

# FOUNDATIONS AND PHILOSOPHY

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ABSTRACT. The *Univalent Foundations* (UF) of mathematics take the point of view that spatial notions (e.g. “point” and “path”) are fundamental, rather than derived, and that all of mathematics can be encoded in terms of them. We will argue that this new point of view has important implications for philosophy, and especially for those parts of analytic philosophy that take set theory and first-order logic as their benchmark of rigor. To do so, we will explore the connection between foundations and philosophy, outline what is distinctive about the logic of UF, and then describe new philosophical theses one can express in terms of this new logic.

Philosophers investigate notions such as “universal” and “proposition”. Analytic philosophers tend to investigate them in terms of *sets*: a *universal*, they may say, is a *set* containing its instances; a *proposition*, they may claim, is a *set* of possible worlds. But why *sets*? What if there were an alternative way to investigate these notions?

A newly-emerged foundation of mathematics – the *Univalent Foundations* (UF) – allows us to consider just such a possibility. UF differs from previous foundations of mathematics in essential ways. For instance, it takes the point of view that spatial notions (e.g. “point” and “path”) are fundamental, rather than derived, and that all of mathematics can be encoded in terms of them. In this paper, we want to argue that this new point of view has important implications for philosophy, and especially for those parts of analytic philosophy that have unquestioningly relied on set theory and first-order logic as their benchmark of rigor. To do so, we will explore the connection between foundations and philosophy, and, granted this connection, we will explore the implications for philosophy of the Univalent Foundations.

## 1. FOUNDATIONS AND ANALYTIC PHILOSOPHY

The dominant narrative in the foundations of mathematics goes roughly as follows: the increasing complexity of mathematical analysis in the 19th century made it imperative to reconstrue all mathematical propositions as logical (i.e. *analytic*) truths about simple entities – indeed, ideally, as logic itself. And this demand – through Cantor, Frege, Russell, Dedekind, Hilbert, Peano, Zermelo, Gödel and many others – eventually gave rise to set theory and first-order logic as we now understand it together with an all-encompassing *foundation* of mathematics in the form of Zermelo-Fraenkel (“ZF”) set theory. Surely it is no exaggeration to say that the fact that all of mathematics could conceivably be encoded in terms of a single binary predicate (“ $\in$ ”) constitutes one of the major discoveries of the 20th century, in any field of knowledge. This monumental achievement also played a key role in bringing about the school of philosophy that has come to be known as *analytic philosophy*.

The story, as with many stories, begins with Kant. In the First Critique’s Doctrine of Method (especially [A713/B742]) Kant outlines his view that mathematics proceeds by *constructing* intuitions adequate to *a priori* concepts, not by *analyzing* such concepts (which, he says, is the task of philosophy). Russell’s very influential criticism of Kant was that a strong enough logic, like the logic of his *Principia*, could *compensate* for this element of “construction” without invoking some kind of pure intuition provided by the human subject. Russell concludes that Kant’s doctrine of construction in pure intuition was metaphysical froth added to the mix in order to *compensate* for the deficient (Aristotelian) logic that Kant was working with.<sup>1</sup> This point of view, which takes Kant to have invented the synthetic *a priori* only in order to compensate for an insufficiently expressive logic, came to be known as the “compensation thesis”.<sup>2</sup>

<sup>1</sup>Russell maintained this criticism of Kant, in one form or another, throughout his life, cf. in particular [46] and [45], quoted below. As Friedman writes: “Russell [...] habitually blamed all the traditional obscurities surrounding space and geometry – including Kant’s views of course – on ignorance of the modern theory of relations and uncritical reliance on [Aristotelian logic].” ([15], p.457)

<sup>2</sup>For more discussion on Kant’s views on geometry see the Parsons-Hintikka debate [39, 23] as well as Friedman’s very influential recasting of the compensation thesis in [15, 16]. It should also be made clear that the issue of

Eventually, at least in the Anglo-Saxon tradition, the compensation thesis comes to be accepted as a more or less definitive objection against Kant's conception of mathematics.<sup>3</sup>

From there, it is very natural to think that if a logic is powerful enough to provide a foundation for mathematics that does not rely on "metaphysical froth" then it should be powerful enough to be deployed to tackle other philosophical problems. It is unclear whether Frege himself had such ambitions, but Russell undoubtedly did, and it is exactly this step that Russell took.<sup>4</sup> What Russell had in mind is perhaps best captured by the following passage of his programmatic [45]:<sup>5</sup>

The proof that all pure mathematics, including Geometry, is nothing but formal logic, is a fatal blow to the Kantian philosophy. [...] The whole doctrine of *a priori* intuitions, by which Kant explained the possibility of pure mathematics, is wholly inapplicable to mathematics in its present form. The Aristotelian doctrines of the schoolmen come nearer in spirit to the doctrines which modern mathematics inspire; but the schoolmen were hampered by the fact that their

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whether or not Kant thought that demonstrations themselves also involved pure intuition (rather than just the construction of the concepts that they were to be applied to) remains a topic of controversy. For the latest installment, see Hogan [24].

