Expanding the Universe of Universal Logic[*]

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ABSTRACT: In (Béziau 2001), Béziau provides a means by which Gentzen's sequent calculus can be combined with the general semantic theory of bivaluations. In doing so, according to Béziau, it is possible to construe the abstract "core" of logics in general, where logical syntax and semantics are "two sides of the same coin". The central suggestion there is that, by way of a modification of the notion of maximal consistency, it is possible to prove the soundness and completeness for any normal logic (without invoking the role of classical negation in the completeness proof). However, the reduction to bivaluation may be a side effect of the architecture of ordinary sequents, which is both overly restrictive, and entails certain expressive restrictions over the language. This paper provides an expansion of Béziau's completeness results for logics, by showing that there is a natural extension of that line of thinking to n-sided sequent constructions. Through analogical techniques to Béziau's construction, it is possible, in this setting, to construct abstract soundness and completeness results for n-valued logics.

Keywords: Universal logic; Bivaluation; Galois connection; n-sequents

RESUMEN: En (Béziau 2001), Béziau ofrece un recurso para combinar el cálculo de secuentes de Gentzen con la teoría semántica general de bivaluaciones. Al hacer esto, según Béziau, es posible construir el "núcleo" abstracto de la lógica en general, donde sintaxis y semántica son las dos caras de una misma moneda. La sugerencia clave es que, mediante una modificación de la noción de consistencia máxima, es posible probar la corrección y completud de cualquier lógica normal (sin invocar la función de la negación clásica en la prueba de completud). Sin embargo, la reducción a bivaluaciones puede ser un efecto colateral de la arquitectura de los secuentes ordinarios, que es abiertamente restrictiva y entraña determinadas restricciones expresivas sobre el lenguaje. Este artículo ofrece una expansión de los resultados de completud de Béziau para la lógica, mostrando que existe una extensión natural de esta línea de pensamiento a construcciones de secuentes de n lados. Mediante técnicas análogas a la construcción de Béziau, en este marco es posible construir resultados abstractos de corrección y completud para la lógica n-valuada.

Palabras clave: Lógica universal; bivaluación; conexión de Galois; n secuentes

1. Sequents and bi-valuations

In (Béziau 2001), Béziau provides a means by which Gentzen's sequent calculus can be combined with the general semantic theory of bivaluations. In doing so, according to Béziau, it is possible to construe the abstract "core" of logics in general, where logical syntax and semantics are "two sides of the same coin".[1] The central suggestion there is that, by way of a modification of the notion of maximal consistency, it is possible to

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[*] Thanks to Ole Hjortland and Alex Tillas for helpful discussion on many of the issues discussed here, as well as two anonymous referees for their constructive comments.

[1] See (Béziau 2005) for a general discussion of universal logic in which these results are positioned.

prove the soundness and completeness for any normal logic (without invoking the role of classical negation in the completeness proof). The process of abstraction is typically taken to be a continuation of the work of Tarski's theory of (generalised) logical consequence; Suszko's notion of abstract logic; Gentzen's attempt to construct a calculus in which a wide variety of proof-systems can be shown to share a set of formal properties. What follows also adds the work of Dunn and Hardegree (Dunn and Hardegree 2001, Hardegree 2005) to this canon, primarily their use of generalised Galois connections to consider the relation between logics and valuational semantics.

We should note, for clarification, that the process of abstraction at work in the project of Béziau's universal logic is not an attempt to construct the "one true logic" (whatever that means). Rather, it is a programme allowing both for the navigation of different logics on a common plane, and also for the construction of a substantive set of tools for their analysis. In this sense, it is supposed that it should be possible to analyse the concrete variants of different logics, such as paraconsistent logics, paracomplete logics, classical propositional logic, and so on, from this abstract perspective.

The conjoining of the sequent calculus with bivaluation theory allows for a generalisation of completeness results, surpassing what is achieved in Tarski's generalised theory of consequence. It is in this way that the results are taken by Béziau to substantiate Suszko's (Suszko 1977) argument to the effect that "every logic is two-valued". That is to say, every set of truth-values characterising the semantics for a logic (even if algebraically many-valued) can be reduced to the logical values $\{1,0\}$, which are taken to represent truth and falsity. The chief result is that, relatively maximal theories form a sound and complete semantics (by taking valuations as characteristic functions) for any normal logic.

But, in this respect, the reduction to bivaluations may be a hindrance to the process of abstraction, rather than an analysis of the universal core of logical structures (or so I will suggest). What follows should be taken as providing an expansion of Béziau's completeness results for logics constructed in terms of Gentzen's sequent calculus, by showing that there is a natural extension of that line of thinking to n-sided sequent constructions. Through analogical techniques to Béziau's construction, it is possible to construct abstract soundness and completeness results for n-valued logics. It is, of course, possible, to provide a bivalent semantics for each n-valued logic, but I will argue that the resulting semantics is deficient, particularly since it fails to have the property of absoluteness (Hardegree 2005).

1.1.   *The intuitive duality of sequents*

In essence, Béziau's construction expands upon Gentzen's suggestion that sequents may be read dually, both in terms of derivability and truth. Gentzen's sequents are expressions of the form $\Gamma \vdash \Delta$, where Γ and Δ are both (possibly empty) sets of formulas. Intuitively, as Gentzen (Gentzen 1970) notes, the sequent calculus provides a means by which the distinction between the syntactical definition of derivability, and the semantical account of consequence is broken down. A sequent $\Gamma \vdash \Delta$ can be read in two ways. $\Gamma \vdash \Delta$ may be read as saying that the disjunction of formulas in Δ is derivable from the conjunction of the formulas in Γ. But, equally, we can read $\Gamma \vdash \Delta$ as saying that, either one of

the formulas in Γ if false, or one of the formulas in Δ is true. In classical logic, the two interpretations coincide, since the semantic interpretation is equivalent to the derivability of $\bigwedge \Gamma \supset \bigvee \Delta$, by which we can derive $\bigwedge \Gamma \vdash \bigvee \Delta$.

Take, for example, the classical sequent rules for conjunction, where, following Gentzen's suggestion, the commas on the left of $\vdash$ are interpreted as "and", and those on the right as "or":

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge\text{-L} \qquad\qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \wedge\text{-R}$$

We can read these as telling us the conditions under which it is legitimate to derive a conjunction from its conjuncts (and a conjunct from a conjunction). But, we can equally read the rules as placing constraints upon when a conjunction may be judged to be true or false, given the truth or falsity of its conjuncts. On the latter, we can extract, from the rules, the classical truth-function for $\wedge$:

$$f^{\wedge}(x, y) = \begin{cases} 1 & \text{if } x = 1 \text{ and } y = 1 \\ 0 & \text{otherwise} \end{cases}$$

What follows is a generalisable formalisation of this intuitive dual interpretation, without appeal to the resources of classical propositional logic.[2]

1.2. Generalising "logic"

For reasons that will become obvious later, unlike Béziau, I will consider a logic in its generalised form as follows.

Definition 1. (Logic) We define a logic L any structure, $\langle S, \vdash_L \rangle$, where S is the enumerable set of wff's in a language $\mathscr{L}$, and $\vdash_L$ is a binary derivability relation between a subset of formulas of S (denoted (Γ, Δ)), and a subset of formulas of S, such that $\vdash_L \subseteq \mathcal{P}(\Gamma^S) \times \mathcal{P}(\Delta^S)$, (elements of S are single wff's denoted (α, β)).[34]

Say that $\vdash_L$ is normal when the following conditions hold for all formulas, $\alpha, \beta \in S$, and all subsets $\Sigma, \Delta, \Gamma, \Theta \in S$:

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[2] It is sometimes charged that there is an implicit reliance on the classical interpretation of the commas in sequent rules. However, on certain interpretations of derivability over sequents, the criticism loses its bite. For example, if (following (Restall 2005)) we think of a sequent as imposing rational normative constraints, then we interpret $\Gamma \vdash \Delta$ as saying that an agent who simultaneoulsy asserts all $\alpha \in \Gamma$, and denies all $\beta \in \Delta$ has made a mistake. This construal does not require disjunction, since denial of each $\beta \in \Delta$ may be read conjunctively. In what follows, I will assume that these issues can be assuaged, and so, continue to use the more perspicuous notation.

