Modal logic of time division

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ABSTRACT. A logic $\mathcal{L}_{TD}$ is defined, inspired by [37]. It is syntactically like basic modal logic with an additional unary operator but it has an interval-based semantics on structures with arbitrary linear frames. $\boxminus\psi$ is interpreted as meaning 'the current interval has a finite partition whose all members satisfy ψ.' $\mathcal{L}_{TD}$ is translatable into weak monadic second-order logic but not into first-order logic. The expressive power and the decidability properties of $\mathcal{L}_{TD}$ and its fragments are studied.

Keywords: decidability, expressive power, interval tense logic, linear order, negation, order type, von Wright, weak monadic second-order logic.

1 Introduction

G. H. von Wright suggested in his essay "Time, Change, and Contradiction" [37] an original approach to the logical investigation of time, where the basic objects of study are temporal intervals and their internal structure. In the present paper a formal semantics doing justice to von Wright's informally presented semantic ideas is formulated;[1] the resulting logic and its fragments are then studied for their expressive power and decidability properties.

Background. The guiding idea in von Wright's essay is to examine the relation between *time and change* on the one hand, and *time and contradiction* on the other. As von Wright saw it, in his paper — presented as the 22nd Eddington Memorial Lecture at Cambridge in 1968 — a new avenue in tense logic was opened up, leading to a study of the logic of the division of time into 'bits' of ever shorter duration [38, pp. xi–xii]. While Prior's tense logic [29] studies instants (time points) and their relationships, the basic relation being *succession* in time, in von Wright's approach one takes as the point of departure 'bits' of time and proceeds to the analysis of their internal structure; here the basic relation is *division* of time [39, p. 862].

Von Wright approaches the relation of time and contradiction by reference to the following modal-logical axioms: **(A1)** or $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$, **(A2)** or $\Box(p \vee q) \rightarrow (\Box p \vee \Box q)$, **(A3)** or $\Box(p \vee \sim p)$ and **(A4)** or $\sim\Box(p \wedge \sim p)$. Noting that one interpretation of $\Box$ satisfying the axioms is 'it will next be the case that,' he however proposes to use the axioms to speak of temporal occasions and read $\Box$ as 'it is completely or throughout the case that.' In this framework he formulates the notion of 'contradiction in nature,' which he

[1] For a different formal semantics based on von Wright's relevant logical ideas, see Dalla Chiara [3]; the logic she formulates is many-valued, actually a variant of Łukasiewicz logic.

relates to the analysis of the notion of change. But what is a contradiction in nature? And how is von Wright led to recognize that type of contradiction?

Von Wright considers different cases arising from accepting some axioms and rejecting the others. He takes (A3) to state of an occasion that it can be divided into parts during each of which either p or else $\sim p$ holds throughout. He notes that hence not-(A3) is true of an occasion *any* partition of which has at least one member in which both p and $\sim p$ occur. Such an occasion is said to incorporate a *real contradiction* or a *contradiction in nature* — an idea that has, as von Wright notes, a rather Hegelian flavor. What is at stake, however, is not a *logical* contradiction, but the impossibility of an analysis of change without at least one of the building blocks of the analysis carrying logically contradictory but non-simultaneous constituents. Yet if ontological priority is given to extended occasions, occasions o making not-(A3) true deserve to be called contradictory in a sense: any analysis of o into sub-occasions has a component in which both p and $\sim p$ are present.

When discussing occasions not satisfying (A3), von Wright goes on to say that they positively satisfy $\sim\Box(p \vee \sim p)$, from which he further deduces that they satisfy $\Diamond(p \wedge \sim p)$. This reasoning is mistaken and illustrates problems to which one may be led when not being explicit about the semantics of the expressions involved. Negating (A3) cannot mean affirming $\sim\Box(p \vee \sim p)$ (unless the force of $\sim$ depends on its syntactic position vis-à-vis $\Box$). Namely, von Wright takes $\sim p$ to state at occasion o that not-p holds throughout o, whence $\sim\Box(p \vee \sim p)$ must mean that $\Box(p \vee \sim p)$ fails throughout o, while not-(A3) just says that $\Box(p \vee \sim p)$ fails at o. The temporal ontology of time intervals calls for a distinction between two negations:[2] $\neg\psi$ states at an occasion o that ψ fails at o, while $\sim\psi$ states that ψ fails at each time point in o. With this distinction at our disposal, it is seen that occasions incorporating a 'contradiction in nature' — i.e., occasions at which (A3) fails — satisfy $\neg\Box(p \vee \sim p)$. By the formal semantics to be given in the present paper, this formula is merely equivalent to $\Diamond(\neg\sim p \wedge \neg p)$, not to $\Diamond(p \wedge \sim p)$. (The formula $\neg\sim p$ is true at o iff p holds at least once during o, and $\neg p$ is true at o iff p fails at least once during o.) By contrast, $\Diamond(p \wedge \sim p)$ is logically contradictory (w.r.t. all occasions consisting of at least two instants). Von Wright overlooked the need for making the distinction between the two negations.[3] Also, he paid no attention to nested modal operators; and he did not note that the axioms may hold for *atomic* substitution instances of p and q without holding for arbitrary substitution instances.

Basic definitions. Let T be a set and $R \subseteq T^2$. R is *reflexive* (*irreflexive*) if $R(t,t)$ holds for all (no) t in T; *antisymmetric* if $R(s,t)$ and $R(t,s)$ implies $s = t$; *transitive* if $R(s,t)$ and $R(t,u)$ implies $R(s,u)$; *dichotomous* if $R(s,t)$ or $R(t,s)$ holds for every s,t; *trichotomous* if $R(s,t)$ or $R(t,s)$ or $s = t$ holds for every s,t; *linear order* if it is irreflexive, transitive and trichotomous. All

[2] Dalla Chiara [3] does not distinguish between two negations, but in a sense allows violations of the law of non-contradiction.

[3] Neither Prior [30] nor Smith [35] nor Mortensen [25] realizes that von Wright attaches two incompatible meanings to one and the same negation sign.

linear orders are antisymmetric and fail to be dichotomous. Somewhat confusingly, some authors use 'linear' as a synonym for 'trichotomous' [28]. By 'linear order' some authors mean antisymmetric, transitive and dichotomous — therefore reflexive — binary relation [22] (*reflexive linear orders*), while others mean what was above termed linear order [23]. When quoting other people's results, one must be careful about what they actually proved. A linear order $<$ is *dense* if $s < t$ implies the existence of u with $s < u < t$. Given a linear order $<$, write $\leq$ for its reflexive closure. It is assumed that the reader is familiar with propositional logic (**PL**), first-order logic (**FO**), basic modal logic (**ML**) and basic tense logic (**TL**) (see, e.g., [7, 1, 9]). The symbol $\top$ ($\bot$) denotes a propositional atom by stipulation true (false) under all valuations. The *quantifier rank* of an **FO**-formula is its maximum number of nested quantifiers. For the technique of using Ehrenfeucht-Fraïssé games to prove the elementary equivalence of two structures up to a given quantifier rank, see [6]. Recall that the future tense operators of **TL** are F and G. The reader is reminded of the logic **US** of *Until* and *Since* introduced by Kamp [19]. (For **US**, see [9].) *Weak monadic second-order logic* ($\mathcal{L}_{\mathrm{w}}^{\mathrm{mon}}$) [7, 21, 24] is obtained from **FO** by allowing atomic formulas $X(t)$ and complex formulas $\exists X\phi$, where X is a unary relation variable and t is a term. Crucially, the unary relation variables range over *finite* subsets of the domain. Allowing quantification over arbitrary subsets leads to monadic second-order logic ($\mathcal{L}^{\mathrm{mon}}$). $\mathcal{L}_{\mathrm{w}}^{\mathrm{mon}}$ is decidable over reflexive linear orders [22]. This implies the decidability of $\mathcal{L}_{\mathrm{w}}^{\mathrm{mon}}$ over linear orders, because the latter are (first-order) definable from the former.

We write $\phi \in L$ to indicate that ϕ is a formula of a logic L: we do not notationally distinguish a logic from its set of formulas. Henceforth, 'iff' abbreviates 'if and only if.' If S is a set of formulas, $\mathbf{Cl}_{\wedge,\vee}(S)$ is its closure under $\wedge$ and $\vee$: the set of formulas obtained from S by finitely many applications of $\wedge$ and $\vee$. Given a logic L, its satisfiability (validity) problem is denoted by L-SAT (L-VAL). If for every $\phi \in L$ there is $neg(\phi) \in L$ true precisely when ϕ is not true, and if $neg(\phi)$ is computed from ϕ in **PTIME**, L-SAT is decidable using an algorithm from complexity class C iff L-VAL is; for such L we may speak of decidability without specifying whether we mean L-SAT or L-VAL. **3-CNF** denotes the **NP**-complete problem of deciding whether a **PL**-formula in conjunctive normal form is satisfiable, given that each conjunct consists of just 3 disjuncts (each of which is a literal). In complexity results, time bounds are measured relative to the *length* of the input: its number of symbol tokens. If p_i is an atom ($i < \omega$), its length is $1 + b(i)$, where $b(i)$ is the number of digits of the numeral representing i in binary. If L, L' are (modal or abstract) logics defined over the same class of structures $\mathcal{K}$, a syntactic map $t : L \to L'$ is a *translation* of L into L' if for all $\phi \in L$ and $\mathfrak{M} \in \mathcal{K}$: ϕ is true in $\mathfrak{M}$ iff $t(\phi)$ is true therein. $L \leq L'$ means: a translation of L into L' exists; and $L < L'$ means: $L \leq L'$ but not $L' \leq L$. If $f : A \to B$ is a map, its *image* $Im(f)$ is the set $\{f(a) : a \in A\}$. Basic knowledge of order types and ordinals is assumed [8, 18, 34]. ω is the order type of natural numbers, ω^* is the dual of ω (having the order of ω

reversed), and η is the order type of the set of rational numbers (non-empty countable dense linear orders without end-points).

Plan of the paper. *Section* 2 introduces the *logic of time division* ($\mathcal{L}_{TD}$). Its expressive power is discussed in *Section* 3. Three fragments of $\mathcal{L}_{TD}$ are studied in some detail in *Sections* 4, 5 and 6. *Section* 7 concludes the paper by pointing out related work and questions for future research.

2 Logic of time division

Syntax. Let **prop** be a set of propositional atoms containing $\top$ and $\bot$. The syntax of the *logic of time division* (or $\mathcal{L}_{TD}$) is given by the grammar $\phi ::= p \mid \neg\phi \mid {\sim}\phi \mid (\phi\vee\phi) \mid (\phi\wedge\phi) \mid \Diamond\phi \mid \Box\phi$, with $p \in$ **prop**. Syntactically, $\mathcal{L}_{TD}$ is **ML** with an additional unary operator (${\sim}$). $\Box$ and $\Diamond$ are *modal operators*. The *modal depth* $md(\phi)$ of ϕ is the maximum number of nested modal operators in ϕ. Formulas of the forms $p, \neg p, {\sim}p, \neg{\sim}p$ ($p \in$ **prop**) are termed *literals*. The notion of subformula is defined in the expected way.

Semantics. Only linear flows of time will be considered.[4] Von Wright [37] takes 'occasions' or intervals to be primary in relation to extensionless time points. These latter he views as 'idealizations'. The former he characterizes in the strict sense as 'bits' or 'stretches' of time during which no change takes place, but allows for a generalized sense in which an occasion is any interval offering a medium within which changes may occur. In the subsequent formal development this view on time results in letting a domain T consist of instants, but making evaluation, primarily, relative to extended intervals. (So the domain, as it is given, is a result of idealization, but the primary mode of evaluation reflects the conceptual priority of intervals over instants.) However, since we do not wish to outright exclude idealizations, evaluation relative to instants (or, singleton intervals) is admitted as well.

Frames are pairs $(T, <)$, where $T \neq \varnothing$ and $<$ is a linear order on T. For later purposes, we assume that for each frame an element $t^* \in T$ has been fixed. *Models* are triples $\mathcal{M} = (T, <, V)$ with $(T, <)$ a frame and $V : \mathbf{prop} \to \mathcal{P}(T)$ a valuation. Always $V(\top) = T$ and $V(\bot) = \varnothing$. Models are, then, modal structures with a linear accessibility relation. However, formulas are evaluated relative to *certain kinds of subsets* of the domain (in **ML** all evaluation is relative to single elements of the domain). We define an *occasion* o in a frame $(T, <)$ to be a subset of T which either is of the form $]s, t] = \{x : s < x \leq t\}$ for some $s, t \in T$ or of the form $\{t\}$ for some $t \in T$. Occasions of the former kind are *occasions proper*; $\{t\}$ is an *idealized occasion* if t has no immediate predecessor in T (while if t has one, t', then $\{t\} =]t', t]$ is an occasion proper). The empty interval $\varnothing$ is an occasion proper: if $s \geq t$, then $]s, t] = \varnothing$.[5] The cardinality of o is denoted by $|o|$. If $o =]s, t]$ is non-empty, s is the *left bound* of o, denoted $l(o)$, and t its *right bound*, denoted $r(o)$. If o is empty, by stipulation $l(o) = t^* = r(o)$.

[4] This restriction reflects von Wright's interest in experienced time; certain time-related phenomena are better studied by reference to tree-like flows of time.

[5] We prefer not to preclude empty intervals at the outset. It will turn out that the emptiness and non-emptiness of an interval are properties definable in $\mathcal{L}_{TD}$.

(A stipulation is needed: we may have $]s, t] = \varnothing =]s', t']$ while $s \neq s'$ or $t \neq t'$.) Finally, $l(\{t\}) = t = r(\{t\})$. If $\mathcal{M}$ is a model and o is an occasion in its frame, the pair $(\mathcal{M}, o)$ is an *anchored model*.