<sup>3</sup>Needless to say, the neo-Kantians of the Marburg school did not share this sentiment. In particular, Cassirer thought that "Russell's logicistic" not only failed to undermine the critical philosophy, but indeed could provide it exactly with the kind of raw material that it needs for its future development. For an illuminating analysis of the Neo-Kantian "absorption" of Russell and Frege's work on the foundations of mathematics by Cassirer see Heis [20, 21].

<sup>4</sup>Even at his most philosophical, e.g. [14], it seems unclear to us whether Frege is writing with the purpose of *applying* symbolic logic to philosophical questions, rather than using philosophical questions to *clarify* the principles of his system. There is little doubt that Frege's initial and overarching interest was in the foundations of mathematics (especially arithmetic) as is evident throughout the *Grundlagen* and also apparent from his own academic trajectory. Whether or not he thought of his foundational work as a way of attaining definitive solutions to philosophical problems is less clear, and in our opinion probably false. Perhaps Frege was restrained in this regard by the near-total acceptance of the Kantian system within German academia – as evidenced, for example, by the excessive (perhaps even ironic) caution he exercises when raising objections to Kantian doctrine:

In order not to open ourselves up to the criticism of carrying on a picayune search for faults in the work of a genius whom we look up to only with thankful awe [...] If Kant erred with respect to arithmetic, this does not detract essentially, we think, from his merit. (*Grundlagen*, §89, as translated in [6])

As this last sentence makes clear, and his correspondence with Hilbert even clearer, Frege never really abandoned a Kantian view of (Euclidean) geometry, whose statements he regarded as synthetic *a priori*. As a result, it seems implausible that he would have come to hold a view of philosophy similar to Russell's, at least insofar as Russell's own view was primarily motivated, as we want to claim here, from his compensation objection against Kant.

<sup>5</sup>It is also worth noting that when Russell first wrote these words, he was not yet familiar with Frege's work.

formal logic was very defective, and that the philosophical logic based upon the syllogism showed a corresponding narrowness. *What is now required is to give the greatest possible development to mathematical logic, to allow to the full the importance of relations, and then to found upon this secure basis a new philosophical logic, which may hope to borrow some of the exactitude and certainty of its mathematical foundation.* [our emphasis]

To be sure, this is Russell at his most programmatic (if not propagandistic), writing at perhaps the height of his optimism about what had been achieved with the *Principia*.<sup>6</sup> Nevertheless, we do believe the highlighted sentence in the above passage contains the core founding idea of analytic philosophy: given a *mathematical* logic associated to a foundation for mathematics, one can then “found upon” it a *philosophical* logic that can be applied to clarify and tackle philosophical problems.

Russell, of course, speaks simply of a “mathematical logic”, without mentioning a foundation for mathematics. This is because, from Russell’s logicistic viewpoint, a mathematical logic simply *is* a foundation of mathematics: since all of mathematics is reducible to logic, a mathematical logic is simply the logic to which all of mathematics is reducible. For Russell, this “mathematical logic” is the logic presented in *Principia Mathematica* which is more or less what is now known as a (ramified) theory of types.<sup>7</sup>

Eventually, the notion of a foundation of mathematics is separated from the notion of a mathematical logic. The modern picture of a foundation for mathematics consists, rather, of three components: (i) a *core logic* (e.g. classical first-order predicate logic) which is used to express (ii) a *theory* (e.g. the axioms of ZF set theory expressed using “ $\in$ ”) which we take to be describing (iii) a *universe of basic objects* (e.g. the cumulative hierarchy of sets, usually

<sup>6</sup>Russell must have approved the above view at least until 1917, the year in which [45] was included in an anthology of his writing.

<sup>7</sup>The fact that Russell identified a foundation for mathematics with a mathematical logic is captured even by the title of [47]: *Mathematical Logic as Based on the Theory of Types*.

denoted by  $V$ ). In these modern terms we therefore take the relationship between foundations of mathematics and philosophy that Russell envisioned to be the following: first-order predicate logic is the core logic associated to set-theoretic foundations of mathematics and upon it can be founded a philosophical logic with which to clarify and tackle philosophical problems. This philosophical logic constitutes what Russell called the “philosophy of logical analysis” – or, as it is now known, *analytic philosophy*.

Thus, what was new in the “philosophy of logical analysis” as Russell envisioned it was neither the idea of *analysis*, nor the idea of this analysis being carried out in terms of *logic*. What is new, rather, is the *particular* core logic on which this “logical analysis” is to be based, namely first-order logic. Therefore, if one could find an alternative core logic associated to an alternative foundation of mathematics, then there is just as much reason (even by Russell’s lights) to build a “philosophical logic” on top of this alternative logic as there was to build the philosophical logic that we now call Analytic Philosophy on top of first-order predicate logic. The Univalent Foundations provide just such an alternative foundation.