[3] Obviously, we are working with a formal language here, so we probably don't strictly need to use the phrase "well-formed formula" (wff) of $\mathscr{L}$. I should flag up that I interpret the phrase quite liberally, allowing, for example "$\{\phi|...\}$" to be a wff.

[4] Whilst the multiple-succedent (**SET-SET**) formulation of $\vdash_L$ has received less attention in the literature (though see (Shoesmith and Smiley 1978)), it can be understood as a generalisation of a single-conclusion framework (**SET-FMLA**), allowing multiple formulas for succedents. A **SET-SET** derivability relation "contains" a corresponding **SET-FMLA** relation, since $\Gamma \vdash \beta$ iff $\Gamma \vdash \{\beta\}$ for $\Gamma \in \mathbb{P}(S)$, and $\beta \in S$.

$(\mathbb{R}) : \alpha \vdash_L \alpha$

$(\mathbb{M}) : \Sigma \vdash_L \Delta$ implies $\Sigma, \Gamma \vdash_L \Delta, \Theta$

$(\mathbb{T}) :$ If $\Gamma, \alpha \vdash \Delta$ and $\Gamma \vdash \alpha, \Delta$ then $\Gamma \vdash \Delta$.

Furthermore, say that an inference structure is finitary, when, for all $\Gamma, \Delta \subseteq WFF$, if $\Gamma \vdash_L \Delta$, then there are finite subsets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ where $\Gamma' \vdash \Delta'$.

Definition 2. (Valuation structure) Define a valuation structure as a structure $\langle S, V \rangle$, where S is as above, and V is a valuation space where $V \subseteq U$. Think of U as the entire universe of possible valuations over the language $\mathscr{L}$, then any valuation-space $V \subseteq U$ will represent a particular choice of admissible valuations on the language.[5] As is usual, we let, $\mathcal{V}$ be a set of truth-values, with $\mathcal{D} \subseteq \mathcal{V}$ designated values. Then, a valuation v is a function on $\mathscr{L}$ assigning a truth-value $\in \mathcal{V}$ to a formula $\alpha \subseteq S$ where $v : S \rightarrow \{\mathcal{V}\}$.

Definition 3. (Bivaluation (Béziau 2001)) An adequate bivalent valuation space V for a logic $\langle S, \vdash_L \rangle$ is a set of valuations (functions) from S to $\{1, 0\}$ such that the semantic consequence relation $\models$ is defined in the usual manner. First, we say that any valuation $v \in V$ satisfies $\langle \Gamma, \alpha \rangle$ iff, when $v(\beta) = 1$ for each $\beta \in \Gamma, v(\alpha) = 1$ (and v refutes $\langle \Gamma, \alpha \rangle$ otherwise). Or, where we are working with $\langle \Gamma, \Delta \rangle$, we say that $\langle \Gamma, \Delta \rangle$ refuted by a valuation $v \in V$ iff, when $v(\alpha) = 1$ for each $\alpha \in \Gamma, v(\beta) = 0$ for each $\beta \in \Delta$; otherwise v satisfies the argument.

We use this to define the consequence relation over V: $\Gamma \models \Delta$ iff, for all $v \in V, v$ satisfies $\langle \Gamma, \Delta \rangle$.

When $\vdash_L$ is included in $\models$, we say that the valuation space is sound (for L), and when $\models$ is included in $\vdash_L$, we say that the valuation space is complete (for L).

1.3.   Generalising maximal consistency

It is possible to construct a bivaluation semantics by taking a valuation to be the characteristic function of a closed and consistent theory. It is possible, therefore, to see the two as inter-derivable in substantive sense, since we can see a bivaluation as a theory by taking the set of true formulas under its characteristic function. First, we define a theory in terms of closure under derivability.

Definition 4. (Closure) Say that a theory is some subset, $\Sigma \subseteq S$ which is $\vdash_L$-closed, for some L, when, for all $\langle \Gamma, \Delta \rangle \in L, \Delta \cap \Sigma \neq \emptyset$ whenever $\Gamma \subseteq \Sigma$. Limited to singletons on the right, this is equivalent to saying that a theory $\Sigma \subseteq S$ is $\vdash_L$-closed, for some L, when, for all $\Sigma \vdash \alpha, \alpha \in \Sigma$.

Then, let us define completeness and consistency for a logic L.

Definition 5. A logic L is consistent when there is no formula α, such that $\emptyset \vdash \alpha$ and $\alpha \vdash \emptyset$, (equivalently, in this form, L is consistent if there is some pair of theories $\langle \Gamma, \Delta \rangle$, such that $\Gamma \nvdash \Delta$). A logic is complete when, for each formula $\alpha \in S$, either $\emptyset \vdash \alpha$, or $\alpha \vdash \emptyset$.

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[5] See (Dunn and Hardegree 2001, Hardegree 2005).

We also require a general definition of a maximal theory. The obvious thing to do would be to follow Lindenbaum, but we do not want to prejudice the construction towards any specific set of proof-theoretic constraints on L, so we do not want to employ classical negation. To deal with this, we utilise what Béziau (originally developed in (Wojcicki 1988)) terms a relatively maximal theory.[6]

Definition 6. (Relatively maximal theory) For a theory Γ, a formula α, and a relation $\vdash_L$, say that Γ is relatively maximal with respect to α in L when $\Gamma \nvdash_L \alpha$, and, for all $\beta \notin \Gamma, \Gamma, \beta \vdash_L \alpha$.

The first clause ensures that Γ avoids the formula α, and the second ensures maximality. Since, when Γ is closed, we have the equivalence $\Gamma \vdash \alpha$ iff $\alpha \in \Gamma$, any closed set Γ obeying the above clauses is a relatively maximal theory. Then, any proper superset of Γ will be trivial, such that, for any $\alpha \notin \Gamma, \Gamma \cup \{\alpha\}$ is trivial.

Theorem 7. *For any finite normal logic L, given a formula β, and a theory Γ (where $\Gamma, \beta \in S$) such that $\Gamma \nvdash \beta$, there is an extended set, Γ' (where $\Gamma \subseteq \Gamma'$, such that Γ' is relatively maximal with respect to β (in L).*

Proof. Take the enumerable formulas of S, $\{\alpha_1, \alpha_2 ... \alpha_i, \alpha_{i+1}\}$, and a theory $\langle \Gamma \rangle$ (which is closed under $\vdash_L$, and $\Gamma \subseteq S$), where $\Gamma \nvdash \beta$. Then, for some formula $\alpha_i \in S$, we know that either $\Gamma \cup \{\alpha_i\} \nvdash \beta$, or $\Gamma \nvdash \beta \cup \{\alpha_i\}$. When the former is the case, we can extend Γ to $\Gamma \cup \{\alpha_i\}$. In other words, by induction on the enumerable formulas of S, we can build up the relatively maximal theory Γ', which avoids each $\alpha_i \in S \backslash \Gamma'$ through the construction of a chain:

i. $\langle \Gamma_n \rangle = \langle \Gamma \rangle$

ii. $\langle \Gamma_{n+1} \rangle = \begin{cases} \langle \Gamma_n \cup \{\alpha_{i+1}\} \rangle \text{ if } (\Gamma_n \cup \{\alpha_{i+1}\} \nvdash \beta) \\ \langle \Gamma_{n+1} = \Gamma_n \rangle \text{ otherwise.} \end{cases}$

The limit of this construction is:

$\langle \Gamma' \rangle = \bigcup\limits_{n \in \prime} \Gamma_n$

It is simple to see that Γ' is relatively maximal *w.r.t* β in L. As is clear, on the above chain, we have $\Gamma_n \nvdash \beta$, and, so $\Gamma' \nvdash \beta$. If not, there must be some finite subset $\Gamma \subseteq \Gamma'$, for which $\Gamma \vdash \beta$ (by finiteness). This means that, for some $\Gamma_{n+i} \supseteq \Gamma, \Gamma_{n+i} \vdash \beta$ (by $\mathbb{M}$), which contradicts the definition of Γ_{n+i}, (where $\Gamma_{n+i} \nvdash \beta$). Then, take a formula α_i, where $\alpha_i \notin \Gamma'$. By definition, $\Gamma_{n+1} \subseteq \Gamma'$, so $\alpha_i \notin \Gamma_{n+1}$. Then, we know that $\Gamma_{n+1} = \Gamma_n$, and $\Gamma_n, \alpha_i \vdash \beta$. By $\mathbb{M}$, and the fact that $\Gamma_n \subseteq \Gamma'$, we have $\Gamma', \alpha_i \vdash \beta$.