We are led by aesthetic considerations when opting for intervals of the form $]s, t]$ and refraining to accommodate, e.g., intervals $\{x : s \leq x \leq t\}$ as well. The semantics of the modal operators $\Box$ and $\Diamond$ will be in terms of *divisions* of the current occasion. If for some $n > 0$ there are points $t_0 \notin o$ and $t_1, \ldots, t_{n+1} \in o$ such that $t_0 < t_1 < \ldots < t_{n+1}$ and $(]t_0, t_1], \ldots,]t_n, t_{n+1}])$ is a partition of the occasion o, this partition is termed the *division* of o *by the points* $t_1, \ldots, t_n$ and denoted $\mathbb{D}_o(t_1, \ldots, t_n)$. (Note that necessarily $t_0 = l(o)$ and $t_{n+1} = r(o)$.) It follows that an occasion o has a division iff $|o| \geq 2$. The members of a division are called its *cells*. If $o =]s, t]$ is finite and $|o| = n + 1$, then o has $2^n - 1$ different divisions. While all divisions of an interval determine its partition, not all partitions are divisions; e.g., $(]1, 2] \cup]3, 4],]2, 3])$ is a partition of the real interval $]1, 4]$, but not its division. Among the *desiderata* guiding the definition of division is that all members of the appropriate partition will be non-empty and that whenever a division exists, it will be possible to choose the number of divisors so as to make the members of the partition to have pairwise the same cardinality. For these reasons it is convenient that the members of the partition are occasions of the same form as the occasion divided.

Given a model $\mathcal{M} = (T, <, V)$, define a binary relation $\mathcal{M}, o \models \psi$ among occasions o in T and formulas ψ of $\mathcal{L}_{TD}$ as the smallest set such that:

- $\mathcal{M}, o \models p$ if: $t \in V(p)$ for all $t \in o$
- $\mathcal{M}, o \models {\sim}\psi$ if: $\mathcal{M}, \{t\} \not\models \psi$ for all $t \in o$
- $\mathcal{M}, o \models \neg\psi$ if: $\mathcal{M}, o \not\models \psi$
- $\mathcal{M}, o \models (\psi \wedge \chi)$ if: $\mathcal{M}, o \models \psi$ and $\mathcal{M}, o \models \chi$
- $\mathcal{M}, o \models (\psi \vee \chi)$ if: $\mathcal{M}, o \models \psi$ or $M, o \models \chi$
- $\mathcal{M}, o \models \Box\psi$ if: for some positive integer n there are $t_1, \ldots, t_n$ with $l(o) < t_1 < \ldots < t_n < r(o)$ such that for each cell o' of the division $\mathbb{D}_o(t_1, \ldots, t_n)$, we have $\mathcal{M}, o' \models \psi$
- $\mathcal{M}, o \models \Diamond\psi$ if: for all positive integers n and all $t_1, \ldots, t_n$ with $l(o) < t_1 < \ldots < t_n < r(o)$, there is a cell o' of the division $\mathbb{D}_o(t_1, \ldots, t_n)$ such that $\mathcal{M}, o' \models \psi$.

It can be shown that all implications in the above definition can be reversed, i.e., that the relation $\{(o, \psi) : \mathcal{M}, o \models \psi\}$ is a fixed point of the inductive truth definition. If $\mathcal{M}, o \models \psi$, ψ is *true* in $\mathcal{M}$ at o; else *false* in $\mathcal{M}$ at o.

The symbols $\Box$ and $\Diamond$ have here a meaning very different from their meanings in **ML**. Seen as a generalized quantifier, $\Box$ ($\Diamond$) involves second-order existential (universal) and first-order universal (existential) quantification. $\Box\psi$ serves to assert at o that for all cells of some division of o, ψ holds. Dually, $\Diamond\psi$ asserts at o that for any division of o, some of its cells makes ψ true. $\Box$ is a kind of *chop-star* operator;[6] $\Diamond$ is its dual w.r.t. $\neg$ (the

[6]For a discussion on the relation of $\Box$ to *chop-star* as used in other logics, see *Sect.* 7.

literature seems not to have settled on any name for the dual of *chop-star*). Disjunction ($\vee$) and conjunction ($\wedge$) have their usual meanings and they are each other's duals w.r.t. $\neg$. While $\neg$ is the plain contradictory negation acting on occasions, $\sim$ acts on single points (technically, singleton intervals). $\neg\psi$ holds at o iff ψ does not hold at o, whereas $\sim\psi$ holds at o iff ψ fails separately at each point $t \in o$. In particular $\neg p$ holds at o if the atom p fails at some $t \in o$, while in order for $\sim p$ to hold at o, p must fail at each of its points. $\neg$ is termed the *contradictory negation* and $\sim$ the *universal negation*.[7] When speaking simply of negation, we mean $\neg$. The formula $\neg\sim p$ using both negations states that positively, p holds at some $t \in o$.

Note: Instead of $\sim$, we might consider the positive operator $\oplus$ with the following semantics: $\oplus\psi$ is true at o iff for every $t \in o$, ψ is true at $\{t\}$.[8] Employing $\oplus$, $\sim\psi$ would be definable as $\oplus\neg\psi$. (In the scope of $\oplus$, like in the scope of $\sim$, the difference between $\sim$ and $\neg$ vanishes.) Note that conversely, $\oplus\psi$ is definable as $\sim\sim\psi$, or, equivalently, as $\sim\neg\psi$. Using $\oplus$ could improve readability; e.g. the truth condition of $\sim(\sim p \wedge \sim q)$ may well be more accessible when referred to via $\oplus(p \vee q)$. For technical development it might be advisable to adopt the modified set of primitives. Yet von Wright's argument discussed in *Section* 1 seems to be best elucidated in terms of the two negations. To retain the connection to the proposed analysis of his argument, the syntax with $\neg$ and $\sim$ is kept in the present paper. $\dashv$

Let $\mathcal{K}$ be a class of anchored models and let $\phi, \psi \in \mathcal{L}_{TD}$. ϕ is *satisfiable (valid) over* $\mathcal{K}$ if $\mathcal{M}, o \models \phi$ for some (all) anchored models $(\mathcal{M}, o)$ in $\mathcal{K}$. A finite set of formulas is satisfiable (valid) if their conjunction is satisfiable (valid). ψ is a *logical consequence* of ϕ *over* $\mathcal{K}$, denoted $\phi \Rightarrow_{\mathcal{K}} \psi$, if $(\neg\phi \vee \psi)$ is valid over $\mathcal{K}$. Let $\mathcal{K}_0$ be the class of all anchored models. We write $\Rightarrow$ for the relation $\Rightarrow_{\mathcal{K}_0}$. Formulas ψ and ϕ are *logically equivalent*, denoted $\psi \equiv \phi$, if $\phi \Rightarrow \psi$ and $\psi \Rightarrow \phi$. The formula ϕ *characterizes* a property P on a class $\mathcal{K}$ if for all $(\mathcal{M}, o) \in \mathcal{K}$: $\mathcal{M}, o \models \phi$ iff o satisfies P. E.g., ϕ characterizes infinity on $\mathcal{K}_0$ if ϕ holds at all and only infinite occasions.

2.1 Some features of the semantics

Clearly all formulas of the forms p, $\top$, $\sim\psi$, $\Diamond\psi$ are true at the empty occasion, while no formula of the form $\Box\psi$ or $\bot$ is. Note that $\mathcal{M}, o \models \sim\top$ iff $o = \varnothing$ and thence $\mathcal{M}, o \models \neg\sim\top$ iff $o \neq \varnothing$. So emptiness and non-emptiness of an occasion are definable in $\mathcal{L}_{TD}$. The formula $(\Diamond\bot \wedge \neg\sim\top)$ is true at o iff ($o \neq \varnothing$ but o has no division) iff $|o| = 1$.

Considering $\Box$ and $\Diamond$ applied to literals, in 4 out of the total of 8 cases the resulting formula is definable in simpler terms. If $\theta \in \{p, \sim p\}$, $\Diamond\neg\theta \equiv (\neg\theta \vee \Diamond\bot)$ and $\Box\theta \equiv (\theta \wedge \Box\top)$. On the other hand, $\Diamond p$ says of o that p fails at most once during it, while $\Diamond\sim p$ states that p holds at most once. $\Box\neg p$ says that p fails at least twice and $\Box\neg\sim p$ that p holds at least twice. Interestingly, $\Diamond\Box\neg p$ asserts — of intervals of size at least 2 — that p fails infinitely often (any division has a cell that is further divisible into at least

[7] Relative to atoms, but not generally, $\neg$ could be termed the *existential negation*.

[8] I owe to an anonymous referee the suggestion to take $\oplus$ rather than $\sim$ as a primitive.

two cells so that in each p fails); while $\Box\Diamond p$ says that p fails only finitely many times (there is a division such that in each of its cells p fails at most once). Replacing in these examples p by an arbitrary formula does not, in general, serve to express an analogous property; e.g., $\Diamond\Box\neg p$ does not assert that $\Box\neg p$ fails at most once (actually, $\Box\neg p$ fails at each point). We may observe that there is a formula satisfiable exactly on occasions of even size: $|o|$ is even iff there is V such that $(T,<,V),o \models \Box(\neg{\sim}p \wedge \Diamond{\sim}p \wedge \neg p \wedge \Diamond p)$.

2.2 Negation normal form

Some peculiarities in the behavior of the universal negation ($\sim$) are worth noting. First, $\sim$ and $\neg\sim$ both distribute over $\vee$: ${\sim}(\phi \vee \psi) \equiv ({\sim}\phi \wedge {\sim}\psi)$ and $\neg{\sim}(\phi \vee \psi) \equiv (\neg{\sim}\phi \vee \neg{\sim}\psi)$ for any $\phi, \psi \in \mathcal{L}_{TD}$. However, neither distributes over $\wedge$. E.g., ${\sim}({\sim}p \wedge {\sim}q)$ is true at a real interval $o =]s,t]$ whose every irrational point makes p true (but q false) and every rational point makes q true (but p false), while $(p \vee q)$ of course fails at o. Second, while ${\sim}{\sim}p \equiv p$ for any atom p, in general ${\sim}{\sim}\phi$ is not equivalent to ϕ. E.g., ${\sim}{\sim}(p \vee q) \equiv {\sim}({\sim}p \wedge {\sim}q) \not\equiv (p \vee q)$. Third, while ${\sim}\neg\phi \equiv {\sim}{\sim}\phi$ for any $\phi \in \mathcal{L}_{TD}$ (cf. the proof of Prop. 1), $\neg{\sim}\phi$ asserts of o that ϕ is true at some singleton interval $\{t\} \subseteq o$; it is equivalent neither to ϕ nor to ${\sim}{\sim}\phi$.

It will be convenient to deal with $\mathcal{L}_{TD}$ by reference to a normal form, where negation symbols $\neg$ are driven as deep as they go. Let the grammar

$$\psi \ ::= \ p \mid {\sim}p \mid (\psi \vee \psi) \mid (\psi \wedge \psi)$$

define the class of formulas $\mathcal{L}_{zero}$, and let the grammar

$$\chi \ ::= \ p \mid {\sim}p \mid \neg p \mid \neg{\sim}p \mid {\sim}(\mathbf{a} \wedge \mathbf{a}) \mid \neg{\sim}(\mathbf{a} \wedge \mathbf{a})$$

define the class of formulas $\mathcal{L}_{base}$, where $p \in \mathbf{prop}$ and $\mathbf{a} \in \mathcal{L}_{zero}$. The four tuples of negation signs (empty, $\neg$, $\sim$, $\neg\sim$) serve to represent the four basic modes of syllogistic assertions applied to time points in an interval: p is universal affirmative, ${\sim}p$ universal negative, $\neg{\sim}p$ particular affirmative and $\neg p$ particular negative. $\mathcal{L}_{zero}$ equals $\mathbf{Cl}_{\wedge,\vee}(\mathbf{prop} \cup \{{\sim}p : p \in \mathbf{prop}\})$; and $\mathcal{L}_{base}$ is obtained from $\mathbf{prop} \cup \{\neg p : p \in \mathbf{prop}\}$ by adding to it the universal negations of all $\mathcal{L}_{zero}$-formulas (except disjunctions), and the contradictory negations of the universal negations of all $\mathcal{L}_{zero}$-formulas (except disjunctions). Let $\mathcal{L}_{nnf}$ be the class of formulas produced by the grammar

$$\theta ::= \mathbf{b} \mid (\theta \vee \theta) \mid (\theta \wedge \theta) \mid \Diamond\theta \mid \Box\theta,$$

where $\mathbf{b} \in \mathcal{L}_{base}$. There is a truth-preserving map of type $\mathcal{L}_{TD} \to \mathcal{L}_{nnf}$.