## 2. GRASSMANN’S DREAM

The universe of basic objects of the Univalent Foundations is best understood as the universe of abstract shapes. They are “shapes” because they have *spatial* features, namely points and lines, possibly with edges between points and faces enclosed by lines etc. They are “abstract” because we do not care about them up to concrete presentations, but are rather able to “deform” them freely with respect to their geometric properties. Two key questions arise. Firstly, what exactly are these abstract shapes? Secondly, why would one want to ever think of the objects of mathematics as shapes?

The answer to both questions begins with an idea best traced back to a letter Leibniz wrote to Huygens in 1679. In that letter, Leibniz described a “Geometric Characteristic” which would

“have great advantages for representing to the intellect everything that depends on the imagination precisely and naturally, but without figures.”<sup>8</sup> Leibniz’s idea, roughly, was to come up with a symbolic notation (i.e. using letters such as  $x, y, z$  rather than figures *pace* Euclid) whose aim would be to represent shapes (e.g. curves) and constructions on shapes (e.g. drawing tangents on curves) exactly like numbers and calculations on numbers are used in algebra (e.g.  $x + y = y + x$ ). This system was never fully worked out, and remained confined to a few sketchy remarks found in Leibniz’s letters and some unpublished papers.<sup>9</sup> In the 19th century, however, this fundamental idea of Leibniz’s is revived through the work of two key figures: B. Riemann and H. Grassmann.

In his remarkable 1856 paper “On the hypotheses which lie at the foundation of geometry” (cf. [43]) Riemann establishes the definition of the modern notion of a *manifold*, thus setting out a clear distinction between the notion of a *space* and the notion of a *geometry*. Non-Euclidean geometries, Riemann argues, emerge as a perfectly natural consequence of the fact that there are many incompatible geometries one can impose on three-dimensional space – and that whether or not the real space of our sense perception is or is not Euclidean is a matter best decided by empirical, not mathematical, investigation:

[T]he propositions of geometry cannot be derived from general notions of magnitude, but [...] the properties which distinguish space from other conceivable [three-dimensional manifolds] are only to be deduced from experience. ([12], p. 652)

<sup>8</sup>*GM. II, 20-27*, translated by Kannenberg in [17].

<sup>9</sup>A possible earlier source for this kind of idea is Euclid himself. In Book II of his *Elements*, Euclid seems to give demonstrations to “obvious” geometric facts that can be seen (from a modern perspective) to encode well-known algebraic identities. Indeed, there was a popular line of interpretation that took Euclid to have used geometry as a way of doing algebra, i.e. of regarding diagrams themselves as playing the role of algebraic formulas. This is almost the converse of the Leibnizian idea but at the end of the day they amount to the same suggestion: an organic synthesis of algebra and geometry. This point of view on Euclid, although popular with mathematicians, was highly controversial and indeed led to an infamous dust-up between the historian S. Unguru [52] and the mathematician A. Weil [59] (among others) played out in the pages of the journal *Archive for History of Exact Sciences*.

In doing so, Riemann sets the stage for the study of *space* abstracted from its geometric properties and almost single-handedly gives birth to the field of *topology*: the study of spaces without a notion of “distance” or “length”.

Eventually, this idea would give rise to *algebraic topology*, a field which captures much of the spirit of Leibniz’s Characteristic since it provides a way to encode shapes and constructions on shapes through algebraic symbolism.<sup>10</sup> In algebraic topology one captures a notion of a shape that is invariant under certain geometric deformations (e.g. stretching or shrinking) and is best handled by logical-algebraic operations (e.g. adding and multiplying) rather than diagrammatic reasoning. Throughout the 20<sup>th</sup> century, this notion of a shape undergoes tremendous developments, mainly as a result of the mathematician A. Grothendieck’s monumental work on the foundations of algebraic geometry.<sup>11</sup> Through these developments we obtain the modern formal notion of a *homotopy type*, which provides an answer to our first question: the “abstract shapes” of UF are best understood as the homotopy types of algebraic topology.<sup>12</sup>

With respect to our second question, we need to turn to the work of H. Grassmann, another 19<sup>th</sup> century mathematician. Grassmann, unlike Riemann, explicitly regarded his own work as the realization of the Leibnizian Characteristic. In his prize-winning *Geometric Analysis* of 1847, Grassmann believes that he has

[formulated], at least in outline, an analysis which in general actually accomplishes what [Leibniz] regarded as the goal of [his Geometric Characteristic].

([17], p. 318)

Later, in his *Ausdehnungslehre*, this basic idea is then to be applied not only to geometry, but to the whole of mathematics:

<sup>10</sup>It must be noted, however, that Riemann’s contribution was almost certainly not (intentionally) motivated by the Leibnizian characteristic, nor was the development of the field of algebraic topology as we understand it today.

<sup>11</sup>For more on Grothendieck’s work and life cf. [8, 25] and for his role in mathematical thinking from a philosophical point of view cf. [34, 60].

<sup>12</sup>For more on the notion of a homotopy type, its role in the Univalent Foundations, and its relation to the more technical notion of an  $\infty$ -groupoid, cf. e.g. [35].