It will become important later (though Béziau does not note this), that the proof can be read symmetrically as constructing, not just the relatively maximal theory Γ', but also its complement, the "maximally avoided" theory $\Delta' =_{df} \forall \alpha \notin \Gamma'$, with $\Gamma' \cup \Delta' = S$, and $\Gamma' \cap \Delta' = \emptyset$. We call these bipartitions.

Lemma 8. *For a finite normal logic L, and for any pair of theories $\langle \Gamma, \Delta \rangle$ (in S), where $\Gamma \nvdash \Delta$, there exists a bipartition extending $\langle \Gamma, \Delta \rangle$ to Γ', and its complement, Δ', where $\Delta' =_{df} \forall \alpha \notin \Gamma'$, with $\Gamma' \cup \Delta' = S$, and $\Gamma' \cap \Delta' = \emptyset$.*

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[6] Thanks to an anonymous referee for pointing me towards the original source.

Proof. Take the enumerable formulas of S, $\{\alpha_1, \alpha_2 \ldots \alpha_i, \alpha_{i+1}\}$, and any Γ, Δ for which $\Gamma \nvdash \Delta$, and construct the following chain of pairs of theories:

i. $\langle \Gamma_n, \Delta_n \rangle = \langle \Gamma, \Delta \rangle$

ii. $\langle \Gamma_{n+1}, \Delta_{n+1} \rangle = \begin{cases} \langle \Gamma_n \cup \{\alpha_{i+1}\}, \Delta_n \rangle \text{ if } (\Gamma_n, \alpha_{i+1} \nvdash \Delta_n) \\ \langle \Gamma_n, \Delta_n \cup \{\alpha_{i+1}\} \rangle \text{ if } (\Gamma_n, \alpha_{i+1} \vdash \Delta_n) \end{cases}$

The limit of this construction is:

$\langle \Gamma' \rangle = \bigcup_{n \in I} \Gamma_n$, and $\langle \Delta' \rangle = \bigcup_{n \in I} \Delta_n$.

The pair $\langle \Gamma', \Delta' \rangle$ form a bipartition over formulas of S, with $\Gamma' \nvdash \Delta'$. Then, (by the fact that L is finitary), for any subsets of $\langle \Gamma', \Delta' \rangle$, we have $\Gamma \nvdash \Delta$, when $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$.

Lemma 9. *For a logic L where S is closed under bipartitions $\langle \Gamma', \Delta' \rangle$, for every $\alpha \notin \Gamma'$, $\alpha \in \Delta'$, where every $\alpha \notin \Gamma'$, $\alpha \vdash \emptyset$, and, every $\alpha \in \Gamma'$, $\emptyset \vdash \alpha$.*

Proof. (Right to left). First, show that if Γ' is closed under L, then $\Gamma' = S - \Delta'$. Take a set of sequents (of L), $\theta_1, \theta_2, \ldots \theta_i \ldots \theta_n$, where $\theta_i = \langle \Gamma^i, \Delta^i \rangle$. By induction, it is simple to show that, whenever $\Gamma^i \subseteq \Gamma'$, $\Delta^i \cap \Gamma' \neq \emptyset$, and so, for any $\alpha \notin \Gamma'$, we have $\alpha \vdash \emptyset$. For this, just note that, if $n = 1$, then $\theta_1 \in L$ since Γ^n is closed under L. If $n \geq 2$, then $\theta_i = \langle \Gamma^i, \Delta^i \rangle$ (since Γ' is closed under $\mathbb{T}$). Take $\Gamma^i \subseteq \Gamma'$, then $\Delta^{i+1} \cap \Gamma' \neq \emptyset$, where $\Delta^{i+1} = \Delta^i \cup \{\alpha\}$ so either $\alpha \in \Gamma'$, or $\Delta^i \cap \Gamma' \neq \emptyset$. If the former ($\alpha \in \Gamma'$), then $\Gamma^i \cup \alpha \subseteq \Gamma'$, so $\Delta^i \cap \Gamma' \neq \emptyset$.

Now, suppose that Γ' is closed under L, but Γ' is not maximal. Then, there must be some theory $\Gamma^i \supset \Gamma'$ (that is also closed under L), where there is a formula $\alpha \in \Gamma^i - \Gamma'$. But, since Γ^i is closed under L, $\Gamma^i \subseteq \Gamma'$ (by the above), and so $\alpha \notin \Gamma^i$.

(Left to right). Suppose that Γ' is maximal and closed under L, but there is some $\alpha \notin \Gamma'$ that is not in the complement theory Δ'. Then, we would have that $\alpha \nvdash \emptyset$, and so $\Gamma', \alpha \nvdash \emptyset$. We need only recall the above to see that it is possible to construct the set Γ^i, where $\Gamma' \cup \{\alpha\} \subseteq \Gamma^i$, and where $\Gamma^i \nvdash \Delta^i$, where $\Delta^i = S - \Gamma^i$. Thus, Γ^i is closed under L. For example, suppose that $\langle \Gamma^1, \Delta^1 \rangle \in L$, and $\Gamma^1 \subseteq \Gamma^i$, but $\Delta^1 \cap \Gamma^i = \emptyset$. Then, $\Gamma^1 \vdash \Delta^1$, and $\Delta^1 \subseteq \Delta^n$, so $\Gamma^i \vdash \Delta^n$, which is impossible. So, Γ^i is closed under L such that $\Gamma' \subseteq \Gamma^i$, which contradicts our assumption that Γ' is maximal.

Definition 10. (Characteristic function) For any relatively maximal theory, Γ', we take a valuation to be the characteristic function of Γ' such that:

$v(\Gamma') = \{v \in U : v(\alpha) = 1 \text{ for each } \alpha \in \Gamma'\}$;

and, for the complement

$v(\Delta') = \{v \in U : v(\alpha_i) = 0 \text{ for each } \alpha_i \in \Delta'\}$.

Then, a valuation space V that is determined by a logic L in this way is defined as the collection of valuations which are characteristic functions for each bipartition of L. Call this $\mathbb{V}(L)$ to denote this formulation of a valuation space.

Corollary 11. *Any valuation-space V constructed from the characteristic functions of the bipartitions of a logic L is an adequate bivaluation semantics for L.*

1.4. Logics as structures of sequents

So far, we have worked with only a minimal definition of logics as antisymmetric pre-orders (posets) over $\mathscr{L}$, leaving a great deal of flexibility in the construction.[7] We can think of the development of a full proof-theory as carving out a specific logic L from the category of possible logics over $\mathscr{L}$, $L' \supseteq L$, by placing constraints on L' with structural and operational rules (where each connective defined in L has a pair of rules determining its behaviour).

Definition 12. (Sequent) A sequent in any logic L is an ordered pair $\langle \Gamma, \Delta \rangle$ where Γ, Δ are sets of formulas of S. As usual, we write a sequent as $\Gamma \vdash_L \Delta$.

Definition 13. (Sequent rule) A sequent rule $\mathcal{R}$ in any logic L is an ordered pair consisting of a set of sequent premises and a sequent conclusion $\mathbb{R} = \langle \{ SEQ^P \}, SEQ^C \rangle$.