PROPOSITION 1. *There is a map $t : \mathcal{L}_{TD} \to \mathcal{L}_{nnf}$ such that for all $\phi \in \mathcal{L}_{TD}$ and anchored models $(\mathcal{M}, o)$: $\mathcal{M}, o \models \phi$ iff $\mathcal{M}, o \models t(\phi)$.*

Proof. Think of formulas ${\sim}\psi$ first. If ψ' is the result of replacing $\neg$ by $\sim$ in ψ, then ${\sim}\psi \equiv {\sim}\psi'$: in the scope of $\sim$ all evaluation is w.r.t. single points, and $\mathcal{M}, \{t\} \models \neg\phi$ iff $\mathcal{M}, \{t\} \models {\sim}\phi$. Moreover, if ψ'' results from replacing in ψ' subformulas $\Box\phi$ by $\bot$ and $\Diamond\phi$ by $\top$, then ${\sim}\psi' \equiv {\sim}\psi''$. ($\Box\phi$ cannot hold at $\{t\}$: there is no s with $t < s < t$; dually, $\Diamond\phi$ is trivially true at $\{t\}$.) These observations motivate defining the following maps $[\,]^{\sim}$ and $[\,]^{zero}$, to be used when translating a formula of $\mathcal{L}_{TD}$ containing occurrences of $\sim$. First, if $\psi \in \mathcal{L}_{TD}$, let $[\psi]^{\sim}$ be the result of replacing all occurrences of $\neg$ in ψ by $\sim$, and putting the resulting formula in negation normal form (like in **ML**). So $[\psi]^{\sim} \in \mathcal{L}_{TD}$, $[\psi]^{\sim}$ contains no occurrences of $\neg$, and $\sim$ occurs in $[\psi]^{\sim}$ only prefixed to an atom. Second, define a map $[\,]^{zero}$ on the set $\{[\psi]^{\sim} : \psi \in \mathcal{L}_{TD}\}$ as follows: $[p]^{zero} = p$, $[\sim p]^{zero} = \sim p$, $[\Diamond\theta]^{zero} = \top$, $[\Box\theta]^{zero} = \bot$, $[(\theta_1 \circ \theta_2)]^{zero} = ([\theta_1]^{zero} \circ [\theta_2]^{zero})$ for $\circ \in \{\wedge, \vee\}$. Note that $Im([\,]^{zero}) \subseteq \mathcal{L}_{zero}$. The idea behind the maps $[\,]^{\sim}$ and $[\,]^{zero}$ is that if ψ appears in ψ' in the scope of $\sim$ (the maps will be applied in such cases), ψ can be replaced by $[[\psi]^{\sim}]^{zero}$ *salva veritate*. Let $[\,]^{*}$ be the composite map $([\,]^{zero} \circ [\,]^{\sim}) : \mathcal{L}_{TD} \to \mathcal{L}_{zero}$. Define a map $[\,]^{nnf} : \mathcal{L}_{TD} \to \mathcal{L}_{nnf}$ as follows:

$$
\begin{aligned}
[\theta]^{nnf} &= \theta \quad \text{for } \theta \in \{p, \neg p\}\\
[\neg\neg\phi]^{nnf} &= [\phi]^{nnf}\\
[\sim(\phi_1 \wedge \phi_2)]^{nnf} &= \sim[(\phi_1 \wedge \phi_2)]^{*}\\
[\neg\sim(\phi_1 \wedge \phi_2)]^{nnf} &= \neg\sim[(\phi_1 \wedge \phi_2)]^{*}\\
[\bigcirc\phi]^{nnf} &= \bigcirc[\phi^{nnf}] \quad \text{for } \bigcirc \in \{\Diamond, \Box\}\\
[(\phi_1 \circ \phi_2)]^{nnf} &= ([\phi_1]^{nnf} \circ [\phi_2]^{nnf}) \quad \text{for } \circ \in \{\vee, \wedge\}\\
[\neg\!\!\!\neg(\phi_1 \vee \phi_2)]^{nnf} &= ([\neg\!\!\!\neg\phi_1]^{nnf} \wedge [\neg\!\!\!\neg\phi_2]^{nnf}) \quad \text{for } \neg\!\!\!\neg \in \{\sim, \neg\}\\
[\neg\sim(\phi_1 \vee \phi_2)]^{nnf} &= ([\neg\sim\phi_1]^{nnf} \vee [\neg\sim\phi_2]^{nnf})\\
[\neg(\phi_1 \wedge \phi_2)]^{nnf} &= ([\neg\phi_1]^{nnf} \vee [\neg\phi_2]^{nnf})\\
[\neg\Diamond\phi]^{nnf} &= \Box[\neg\phi]^{nnf}\\
[\neg\Box\phi]^{nnf} &= \Diamond[\neg\phi]^{nnf}
\end{aligned}
$$

Any $\phi \in \mathcal{L}_{TD}$ can be thought of as being built from components of the forms $p, \sim\psi$ by applying $\neg, \vee, \wedge, \Box, \Diamond$. (A formula ψ prefixed by $\sim$ can be assumed to be an atom or a conjunction of $\mathcal{L}_{zero}$-formulas.) Relative to such 'atoms', $[\,]^{nnf}$ acts like a transformation producing negation normal form in **ML** (w.r.t. $\neg$). As the component formulas $\sim\psi$ have their inner structure, they will be processed further using the map $[\,]^{*}$ (applied to formulas of the forms $\sim\phi$, $\neg\sim\phi$). Doing so gets rid of all occurrences of $\neg$ in ψ, and drives the resulting formula in negation normal form (w.r.t. $\sim$). It is easy to show that $[\,]^{*}$ and $[\,]^{nnf}$ are truth-preserving. So we may take $t = [\,]^{nnf}$. ■

If $\phi \in \mathcal{L}_{TD}$, the *negation normal form* of ϕ is by definition the formula $[\phi]^{nnf}$, where $[\,]^{nnf}$ is the map defined in the proof of Proposition 1.

3 Expressive power

It turns out that $\mathcal{L}_{TD}$ is a very powerful logic.

EXAMPLE 2. $\mathcal{M}, o \models (\Box\top \wedge \Diamond\Box\top)$ iff $|o|$ is infinite. First assume $\mathcal{M}, o \models (\Box\top \wedge \Diamond\Box\top)$, supposing for contradiction that $|o| < \aleph_0$. Since $\Box\top$ holds at o, $|o| \geq 2$. Consider a division of o whose all cells are singletons. Then $\Box\top$ holds at one of these cells: a contradiction. For the converse, suppose $|o|$ is infinite. Then $\Box\top$ holds trivially at o. Choose any finite number $n > 0$ of points $t_1 < \ldots < t_n < r(o)$ in o, and consider the division $\mathbb{D}_o(t_1, \ldots, t_n)$.

At least one of its cells is infinite. At that cell, then, the formula $\Box\top$ holds. By what just established, $\mathcal{M}, o \models (\Diamond\bot \vee \Box\Diamond\bot)$ iff $|o|$ is finite. $\dashv$

A subset $S \subseteq T$ is *dense in* T if for all $x, y \in T$ with $x < y$, there is $z \in S$ such that $x < z < y$. The following fact will be used subsequently:

FACT 3. Let $(T, \prec)$ be of order type $\eta + 1$. There is a subset S of T such that both S and its complement $T \setminus S$ are dense in T.

Proof. W.l.o.g. consider the interval $T :=]0, 1] \cap \mathbb{Q}$. Put $S := \{\frac{m}{2^n} : n \geq 1$ and $m < 2^n$ and m is odd$\}$. Then both S and $T \setminus S$ are dense in T. ■

EXAMPLE 4. Consider the conjunction $\chi := (\neg{\sim}p \wedge \neg p)$, with $p \in$ **prop**. Clearly χ can only be true at an occasion of size at least two. Then the formula $(\Box\top \wedge \Diamond\chi)$ is satisfiable, but not true at any finite occasion. To see that there is an infinite occasion at which $(\Box\top \wedge \Diamond\chi)$ holds, let $o =]0, 1] \cap \mathbb{Q}$. By Fact 3 there is a subset S of o such that both S and its complement are dense in o. Define a model $\mathcal{M} = (T, <, V)$ with $T = [0, 1] \cap \mathbb{Q}$ by letting $<$ be the order of rationals in T by magnitude and $V(p) = S$. Now $\mathcal{M}, o \models (\Box\top \wedge \Diamond\chi)$. For, take any division of T by points $t_1 < \ldots < t_n$. Then the formula χ holds actually at any cell (while one would suffice). $\dashv$

The formulas calling for infinite models discussed in Examples 2 and 4 have extremely low modal depth: 2 *resp.* 1. Let us look at further examples that give some measure of the power of nesting modal operators.

EXAMPLE 5. If $\bigcirc \in \{\Box, \Diamond\}$, write $\bigcirc^n$ for the string consisting of n tokens of $\bigcirc$. Let n, m be positive integers.

- (i) $\mathcal{M}, o \models \Box^n\top$ iff $|o| \geq 2^n$.
- (i') $\mathcal{M}, o \models \Diamond^n\bot$ iff $|o| \leq 2^n - 1$.
- (ii) $\mathcal{M}, o \models \Box^n\Diamond^m\bot$ iff $2^n \leq |o| < \aleph_0$.
- (ii') $\mathcal{M}, o \models \Diamond^n\Box^m\top$ iff $2^n < |o|$ or $|o| \geq \aleph_0$.

(i') and (ii') are immediate from (i) *resp.* (ii). For (i), the estimate on $|o|$ is computed as $1 + \sum_{k=0}^{n-1} 2^k = 2^n$; it is the smallest number of time points allowing n iterated evaluations of $\Box$ in *each* minimal sub-occasion triggered by the previous evaluation step. For (ii), suppose first that $|o|$ is finite but at least 2^n. So we are sure to be able to evaluate $\Box^n$. Moreover, since $|o|$ is finite, we may choose the divisors of o so that the cells of the resulting division are all singletons. But then, by (i'), $\Diamond^m\bot$ is true at all those cells. Further, the value of the parameter $m > 0$ is irrelevant. Conversely, if the formula holds at o, it is possible to evaluate $\Box^n$, whence $|o| \geq 2^n$. Suppose for contradiction that $|o|$ is infinite. Then any division chosen to witness $\Box$ will have at least one infinite cell; at that cell, then, $\Diamond^m\bot$ cannot hold. $\dashv$

EXAMPLE 6. Let $\mathcal{O} = (\omega + 1, <, V)$, where V satisfies $\alpha \in V(p)$ iff $\alpha < \omega$ and α is odd. Let $o =]0, \kappa]$ with $\kappa \leq \omega$. Then $\mathcal{O}, o \models \Box(p \vee {\sim}p)$ iff $\kappa \neq \omega$, but $\mathcal{O}, o \models \Box(p \vee \neg p)$. If $\kappa = \omega$, $\Box(p \vee {\sim}p)$ fails at o, since whichever finite set of divisors is chosen to witness $\Box$, the rightmost of the cells of the corresponding division is infinite, and thus neither p nor $\sim p$ holds at it. On the other hand, $\Box(p \vee \neg p)$ indeed holds irrespective of the value of κ: if o' is a sub-occasion of o which does not satisfy p, there is at least one point in o' at which p does not hold; but then $\neg p$ is true at o'. For a further example, let $\mathcal{N} = (\omega, <, V)$, with V defined as follows: $n \in V(p)$ iff $n \equiv 1 \pmod 3$; $n \in V(q)$ iff $n \equiv 2 \pmod 3$; and $n \in V(r)$ iff $n \equiv 0 \pmod 3$. Let $o =]2, k]$ with $k < \omega$. Then $\mathcal{N}, o \models \Box(\neg\sim p \wedge \neg\sim q \wedge \neg\sim r \wedge \Diamond\Diamond\bot)$ iff $|o|$ is divisible by 3. The formula $\chi := (\neg\sim p \wedge \neg\sim q \wedge \neg\sim r \wedge \Diamond\Diamond\bot)$ states of an occasion o' that its size is at most 3 (cf. Ex. 5) and that each of the atoms p, q, r is true at o' at least once. Given the definition of the model, this only leaves open the possibility that $|o'| = 3$. But then, the formula $\Box\chi$ can only be true in $\mathcal{N}$ at an occasion whose size is divisible by 3. ⊣

EXAMPLE 7. Consider the model $\mathcal{Q} = (\mathbb{Q}, <, V)$ and the rational interval $o =]1, 2]$, given that $V(p) = \{t : t^2 < 2\}$. (Hence $\varnothing \neq V(p) \cap o \neq o$.) Then $\mathcal{Q}, o \models \neg\Box(p \vee \sim p)$. To see this, let $\mathbb{D}_o(t_1, \ldots, t_n)$ be any division of o. Then the divisors satisfy $1^2 < (t_1)^2 < \ldots < (t_n)^2 < 2^2$. Evidently, $(p \vee \sim p)$ can only be true at all cells of the division if for some $1 \leq i \leq n$, we have $(t_i)^2 = 2$. But this is impossible, as the t_i are rational. By contrast, if we move on to look at the model $\mathcal{R} = (\mathbb{R}, <, V)$ and the real interval $o =]1, 2]$, letting $V(p) = \{t : t^2 < 2\}$, we indeed have: $\mathcal{R}, o \models \Box(p \vee \sim p)$. There is exactly one witnessing division, namely $\mathbb{D}_o(\sqrt{2})$. ⊣

EXAMPLE 8. In Example 7, the truth of $\Box(p \vee \sim p)$ failed in a model based on rational numbers due to a gap — the non-existence of the supremum of the set of points making p true. This formula can fail in a model $\mathcal{Q}^* = (\mathbb{Q}, <, V^*)$ also for a different reason. By Fact 3 there is a subset S of the rational interval $o =]1, 2]$ such that both S and $]1, 2] \setminus S$ are dense in $]1, 2]$. Put $V^*(p) := S$. Then $\mathcal{Q}^*, o \models \neg\Box(p \vee \sim p)$. ⊣

Basic negative properties of $\mathcal{L}_{TD}$. Directly by Example 2, we have:

FACT 9. $\mathcal{L}_{TD}$ lacks the finite model property. ■

By contrast, **ML** has (strong) finite model property over the class of all pointed models [1, Thm. 2.34, Cor. 6.8]. Thinking of evaluation, in **ML** we climb up a tree (or more generally, a directed graph), while in $\mathcal{L}_{TD}$ we dig deeper into a given occasion. In view of the above examples, modal depth is a very bad measure of the structural requirements that an $\mathcal{L}_{TD}$-formula can impose on an occasion. A formula of modal depth 2 can characterize infinity (Ex. 2) and there are formulas of modal depth 1 true only at infinite occasions (Ex. 4). A simple formula such as $\Box^n\top$ forces its verifying occasion to be at least of size 2^n, while the same string of symbols, as a formula of **ML**, only requires of a pointed modal structure that its height be n. A property central to a great variety of modal languages $\mathcal{L}$ is having a *notion of finite degree* [1, Def. 7.58]: the existence of a function $f : \mathcal{L} \to \omega$ such that if ϕ is true in a pointed model at all, it is true in the result of removing from that pointed model everything that transcends the height $f(\phi)$. For **ML** we may take $f(\phi) = md(\phi)$. (Not all modal languages have a notion of finite degree, a much-studied case in point is modal mu-calculus.) $\mathcal{L}_{TD}$ fails to have a notion of finite degree, if by 'degree' of a formula of $\mathcal{L}_{TD}$ we understand a measure of the size of an occasion required to verify it.