Pure mathematics is [...] the science of the particular existent that has come to be by thought. The particular existent, viewed in this sense, we call a thought form or simply a form; thus pure mathematics is the *theory of forms*. ([17], p. 24)

It is here that we find the kernel of a revolutionary idea that is entirely independent of Cantorian set theory, namely the idea that all of mathematics can be encoded in terms of *forms* that have some kind of intrinsic spatial meaning.

To be realized, this idea would require some kind of system in which the basic objects can be understood as shapes, but such that they are described at a level of abstraction that makes them amenable to *logical* (rather than diagrammatic) manipulation. In other words: a *Begriffsschrift* for shapes. In describing his formalization of Grassmann's work, Peano in his *Geometric Calculus* explains very well what this idea would entail:

The geometric calculus, in general, consists in a system of operations on geometric entities, and their consequences, analogous to those that algebra has on the numbers. [...] [It] exhibits analogies with analytic geometry; but it differs from it in that, whereas in analytic geometry the calculations are made on the numbers that determine the geometric entities, *in this new science the calculations are made on the geometric entities themselves*. ([40], p. ix) [my emphasis]

But in what way exactly such an idea could materialize into a usable *formal* system in the modern sense remained unclear until very recently, and was certainly not something that Grassmann himself could have envisioned.<sup>13</sup>

To summarize, the key idea that emerges from Grassmann is that of having a foundation of mathematics based on *spatial* basic objects rather than on Cantorian-Fregean *collections* – let

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<sup>13</sup>At around Grassmann's time even the very idea of a "formal system" was lacking in the precision which it now enjoys. But Peano certainly had a conception close to the modern one and he certainly believed it was applicable to Grassmann's system, as evidenced by his [40].



us call this idea *Grassmann's dream*. From Riemann and the development of algebraic topology we get plausible answers concerning what these “spatial basic objects” could be, namely abstract shapes as captured by the notion of a homotopy type. The Univalent Foundations incorporate both ideas: they offer a foundation of mathematics whose basic objects are homotopy types. But what, one might wonder, is the motivation for taking such a point of view?

### 3. BENACERRAF'S ANTINOMY

A possible answer comes from a problem close to the hearts of many analytic philosophers: P. Benacerraf's [5] observation that if numbers are formalized as sets, set theory will still be able to distinguish between equally good (i.e. *isomorphic*) such formalizations. For example, even though the two-element sets  $\{\emptyset, \{\emptyset\}\}$  and  $\{\pi, \mathbb{N}\}$  are equally good formalizations of the number 2 (since they both contain two elements) the formula “ $\emptyset \in x$ ” is true of the former but not of the latter. Put simply: in set theory there are objects that should be indiscernible, but which are not identical. Let us call this problem *Benacerraf's antinomy*.

The usual reaction to Benacerraf's antinomy is to take a so-called *structuralist* view of mathematical objects.<sup>14</sup> In such a view, the objects of mathematics are understood as bearing only those properties that are relevant to the particular mathematical structure in question. For example, we would like to understand the natural numbers as composed of objects  $x$  that are bearers only of properties such as “ $x$  is a prime number” or “ $x$  is greater than 3”, rather than “ $\emptyset \in x$ ”. The inevitable question, then, is what could such structures be?

One way to make this wide-ranging question precise is the following: how could the objects of mathematics be *formalized*, so that Benacerraf's antinomy is avoided? Is there, in other words, a formal system in which no “non-structural” property is stateable? This way of posing

<sup>14</sup>Several such positions have been explored in the literature. For example, there have been positions that take structures to be *possibilia* (Hellman [22] and, more obliquely, Putnam [42]) or Platonic natureless abstractions (Shapiro's *ante rem* structuralism in [48]). More recently, Burgess [7] has advanced a more minimalist position in which he takes Benacerraf's observation to be best understood simply as an (accurate) observation about mathematical practice.

the question takes Benacerraf’s antinomy not as a call for a new ontology of mathematics, but rather as a design constraint for a new foundation of mathematics. If axiomatic set theory is understood as an attempt to solve the paradoxes of naive set theory, then why not also attempt to come up with some alternative axiomatic system that resolves Benacerraf’s antinomy?

A formal system for the Univalent Foundations<sup>15</sup>, then, can be understood as just such an axiomatic system – and therein lies a key philosophical motivation to pursue Grassman’s dream.<sup>16</sup>

In any formal system that realizes UF one achieves an essentially complete resolution of Benacerraf’s antinomy.<sup>17</sup> Formally, this is done through V. Voevodsky’s *axiom of univalence* (UA) which can be paraphrased as follows:

(UA) Identity is isomorphic to isomorphism

At this point, however, it is not quite clear even how to make sense of (UA), let alone how to understand how it resolves Benacerraf’s antinomy. For identity, surely, is a *proposition*: when we write  $a = b$  we are asking *whether or not*  $a$  is identical to  $b$ ; we are not defining some structure, as for example when we write “Let  $G$  be a group...”. On the other hand, it is only *structures* that can be isomorphic one to the other: when we ask “Is  $G$  isomorphic to  $H$ ?” we do not regard  $G$  and  $H$  as propositions, but rather as structures like groups or rings. And since identity is not a structure, but a proposition, how are we to understand (UA)?