Definition 14. (Sequent bivaluation) By the above definition of a bivaluation, and that we may rewrite a sequent in L, $\langle \Gamma, \Delta \rangle$, as $\{ \alpha_1, ..., \alpha_n \} \vdash \{ \beta_1, ..., \beta_m \}$, we say that a valuation $v \in \mathbb{V}(L)$ satisfies $\langle \Gamma, \Delta \rangle$ iff $v(\alpha_i) = 0$ for some $i(1 \leq i \leq n)$, or $v(\beta_i) = 1$ for some $i(1 \leq i \leq m)$. Say that such a valuation $v \in U$ is L-consistent iff v satisfies each sequent in L. Otherwise, v refutes $\langle \Gamma, \Delta \rangle$. Furthermore, we say that a valuation v respects a rule (or a set of rules) R iff it satisfies the conclusion of the rule (SEQ^C) whenever its satisfies the premises of the rule $(\{ SEQ^P \})$. Given that we are dealing with derivability relations which are closed, this is exactly analogous to saying that a relatively maximal theory Γ' satisfies a sequent (and, by extension, a rule), since Γ' satisfies $\langle \Gamma, \Delta \rangle$ iff $(\Gamma' \nvdash \alpha_1$ or $\Gamma' \nvdash \alpha_2$ or...$\Gamma' \nvdash \alpha_n)$ or $(\Gamma' \vdash \beta_1$ or $\Gamma' \vdash \beta_2$ or...$\Gamma' \vdash \beta_m)$. In our construction (though not necessarily in Béziau's), this is equivalent to $(\Delta' \vdash \alpha_1$ or $\Delta' \vdash \alpha_2$ or...$\Delta' \vdash \alpha_m)$ or $(\Gamma' \vdash \beta_1$ or $\Gamma' \vdash \beta_2$ or...$\Gamma' \vdash \beta_m)$. .

1.5. Generalising the result

The relation between the two induces a Galois connection between valuation spaces and logics (Dunn and Hardegree 2001, Hardegree 2005, Hjortland Forthcoming b, Humberstone 2011). Formally, we define the map sending $L \rightarrow V$:

$\mathbb{V}(L) =_{df} \Gamma \vdash_L \Delta : \forall v \in \mathbb{V}(L) \{ v(\Gamma) = 0$ or $v(\Delta) = 1 \}$. And, in the other direction, the map $V \rightarrow L$:

$\mathbb{L}(V) =_{df} v \in V : \forall \langle \Gamma, \Delta \rangle \{ v(\Gamma) = 0$ or $v(\Delta) = 1 \}$.

Thinking of L and $\mathbb{V}(L)$ in the abstract (as not yet constrained by any proof-system), we have, in effect, two partially ordered sets defined over $\mathscr{L}$ (Hardegree 2005):

(P1)   The set of all valuation-spaces $V \subseteq U$ on $\mathscr{L}$, ordered by set-inclusion;

(P2)   The set of all logics $L \subseteq L'$ on $\mathscr{L}$, ordered by set-inclusion.

The relation between the two induces an antitone Galois connection between valuation spaces and logics (Dunn and Hardegree 2001, Hardegree 2005, Hjortland Forthcoming b, Humberstone 2011), where a generalised Galois connection is an adjunction of maps between partially ordered sets in terms of order preservation functions.

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[7] See (Straßurger 2007) for further details.

Definition 15. A Galois connection between posets P, Q is a map: $f_1 : P \to Q$ and $f_2 : Q \to P$ where the following conditions hold for all subsets P_n, Q_n of P, Q:

$$P_0 \subseteq f_2(f_1(P_0)) \tag{1.1}$$

$$Q_0 \subseteq f_1(f_2(Q_0)) \tag{1.2}$$

$$P_0 \subseteq P_1 \Rightarrow f_1(P_1) \subseteq f_1(P_0) \tag{1.3}$$

$$T_0 \subseteq T_1 \Rightarrow f_2(T_1) \subseteq f_2(T_0) \tag{1.4}$$

It follows that $f_1 \subseteq f_1 f_2 f_1 \subseteq f_1$, so $f_1 = f_1 f_2 f_1$ and also $f_2 = f_2 f_1 f_2$.

For our purposes, here P is the set of all valuations over $\mathscr{L}$, and Q the set of all sequents in L, with satisfaction being the relation defining the functions between them. For any valuation space V, $f_1(V)$ consists of the set of sequents satisfied by each $v \in V$, i.e. $f_1(V) = \mathbb{L}(V)$. For any logic L, $f_2(L)$ will consist of the set of valuations that satisfy every sequent in L, i.e. $f_2(L) = \mathbb{V}(L)$.

With this, we can define a closure operator cl as a function on the posets $\langle V, L \rangle$, given that cl obeys the following clauses for all x, y on $\langle V, L \rangle$:

(c1) $x \leq cl(x)$

(c2) $cl(cl(x)) \leq cl(x)$

(c3) $x \leq y \to cl(x) \leq cl(y)$

This ensures that, where cl is a closure operator on a poset $\langle P, \leq \rangle$, and x is an element of P, then x is closed iff $cl(x) = x$. In our context, this gives us an abstract completeness theorem over $\langle \mathbb{V}, \mathbb{L} \rangle$.

Fact 16. *For each $V \subseteq U$ and $L \subseteq L'$ (for some S):*

$$L \subseteq \mathbb{L}(\mathbb{V}(L)) \tag{1.5}$$

$$V \subseteq \mathbb{V}(\mathbb{L}(V)) \tag{1.6}$$

$$L \subseteq L' \Rightarrow \mathbb{V}(L') \subseteq \mathbb{V}(L) \tag{1.7}$$

$$V \subseteq U \Rightarrow \mathbb{L}(U) \subseteq \mathbb{L}(V) \tag{1.8}$$

Proof. Given at length in (Hardegree 2005).

(1.5) indicates that when we determine $\mathbb{V}(L)$, and then induce a logic $\mathbb{L}$ from the valuation space determined, then $\mathbb{L}$ will contain L. Similarly, (1.6) tells us that, when we determine $\mathbb{L}(V)$, and then determine a valuation space $\mathbb{V}$ from the logic determined, that $\mathbb{V}$ will contain V. With this, we can formulate abstract soundness and completeness theorems, which, following Dunn and Hardegree (Dunn and Hardegree 2001), we call *absoluteness*.

Fact 17. (Hardegree 2005) *For any L, V;*

- *L is absolute iff $L = \mathbb{L}(\mathbb{V}(L))$*

- *V is absolute iff $V = \mathbb{V}(\mathbb{L}(V))$.*

Proof. By the fact that L, V form a Galois map, and the definition of Galois closure (c1-3).

Resultantly, we can give general soundness and completeness theorems for the construction of any normal, finite logic.

Lemma 18. *Let Γ be any set of formulas in S. Define v_Γ, as: $v_\Gamma(\alpha) = 1$ if $\Gamma \vdash \alpha$, and $v_\Gamma(\alpha) = 0$ otherwise. Then v_Γ is L-consistent and $v_\Gamma \in \mathbb{V}(L)$.*

Proof. (Hardegree 2005) If not, there must be a sequent, $\Delta \vdash \beta$ in L that is refuted by v_Γ, so that $v_\Gamma(\Delta) = 1$ and $v_\Gamma(\beta) = 0$. Given the way in which v_Γ is defined, this means that $\Gamma \vdash \Delta$. But, $\Delta \vdash \beta$ is L-valid, and given that the $\vdash$ associated with L is closed under transitivity, it follows that $\Gamma \vdash \beta$, so by the definition of v_Γ, $v_\Gamma(\beta) = 1$, so v_Γ does not refute $\Gamma \vdash \beta$.

Theorem 19. *In general, for any finite normal logic L, $L = \mathbb{L}(\mathbb{V}(L))$.*

Proof. (Hardegree 2005) Suppose that some $\langle \Gamma \vdash \beta \rangle \notin L$, to show that $\langle \Gamma \vdash \beta \rangle \notin \mathbb{L}(\mathbb{V}(L))$ (in other words, it is refuted by $\mathbb{V}(L)$). Take the valuation v_Γ, which by Lemma 17 is in $\mathbb{V}(L)$. By definition, v_Γ satisfies all derivable sequents of L. Since L is reflexive, each element of Γ is derivable in L, so v_Γ satisfies Γ. But, since $\langle \Gamma \vdash \beta \rangle$ is not L-valid, $\beta \notin \Gamma$, so v_Γ refutes β. Then v_Γ refutes $\langle \Gamma \vdash \beta \rangle$, and so too does $\mathbb{V}(L)$, thus $\langle \Gamma \vdash \beta \rangle \notin \mathbb{L}(\mathbb{V}(L))$.