If **prop** $= \{p_i : i < \omega\}$ and $\mathcal{M} = (T, <, V)$ is a model, let $\tau = \{\prec\} \cup \{P_i : i < \omega\}$ and define $\mathfrak{M}$ to be the τ-structure $(T, \prec^{\mathfrak{M}}, \langle P_i^{\mathfrak{M}} \rangle_{i<\omega})$, where $\prec^{\mathfrak{M}} = <$ and $P_i^{\mathfrak{M}} = V(p_i)$. ($\mathfrak{M}$ will be termed the *abstract structure induced by* $\mathcal{M}$.) If $\mathcal{L}$ is an abstract logic, say **FO** or $\mathcal{L}_{\mathrm{w}}^{\mathrm{mon}}$, $\mathcal{L}_{TD}$ is *translatable into* $\mathcal{L}$ if the following two conditions hold: (a) for every $\phi \in \mathcal{L}_{TD}$ there is a formula $\psi_\phi^1(x) \in \mathcal{L}[\tau]$ of one free variable such that for all $\mathcal{M}$ and $t \in T$: $\mathcal{M}, \{t\} \models \phi$ iff $\langle \mathfrak{M}, t \rangle \models \psi_\phi^1(x)$; and ($b$) for every $\phi \in \mathcal{L}_{TD}$ there is a formula $\psi_\phi^2(x, y) \in \mathcal{L}[\tau]$ of two free variables such that for all $\mathcal{M}$ and all occasions proper $o =]s, t] \subseteq T$: $\mathcal{M},]s, t] \models \phi$ iff $\langle \mathfrak{M}, s, t \rangle \models \psi_\phi^2(x, y)$. The formula ψ_ϕ^i will be called a *translation of* ϕ *of kind* i ($i := 1, 2$).

THEOREM 10. *$\mathcal{L}_{TD}$ is not translatable into* **FO**.

Proof. Let $\tau = \{<\}$. Given $n \in \mathbb{N}$, let $\mathfrak{M}_n$ be the τ-structure $(M_n, <^{\mathfrak{M}_n}, \varnothing)$ with $M_n = \{0, \ldots, 2^n\}$ and $<^{\mathfrak{M}_n}$ the order of $0, \ldots, 2^n$ by magnitude. Write $\prec$ for the order of integers by magnitude. Let $M = \mathbb{Z}$ and define a τ-structure $\mathfrak{M} = (M, <^{\mathfrak{M}}, \varnothing)$ by setting $z <^{\mathfrak{M}} z'$ iff $0 \preceq z \prec z'$ or $z \prec z' \prec 0$ or ($z \succ 0$ and $z' \prec 0$). So $<^{\mathfrak{M}}$ has the order type $\omega + \omega^*$. Note that $\min(<^{\mathfrak{M}}) = 0$ and $\max(<^{\mathfrak{M}}) = -1$. An easy Ehrenfeucht-Fraïssé game argument shows that for any $n \in \mathbb{N}$, the structures $\langle \mathfrak{M}, 0, -1 \rangle$ and $\langle \mathfrak{M}_n, 0, 2^n \rangle$ agree on all **FO**$[\tau]$-formulas of two free variables with quantifier rank at most n. Consider, then, the $\mathcal{L}_{TD}$-formula $\phi_{\mathrm{inf}} := (\Box\top \wedge \Diamond\Box\top)$, true in $\mathcal{M}$ at o iff $|o| \geq \aleph_0$. For contradiction, suppose ϕ_{inf} has a translation $\chi(x, y)$ of kind 2 into **FO**$[\tau]$. Write q for the quantifier rank of χ. By what just observed, the structures $\langle \mathfrak{M}, 0, -1 \rangle$ and $\langle \mathfrak{M}_q, 0, 2^q \rangle$ are indistinguishable by $\chi(x, y)$. Note that the τ-structures $\mathfrak{M}_n$ and $\mathfrak{M}$ are, formally, also models for $\mathcal{L}_{TD}$. Now $\mathfrak{M}_q,]0, 2^q] \not\models \phi_{\mathrm{inf}}$ and $\mathfrak{M},]0, -1] \models \phi_{\mathrm{inf}}$. Since $\chi(x, y)$ is a translation of ϕ_{inf}, $\langle \mathfrak{M}_q, 0, 2^q \rangle \not\models \chi(x, y)$ and $\langle \mathfrak{M}, 0, -1 \rangle \models \chi(x, y)$, so after all $\chi(x, y)$ distinguishes the two structures: a contradiction. ■

Basic positive properties of $\mathcal{L}_{TD}$. As is immediate from the semantics, $\mathcal{L}_{TD} \leq \mathcal{L}_{\mathrm{w}}^{\mathrm{mon}}$. Write $ST_{x,y}$ for a translation (of kind 2) of $\mathcal{L}_{TD}$ into $\mathcal{L}_{\mathrm{w}}^{\mathrm{mon}}$. Note that **FO** $\not\leq \mathcal{L}_{TD}$; clearly e.g. the simple (**TL**-definable) formula $\exists z \exists v (x < z < v \leq y \wedge P(z) \wedge Q(v))$ has no translation into $\mathcal{L}_{TD}$.

PROPOSITION 11. *$\mathcal{L}_{TD} < \mathcal{L}_{\mathrm{w}}^{\mathrm{mon}}$.* ■

COROLLARY 12. *$\mathcal{L}_{TD}$ has downwards Löwenheim-Skolem property: any satisfiable formula of $\mathcal{L}_{TD}$ is true in a countable anchored model.*

Proof. Let $\phi \in \mathcal{L}_{TD}$. Suppose $\mathcal{M}, o \models \phi$, where $o =]s, t]$ and $s < t$. (If o is empty or a singleton there is nothing to prove.) Let ϕ_{lin} be an **FO**$[\{<\}]$-formula stating that the interpretation of $<$ is linear; then the sentence $\chi := (\exists x \exists y (x < y \wedge ST_{x,y}(\phi)) \wedge \phi_{\mathrm{lin}})$ of $\mathcal{L}_{\mathrm{w}}^{\mathrm{mon}}$ is true in the structure $\mathfrak{M}$ induced by $\mathcal{M}$. By the downwards Löwenheim-Skolem property of $\mathcal{L}_{\mathrm{w}}^{\mathrm{mon}}$ [24], there is a *countable* $\langle \mathfrak{M}', s', t' \rangle$ such that $\langle \mathfrak{M}', s', t' \rangle \models ST_{x,y}(\phi)$, the relation $<^{\mathfrak{M}'}$ is linear, and $s' <^{\mathfrak{M}'} t'$. Let $\mathcal{M}'$ be the model inducing $\mathfrak{M}'$. Then $\mathcal{M}',]s', t'] \models \phi$. ■

COROLLARY 13. *$\mathcal{L}_{TD}$ is decidable.*

Proof. Given $\phi \in \mathcal{L}_{TD}$, apply an algorithm solving $\mathcal{L}_{\mathrm{w}}^{\mathrm{mon}}$-SAT over linear orders [22] to the sentence $\exists x \exists y ST_{x,y}(\phi)$. (Note that ϕ is true at a singleton iff $ST_{x,y}(\phi)$ is satisfied in some $\langle \mathfrak{M}, t, t' \rangle$, where $]t, t']$ is a singleton.) ■

Further observations. Write ϕ_{inf} for the formula $(\Box\top \wedge \Diamond\Box\top)$ that was seen to hold at o iff $|o| \geq \aleph_0$ (Ex. 2). Using ϕ_{inf} we can build further formulas which will serve to capture classes of ordinals. (For any successor ordinal $\alpha = \beta + 1 = [0, \beta]$, the interval $]0, \beta]$ is an occasion proper in the class of all ordinals. However, we may consider α itself as an occasion proper.) E.g., $\Box\phi_{\mathrm{inf}}$ is true at an ordinal $\beta + 1$ iff $\beta \geq \omega \cdot 2$, while $(\Box\top \wedge \Diamond\Box\phi_{\mathrm{inf}})$ is true at $\beta + 1$ iff $\beta \geq \omega^2$. To see that the latter holds, suppose for contradiction that $\beta + 1 \models (\Box\phi_{\mathrm{inf}} \wedge \Diamond\Box\phi_{\mathrm{inf}})$ for some $\beta < \omega^2$. So $\beta = \omega \cdot n + k$ for some $n, k < \omega$. Consider a division of β by divisors $\omega, \omega \cdot 2, \ldots, \omega \cdot n$. Each cell of this division is of order type $\gamma_i \in \{k, \omega\}$ and yet one of them satisfies $\Box\phi_{\mathrm{inf}}$. This is impossible. More generally, we may define recursively formulas ϕ_{ω^n} by putting $\phi_{\omega^1} := (\Box\top \wedge \Diamond\Box\phi_{\mathrm{inf}})$ and $\phi_{\omega^{n+1}} := (\Box\top \wedge \Diamond\Box\phi_{\omega^n})$. Then we have for an ordinal β that $\beta + 1 \models \phi_{\omega^n}$ iff $\beta \geq \omega^n$. An ordinal $\beta + 1$ makes true *all* formulas of the set $\{\phi_{\omega^n} : n < \omega\}$ iff $\beta \geq \omega^\omega$.

Formula χ of $\mathcal{L}_{TD}$ is *idempotent* if $\Box\chi \Rightarrow \chi$: any occasion that can be cut into finitely many pieces so that each piece satisfies χ, itself satisfies χ. Not all formulas are idempotent. E.g., neither $(p \vee q)$ nor $(p \vee \sim p)$ nor $\Diamond\bot$ is. By contrast, all formulas p, $\neg p$, $\neg\sim p$, $\sim\phi$, $\Box\phi$, $(p \vee \neg p)$ are idempotent. Formula χ is *hyperconsistent* if $\Box\chi \Rightarrow \Diamond\chi$, i.e., if the truth of $\Box\chi$ excludes the truth of $\Box\neg\chi$. In other words, χ is hyperconsistent iff $(\Diamond\chi \vee \Diamond\neg\chi)$ is valid. Dually, formula χ is a *hypoantilogy* if $(\Box\chi \wedge \Box\neg\chi)$ is satisfiable. Hypoantilogies χ have the property that at least one occasion can be divided, in two ways, into at least two cells so that all cells of one division make χ true while all cells of the other division make $\neg\chi$ true. Hypoantilogies are formulas which can, by multiplication so to say, be fitted into one and the same interval together with their contradictories. A formula is hyperconsistent (hypoantilogy) iff its negation is. Directly by definitions, if a formula χ and its negation both are idempotent, χ is hyperconsistent. Hyperconsistency is a very restrictive condition. E.g., $(p \vee q)$ is not hyperconsistent, as witnessed by a rational interval $]1, 4]$ where q holds throughout $]1, 2]$ and $]3, 4]$, p holds throughout $]2, 3]$, and p fails at $1\frac{1}{2}$ and $3\frac{1}{2}$ while q fails at $2\frac{1}{4}$ and $2\frac{3}{4}$. Then $\Box(p \vee q)$ holds at $]1, 4]$ as witnessed by the division $\mathbb{D}_{]1,4]}(2, 3)$. Yet also $\Box\neg(p \vee q)$ holds at $]1, 4]$. This is witnessed by the division $\mathbb{D}_{]1,4]}(2\frac{1}{2})$. In both cells $]1, 2\frac{1}{2}]$ and $]2\frac{1}{2}, 4]$, both literals $\neg p$ and $\neg q$ hold. Other examples of hypoantilogies are $(r \vee (q \wedge s))$, $\Diamond\bot$ and $\Box\top$. E.g., in order for $\Diamond\bot$ to be hyperconsistent, the formula $(\Diamond\Diamond\bot \vee \Diamond\neg\Diamond\bot)$ must be valid; however, this formula is false at all finite occasions of size at least 4. The existence of hypoantilogies shows immediately that the distribution laws $(\Box\chi \wedge \Box\theta) \equiv \Box(\chi \wedge \theta)$ and $(\Diamond\chi \vee \Diamond\theta) \equiv \Diamond(\chi \vee \theta)$ *fail* in $\mathcal{L}_{TD}$.