The answer, in short, is that in order for (UA) to make sense, identity must indeed *not* be understood as a proposition, but rather as a structure. It is exactly at this point that we begin to diverge from the usual way philosophers and mathematicians have thought about these matters.

This is the key new idea in (and also the necessary precondition for) stating (UA) and therefore

<sup>15</sup>Such a formal system need not necessarily be a dependent type theory, just as a formal system for set-theoretic foundations need not necessarily be a certain first-order theory.

<sup>16</sup>To be clear, UF was not the first such attempt. This honour belongs to the Elementary Theory of the Category of Sets (“ETCS”) due to F. W. Lawvere in [31] which, coincidentally, was published in exactly the same year as Benacerraf’s paper. Lawvere’s work set off a long-running debate on so-called “categorical” foundations of mathematics, which aimed to formally capture a structural view of the objects of mathematics using category theory. For a wide-ranging and thorough summary of the role of category theory in the foundations of mathematics, cf. [33, 36].

<sup>17</sup>For an explanation and precisification of how this is achieved, see [2] and [51].

also for resolving Benacerraf’s antinomy. Indeed, the two strains of thought examined in Sections 2 and 3 converge on exactly this idea: a foundation that completely resolves Benacerraf’s antinomy must include an identity that behaves like a structure, and the only way to make sense of such an identity is in terms of spatial basic objects, thus fulfilling Grassman’s dream. But beyond even the issue of how to formalize it, is the idea of identity-as-structure philosophically salient?

#### 4. IDENTITY AS STRUCTURE

So far, the formalization of the axiom of univalence has been carried out in formal systems that go under the general umbrella terms *Homotopy Type Theory* (“*HoTT*”) or *Univalent Type Theory* (“*UTT*”). These formal systems extend a certain well-known and well-studied (by logicians and computer scientists) family of formal systems called *Martin-Löf Type Theories* (or simply *Dependent Type Theories*).<sup>18</sup> But the precise details of HoTT, fascinating though they are, would take us too far afield, and are not directly relevant to our purposes here. Indeed, in a way, they are no more necessary to understanding UF as Russell’s Theory of Types is necessary to understanding Cantorian set theory. We will therefore focus on one particular feature that any such formal system must include, and which provides, as indicated above, the crucial and most philosophically interesting component of the core logic of UF: *identity is a structure, not a proposition*.

The best way to introduce the idea of identity as a structure is perhaps through the historical accident that led to its definition, namely through the constructive variant of type theory alluded to above, due to P. Martin-Löf. The constructivist viewpoint disallows the approach to explaining the meanings of logical compounds taken in classical logic, involving truth-conditions,

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<sup>18</sup>HoTT originates from the surprising observation that Martin-Löf Type Theory [37] – whose original aim was that of providing an intuitionistic foundation for mathematics – can be given a semantics in which its basic objects are interpreted as abstract shapes, i.e. as homotopy types. For standard introductions to HoTT, see [53, 41, 49, 57]. For the ideas that led to the homotopy interpretation of type theory see [3, 58]. For some of the earlier writing that led to the ideas for UF see [56, 55, 54] and for some philosophical issues associated to HoTT see [26, 11, 10, 2].

taking an approach involving proof-conditions instead. In the case of a conditional  $\phi \rightarrow \psi$ , for instance, where the classicist will say that a conditional is true unless the antecedent is true and the consequent is false, the constructivist will say that a proof of a conditional is a function taking proofs of the antecedent as inputs and giving proofs of the consequent as outputs. In the setting of constructive type theory statements like  $\phi$  and  $\psi$  are understood themselves as collections (or *types*) containing proofs (or *witnesses*) of their truth. As a result, the meaning of a conditional such as  $\phi \rightarrow \psi$  is taken itself as collection of functions that take a proof of  $\phi$  to a proof of  $\psi$ . If one now applies the same kind of thinking to an identity  $a = b$  we get the idea that identity itself should be understood as the collection of its proofs, and therefore not merely as a proposition that is either true or false, but rather as the carrier of additional information, namely as a structure. Thus, whether or not two things are identical is not a matter of “yes” or “no”, but rather a question of *in what ways* these things are identical – which allows for there to be many such ways, themselves also related in other ways, and so on.<sup>19</sup>

Nevertheless, constructive type theories of this kind never intended to have a notion of identity-as-structure. Indeed, axioms were added to these type theories in order to effectively “kill off” the structure of identity types and to make it behave exactly like the usual notion of identity. In other words, identity-as-structure was regarded as a bug, not a feature (to invert a common phrase) of these formal systems. If we are to make the idea salient, then, it is not going to be by simply relying on the fact that the idea was accidentally captured in some formal system.