Theorem 20. *For any $V \subseteq U$, built-up over quasi-partitions as above, $V = \mathbb{V}(\mathbb{L}(V))$.*

Proof. (Dunn and Hardegree 2001, p.200) We prove contra-positively by defining a valuation $v_0 \notin V$ (in order to show that $v_0 \notin \mathbb{V}(\mathbb{L}(V))$). Then define $T = \{\alpha \in S : v_0(\alpha) = 1\}$ and $F = \{\alpha \in S : v_0(\alpha) = 0\}$. For any $v \in V$, $v \neq v_0$, so either $v(\alpha) = 0$ for some $\alpha \in T$ or $v(\alpha) = 1$ for some $\alpha \in F$. Then v satisfies $T \vdash F$, and it follows that $T \vdash F$ is valid on V. But, by definition, v_0 refutes $T \vdash F$, so $v_0 \notin \mathbb{V}(\mathbb{L}(V))$.

2.   Applying the result

Whilst the above way of construing abstract soundness and completeness proofs does not follows Béziau's, it is in clear accord with his project. Moreover, it gives us a clear way of assessing completeness results for specific logics. For example, asymmetric logics (where $\vdash_L \subseteq \mathcal{P}(S) \times S$) fail to be absolute.

Example 21. Absoluteness does not hold for V_{CPL} because the classical proof-system defining the connectives $\neg, \vee$ is compatible with valuations $\notin V_{CPL}$. For example, say we define negation in this framework as:

$$\frac{\Gamma, A \vdash B \wedge \neg B}{\Gamma \vdash \neg A} \; (Reductio) \qquad\qquad \frac{\Gamma \vdash A \quad \Gamma \vdash \neg A}{\Gamma \vdash B} \; (EFQ)$$

From this, induce $L_\neg$, and determine a corresponding valuation space $V_\neg$. Then $V_\neg \neq V_{CPL}$ if the latter is supposed accord with the truth-functional definition $f^\neg$:

$$f^\neg(x) = \begin{cases} 1 & if \; x = 0 \\ 0 & if \; x = 1 \end{cases}$$

More precisely, $\mathbb{V} \neq \mathbb{V}(\mathbb{L}(V_{CPL\neg}))$ because, whilst $L_\neg$ is sound and complete *w.r.t* V_{CPL}, it is also sound and complete *w.r.t* alternative semantic structures. For example, (Hardegree 2005) defines a super-valuation associated with a valuation-space V to be the valuation v_V where, for every formula, α, $v_V(\alpha) = 1$ if $v(\alpha) = 1$ for every $v \in V$ and $v_V(\alpha) = 0$ otherwise. Clearly, $L_\neg$ is sound and complete *w.r.t* both V_{CPL}, and $V_{CPL} \cup v_V$.

Similarly, for disjunction defined:

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \; (\vee L)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \; (\vee R_1) \qquad\qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \; (\vee R_2)$$

It is well known (e.g. Carnap 1943) that the induced logic $L_\vee$ can not force the exclusion of a valuation where $v(A \vee B) = 1$ when $v(A) = 0 = v(B)$. Again, the rules are sound and complete *w.r.t* V_{CPL}, but they are also sound and complete *w.r.t* $V_{CPL} \cup v_V$. So, the logic defined asymmetrically *underdetermines* the relevant semantic structure since it fails to determine a unique valuation space.

Remark 22. The point can be made in relation to the known (Belnap and Massey 1990, Carnap 1943, Dunn and Hardegree 2001, Garson 2010, Hardegree 2005, Hjortland Forthcoming b, Humberstone 2011, Shoesmith and Smiley 1978) result that standard formulations of classical logic fail to rule out non-standard valuations.

Proof. Consider the valuation space V_{CPL}. Now, consider v_t, defined such that, for every formula of S, α, $v_t(\alpha) = 1$. Clearly, $v_t \notin V_{CPL}$ since v_t makes everything true at once. However, it is simple to see that extending V_{CPL} to $V_{CPL} \cup v_t$ does not alter $\models_{CPL}$. Take $\models_{CPL*}$ to be the consequence relation for $V_{CPL} \cup v_t$. We prove equivalence by supposition that $\models_{CPL*} \neq \models_{CPL}$. Then there must be formulas $\Gamma \cup \{\alpha\} \subseteq S$ such that $\Gamma \models_{CPL} \alpha$, but $\Gamma \not\models_{CPL*} \alpha$. Hence, for $\models_{CPL*}$ it must be that there is some $v \in V_{CPL} \cup v_t$ where $v(\beta) = 1$ for each $\beta \in \Gamma$ whilst $v(\alpha) = 0$. By the definition of

v_t this can not be the case. Hence, there is a classically inadmissible valuation, which is nonetheless compatible with V_{CPL}.

Insofar as we have constructed the above proofs utilising the generalised, symmetric, construction of logic, these problems are assuaged (in fact, this is due to the simple fact that we allow the presence of $\alpha \vdash \emptyset$).

Remark 23. We should pause to discuss this momentarily, given that we are going to push these issues further in the following discussion of n-valued logics. One may well object to the above that absoluteness is simply too strong a requirement on the relationship between a logic and a valuation space - surely soundness and completeness is enough!. Then, perhaps, we have simply invented a problem where there is none to be found (a task not uncommon to analytic philosophy, perhaps). However, I do not think that this is the case, primarily because Béziau's results are supposed to show that syntax and semantics are "two sides of the same coin". The fact that the two form a Galois-connection is further substantiation of this suggestion, and it is one that offers many nice results for their formalisation, particularly in the context of developing a universal logic. Furthermore, the failure of absoluteness for some logics and valuation-spaces shows that where there is such a failure, a logic and a valuation-space can not be considered as two sides of the same coin, since the transition $V \rightarrow \mathbb{V}(\mathbb{L}(V))$ does not return us to where we began. Flipping the coin, as it were, simply does not work.

Let us turn to how the above results bear out with respect to n-valued logics, bearing in mind Béziau's suggestion that the bivaluation theory upholds Suszko's reduction to bivalent semantics.

Example 24. Consider a three-valued logic such as Graham Priest's Logic of Paradox (LP), which allows for some sentences to be "gluts", that is, both true and false (Priest 2006). This has an algebraic set of truth-values $\mathcal{V} = \{1, b, 0\}$, with designated values $\mathcal{D} = \{1, b\}$, ($\{b\}$ representing "both true and false"). Leaving aside the construction of a proof-theory, we say that a sequent, $\Gamma \vdash \Delta$ in the logic LP, will be refuted by a valuation v iff, when $v(\alpha) \in D$ for each $\alpha \in \Gamma$, $v(\beta) \notin D$ for all $\beta \in \Delta$); and otherwise satisfied by v.[8] This is equivalent saying that a valuation v satisfies $\langle \Gamma, \Delta \rangle$ iff $v(\alpha_i) \notin \mathcal{D}$ for some $i (1 \leq i \leq n)$, or $v(\beta_i) \in \mathcal{D}$ for some $i (1 \leq i \leq m)$. Otherwise, v refutes $\langle \Gamma, \Delta \rangle$. Quite clearly, this will give rise to a bivalent semantics for LP.

This is a simple bi-product of the Suszko reduction, which says that, for any finite n-valued consequence relation, it is possible to define an equivalent two-valued consequence relation, which evaluates each formula $\alpha \in S$ as 1 if $v(\alpha) \in D$, and 0 otherwise (Suszko 1977).

But, and with this in mind, it is easily seen that such a semantics for LP will not be absolute.

Theorem 25. *For any finite normal logic L, and any $V \subseteq U$, (where $\mathcal{V} = \{1, b, 0\}$), $V \neq \mathbb{V}(\mathbb{L}(V))$.*

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[8] For a proof theory in sequent structure, see (Palau and Oller 2014).