Write $\phi \to \psi$ for $(\neg\phi \vee \psi)$. All of the following modal-logical axiom schemata fail in $\mathcal{L}_{TD}$: **4** or $\Box\theta \to \Box\Box\theta$, **T** or $\Box\theta \to \theta$, **B** or $\theta \to \Box\Diamond\theta$, **D** or $\Box\theta \to \Diamond\theta$, **.3** or $(\Diamond\theta \wedge \Diamond\chi) \to \Diamond(\theta \wedge \Diamond\chi) \vee \Diamond(\theta \wedge \chi) \vee \Diamond(\chi \wedge \Diamond\theta)$, **L** or $\Box(\Diamond\neg p \vee p) \to \Box p$ and **K** or $\Box(\neg\theta \vee \chi) \to (\Diamond\neg\theta \vee \Box\chi)$. The failure of **D** follows from the existence of hypoantilogies: it fails in $\mathcal{L}_{TD}$ for a rather 'substantial' reason. (In **ML** an instance of this schema can only fail at a state having no successor states.) The schema **.3** holds for *atomic* θ, χ — in particular $(\Diamond p \wedge \Diamond q) \Rightarrow \Diamond(p \wedge \Diamond q)$ — but fails already for negated atomic formulas. On the positive side, the axiom schema that in basic modal logic corresponds to the density of the accessibility relation, viz. $\Box\Box\theta \to \Box\theta$, actually holds for $\mathcal{L}_{TD}$. Here it has nothing to do with density, but is a simple consequence of the semantics of $\Box$. Since **K** fails in $\mathcal{L}_{TD}$, nominally $\mathcal{L}_{TD}$ is not a normal modal logic. Whether it would do justice to $\mathcal{L}_{TD}$ to call it a non-normal modal logic, or a modal logic at all, is debatable; what is clear is that $\mathcal{L}_{TD}$ is some sort of modal-like temporal logic.

4 Prenex formulas

A reasonable way to get to grips with characteristic features of $\mathcal{L}_{TD}$ is to distinguish its fragments and study them in isolation. What is learned from such case studies can, then, hopefully shed light on the general features of the logic. In this section we take up the study of 'prenex formulas'; in *Sections* 5 and 6 the 'propositional fragment' *resp.* the 'simple fragment' are considered. Already attempts to reach an overview of such relatively straightforward fragments of $\mathcal{L}_{TD}$ lead to rather involved considerations; there would be no realistic hope of understanding the details of the semantic behavior of $\mathcal{L}_{TD}$ from scratch. E.g., should anyone wish to design a decision algorithm specifically for $\mathcal{L}_{TD}$, such a detailed understanding would be needed. From this perspective, the question whether the chosen fragments are 'natural' in a more global setting is immaterial. (The 'propositional fragment' actually turns out to be a natural fragment of **TL**.)

A *prenex formula* is any $\mathcal{L}_{TD}$-formula of one of the forms

$$(\Box\Diamond)^n\ell,\ (\Diamond\Box)^n\ell,\ \Diamond(\Box\Diamond)^n\ell \text{ and } \Box(\Diamond\Box)^n\ell,$$

with $n < \omega$ and ℓ a literal. Prenex formulas with ℓ in their matrix are *ℓ-formulas*. E.g. $\Box\Diamond\Box p$, $\Diamond\Box\neg\sim p$ and $\sim p$ are prenex formulas; the first is p-formula, the second $\neg\sim p$-formula and the third $\sim p$-formula. Write $\mathcal{L}_{PR}$ for the class of all prenex formulas, and let $\mathcal{L}_{BPR} := \mathbf{Cl}_{\wedge,\vee}(\mathcal{L}_{PR})$. E.g., $(\Box\Diamond p \vee (\Diamond\sim q \wedge \neg r))$ is in $\mathcal{L}_{BPR}$ but $\Box(p \vee \Diamond q)$ is not. Note that semantically, $\mathcal{L}_{BPR}$ is closed under $\neg$. We will study $\mathcal{L}_{BPR}$-SAT w.r.t. the class $\mathcal{D}$ of all anchored models whose linear order is dense and whose occasion is proper. By Corollary 13 it is already known that this problem is decidable (density is first-order definable), but here a more fine-grained analysis is attempted. Dropping the assumption of density would complicate the details of the proofs; we leave the study of the more general case for future research.

Over $\mathcal{D}$, we have the equivalences $\Diamond\neg p \equiv \neg p$, $\Diamond\neg\sim p \equiv \neg\sim p$, $\Box p \equiv p$ and $\Box\sim p \equiv \sim p$. (For these it suffices to exclude occasions o with $0 \neq |o| \neq 1$, cf. *Subsect.* 2.1.) So we may, w.l.o.g., restrict attention to prenex formulas of the forms $(\Box\Diamond)^n p$, $(\Box\Diamond)^n \sim p$, $\Diamond(\Box\Diamond)^n p$, $\Diamond(\Box\Diamond)^n \sim p$ and $(\Diamond\Box)^n \neg p$, $(\Diamond\Box)^n \neg\sim p$, $\Box(\Diamond\Box)^n \neg p$, $\Box(\Diamond\Box)^n \neg\sim p$. Recall that $\mathcal{M}, o \models \Box\Diamond p$ iff $\neg p$ occurs only finitely often in o; and $\mathcal{M}, o \models \Diamond\Box\neg p$ iff $\neg p$ occurs infinitely often in o. Similarly, $\Box\Diamond\sim p$ asserts of an occasion that p holds only finitely often in it and $\Diamond\Box\neg\sim p$ that p holds infinitely often in it. We prove some auxiliary results and then show that $\mathcal{L}_{BPR}$-SAT over $\mathcal{D}$ is **NP**-complete. If $\theta \in \mathcal{L}_{PR}$, write $|\theta|$ for the number of modal operator tokens in its prefix.

LEMMA 14. **(a)** *If θ_1 and θ_2 are p-formulas ($\sim$p-formulas) with $|\theta_1| \leq |\theta_2|$, then $\theta_1 \Rightarrow_{\mathcal{D}} \theta_2$.* **(b)** *If θ_1 and θ_2 are $\neg$p-formulas ($\neg\sim$p-formulas) with $|\theta_1| \leq |\theta_2|$, then $\theta_2 \Rightarrow_{\mathcal{D}} \theta_1$.*

Proof. For (a), consider how the complement of $V(p)$ looks like in anchored models $\langle (T, <, V), o\rangle \in \mathcal{D}$ satisfying a given p-formula. It is not difficult to see that the situation is as summarized by Table 1. Note that if θ_1 and θ_2 are p-formulas with $|\theta_1| \leq |\theta_2|$, any occasion satisfying θ_1 is a special case of an occasion satisfying θ_2. Therefore $\theta_1 \Rightarrow_{\mathcal{D}} \theta_2$. The case of $\sim p$-formulas is analogous. Item (b) is immediate from (a). Namely, if χ_1 and χ_2 are $\neg p$-formulas with $|\chi_1| \leq |\chi_2|$, then $\neg\chi_i$ is logically equivalent to a p-formula θ_i ($i := 1, 2$) such that $|\chi_i| = |\theta_i|$. Therefore by (a), $\theta_1 \Rightarrow_{\mathcal{D}} \theta_2$. But this means that $\chi_2 \Rightarrow_{\mathcal{D}} \chi_1$. The case of $\neg\sim p$-formulas is analogous. ■

Formula	*Order type of* $T \setminus V(p)$ *in o*
$\Diamond p$	≤ 1
$\Box\Diamond p$	n_0
$\Diamond\Box\Diamond p$	α_1
$\Box\Diamond\Box\Diamond p$	$\alpha_1 \cdot n_1$
$\Diamond\Box\Diamond\Box\Diamond p$	$\alpha_1 \cdot n_1 \cdot \alpha_2$
$\Box\Diamond\Box\Diamond\Box\Diamond p$	$\alpha_1 \cdot n_1 \cdot \alpha_2 \cdot n_2$
...	...
$(\Box\Diamond)^m p$	$\prod_{i=1}^m \alpha_1 \cdot n_i$
$\Diamond(\Box\Diamond)^m p$	$\prod_{i=1}^m \alpha_1 \cdot n_i \cdot \alpha_{m+1}$
...	...

$1 < n_i < \omega$; α_i finite or $\alpha_i \in \{\omega, \omega^*\}$

Table 1

Formula	*Order type of* $T \setminus V(p)$ *in o*
$\Box\neg p$	α_0
$\Diamond\Box\neg p$	∞_1
$\Box\Diamond\Box\neg p$	$\infty_1 \cdot \alpha_1$
$\Diamond\Box\Diamond\Box\neg p$	$\infty_1 \cdot \alpha_1 \cdot \infty_2$
$\Box\Diamond\Box\Diamond\Box\neg p$	$\infty_1 \cdot \alpha_1 \cdot \infty_2 \cdot \alpha_2$
$\Diamond\Box\Diamond\Box\Diamond\Box\neg p$	$\infty_1 \cdot \alpha_1 \cdot \infty_2 \cdot \alpha_2 \cdot \infty_3$
...	...
$(\Diamond\Box)^m \neg p$	$(\prod_{i=1}^{m-1} \infty_i \cdot \alpha_i) \cdot \infty_m$
$\Box(\Diamond\Box)^m \neg p$	$\prod_{i=1}^m \infty_i \cdot \alpha_i$
...	...

$0 \neq \alpha_i \neq 1$; ∞_i is infinite

Table 2

Table 2 shows how the complement of $V(p)$ looks like in occasions satisfying a given $\neg p$-formula. Indeed if χ_1 and χ_2 are $\neg p$-formulas with $|\chi_1| \leq |\chi_2|$, any occasion satisfying χ_2 is a special case of an occasion satisfying χ_1.

LEMMA 15. **(a)** *Suppose $\langle (T, <, V), o\rangle \in \mathcal{D}$ and $(T, <, V), o \models \theta$. If θ is p-formula, the set $V(p) \cap o$ is dense but the set $o \setminus V(p)$ is not. Similarly, if θ is $\sim$p-formula, the set $o \setminus V(p)$ is and the set $V(p) \cap o$ is not dense.* **(b)** *If θ is p-formula and χ is $\sim$p-formula, $(\theta \wedge \chi)$ is not satisfiable over $\mathcal{D}$.* **(c)** *If θ is p-formula ($\sim$p-formula) and χ is $\neg$p-formula ($\neg\sim$p-formula), $(\theta \wedge \chi)$ is satisfiable iff $|\theta| > |\chi|$.* **(d)** *If θ is p-formula ($\sim$p-formula) and Σ is the set of all $\neg\sim$p-formulas (all $\neg$p-formulas), $\theta \Rightarrow_{\mathcal{D}} \psi$ for all $\psi \in \Sigma$.*

Proof. (a) is immediate from Table 1. For (b), suppose for contradiction that $(T, <, V), o \models (\theta \wedge \chi)$, with $<$ dense and o proper. By (a), the set $V(p) \cap o$ is and is not dense. Item (c) is immediate from Tables 1 and 2. For (d), suppose θ is p-formula and $\mathcal{M}, o \models \theta$ with $(\mathcal{M}, o) \in \mathcal{D}$. From Table 1 it is seen that o can be partitioned into ω cells in each of which p occurs infinitely often, whence all $\neg\sim p$-formulas hold at o. ■

THEOREM 16. *Let* $\Sigma(p)$ *be any set of* p*-formulas,* $\Sigma(\neg p)$ *any set of* $\neg p$*-formulas,* $\Sigma(\sim p)$ *be any set of* $\sim p$*-formulas and* $\Sigma(\neg\sim p)$ *any set of* $\neg\sim p$*-formulas. The union* $\Sigma(p) \cup \Sigma(\neg p) \cup \Sigma(\sim p) \cup \Sigma(\neg\sim p)$ *is satisfiable iff one of the following two conditions holds:* (i) $\Sigma(\sim p) = \varnothing$ *and* $\Sigma(\neg p)$ *is finite and* $\max\{|\theta| : \theta \in \Sigma(\neg p)\} < \min\{|\chi| : \chi \in \Sigma(p)\}$, (ii) $\Sigma(p) = \varnothing$ *and* $\Sigma(\neg\sim p)$ *is finite and* $\max\{|\theta| : \theta \in \Sigma(\neg\sim p)\} < \min\{|\chi| : \chi \in \Sigma(\sim p)\}$.