The way to make it salient, thankfully, is simple enough: simply refuse to regard identity-as-structure as a kind of identity, and regard it instead as a primitive notion of *isomorphism*. But what is “isomorphism”? Etymologically, isomorphism means “equality of form” and that is exactly how we ought to regard it: isomorphism relates structures and two structures are

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<sup>19</sup>The whole of this paragraph, as well as of the idea of explaining the origins of the idea of identity-as-structure in this manner, is due to John Burgess.

isomorphic if they can be shown to have the same (internal) form. To simplify, we can refer to the “form of a structure” as simply its structure, in which case we obtain the almost tautological statement: two structures are isomorphic if they can be shown to have the same structure. But what does it mean, *to show that they have the same structure*?

Consider two sets of two elements, say  $S = \{a, b\}$  and  $T = \{c, d\}$ . The structure of  $S$  and  $T$  is simply the number of elements they contain (a “purely extensional” structure). An isomorphism between  $S$  and  $T$  would then have to show that they have the same structure, i.e. the same number of elements. How can this “be shown”? Well, by exhibiting a function  $f: S \rightarrow T$  that is a one-to-one correspondence, namely which takes distinct elements of  $S$  to distinct elements of  $T$ . But there is not just one way of doing so. We can either define  $f$  as the function which takes

$$a \mapsto c$$

$$b \mapsto d$$

or as the function  $g$  which takes

$$a \mapsto d$$

$$b \mapsto c$$

These distinct functions  $f$  and  $g$  are equally good isomorphisms, which means there is not merely a *fact* of the isomorphism of  $S$  and  $T$ , but rather a *structure* of isomorphisms between them. In other words, to write  $S \cong T$  can be understood not as the *fact* of  $S$  and  $T$  being isomorphic, but rather as the *structure*  $\{f, g\}$  of isomorphisms between them. The structure  $S \cong T$  can then be treated in exactly the same way as  $S$  and  $T$  – and indeed, in this example,  $S \cong T$  will itself be isomorphic to both  $S$  and  $T$  since it too contains only two distinct elements.

The key idea of isomorphism is that there is not just a *unique* way of showing that two structures have the same structure – there is, rather, a *structure* of ways in which to do so.

The observant reader may have noticed that we have applied the term “distinct” to elements of  $S$  and  $T$  (and  $S \cong T$ ) in the above example. This may make one suspicious that there is still a “strict” notion of identity in the background, which is what even allows us to be able to explain what an isomorphism is. This is partly true: there is indeed a “strict” notion of identity, but it is only a degenerate version of isomorphism, namely the isomorphism between objects that are assumed to have no structure, e.g. atoms or *urelemente*. In the case of such structureless objects, to ask whether or not they are isomorphic is the same as (i.e. is isomorphic to) asking of whether or not they are identical. But it is false that such a strict notion of identity is *always* in the background, and “less strict” notions of isomorphism are always defined in terms of it.

To illustrate, consider now the structure  $G = \{S, T\}$ , where we now take seriously the idea that  $S$  and  $T$  are not either distinct or not, but that there is rather a structure of isomorphisms between them. The structure of  $G$  is then not that of a “purely extensional” set: it does not consist of the number of its components. Rather, the structure of  $G$  is best visualized as

$$S \begin{array}{c} \frown \\ \smile \end{array} T$$

namely as two components connected by two isomorphisms. And if we now also consider the structure  $H = \{S', T'\}$  with  $S = \{a', b'\}$  and  $T' = \{c', d'\}$ , we can ask: what is an isomorphism between  $G$  and  $H$ ? And we will similarly obtain another (more complicated) structure  $G \cong H$ , and so on.

Importantly, in this way of thinking about things, the distinctness of  $S$  and  $T$  is not to be regarded as a *property* of either of them. If we did so regard it, then clearly that property would have to be part of their structure, and if it were part of the structure, it would have to be something that only one kind of isomorphism could preserve: strict identity. (In other

words,  $S$  could be isomorphic to  $T$  only if  $S$  was identical to  $T$ ). But since we consider many isomorphisms between  $S$  and  $T$ , this property is no longer part of their structure. The notion of isomorphism (which formalizes the concept of identity-as-structure that we are interested in) does not track the strict identity of the objects it relates: they are, in that regard, *abstract*.

But what exactly *are* they? The answer depends on how exactly we *define* isomorphism, just as, in set theory, exactly what sets are depends on how we axiomatize their identity conditions. A very general way of understanding the objects that isomorphism relates is as abstract structures with components. A more specific way of understanding them is as abstract shapes, in the vein of Grassmann and contemporary algebraic topology. Thus, the axiom of univalence can be understood as the *definition* of the isomorphism between abstract shapes, exactly how the axiom of extensionality in set theory (“ $a = b \leftrightarrow \forall x(x \in a \leftrightarrow x \in b)$ ”) can be understood as the definition of the identity between ZF-style sets (“having the same elements”). But the axiom of univalence, just like the axiom of extensionality, is an *axiom* describing certain objects; it is not a logical law. And just as the logical laws governing identity in first-order logic come before we state the axiom of extensionality, similarly there ought to be logical laws governing isomorphism. Which brings us to the crucial question: what is the underlying logic of isomorphism-as-primitive (or, isomorphically, of identity-as-structure)?