Proof. We prove indirectly, by highlighting the inadequacies of the absoluteness proof when we add intermediate truth-values. First, define a valuation $v_b \notin V_{LP}$ (in order to show that $v_b \in \mathbb{V}(\mathbb{L}(V_{LP}))$). Let v_b be defined such that, for every formula of S, α, $v_b(\alpha) = b$. We do not need to look at the details regarding the construction of V_{LP} to realise this since, clearly, $v_b \notin V_{LP}$ since v_b makes everything a glut at once. Since $v_b \notin V_{LP}$, for each $v \in V_{LP}$, there is some formula β for which $v(\beta) \neq v_b(\beta)$. For such a formula, let $\Gamma = \{\beta \in S : v(\beta) \notin \mathcal{D}\}$ and $\Delta = \{\beta \in S : v(\beta) \in \mathcal{D}\}$. Keep in mind that $v_b(\Gamma) = b$, and $v_b(\Delta) = b$. Where $L = \mathbb{L}(V_{LP})$, $\Gamma \vdash \Delta$, since either $\beta \in \Gamma$, and so $v(\beta) \notin \mathcal{D}$, or $\beta \in \Delta$, and so $v(\beta) \in \mathcal{D}$. However, unlike the case where $\mathcal{V} = \{1, 0\}$, v_b also satisfies $\Gamma \vdash \Delta$, so we can not use that partition to rid ourselves of the problematic valuation.

The obvious issue is that the bivaluation theory is reliant upon partitioning formulas $\in S$ into those taking $v = 1$, $v = 0$. Then, for LP (and any n-valued set of values), it is possible to construct an analogous absoluteness proof, but only by defining a partition over formulas into those that take designated values and those that do not. That is, where, $D = \{\alpha \in S : v(\alpha) \in \mathcal{D}\}$ and $D^- = \{\alpha \in S : v(\alpha) \in \mathcal{D}^-\}$, (where $\alpha \in \mathcal{D}^- =_{equiv} \alpha \notin \mathcal{D}$). But, whilst partitioning into D, D^- tells us something about consequence relations for many-valued logics (i.e. the preservation of $\mathcal{D}$), it involves a loss of grasp on the finer-grained structure of the logic, where we want to distinguish between the designated values, e.g. $\{1, b\}$. As with asymmetric bivalent logics, the failure of absoluteness shows us that the relationship between logics and valuation-spaces is not so tight as we would like.

3.   *Expanding sequents*

On Béziau's (and the above) presentation, the inter-definability of the dual reading of sequents underlies the construction of a bivaluation semantics. Accepting this, however, allows for a fairly simple maneuver which allows us to expand the results to deal with n-valued logics, when considered in terms of n-sided sequents.

Recall that the theory of bivaluation semantics rests upon the symmetry of interpreting a sequent in terms of derivability and in terms of truth. We have utilised this feature throughout, by considering the left side of a sequent to be false, and the right side to be true (disregarding contexts). It is just this intuitive feature that allows for the substantive results outlined above. Let us be perspicuous about this, then, and rewrite a sequent $\Gamma \vdash \Delta$ (given that we are considering it in relation to truth-valuations) as $\Gamma_0 \vdash \Delta_1$, with the indexes indicating truth-values. In making this transparent, it is also clear that, if we are dealing with many-valued logics, there will be an inevitable reduction to bivalence since a sequent will be interpreted analogously as $\Gamma_{D^-} \vdash \Delta_D$. It is for this reason that it is possible to provide a bivalent semantics for many-valued logics. But, in this context (and given the failure of absoluteness for many-valued logics), such a restriction on the construction of sequents is unduly restrictive. Following the suggestion in (Baaz, Fermueller and Zach 1993, Baaz, Fermuller and Zach 1993, Baaz et al. 1998, Zach 1993), we can expand the structure of a sequent written $\Gamma_0 \vdash \Delta_1$ to one in which there is a

"place" for every truth-value in the logic: $\Gamma_0|\Gamma_1|...|\Gamma_n$, where n is the number of truth values.

Definition 26. An n-sided sequent is an ordered n-tuple of finite formulas $\Gamma_1|...|\Gamma_n$ where Γ_n is the n-th component of the sequent.

Where Γ is a sequent, Γ_i denotes the i-th component of the sequent, with the sequent interpreted as a disjunction of statements saying that a particular formula takes a particular location in the structure of the sequent. This gives rise to the following definitions of valuation satisfaction and refutation.

Definition 27. For a (finite) set of values $\mathcal{V} = \{v_1,...v_n\}$, and for an n-sequent $\{\Gamma_1|\Gamma_i|...|\Gamma_n\}$, for each location Γ_i (which is a possibly empty set of formulas $\in S$), we say that a valuation v satisfies an n-sided sequent iff for some Γ_i, when $i, 1 \leq i \leq n$, and some formula $\alpha \in \Gamma_i, v(\alpha) = i$. Otherwise, v refutes the sequent.

We may construct an n-sided sequent calculus in a uniform way (Baaz, Fermueller and Zach 1993). First, we expect it to obey the usual structural constraints (ensuring normality):

$$(\mathcal{R}) : A|...|A$$

For each sequent location i:

$$(\mathcal{M}) \; \frac{\Gamma}{\Gamma, [i : A]}$$

For each couple of truth-values where $v_i \neq v_j$:

$$(\mathcal{T}(i,j)) \; \frac{\Gamma, [i : A] \qquad \Delta, [j : A]}{\Gamma, \Delta}$$

Moreover, the set of operational rules for each connective are defined as one per "place" in the sequent.

Definition 28. n-sided sequent rules for a connective $\square$ at location i take the form:

$$\frac{\langle \Gamma, \Delta_{\square:i}(j) \rangle_{j \in I}}{\Gamma_i(\square A_1...A_n)} \square : i$$

where the arity of $\square$ is n, I is a finite set, $\mathrm{frm}(\Delta_{\square:i}(j)) \subseteq \{A_1, ..., A_n\}$ ($\mathrm{frm}(\Delta)$ denotes the set of formulas $\in \Delta$), and the following condition holds:

For a valuation v, the following are equivalent:

i. $\square(A_1, ..., A_n)$ takes the value v_i under v,

and;

ii. For $j \in I$, v satisfies the sequents $\Delta_{\square:i}(j)$.

The definition, taken for each connective of the language, provides a sequent rule for each truth-value. It should be noted that the connective rules are not unique. In fact, any conjunctive normal form $(\bigwedge_{j=1}^{k} \bigwedge_{l=1}^{m} \bigvee_{A \in \Delta_{\square:i}(j)_{(l)}})$, where A^{v_l} denotes that $v(A) = v_l$, will provide a set of sequent rules.

Definition 29. In general in this setting, a rule $\mathbb{R}$, consists of a set of n-sequent premises and an n-sequent conclusion $\mathbb{R} = \{NSEQ^P\} \to NSEQ^C$. In general, this is written:

$$\frac{(\Gamma_1^1|...|\Gamma_n^1) \qquad (\Gamma_1^m|...|\Gamma_n^m)}{(\Gamma_1|...|\Gamma_n)}$$

Example 30. A three-sided sequent for L_{LP} is written as:

$$\Gamma_1|\Gamma_b|\Gamma_0$$

. The structural rules are as above, and operational rules for each connective are given for each location in the three-sided sequent as in the following examples:

$$\frac{\Gamma_0|\Gamma_b|\Gamma_1, A \qquad \Gamma_0|\Gamma_b|\Gamma_1, B}{\Gamma_0|\Gamma_b|\Gamma_1, A \wedge B} \ (\wedge 1) \qquad\qquad \frac{\Gamma_0, A, B|\Gamma_b|\Gamma_1}{\Gamma_0, A \wedge B|\Gamma_b|\Gamma_1} \ (\wedge 0)$$

$$\frac{\Gamma_0, A|\Gamma_b, A|\Gamma_1 \qquad \Gamma_0, B|\Gamma_b, B|\Gamma_1 \qquad \Gamma_0|\Gamma_b, A, B|\Gamma_1}{\Gamma_0, |\Gamma_b, A \wedge B|\Gamma_1} \ (\wedge b)$$

$$\frac{\Gamma_0, A|\Gamma_b|\Gamma_1 \qquad \Gamma_0, B|\Gamma_b|\Gamma_1}{\Gamma_0, A \vee B|\Gamma_b|\Gamma_1} \ (\vee 0) \qquad\qquad \frac{\Gamma_0|\Gamma_b|\Gamma_1 A, B}{\Gamma_0|\Gamma_b|\Gamma_1, A \vee B} \ (\vee 1)$$

$$\frac{\Gamma_0, A|\Gamma_b, A|\Gamma_1 \qquad \Gamma_0, B|\Gamma_b, B|\Gamma_1 \qquad \Gamma_0|\Gamma_b, A, B|\Gamma_1}{\Gamma_0|\Gamma_b, A \vee B|\Gamma_1} \ (\vee b)$$

$$\frac{\Gamma_0|\Gamma_b|\Gamma_1, A}{\Gamma_0, \neg A|\Gamma_b|\Gamma_1} \ (\neg 0) \qquad\qquad \frac{\Gamma_0, A|\Gamma_b|\Gamma_1}{\Gamma_0|\Gamma_b|\Gamma_1, \neg A} \ (\neg 1)$$

$$\frac{\Gamma_0|\Gamma_b, A|\Gamma_1}{\Gamma_0|\Gamma_b, \neg A|\Gamma_1} \ (\neg b)$$

In this setting, the logic L_{LP} comprises the set of valid sequents recursively determined over the formulas in S by above the proof-system.