Proof. By Lemma 14(a), the set $\Sigma(p)$ alone is satisfiable. By Lemmas 14 and 15(c), $\Sigma(p) \cup \Sigma(\neg p)$ is satisfiable iff $\Sigma(\neg p)$ is finite and $\max\{|\theta| : \theta \in \Sigma(\neg p)\} < \min\{|\chi| : \chi \in \Sigma(p)\}$. By Lemma 15$(d)$, again, $\Sigma(p) \cup \Sigma(\neg p)$ is satisfiable iff $\Sigma(p) \cup \Sigma(\neg p) \cup \Sigma(\neg\sim p)$ is satisfiable. So if $\Sigma(\sim p) = \varnothing$, the union $\Sigma(p) \cup \Sigma(\neg p) \cup \Sigma(\sim p) \cup \Sigma(\neg\sim p)$ is satisfiable iff condition (i) holds. Suppose, then, $\Sigma(\sim p) \neq \varnothing$. By Lemma 15$(b)$, no occasion satisfying $\Sigma(\sim p)$ can satisfy $\Sigma(p)$ unless $\Sigma(p) = \varnothing$. Dually to what noted above, $\Sigma(\sim p) \cup \Sigma(\neg\sim p) \cup \Sigma(\neg p)$ is satisfiable iff $\Sigma(\neg\sim p)$ is finite and $\max\{|\theta| : \theta \in \Sigma(\neg\sim p)\} < \min\{|\chi| : \chi \in \Sigma(\sim p)\}$. That is, if $\Sigma(\sim p) \neq \varnothing$, $\Sigma(p) \cup \Sigma(\neg p) \cup \Sigma(\sim p) \cup \Sigma(\neg\sim p)$ is satisfiable iff condition (ii) holds. ■

THEOREM 17. *Over* $\mathcal{D}$, $\mathcal{L}_{BPR}$-SAT *(*$\mathcal{L}_{BPR}$-VAL*) is* **NP***-complete.*

Proof. It suffices to consider satisfiability ($\mathcal{L}_{BPR}$ is semantically closed under $\neg$ and a formula expressing the negation of a given formula is computed in constant time). **Inclusion:** Given $\theta \in \mathcal{L}_{BPR} \setminus \mathcal{L}_{PR}$, non-deterministically guess a map $d : \cup_{i \leq n}\{0,1\}^i \to \{0,1\}$, with $n+1$ the maximum number of nested disjunctions and conjunctions in θ; the map d can be used to determine for every disjunctive subformula of θ one of the disjuncts in an obvious way. Starting with $S_0 := \{\theta\}$, generate a set $S_n \subseteq \mathcal{L}_{PR}$ by letting S_{i+1} contain $S_i \cap \mathcal{L}_{PR}$, both conjuncts of every conjunction in S_i, for every disjunction in S_i, the disjunct determined by d, and no other formulas. (If $\theta \in \mathcal{L}_{PR}$, proceed with $S_0 := \{\theta\}$.) Using Theorem 16, we determine whether S_n is satisfiable over $\mathcal{D}$: first see if S_n contains both a p-formula and a $\sim p$-formula. If so, S_n is not satisfiable over $\mathcal{D}$. Else, if S_n contains no $\sim p$-formula, check if S_n contains a $\neg p$-formula whose prefix is not shorter than the prefixes of all p-formulas in S_n; if it does, S_n is not satisfiable over $\mathcal{D}$, otherwise it is. If, again, S_n contains no p-formulas, similarly check if S_n contains a $\neg\sim p$-formula whose prefix is not shorter than the prefixes of all $\sim p$-formulas in S_n; if so, S_n is not satisfiable over $\mathcal{D}$, else it is. The induced algorithm runs in **NP**: the non-deterministically guessed map d was employed to generate in constant time the set S_n, whose satisfiability over $\mathcal{D}$ was then checked in polynomial time. **Hardness:** The **NP**-complete problem **3-CNF** can be simulated in $\mathcal{L}_{BPR}$ w.r.t. $\mathcal{D}$: the **PL**-formula $\bigwedge_{i<n}(\ell_{i1} \vee \ell_{i2} \vee \ell_{i3})$ is satisfiable iff the formula $(\bigwedge_{i<n}(\ell_{i1} \vee \ell_{i2} \vee \ell_{i3})) \wedge \neg{\sim}\top$ of $\mathcal{L}_{PR}$ is satisfiable over $\mathcal{D}$, given that each ℓ_{ij} equals p or $\neg p$ for some atom p. (The conjunct $\neg{\sim}\top$ serves to exclude the empty occasion). ■

5 Propositional fragment

Define the *propositional fragment* $\mathcal{L}_{prop}$ of $\mathcal{L}_{TD}$ to be $\mathbf{Cl}_{\wedge,\vee}(\mathcal{L}_{base})$. Now $\mathcal{L}_{prop}$ has genuine tense-logical content — despite its 'propositional' nature. Use $\sim$ as the negation symbol in **TL**; let **SLF** or *simple logic of future* be the fragment of **TL** whose syntax is generated thus:

$$\gamma \ ::= Fp \mid F{\sim}p \mid Gp \mid G{\sim}p \mid F(\beta \wedge \beta) \mid G(\beta \vee \beta) \mid (\gamma \wedge \gamma) \mid (\gamma \vee \gamma),$$

with $p \in$ **prop** and $\beta \in \mathcal{L}_{zero}$. (Syntactically, $\mathcal{L}_{zero} \subset$ **TL**.) Semantically, **SLF** coincides with the fragment of **TL** whose formulas have modal depth at most one and make no use of the past tense operators. $\mathcal{L}_{prop}$ and **SLF** are intertranslatable, in the sense expressed by Fact 18. If $\mathcal{M} = (T, <, V)$ is a model, $t, t' \in T$ and $t < t'$, let $\mathcal{M}_{t,t'}$ be the model $([t, t'], <_{t,t'}, V_{t,t'})$, where $<_{t,t'}$ ($V_{t,t'}$) is the restriction of $<$ (*resp.* V) to the interval $[t, t']$.

FACT 18. There are **PTIME**-computable functions $T : \mathcal{L}_{prop} \to$ **SLF** and $S :$ **SLF** $\to \mathcal{L}_{prop}$ such that for all $\chi \in \mathcal{L}_{prop}$, models $\mathcal{M} = (T, <, V)$ and non-empty $]t, t'] \subseteq T$, we have: $\mathcal{M},]t, t'] \models \chi$ iff $\mathcal{M}_{t,t'}, t \models T[\chi]$. Conversely, for all $\theta \in$ **SLF**, models $\mathcal{N} = (N, <, V)$ and non-empty $]t, t'] \subseteq N$, we have: $\mathfrak{N}_{t,t'}, t \models \theta$ iff $\mathcal{N},]t, t'] \models S[\theta]$.

Proof. If $\psi \in \mathcal{L}_{TD}$, let $[\psi]^{nnf}$ be its negation normal form. Define a map $T : \mathcal{L}_{prop} \to$ **SLF** as follows: $T[p] = Gp$, $T[\sim p] = G{\sim}p$, $T[\neg p] = F{\sim}p$, $T[\neg{\sim}p] = Fp$, $T[\alpha] = \alpha$, $T[\sim(\alpha \wedge \alpha')] = G(T[[\sim\alpha]^{nnf}] \vee T[[\sim\alpha']^{nnf}])$, $T[\neg\sim(\alpha \wedge \alpha')] = F(T[\alpha] \wedge T[\alpha'])$ and $T[(\chi \circ \chi')] = (T[\chi] \circ T[\chi'])$ for $\circ \in \{\wedge, \vee\}$, where $\alpha, \alpha' \in \mathcal{L}_{zero}$ and $\chi, \chi' \in \mathcal{L}_{prop}$. Conversely, define a map $S :$ **SLF** $\to \mathcal{L}_{prop}$ by putting $S[Fp] = \neg{\sim}p$, $S[F{\sim}p] = \neg p$, $S[Gp] = p$, $S[G{\sim}p] = {\sim}p$, $S[\beta] = \beta$, $S[F(\beta \wedge \beta')] = \neg{\sim}(S[\beta] \wedge S[\beta'])$, $S[G(\beta \vee \beta')] = {\sim}([S[\sim\beta]]^{nnf} \wedge [S[\sim\beta']]^{nnf})$ and $S[(\gamma \circ \gamma')] = (S[\gamma] \circ S[\gamma'])$ for $\circ \in \{\wedge, \vee\}$, where $\beta, \beta' \in \mathcal{L}_{zero}$ and $\gamma, \gamma' \in$ **SLF**. Clearly T and S are **PTIME**-computable and translations in the sense required. ■

TL-SAT over linear orders is **NP**-complete. (In [28] **TL**-SAT was proven **NP**-complete over transitive trichotomous orders; this settles also the case of linear orders: a transitive trichotomous order can be turned by 'bulldozing' into an irreflexive transitive trichotomous order, cf. [1, Thm. 4.56].) By Fact 18, then, $\mathcal{L}_{prop}$-SAT is in **NP**. However, let us look at the situation in detail. Let $\mathcal{C}_0 = \{(\mathcal{M}, o) : \mathcal{M}$ is a model and o is non-empty and proper$\}$.

LEMMA 19. *The satisfiability and validity problems of* $\mathcal{L}_{prop}$ *are* **NP**-*complete, both over* $\mathcal{C}_0$ *and over* $\mathcal{D}_0 = \{(\mathcal{M}, o) \in \mathcal{D} : o \neq \varnothing\}$.

Proof. We show that the satisfiability problem of **SLF** is **NP**-complete over all (dense) models $\mathcal{M}_{t,t'}$ with $t < t'$. The claims about $\mathcal{L}_{prop}$ follow due to Fact 18. **Inclusion:** Any $\gamma \in$ **SLF** is obtained from formulas of the forms $F\beta$ or $G\beta$ (for suitable $\beta \in \mathcal{L}_{zero}$) by finitely many applications of $\wedge$ and $\vee$. Non-deterministically guess a map choosing a disjunct for each disjunctive subformula of γ, and use it to produce a set X of formulas of the forms F and G such that γ is satisfiable iff the set X is. There are, then, formulas $\alpha_i, \beta_j \in \mathcal{L}_{zero}$ and numbers x, y such that $X = \{G\alpha_1, \ldots, G\alpha_x, F\beta_1, \ldots, F\beta_y\}$. Evidently X is satisfiable iff all sets $\{\alpha_1, \ldots, \alpha_x, \beta_j\}$ are satisfiable ($j := 1, \ldots, y$). This, again, is the case iff for a non-deterministically chosen number l, the set $X^l := \{\alpha_1, \ldots, \alpha_x, \beta_l\}$ is satisfiable. To decide whether it is, non-deterministically resolve the disjunctions in the formulas $\alpha_1, \ldots, \alpha_x, \beta_l$ and obtain a set $A_1^l \cup \ldots \cup A_x^l \cup B^l$ of literals. If this set contains an atom and its negation, X^l is not satisfiable, otherwise it is. The algorithm runs in **NcoP** = **NP**. If it accepts the input γ, the induced model $\mathcal{M}_{t,t'}$ satisfies: $|\{z : t \leq z \leq t'\}| = y+1$, where y is the number of formulas of the form F in the non-deterministically guessed set X. $\mathcal{M}_{t,t'}$ can then be turned into a model $\mathcal{M}'_{t,t'}$ with a dense linear order, e.g. by adding an isomorphic copy of the rational interval $]0, 1[$ between each point in the domain of $\mathcal{M}_{t,t'}$ and making all α_i with $G\alpha_i \in X$ true throughout those intervals. So $\mathcal{L}_{prop}$-SAT is **NP**-decidable also over $\mathcal{D}_0$. **Hardness: 3-CNF** can be simulated in **SLF**: the **PL**-formula $\bigwedge_{i<n}(\ell_{i1} \vee \ell_{i2} \vee \ell_{i3})$ is satisfiable iff the formula $(\bigwedge_{i<n}(\ell'_{i1} \vee \ell'_{i2} \vee \ell'_{i3}))$ of **SLF** is satisfiable over arbitrary (*resp.* dense) models $\mathcal{M}_{t,t'}$, given that each ℓ_{ij} equals p or $\neg p$ for some atom p, and $\ell'_{ij} = Gp$ if $\ell_{ij} = p$ and $\ell'_{ij} = F{\sim}p$ if $\ell_{ij} = \neg p$. ■

6 Simple fragment

The syntax of the *simple fragment* $\mathcal{L}^1_{TD}$ of $\mathcal{L}_{TD}$ is generated by the grammar

$$\chi \ ::= \ \mathbf{b} \mid \Box\mathbf{b} \mid \Diamond\mathbf{b} \mid (\chi \wedge \chi) \mid (\chi \vee \chi),$$

with $\mathbf{b} \in \mathcal{L}_{prop}$. The syntax excludes nested modal operators; semantically $\mathcal{L}^1_{TD}$ equals $\{\phi \in \mathcal{L}_{TD} : md(\phi) \leq 1\}$. We observe some facts about $\mathcal{L}^1_{TD}$.

Diamond formulas. Let us call $\mathcal{L}^1_{TD}$-formulas of the form $\Diamond\theta$ *diamond formulas*. They can impose quite strong requirements. E.g., $\Diamond(\neg p \wedge \neg{\sim}p)$ has only infinite models. Restricting attention to anchored models from $\mathcal{D}$, consider the kinds of statements that can be made using formulas $\Diamond\theta$.

Fix some auxiliary notation. If the χ_i are $\mathcal{L}_{base}$-formulas, let $\mathcal{M},]s, s'] \models \mathsf{low}(\chi_1, \ldots, \chi_n)$ state that for any given $t \in]s, s']$ and any i, there is $t'_i \in]s, t[$ such that χ_i holds at t'_i. Similarly, $\mathsf{high}(\chi_1, \ldots, \chi_n)$ states that above any given point each χ_i is true. Let $\mathcal{M}, o \models \mathsf{close}(\chi_1, \ldots, \chi_n)$ mean that there is $t \in o$ such that either every χ_i is true arbitrarily close to t above t or every χ_i is true arbitrarily close to t below t. If $\psi \in \mathcal{L}_{prop}$, let $]s, s'[\models \psi$ be otherwise the same statement as $]s, s'] \models \psi$ except that the metalanguage first-order quantifiers are not allowed to range over the right bound s' of the occasion $]s, s']$. Then if $\psi, \chi, \phi \in \mathcal{L}_{prop}$, let $\sharp(\psi, \chi, \phi)$ state at $]t, t']$ that there is a point $s \in]t, t'[$ such that $]t, s[\models \psi$ and $\{s\} \models \chi$ and $]s, t'] \models \phi$. Finally, define $\S(\psi)$ to be true at $]t, t']$ if there is s such that $]t, s] \models \psi$ or $]s, t'] \models \psi$. The abbreviated statements are not themselves definable in

$\mathcal{L}_{TD}$; they will be used in analyzing what can be stated in terms of diamond formulas relative to $\mathcal{D}$. (Incidentally, all statements are **US**-definable.)

EXAMPLE 20. The following equivalences are relative to $\mathcal{D}$; they are straightforward consequences of the semantics of $\Diamond$.

(*a*) $\Diamond(p \vee q)$ iff $\sharp(p \vee q, \top, p \vee q)$.

(*b*) $\Diamond(\neg p \vee \neg q)$ iff $(\neg p \vee \neg q)$.