## 5. A NEW MATHEMATICAL LOGIC

A mathematical logic for the Univalent Foundations would need to formalize the idea of treating isomorphism as a primitive, rather than derived, notion of identity. Such a mathematical logic has been proposed in [50] and the proposal therein can be thought of as “first-order predicate logic *with isomorphism*” (instead of *equality*). The basic idea is to modify traditional first-order logic in two ways.

Firstly, we need to work in *multi-sorted* first-order logic to which we also add a notion of sort *dependency*. To illustrate: in the usual framework of first-order logic one may speak of a two-sorted signature for graphs, with a sort  $V$  for vertices and a sort  $E$  for edges between vertices. In this new system, we will be able to say that  $E$  depends on  $V$ , in the following sense: we cannot speak of a variable  $f$  of  $E$  until we have declared variables  $v, w$  of  $V$  (the “source” and “target” of  $f$ ). On top of this novel syntax, we add three new kinds of sorts that are *logical* (in the sense that they have a fixed denotation) and which gives us the syntax of identity-as-structure: isomorphism sorts  $x \cong y$  (dependent on  $x$  and  $y$ ), reflexivity predicates  $r_x(q)$  (dependent on  $q: x \cong x$ ) and transport structure (dependent on  $p: x \cong_K y$  and a dependent sort  $A$  over  $K$ ).

Secondly, and more substantially, we supplement the usual laws of (classical or intuitionistic) first-order logic ( $\wedge$ -introduction,  $\exists$ -elimination etc.) with a new logical law governing isomorphism sorts. This new logical law – which simplifies the “identity elimination” rule of Martin-Löf Type Theory – can be written as the following inference rule (in natural deduction style):

$$\frac{\begin{array}{c} \vdots \\ \phi[x, x, q] \end{array} \quad \begin{array}{c} \vdots \\ r(q) \end{array}}{\phi(x, y, p)} \quad p: x \cong y, q: x \cong x$$

This rule can be understood as saying the following: the only way you can deduce a property  $\phi$  of two isomorphic things is if you can already deduce it for just one of them. After all, if two structures are isomorphic ( $p: x \cong y$ ) then they must be indistinguishable, and therefore if there is some statement we can make involving both of them ( $\phi(x, y, p)$ ) then that very statement can be made of just one of them ( $\phi[x, x, q]$ ). As such, quite simply, the new logical law can be justified as follows: isomorphic structures can no more be separated by statements we make about them than can a structure be separated from itself.



Formally, this rule – together with some obvious axioms, e.g. that  $x \cong x$  is always inhabited by a “trivial isomorphism” – ensures that all the usual laws of identity hold. For example, we can prove the indiscernibility of isomorphs: for any formula  $\phi$  we can show that given an isomorphism between  $x$  and  $y$  then  $\phi(x)$  implies  $\phi(y)$ . More importantly, however, isomorphism sorts  $x \cong y$  can be treated as if they were structures. For example, over appropriate signatures, we can state axioms such as

$$\text{Iso}(x, y) \simeq (x \cong y)$$

which says that the collection of “non-logical” isomorphisms  $\text{Iso}(x, y)$  between  $x$  and  $y$  is itself (non-logically) isomorphic ( $\simeq$ ) to the collection of “logical” isomorphisms between them ( $x \cong y$ ). If, in particular, we think of  $\text{Iso}(x, y)$  as a set of functions that show  $x$  and  $y$  to be isomorphic, then the above sentence says that there are just as many “functional” isomorphisms between  $x$  and  $y$  as there are “logical” isomorphisms between them. In other words: that given an arrow-like isomorphism, there is an underlying isomorphism that ensures that  $x$  and  $y$  are indistinguishable. The axiom of univalence is itself an instance – the most general one – of this kind of correspondence.

This new mathematical logic, with isomorphism as its primitive notion of “identity”, can be thought of as the core logic of UF similar to how first-order logic is regarded as the core logic of set-theoretic foundations. As such, we obtain an entirely new mathematical logic in the sense of Section 1 and one which is moreover a proper expansion of first-order logic.<sup>20</sup> Following Russell, we may now envision building an entirely new philosophical logic on top of it.

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<sup>20</sup>To be precise, this means that all first-order logic can be recovered as the “classical limit” of the new core logic, but not the other way around.

## 6. NEW PHILOSOPHICAL LOGIC

The new philosophical logic that can be built upon the new mathematical logic opens up new vistas for investigation and experimentation in formal philosophy. There are several directions one might pursue, and we want to highlight here two in particular.

Firstly, the reasoning and methods of *category theory* will certainly be included in this new philosophical logic.<sup>21</sup> The idea that category theory has some role to play in philosophy is not new. On the one hand, philosophers of a more analytic bent (e.g. David Corfield [9], J.-P. Marquis [34] and E. Landry [27]) and logicians (e.g. S. Feferman [13] and C. McLarty [38]) have long explored its implications for the philosophy and foundations of mathematics. But they have done so mainly by regarding category theory as situated *within* mathematics rather than as a general methodology that can be applied to philosophical issues not directly related to mathematics and logic. On the other hand, philosophers of a more continental bent (e.g. A. Badiou [4]), category theorists (e.g. F. W. Lawvere [29, 30]) and, most importantly, the hard-to-classify and very courageous work of F. Zalamea [60] all have come closer to intimating a role for category theory as a general philosophical methodology. But their justification for doing so has so far mainly been based on the prominent role of category theory within the contemporary practice of mathematics.