3.1. *Semantics for n-sequents*

Since the structure has more than two locations for formulas, $\mathcal{R}$ makes sure that each formula takes a valuation, and $\mathcal{T}$ operates on pairs of truth-values. So, $\mathcal{T}$ allows us to partition formulas so that we can construct relatively maximal theories by ensuring that each formula takes a single position in the structure of a sequent. For any $v_i \neq v_j$ for a formula A such that $v(A) = v_i = v_j$, A is removed:

$$\frac{\Gamma_1|...|\Gamma_i, A|...|\Gamma_n \qquad \Delta_1|...|\Delta_j, A|...|\Delta_n}{\Gamma_1, \Delta_1|...|\Gamma_n \Delta_n}$$

Then, since we are dealing with finite logics, we extend any subset of formulas, in much the same way as above, by extending the notion of a bipartition to an n-partition over n-sequents of the form $\{\Gamma_1|\Gamma_i|...|\Gamma_n\}$. For simplicity, let us work with tripartitions over a three-sided sequent structure, $\{\Gamma_1|\Gamma_b|\Gamma_0\}$, but note that the below construction can be extended to any (finite) n-sided construction. Above, we saw that the extension of a theory to a relatively maximal theory can equally be understood as a bipartition since the complement of a relatively maximal theory is defined simultaneously, (built out of those formulas avoided by the former). It is similarly possible to form a generalised n-partition over the formulas of S that cuts along the grain of the positions that formulas can take in n-sequents.

Theorem 31. *For any finite n-value normal logic L (in n-sequent form), given any set of theories $\langle\Gamma_1,\Gamma_b,\Gamma_0\rangle$, where the sequent $\{\Gamma_1|\Gamma_b|\Gamma_0\}$ is valid (for L), there are extensions of $\langle\Gamma_1,\Gamma_b,\Gamma_0\rangle$ to $\langle\Gamma_1',\Gamma_b',\Gamma_0'\rangle$, where $\{\Gamma_1'|\Gamma_b'|\Gamma_0'\}$ is valid (for L) (the construction of n-sequents allows us to work directly with sequents in L). Call these n-partitions since $\Gamma_1'\cup\Gamma_b'\cup\Gamma_0' = S$ and $\Gamma_1'\cap\Gamma_b'\cap\Gamma_0' = \emptyset$.*

Proof. Take the enumerable formulas of S, $\{\alpha_1,\alpha_2...\alpha_i,\alpha_{i+1}\}$, and sets of formulas (theories) closed under L, of the form $\langle\Gamma_1,\Gamma_b,\Gamma_0\rangle$, where the sequent $\{\Gamma_1|\Gamma_b|\Gamma_0\}$ is valid (for L). Then, for any formula $\alpha_i \in S$, we know that there is a valid sequent where either $\{\Gamma_1 \cup \{\alpha_i\}\,|\Gamma_b|\Gamma_0\}$ or $\{\Gamma_1|\Gamma_b \cup \{\alpha_i\}\,|\Gamma_0\}$ or $\{\Gamma_1|\Gamma_b|\Gamma_0 \cup \{\alpha_i\}\}$.

By induction on the formulas of S, we construct the set of tripartitions $\langle\Gamma_1',\Gamma_b',\Gamma_0'\rangle$ over formulas of S:

i. $\langle\Gamma_1^n\rangle = \langle\Gamma_1\rangle$
$\langle\Gamma_b^n\rangle = \langle\Gamma_b\rangle$
$\langle\Gamma_0^n\rangle = \langle\Gamma_0\rangle$
ii. $\langle\Gamma_1^{n+1}\rangle = \langle\Gamma_1 \cup \{\alpha_{i+1}\}\rangle$ if $\{\Gamma_1,\alpha_{i+1}|\Gamma_b|\Gamma_0\}$
$\langle\Gamma_b^{n+1}\rangle = \langle\Gamma_b \cup \{\alpha_{i+1}\}\rangle$ if $\{\Gamma_1|\Gamma_b,\alpha_{i+1}|\Gamma_n\}$
$\langle\Gamma_0^{n+1}\rangle = \langle\Gamma_0 \cup \{\alpha_{i+1}\}\rangle$ if $\{\Gamma_1|\Gamma_i|\Gamma_n,\alpha_{i+1}\}$

Then the limit of this process is: $\langle\Gamma_1',\Gamma_b',\Gamma_0'\rangle = \left(\bigcup_{n\in\prime}\Gamma_1, \bigcup_{n\in\prime}\Gamma_b, \bigcup_{n\in\prime}\Gamma_0\right)$.

Again, $\Gamma_1'\cup\Gamma_b'\cup\Gamma_0' = S$, and $\Gamma_1'\cap\Gamma_b'\cap\Gamma_0' = \emptyset$. Then, as above, a tripartition can, therefore, be understood as partitioning the wff's $\in S$ (according to L). As is clear, at each stage of the construction, a sequent $\{\Gamma_1^n|\Gamma_b^n|\Gamma_0^n\}$ will be L-valid, as will the sequent $\{\Gamma_1'|\Gamma_b'|\Gamma_0'\}$.

Definition 32. (Characteristic function) For any set of n-partitions, we take a valuation to be the characteristic function of each section of the partition (taking the valuation indexed by each position). That is, for a (finite) set of values $\mathcal{V} = \{v_1,...v_n\}$, and for each n-partition, Γ_i', we define its characteristic function as $v(\Gamma_i') = \{v \in U : v(\alpha) = i$ for each $\alpha \in \Gamma'^i\}$, (where $i, 1 \leq i \leq n$). Then, a valuation space V that is determined by a logic L in this way is defined as the collection of valuations which are characteristic functions for each n-partition of L. Again, call this $\mathbb{V}(L)$ to denote this formulation of a valuation space.

Corollary 33. *Any valuation-space V constructed from the characteristic functions of the generalised n-partitions of an n-sequent logic L is an adequate (n-valued) semantics for L.*

Definition 34. (*N*-sequent valuation) For a (finite) set of values $\mathcal{V} = \{v_1, ...v_n\}$, and for an *n*-sequent $\{\Gamma_1|\Gamma_i|...|\Gamma_n\}$, for each location Γ_i (which is a possibly empty set of formulas $\in S$), we say that a valuation $v \in \mathbb{V}(L)$ satisfies an *n*-sided sequent iff for some Γ_i, when $i, 1 \leq i \leq n$, and some formula $\alpha \in \Gamma_i, v(\alpha) = i$. Otherwise, v refutes the sequent. This is equivalent to an expansion of the intuitive semantic reading of ordinary sequents. For example, it is simple to see that a valuation v will satisfy a sequent of the form $\{\Gamma_1|\Gamma_b|\Gamma_0\}$ iff, for some $\alpha \in \Gamma_1, v(\alpha) = 1$, or, for some $\beta \in \Gamma_b, v(\beta) = b$, or, for some $\phi \in \Gamma_0, v(\phi) = 0$.

Furthermore, we say that a valuation v respects a rule (or a set of rules) $\mathbb{R}$ iff it satisfies the conclusion of the rule $(NSEQ^C)$ whenever its satisfies the premises of the rule $(\{NSEQ^P\})$.

Definition 35. Say that a valuation $v \in U$ is *L*-consistent iff v satisfies each *n*-sequent in *L*. Then, $\mathbb{V}(L) =_{df} \{v \in U : v \text{ is } L\text{-consistent}\}$.