(*c*) $\Diamond(p \wedge q)$ iff $\sharp(p \wedge q, \top, p \wedge q)$.

(*d*) $\Diamond(\neg p \wedge \neg q)$ iff $\mathsf{close}(\neg p, \neg q) \vee \mathsf{low}(\neg p, \neg q) \vee \mathsf{high}(\neg p, \neg q)$.

(*e*) $\Diamond(p \vee \neg q)$ iff $\Diamond p \vee \Diamond\neg q$ iff $\Diamond p \vee \neg q$.

(*f*) $\Diamond(p \vee [\neg q \wedge \neg r])$ iff $\Diamond p \vee \Diamond(\neg q \wedge \neg r) \vee \S(p \wedge \neg q \wedge \neg r)$.

(*g*) $\Diamond(p \vee \neg q \vee \neg r)$ iff $\Diamond p \vee \Diamond(\neg q \wedge \neg r) \vee \S(p \wedge [\neg q \vee \neg r])$.

(*h*) $\Diamond(p \wedge \neg q)$ iff $(p \wedge \neg q)$ or $[\sharp(p, \neg p, p) \wedge \mathsf{low}(\neg q) \wedge \mathsf{high}(\neg q)]$.

(*i*) $\Diamond[(p \wedge \neg q) \vee (r \wedge \neg s)]$ iff $\Diamond(p \wedge \neg q)$ or $\Diamond(r \wedge \neg s)$ or $[\sharp(p, \neg p, r) \wedge \mathsf{low}(\neg q) \wedge \mathsf{high}(\neg s)]$ or $[\sharp(r, \neg r, p) \wedge \mathsf{low}(\neg s) \wedge \mathsf{high}(\neg q)]$.

(*j*) $\Diamond[(p \wedge \neg q) \vee (r \wedge \neg s) \vee (t \wedge \neg u)]$ iff $\Diamond[(p \wedge \neg q) \vee (r \wedge \neg s)] \vee \Diamond[(p \wedge \neg q) \vee (t \wedge \neg u)] \vee \Diamond[(r \wedge \neg s) \vee (t \wedge \neg u)]$. ⊣

An $\mathcal{L}_{base}$-formula is *universal* if it is of the form p, $\sim p$ or $\sim(\chi \wedge \chi')$, and *existential* if of the form $\neg p$, $\neg\sim p$ or $\neg\sim(\chi \wedge \chi')$. By their semantics, universal formulas make a universal statement about an occasion, while existential formulas are witnessed by a single point in an occasion.

LEMMA 21. *Let the u_i (resp. e_i) range over universal (existential) formulas of $\mathcal{L}_{base}$. The following equivalences hold relative to $\mathcal{D}$.*

(*a*) $\Diamond(\bigvee_i u_i \vee \bigvee_j e_j)$ *iff* $\bigvee_j e_j \vee \sharp(\bigvee_i u_i, \top, \bigvee_i u_i)$.

(*b*) *Suppose that $n \geq 1$ and that there are i, j with $e_i \neq e_j$. Then* $\Diamond(\bigwedge_{i<n+1} e_i)$ *iff* $\mathsf{close}(e_0, \ldots, e_n) \vee \mathsf{low}(e_0, \ldots, e_n) \vee \mathsf{high}(e_0, \ldots, e_n)$.

(*c*) $\Diamond[\bigwedge_i u_i \wedge \bigwedge_j e_j]$ *iff* $\bigwedge_i u_i \wedge \bigwedge_j e_j \vee [\sharp(\bigwedge_i u_i, \top, \bigwedge_i u_i) \wedge \mathsf{low}(e_0, \ldots, e_n) \wedge \mathsf{high}(e_0, \ldots, e_n)]$.

(*d*) $\Diamond(\bigwedge_i u_i \vee \bigwedge_j e_j)$ *iff* $\Diamond(\bigwedge_i u_i) \vee \Diamond(\bigwedge_j e_j) \vee \S(\bigwedge_i u_i \wedge \bigwedge_j e_j)$.

(*e*) $\Diamond([\bigwedge_i u_i \wedge \bigwedge_j e_j] \vee [\bigwedge_k u'_k \wedge \bigwedge_l e'_l])$ *iff* $\Diamond[\bigwedge_i u_i \wedge \bigwedge_j e_j] \vee \Diamond[\bigwedge_k u'_k \wedge \bigwedge_l e'_l] \vee [\sharp(\bigwedge_i u_i, \top, \bigwedge_k u'_k) \wedge \mathsf{low}(e_0, \ldots, e_n) \wedge \mathsf{high}(e'_0, \ldots, e'_m)] \vee [\sharp(\bigwedge_k u'_k, \top, \bigwedge_i u_i) \wedge \mathsf{low}(e'_0, \ldots, e'_m) \wedge \mathsf{high}(e_0, \ldots, e_n)]$.

(*f*) $\Diamond\big(\bigwedge_{i_1} \theta^1_{i_1} \vee \ldots \vee \bigwedge_{i_n} \theta^n_{i_n}\big)$ *iff* $\bigvee_{(k,l)\in\{1,\ldots,n\}^2} \Diamond\big(\bigwedge_{i_k} \theta^k_{i_k} \vee \bigwedge_{i_l} \theta^l_{i_l}\big)$.

Proof. Generalizing the observations incorporated in Example 20. ■

Formula θ of $\mathcal{L}_{prop}$ is in *disjunctive normal form* (DNF) if $\theta = \bigvee_i \bigwedge_j \theta_{ij}$, where the θ_{ij} are literals. Formula $\Diamond\theta$ of $\mathcal{L}^1_{TD}$ is in DNF if θ is; and $\Diamond\theta$ is *non-degenerate* if it is in DNF and each disjunct of θ is of the form $(\bigwedge_i u_i \wedge \bigwedge_j e_j)$ for positive i, j. Observe the following decidability result.

THEOREM 22. *The satisfiability (and validity) of finite sets of non-degenerate diamond formulas over $\mathcal{D}$ is decidable in* **NP**.

Proof. Let $\Theta := \{\Diamond\theta_i : 1 \leq i \leq n\}$ with the $\Diamond\theta_i$ non-degenerate. By Lemma 21(f), Θ is satisfiable iff for every θ_i there are disjuncts θ_i^1 and θ_i^2 (having both universal and existential conjuncts) such that the set $\Theta' := \{\Diamond(\theta_i^1 \vee \theta_i^2) : 1 \leq i \leq n\}$ is satisfiable. By Lemma 21(c, e), $\Diamond([\bigwedge_i u_i \wedge \bigwedge_j e_j] \vee [\bigwedge_k u'_k \wedge \bigwedge_l e'_l])$ holds at a dense occasion o iff one of the following six conditions holds at o:

(1) $\bigwedge_i u_i \wedge \bigwedge_j e_j$;
(2) $\bigwedge_k u'_k \wedge \bigwedge_l e'_l$;
(3) $[\sharp(\bigwedge_i u_i, \top, \bigwedge_i u_i) \wedge \mathsf{low}(e_0, \ldots, e_n) \wedge \mathsf{high}(e_0, \ldots, e_n)]$;
(4) $[\sharp(\bigwedge_k u'_k, \top, \bigwedge_k u'_k) \wedge \mathsf{low}(e'_0, \ldots, e'_m) \wedge \mathsf{high}(e'_0, \ldots, e'_m)]$;
(5) $[\sharp(\bigwedge_i u_i, \top, \bigwedge_k u'_k) \wedge \mathsf{low}(e_0, \ldots, e_n) \wedge \mathsf{high}(e'_0, \ldots, e'_m)]$;
(6) $[\sharp(\bigwedge_k u'_k, \top, \bigwedge_i u_i) \wedge \mathsf{low}(e'_0, \ldots, e'_m) \wedge \mathsf{high}(e_0, \ldots, e_n)]$.

Enumerate these options as $1, \ldots, 6$ in the above order and guess an assignment $f : \{1, \ldots, n\} \to \{1, \ldots, 6\}$. If it so happens that $\{1, 2\} \cap Im(f) = \varnothing$, it is easy to decide whether Θ' is satisfiable. Let $\mathfrak{L}_u$ ($\mathfrak{R}_u$) be the set of all universal formulas that appear as conjuncts in the first (third) argument of the metalanguage connective $\sharp$ in a condition $f(x)$ for some $1 \leq x \leq n$. Let $\mathfrak{L}_e$ ($\mathfrak{R}_e$) be the set of all existential formulas that appear as arguments of the metalanguage connective low (high) in a condition $f(x)$ with $1 \leq x \leq n$. Then check, applying the **NP**-algorithm provided by the proof of Lemma 19, whether for all $\epsilon \in \mathfrak{L}_e$ and all $\epsilon' \in \mathfrak{R}_e$, both sets $\{\epsilon\} \cup \mathfrak{L}_u$ and $\{\epsilon'\} \cup \mathfrak{R}_u$ are satisfiable. If they are, then so is Θ', otherwise Θ' is not satisfiable.

Under the assumption $\{1, 2\} \cap Im(f) = \varnothing$, all existential formulas from $\mathfrak{L}_e$ ($\mathfrak{R}_e$) must be compatible with *all* universal formulas from $\mathfrak{L}_u$ ($\mathfrak{R}_u$). By contrast, there is some more room in accommodating $\bigwedge_i u_i \wedge \bigwedge_j e_j$ with formulas of the forms $3, 4, 5, 6$. Perhaps an occasion has two points of division z and v, and A, B, C, D are conjunctions of universal formulas such that A (C) holds to the left of z (v) and B (D) to the right of z (v). Then for satisfying $\bigwedge_j e_j$ simultaneously with A it is perfectly sufficient that some conjuncts are compatible with C and the rest with D — provided that A and D are mutually compatible. For the general case, let $1 \leq x \leq n$. If $1 \leq f(x) \leq 2$, let $\mathfrak{L}_u^x = \mathfrak{R}_u^x$ contain all universal conjuncts of $f(x)$, otherwise let $\mathfrak{L}_u^x$ ($\mathfrak{R}_u^x$) be the set of all universal formulas that appear as conjuncts in the first (third) argument of the metalanguage connective $\sharp$ in the condition $f(x)$. Similarly, if $1 \leq f(x) \leq 2$, let $\mathfrak{L}_e^x = \mathfrak{R}_e^x$ contain all existential conjuncts of $f(x)$, otherwise let $\mathfrak{L}_e^x$ ($\mathfrak{R}_e^x$) be the set of all existential formulas that appear as arguments of the metalanguage connective low (high) in the condition $f(x)$. Using the **NP**-algorithm employed to prove Lemma 19, check if for every $\epsilon \in \bigcup_{f(x) \notin \{1,2\}} \mathfrak{L}_e^x$ and every $\epsilon' \in \bigcup_{f(x) \notin \{1,2\}} \mathfrak{R}_e^x$, the sets

$\{\epsilon\} \cup \bigcup_x \mathfrak{L}_u^x$ and $\{\epsilon'\} \cup \bigcup_x \mathfrak{R}_u^x$ are satisfiable. If not, Θ' is not satisfiable. Otherwise proceed to check if the existential formulas brought in by formulas $f(x) \in \{1, 2\}$ can be accommodated in a model. To this end, think of each formula $f(x)$ as inducing a point dividing an attempted model into a left side and a right side. Distinct formulas may but need not induce the same division point. Let $1 \leq k \leq n$ and let $g : \{1, \ldots, n\} \to \{1, \ldots, k\}$ be a surjection. Intuitively, g determines a linear order for the division points corresponding to the formulas $f(x)$, allowing the identity of points induced by several formulas. Introduce sets S_y $(1 \leq y \leq k+1)$ as follows: S_y includes those $\mathfrak{L}_u^x$ for which $y \leq g(x)$ and those $\mathfrak{R}_u^x$ for which $y > g(x)$; further, for every x guess a partition of $\mathfrak{L}_e^x$, including each member of the partition into one of the sets S_y with $y \leq g(x)$, and guess a partition of $\mathfrak{R}_e^x$, including each member of the partition into one of the sets S_y with $y > g(x)$. Do not include any other elements into the S_y. Then decide using the **NP**-algorithm of Lemma 19 whether all sets S_y are satisfiable. If they are, then so is Θ', otherwise Θ' is not satisfiable. ■

Theorem 22 is clearly generalizable to finite sets of arbitrary diamond formulas in DNF; however, a proof of this fact is left to another occasion.

Box formulas. Any $\mathcal{L}^1_{TD}$-formula $\Box\theta$ is a *box formula*. We content ourselves with a couple of observations concerning finite sets of box formulas. A box formula $\Box\theta$ is in DNF if the $\mathcal{L}_{prop}$-formula θ is; and it is *purely universal* if it contains no existential $\mathcal{L}_{base}$-formulas as components.

FACT 23. Let $\theta_1, \ldots, \theta_n$ be purely universal $\mathcal{L}_{prop}$-formulas in DNF. The set $\{\Box\theta_1, \ldots, \Box\theta_n\}$ is satisfiable iff there are formulas χ_i $(i := 1, \ldots, n)$ such that χ_i is a disjunct of θ_i and $(\chi_1 \wedge \ldots \wedge \chi_n)$ is satisfiable.

