, whether mathematics is to be assertive or hypothetical. Effective semi-recursive marks the boundary. The question remains open.

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