For our part, we agree with the latter that category theory can play the role of a general philosophical methodology – but disagree that the justification for doing so must be that category theory plays a central role in contemporary mathematics. Such justifications smack of false prophecy: to import new formalism into philosophy simply because it is popular in some field outside of it is to throw an ash of jargon in the eyes of those unfamiliar with it, and then expect them to be dazzled. But with the new mathematical logic in hand, we can do better (even if we

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<sup>21</sup>For technical introductions to category theory, cf. Mac Lane's classic textbook [32], as well as the more accessible [1].

cannot entirely avoid the ashen cloud of jargon): category theory as a general-purpose philosophical methodology can now be justified in exactly the same way that analytic philosophers justify the pervasive use of set theory and first-order logic, namely by a mathematical logic associated to a foundation of mathematics. This, for example, is the only reasonable justification we see for the project of applying category theory to the (independent) issue of theoretical equivalence of scientific theories.<sup>22</sup> But such a justification can now be applied to many more such projects, allowing category theory to cast as wide a net over philosophy as first-order logic and set theory has done.<sup>23</sup>

Secondly, and more radically, the new mathematical logic can support a novel formal theory of concepts. To illustrate, consider one of the founding problems of analytic philosophy, that of how to understand identity statements like “ $a = b$ ” as having content while at the same time maintaining the principle that equals can be substituted for equals. For, the problem goes, if  $a = b$  and equals can be substituted for equals, then by substituting  $b$  with  $a$  we get  $a = a$ , which is trivially true – surely, however, not every identity  $a = b$  is trivially true. Frege’s seminal paper attempts to solve this problem by distinguishing between the *sense* of an expression and its *reference*.<sup>24</sup> This distinction is a natural consequence of the dominant view of concepts, namely that they have an *extension* that can be described in multiple ways (*intensions*).

In the new mathematical logic, however, we can re-imagine concepts as abstract shapes composed of points and paths between them (or, more generally, as abstract structures composed of components and isomorphisms between them). This allows us to model concepts in such a way that the distinction between extension and intension (and of sense and reference) no longer appears inevitable. For in an abstract shape we may have distinct points connected by paths, which can be seen as a formalization of the very basic idea that there are distinct intensions

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<sup>22</sup>The original idea is in [18], and a more detailed exposition is in [19].

<sup>23</sup>For an indication of the breadth of such projects, at least within the analytic tradition, see the upcoming volume [28].

<sup>24</sup>See [14].



same distinguishing mark. You can now confidently assert that the *MS* was also in position *X* relative to *S* at a particular point in time. Informally, paths in abstract shapes can be understood exactly as such processes.<sup>26</sup>

“But have you thereby demonstrated that the given property holds of the *same* thing?” The answer is: it is irrelevant! Do we *know* that the *ES* is the same thing as the *MS* because we picked out the same distinguishing mark on both using our telescope? No. What we do know is that the property that we determined held for the *ES* (its position *X* relative to *S*) holds also of *MS* insofar as we are willing to regard the observation of the same distinguishing mark on both as an isomorphism, and therefore as a process along which we can transfer properties from the (intension) *ES* to the (intension) *MS*. In the new mathematical logic, there is no ultimate, once-and-for-all *identity* between things – instead, there are structures and isomorphisms between them, and insofar as we are given an isomorphism between structures, we can transfer properties of one to the other, making them indistinguishable. But isomorphisms are no longer *facts*, they are themselves *structures*. The world, one might say, is not the collection of facts, but of structures.

Of course, the above example of sense and reference is not meant as a complete account, nor is it put forward as a theory. Neither do we want to claim, in any way, that this is the “one true way” of understanding concepts. But it is certainly a *new* way and we offer it here as an illustration that simple philosophical ideas can now be backed up by a view of concepts as abstract shapes. It is in exactly this way, fortified by formal work, that the possibility of carrying out a new kind of “logical analysis” is established within the framework of UF.

So as long as one wishes to remain bound to a “normal” formal philosophy – to be carried out in terms of research projects and subject to pre-existing grand narratives – it is imperative that one explores the implications of this new mathematical logic. Only lack of curiosity would

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<sup>26</sup>Similar thought have been expressed by Andrei Rodin in [44] where he also considers what the UF perspective has to say about the *MS/ES* case.

prevent philosophers from exploring category-theoretic thinking and the mathematical logic of UF as the basis for an alternative, expanded philosophical logic, to be profitably applied to all sorts of rigorous philosophical pursuits. And if, on the other hand, one does not conceive of philosophy as a research project, if one has a more radical calling and more restless disposition, then may at least this new philosophical logic inspire one both to wild novelty and, perhaps, to a novel wildness.

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