Proof. *Right to left:* If $(\chi_1 \wedge \ldots \wedge \chi_n)$ is satisfiable, so is $(\theta_1 \wedge \ldots \wedge \theta_n)$. Thus $\Box(\theta_1 \wedge \ldots \wedge \theta_n)$ is satisfiable and, *a fortiori*, so is $(\Box\theta_1 \wedge \ldots \wedge \Box\theta_n)$. *Left to right:* Write $\theta_i = \bigvee_{j<n_i} \bigwedge_{l<m_{ji}} \theta^i_{jl}$. Suppose $\bigwedge_{1 \leq i \leq n} \Box(\bigvee_{j<n_i} \bigwedge_{l<m_{ji}} \theta^i_{jl})$ is true in $\mathcal{M}$ at o for some $(\mathcal{M}, o)$. Each conjunct $\Box(\bigvee_{j<n_i} \bigwedge_{l<m_{ji}} \theta^i_{jl})$ determines a finite division D_i of o into $N_i \geq 1$ cells such that each cell makes true one of the formulas $\bigwedge_{l<m_{ji}} \theta^i_{jl}$ $(j < n_i)$. Jointly the conjuncts determine, therefore, a partition of o into $(\sum_{i=1}^n (N_i - 1)) + 1$ sets. Each member S'_r of this partition is an intersection $S_1 \cap \ldots \cap S_n$, where S_i is a cell of the division D_i; hence S'_r is an occasion and satisfies a conjunction $C_r := (\chi^r_1 \wedge \ldots \wedge \chi^r_n)$, where χ^r_i is a disjunct of θ_i true at S_i. ■

The case of arbitrary box formulas is considerably more involved. Let $\{\Box\theta_1, \ldots, \Box\theta_n\}$ be any set of box formulas in DNF. To check if it is satisfiable, we should be able to operate with the formulas θ_i as follows. Let Θ_i be the set whose members are the sets of conjuncts of the disjuncts of θ_i. Suppose the attempted model we are constructing has a sub-occasion satisfying $S^i_j \in \Theta_i$. At the moment we introduce S^i_j, we will have to check, separately for every existential formula $e \in S^i_j$, that from each of the remaining sets Θ_l with $l \neq i$ we can find a member $S^l_{k_l}$ such that $\{e\} \cup \bigcup_l \forall(S^l_{k_l}) \cup \forall(S^i_j)$ is satisfiable. (If $S \subseteq \mathcal{L}_{base}$, $\forall(S)$ denotes the set of universal formulas of S.) Checking this for just one existential formula $e \in S^i_j$, we get committed to checking the analogous condition for the total number of existential formulas in the sets $S^l_{k_l}$. In order for this process to terminate, we must, evidently, be able to exhibit finite numbers $N_1, \ldots, N_n$ such that $\{\Box\theta_1, \ldots, \Box\theta_n\}$ is satisfiable iff it is satisfiable when attention is restricted to divisions triggered by $\Box\theta_i$ into at most N_i cells. Without such bounds, we can only show that the set X of finite satisfiable sets of box formulas is recursively enumerable. By Corollary 13, we know that actually X is recursive. We leave it as an open question how to determine the numbers $N_1, \ldots, N_n$ (depending on the sizes of the input formulas). Note that it *can* indeed happen that the model-seeking process described above in general terms never terminates while the relevant set of box formulas is *not* satisfiable.

EXAMPLE 24. Let $\theta_1 := [p \wedge (\neg r \wedge \neg s)] \vee [q \wedge (\neg r \wedge \neg s)]$ and $\theta_2 := [r \wedge (\neg p \wedge \neg q)] \vee [s \wedge (\neg p \wedge \neg q)]$. It is easy to see that $\{\Box\theta_1, \Box\theta_2\}$ is not satisfiable. Yet there *is* $(\mathcal{M}, o)$ such that o has a division into ω cells each of which satisfies θ_1 and another division into ω cells each of which satisfies θ_2. $\dashv$

7 Concluding remarks

Related work. In the literature various temporal logics using intervals in their semantics have been formulated; see [14] for a discussion. (On the whole, interval-based logics remain, however, less studied than point-based ones.) Among different modal logics of intervals, $\mathcal{L}_{TD}$ bears the closest resemblance to the *propositional interval temporal logic* or **PITL** [13, 26, 27] and (the propositional fragment of) the *duration calculus with iteration* or **DC*** [5, 10, 15]. These logics have a strong motivation deriving from computer science, with applications e.g. to hardware description and verification. It is of some interest to note that von Wright's philosophically driven considerations dating from 1968 led largely to the same logical conceptualizations as those that some 15 years later emerged in **PITL** — which, again, has an unmistakable connection to regular expressions, the formulation of which goes back to Kleene's work [20] around 15 years before von Wright's paper. On finite intervals, the operators *chop* and *chop-star* of **PITL** are related to catenation *resp.* catenation closure (Kleene star). (Regular expressions as such have, though, nothing to do with temporal logic.)

PITL uses natural numbers to model time, while **DC*** employs real numbers. Formulas of **PITL** are evaluated relative to finite or infinite maps $\sigma : I \to 2^{\mathbf{prop}}$ $(I \subseteq \mathbb{N})$ called *intervals.* For each $i \in I$, the object $\sigma(i)$ is termed a *state*; hence states are truth-value distributions over a fixed set **prop** of atoms. An atom p holds at σ iff $\sigma(\mathbf{min})(p) = 1$, where **min** is the smallest element in $dom(\sigma)$. Disjunction ($\vee$), negation ($\neg$) and *verum* ($\top$) have their expectable semantics. The modal operators *eventually* ($\Diamond$) and *next* ($\bigcirc$) satisfy: $\sigma \models \Diamond\chi$ iff $\sigma' \models \chi$ for some suffix σ' of σ; and $\sigma \models \bigcirc\chi$ iff $\sigma' \models \chi$, given that $\sigma = \sigma(\mathbf{min})^\frown\sigma'$. The dual of $\Diamond$ is denoted by $\Box$. Characteristic of **PITL** are the binary operator *chop* (written ;) for sequential composition and the unary operator *chop-star* (written *) for the closure of sequential composition. If σ is finite, by definition $\sigma \models \psi;\chi$ iff there is $k \in dom(\sigma)$ such that $\sigma(\mathbf{min})\ldots\sigma(k) \models \psi$ and $\sigma(k)\ldots\sigma(\mathbf{max}) \models \chi$. And $\sigma \models \psi^*$ iff there are $m < \omega$ and $k_0,\ldots,k_m \in dom(\sigma)$ such that $k_0 = \mathbf{min}$ and $k_m = \mathbf{max}$ and $\sigma(k_i)\ldots\sigma(k_{i+1}) \models \psi$ for all $0 \leq i < m$. The clauses for *chop* and *chop-star* are similar in $\mathbf{DC}^*$; the evaluation in $\mathbf{DC}^*$ is relative to *closed real intervals*.[9] The semantics of *chop-star* is obviously very close to the semantics of the operator $\Box$ of $\mathcal{L}_{TD}$. The main difference is that when the latter is evaluated, the relevant interval is divided into finitely many *disjoint* pieces, while when evaluating *chop-star*, adjacent subintervals have exactly one state in common. So it is $\Box$ rather than *chop-star* that has a really straightforward connection to regular expressions: if the 'extension' of ψ is denoted by a regular expression r, the 'extension' of $\Box\psi$ is denoted by rrr^*. (Recall that $\Box\psi$ is only true of intervals divisible into at least two cells satisfying ψ.) On finite intervals, the following map t translates $\mathcal{L}_{nnf}$ into **PITL**, whereas by Proposition 1, $\mathcal{L}_{TD} \leq \mathcal{L}_{nnf}$. Define a map $s : \mathcal{L}_{zero} \to \mathbf{PITL}$ by stipulating that $s(\psi)$ is the result of replacing all occurrences of $\sim$ in ψ by $\neg$. Then put: $t(p) = \Box p$, $t(\sim\phi) = \Box\neg s(\phi)$, $t(\neg\phi) = \neg t(\phi)$, $t(\phi \circ \chi) = t(\phi) \circ t(\chi)$ for $\circ \in \{\vee, \wedge\}$, $t(\Box\phi) = t(\phi);\bigcirc t(\phi);[\bigcirc t(\phi)]^*$, $t(\Diamond\phi) = \neg(\neg t(\phi);\bigcirc\neg t(\phi);[\bigcirc\neg t(\phi)]^*)$. It is not immediately clear whether $\mathcal{L}_{TD} \leq \mathbf{DC}^*$; relative to real intervals the *next* operator is not available to help expressing the semantics of $\Box$. Both **PITL** and $\mathbf{DC}^*$ have been extensively studied from the proof-theoretic viewpoint. **PITL** is known to admit of a complete proof system both over finite and over infinite time; see the bibliography of [27]. A complete proof system for $\mathbf{DC}^*$ relative to so-called abstract-time semantics has likewise been presented [10]. For more information about duration calculus, see [15, 16]. Unlike **PITL** and $\mathbf{DC}^*$, $\mathcal{L}_{TD}$ is not for its semantics restricted to any particular class of linear orders. On the other hand, relative to the appropriate classes of linear orders, $\mathcal{L}_{TD}$ is less expressive than either **PITL** or $\mathbf{DC}^*$: in particular the operator *chop* is neither syntactically given nor definable in $\mathcal{L}_{TD}$.

Questions for future research. The present paper leaves it for future research to estimate the complexity of an optimal algorithm solving $\mathcal{L}_{TD}$-SAT. Läuchli's proof [22] does not yield an explicit upper bound on the time complexity of $\mathcal{L}_{\mathrm{w}}^{\mathrm{mon}}$-SAT, while Rabin's proof [31] provides a non-elementary upper bound (i.e., the time complexity is not bounded by any stack of exponentials of a fixed height). Another obvious task is to provide a complete proof system for $\mathcal{L}_{TD}$. One might also attempt relating fragments of $\mathcal{L}_{TD}$ to independently interesting fragments of $\mathcal{L}_{\mathrm{w}}^{\mathrm{mon}}$.

The study of $\mathcal{L}_{TD}$ can be pursued in different directions. One option is to study its finite models; technically, these are *word models*. By Büchi's theorem, cf. [6], $\mathcal{L}^{\mathrm{mon}}$ can define over word models precisely the denotations of regular expressions. (Over finite models, of course $\mathcal{L}_{\mathrm{w}}^{\mathrm{mon}} = \mathcal{L}^{\mathrm{mon}}$.) One can ask in which precise way $\mathcal{L}_{TD}$ falls short of capturing regular languages;

[9] For *chop* as a modal operator employing a ternary accessibility relation, see [36].

cf. [17] for related research. The question of whether $\mathcal{L}_{TD}$ has 0-1 law could also be studied. (Evenness is, apparently, not definable.) It might be possible to link $\mathcal{L}_{TD}$ to certain logics of trees with a *yield* operator [4].[10]

$\mathcal{L}_{TD}$ could be modified by adding an operator $\Box'$ otherwise like $\Box$ but involving a division into a fixed finite number of cells, say 2; this might be further strengthened into a binary *chop* modality: ϕ holds first and then ψ. The semantics of $\Box$ itself can also be varied. We might, e.g., allow the number of divisors in the semantic clause for $\Box$ to be countably infinite or arbitrary. Without further restrictions, however, the resulting logic would be very expressive indeed and the whole idea of division would take a somewhat unintended form: e.g. a rational interval would have a division into singleton cells. A more interesting variant is obtained by imposing a condition on the *induced order* of the divisors. We might, say, restrict attention to sets of divisors whose induced order is of type ω. Actually von Wright did not forbid infinite divisions but still took occasions to be always divided into discrete 'bits' or 'stretches' [37, p. 127]. If $\mathcal{C}$ is a class of *discrete* order types, we might study the logic $\mathcal{L}_{TD}^{\mathcal{C}}$ with $\Box$ strengthened so as to allow any sets of divisors the induced order of which has a type $\alpha \in \mathcal{C}$. The semantics of $\Box$ in $\mathcal{L}_{TD}$ results from letting $\mathcal{C}$ consist of all finite order types $n \geq 1$. On arbitrary linear orders, $\mathcal{L}_{TD}^{\mathcal{C}}$ can merely be translated into $\mathcal{L}^{\text{mon}}$ if $\mathcal{C}$ admits of infinite order types; more specifically, into the fragment $\mathcal{L}_{\mathcal{C}}^{\text{mon}}$ of $\mathcal{L}^{\text{mon}}$ whose second-order quantifiers range over subsets meeting the appropriate order type requirement. There is no immediate way of settling whether $\mathcal{L}_{TD}^{\mathcal{C}}$ is decidable: $\mathcal{L}^{\text{mon}}$ is undecidable over arbitrary linear orders — notably over the real line [33, 12]. Over some classes of linear orders $\mathcal{L}^{\text{mon}}$ is decidable, however, e.g., countable linear orders [31], $\{\omega_1\}$ and $\{\alpha : \alpha < \omega_2\}$ [2]. As to $\{\omega_2\}$, ZFC does not determine which sentences of $\mathcal{L}^{\text{mon}}$ are true of ω_2 [11, 33]. Evidently $\mathcal{L}_{TD}^{\mathcal{C}} < \mathcal{L}_{\mathcal{C}}^{\text{mon}} < \mathcal{L}^{\text{mon}}$; it might happen that $\mathcal{L}_{TD}^{\mathcal{C}}$ is decidable over arbitrary linear orders.

Finally, one might experiment with studying $\mathcal{L}_{TD}$ relative to a larger class of models, say tree structures. This would require finding a suitable interpretation to the idea of division relative to trees. Conceivably divisor points might be replaced by *bars* (i.e., sets B such that every branch of the tree intersects B exactly once). Or by non-comparable nodes such that every maximal branch belongs to the 'neighborhood' determined by one of the nodes. Or the cells of divisions might be taken to be subtrees of a tree.

Acknowledgments. The author wishes to thank the three anonymous referees for their detailed, useful comments. The research was carried out within the project "Modalities, Games and Independence in Logic" funded by the Academy of Finland.

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