We are now in a position to state abstract absoluteness results for finite normal *n*-sequent logics, utilising the Galois connection discussed above.

Theorem 36. *In general, for any finite normal logic, $L = \mathbb{L}(\mathbb{V}(L))$.*

Proof. We know that $V = \mathbb{V}(L)$ by definition 31, so we need only consider the "only if" clause. Consider an *n*-sided sequent Σ, of the form $\Gamma_1, \Delta_1|...|\Gamma_n\Delta_n$, where $\Sigma \notin L$. Then there must be some $v \in \mathbb{V}(L)$ that refutes Σ, (where v refutes Σ iff, for some $\Gamma_i \in \Sigma$ there is a formula $\alpha \in \Gamma_i$ such that $v(\alpha) \neq v_i$). Recall that we are dealing with relations between *finite* subsets of *n*-sequents. By appeal to $\mathcal{T}$, we can form *n*-partitions of the set of formulas of S as above, into $\langle \Gamma'_1, \Gamma'_i, ..., \Gamma'_n \rangle$ where it is not the case that any formula α_i takes more than one location (by $\mathcal{T}$). Define a valuation v' over *n*-partitions as: $\Gamma'_1[\alpha_i \in \Gamma'_1 : v'(\alpha_i) = v_1]; \Gamma'_i[\alpha_i \in \Gamma'_i : v'(\alpha_i) = v_i]; ...; \Gamma'_n[\alpha_i \in \Gamma'_n : v'(\alpha_i) = v_n]$. Clearly, $v' \in \mathbb{V}(L)$ since v' satisfies all valid sequents $\in L$ (i.e. v' is *L*-consistent by $\mathcal{R}, \mathcal{M}$). But v' refutes Σ by definition, since v' refutes each sequent for which $\alpha \in \Gamma_i$ is such that $v(A) \neq v_i$.

Theorem 37. *In general, for a valuation space $V \subseteq U$ built up over generalised quasi-partitions as above, $V = \mathbb{V}(\mathbb{L}(V))$ (Hjortland Forthcoming b).*

Proof. We proceed by supposing that $v_0 \notin V$, with the intent to show that $v_0 \notin \mathbb{V}(\mathbb{L}(V))$. First, define $\Gamma_1 = \{\alpha \in WFF : v_0(\alpha) \neq v_1\}; \Gamma_2 = \{\alpha \in S : v_0(\alpha) \neq v_2\}; ...; \Gamma_n = \{\alpha \in WFF : v_0(\alpha) \neq v_n\}$. As is clear, for each $v \neq v_0$, v satisfies the sequent $\Gamma_1|\Gamma_2|...|\Gamma_n$, since $v \neq v_0$, there is a formula α where $v(\alpha) \neq v_0(\alpha)$. If we assume that $v_0(\alpha) = v_i$, then, for some $j \neq i$, $v(\alpha) = v_j$. By definition, $\alpha \in \Gamma_k$ for each Γ_k where $k \neq i$, so $\alpha \in \Gamma_j$. Hence v satisfies $\Gamma_1|\Gamma_2|...|\Gamma_n$. However, v_0 fails to satisfy $\Gamma_1|\Gamma_2|...|\Gamma_n$, since if it did, then for some i there is a formula $\alpha \in \Gamma_i$ such that $v_0(\alpha) = v_i$. But, by definition, if $\alpha \in \Gamma_i$, then $v_0(\alpha) \neq v_i$. Hence, $\Gamma_1|\Gamma_2|...|\Gamma_n$ is *V*-valid since it is satisfied by each $v \neq v_0$, and, because it fails to be satisfied by v_0, $v_0 \notin \mathbb{V}(\mathbb{L}(V))$.

   With this we can immediately prove absoluteness for the 3-sided construction for LP. An immediate advantage of this is that it offers a way of maintaining fine-grained distinctions across sequents. By way of illustration, we can see that there is, despite absoluteness, no way of retaining these distinctions in the ordinary sequent set-up.

Example 38. For $\mathbb{L}_{LP}(V)$, in the ordinary sequent construction, as we saw earlier, we say that $\Gamma \vdash \Delta$ is valid iff when $v(\alpha) \in \mathcal{D}$ for each $\alpha \in \Gamma$, $v(\beta) \in \mathcal{D}$ for some $\beta \in \Delta$, or equivalently, either $v(\alpha) \notin \mathcal{D}$ for some $\alpha \in \Gamma$ or $v(\beta) \in \mathcal{D}$ for some $\beta \in \Delta$. We can spell out the latter by saying that either $v(\alpha) \neq 1$ and $v(\alpha) \neq b$ for some $\alpha \in \Gamma$ or $v(\beta) = 1$ or $v(\beta) = b$ for some $\beta \in \Delta$. As we saw above, the differentiation between designated values is lost. This is clear when we consider that, $\Gamma \models_{LP} \Delta$ when $\Gamma|\Gamma|\Delta$ is L_{LP}-valid. In other words, we can switch back from $\Gamma_1|\Gamma_b|\Gamma_0$ to the two-sided $\Gamma_0 \rightarrow \Gamma_b \cup \Gamma_1$. But, there is no route back from the latter sequent in its two-sided incarnation to any specific 3-sided sequent: the reduction to bivalent semantics cuts the tie between syntax and semantics.

Example 39. It is not difficult to see that the 3-sided construction for LP is analogous to a construction for the Kleene 3-valued logic, K_3.[9] K_3 similarly has three algebraic truth-values, with the middle value typically denoted i for indeterminate, so $\mathcal{V} = \{1, i, 0\}$. In distinction with LP, K_3 has $\mathcal{D} = \{1\}$. It is well known that K_3 is paracomplete (law of excluded middle may not hold), whereas LP is paraconsistent (law of non-contradiction may not hold). So, the two logics have distinct consequence relations, since, whilst they share the same interpretation of standard connectives, they differ with respect to the interpretation of the truth-values. This fact is typically reflected in standard proof-theoretic constructions of the two logics.[10] However, in an n-sequent construction, the two coincide apart from the decoration of the middle sequent. Nonetheless, whilst the decoration is arbitrary, it reflects a distinction between the two structures at the level of provability.[11] By the translation in Example 38, we say that $\Gamma \vdash_{LP} \Delta$, whenever $\Gamma|\Gamma|\Delta$ is derivable in L_{LP}; in distinction, $\Gamma \vdash_{K_3} \Delta$, whenever $\Gamma|\Delta|\Delta$ is derivable in L_{K_3}, where L_{K_3} is equivalent to L_{LP} (just decorate the middle sequent with i in place of b). This difference allows us to distinguish between the two structures, so that, for example, law of excluded middle is derivable in L_{LP}, but not in L_{K_3}. Additionally, we know, by Theorems 36 and 37, that the semantics for L_{K_3}, $\mathbb{V}(L_{K_3})$, will be absolute, and, since the designated values differ from that of $\mathbb{V}(L_{LP})$, the consequence relation differs accordingly.

4.   Expanding the universe?

Béziau's results formalise the intuitive connection between Gentzen's syntax and semantics (bivaluations). Above, we have developed the abstract completeness results given in (Béziau 2001) by reference to the absoluteness results in (Hardegree 2005). In the light of

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[9] Thanks to an anonymous referee for pressing me to discuss this issue.

[10] See, for example, (Priest 2008).

[11] See, for example, (Hjortland Forthcoming a), where the relation between the two structures is discussed in the context of logical pluralism.

this, it looks possible that the reduction to bivalence may be a side-effect of one form of sequent construction, in which the left side is considered as falsity, and the right, as truth. With this in hand, and by showing that there is a natural way of expanding the results for ordinary sequents to n-sided sequents, we have shown that abstract soundness and completeness results can also be given for n-valued logics. As a benefit, when the n-sided construction is employed we have access to greater expressibility over the language. Before I finish, I should say that I do not take the above to be anything like conclusive argument for the multiplication of truth-values. Rather, this is more a promissory note. Insofar as we have reason to employ valuations between $\{1, 0\}$, we have a generalisable means by which to expand the duality of syntax and semantics in order to deal with those values adequately.

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