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Physical Methodology for Economic Systems Modeling

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Abstract: The paper discusses the possibility of constructing economic models using the methodology of model construction in classical mechanics. At the same time, unlike the "econophysical" approach, the properties of economic models are derived without involvement of any equivalent physical properties, but with account of the types of symmetry existing in the economic system. It has been shown that at this approach practically all known mechanical variables have their "economic twins". The variational principle is formulated on the basis of formal mathematical construction without involving the subjective factor common to the majority of models in economics. The dynamics of interaction of two companies has been studies in details, on the basis of which we can proceed to modeling of more complex and realistic economic systems. Prediction of the possibility of constructing economic theory on the basis of primary principles analogously to physics has been made.

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1. Introduction

In these days the economists are using physical analogies for modeling economic systems more and more often. The complex of models using various analogies with the behavior of physical systems is called econophysics. At the same time its sphere of application varies from statistical methods [1] to quantum mechanical models [2], and also includes the classic dynamics [3]. Nevertheless, most of these approaches lay emphasize on the formal analogy of interaction mechanisms of the system elements, thus significantly lim-

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iting the sphere of application of such models.
Thus, for instance, the model of Brownian motion and the model of share price dynamics, derived by Bachelier [4], use the same assumption of the independent increments of object "coordinate", however the reasons for its realization in these two systems are of completely different nature. While in physics this phenomenon results from roughening of description of the classical system trajectory a multidimensional phase space, in economic models it is due to the principle of prohibition for the possibility of arbitration and, as a consequence, the unpredictability of the share price. As a result of this, parameters of a random process in economics are introduced phenomenologically [5], rather than calculated on the basis of analysis of microdescription of particle collision, as it is made in physics.
In less-developed models of econophysics the substantiation of assumptions validity is often limited to qualitative analogy of processes taking place in an economic system and in its physical analog [6]. As a result, authors can introduce into an economic model the properties of its physical analog without sufficient grounds, and vice versa, some features specific only to an economic model can be missed.
Due to the established tendencies in econophysics, we are convinced there is an insistent necessity to clearly trace the correspondence of the fundamental principles of dynamics, on which the models of physical systems are based, and the main econophysical models. The same aim is pursued in paper [7], in which the author comes to similar conclusions, but at the same time skeptically considers the possibility of creating a fundamental economic model analogously to physical models. With this property of the main argument he claims that unlike the physical approach, the "atoms" in economic models (companies, brokers, individual sellers and buyers) are not identical and the methods of statistical physics can not be applied to them. We agree with this statement, however we assume that such difference is not of principal nature, it rather results in additional "technical" complications, which can be solved using modern mathematical methods.
Our theory is based on the postulate that the economics as a science is a natural science in the standard gnosiological interpretation of this word. As the physics is currently the most harmonious and fundamental natural scientific theory, we will follow the pattern of derivation of a physical theory in the process of constructing the theory of economics.
The essence of constructing physical models is based on two fundamental principles – the principle of symmetry and the variational principle, which allow formulating dynamic equations of physical systems. After that further theory derivation is limited to mathematical computations and, in case of a large number of degrees of freedom, to transition from full (microscopic) to approximated (macroscopic) description. Let us emphasize that with such an approach, unlike the models of the generally-accepted econophysics, the properties of physical analogs are not used on any stages model construction. We can only state the similarity of model properties after its construction with a certain physical analog, fully tracing those symmetric properties of both systems, which actually produce these analogs. A similar situation often originates in physics itself. For instance, electromechanical analogs allow modeling the dynamics of mechanical systems using elec-

tronic circuits, but do not provide the interpretation of mechanics via electrodynamics or vice versa.

In this paper we will demonstrate the efficiency of this approach by the example of a simple (illustrative) economic system of interaction of technologically connected companies. At the same time we will not introduce the notions of the main dynamic parameters of the system, such as the coordinate, mass, power, energy, analogously to classical mechanics, but we will obtain a number of such analogies as the consequence of the general symmetry properties of the analyzed systems.

Let us formulate the theses of the proposed approach to description of economic systems:

- the existence of fundamental principles determining the properties of elementary interaction (transaction) in economics;
- as consequence there is a principal possibility of micro description of an economic system of any complexity in the form of a multitude of such elementary transactions;
- various types of transactions can have different types of symmetry, which is determining the diversity of their dynamical properties;
- the properties of economic systems cab be formulated in the form of initial variation principles;
- practically significant observed macroscopic properties of economic systems can be obtained as a result of statistical averaging (roughening of micro description) and analyzing of open systems as a particular case of such a roughening;
- participation of a subject, as an element of economic system, requires description of its state in the framework of quantum mechanical formalism due to principal unpredictability of its dynamics, resulting from prohibition of arbitrage.

The latter two theses have been partially used by us in our works [8, 9] and will be discussed in details in our further publications.

2. The Analog of Point Mass and its State in The Models of Classical Dynamics of Economic Systems

The main notion which is to be defined in order to make it possible to construct dynamic models is the system state. We will discuss a model of economic system based on the principles of classical mechanics. In this model the state is characterized by a set of parameters, which fully determines its further trajectory. One of the basic elements of the classical description is the notion of point mass.

This notion is introduced for simplification of the description in case when the set of parameters characterizing the internal system properties does not effect in any way its evolution as a whole relative to other systems. Limiting ourselves to a macro description of stock-exchange dynamics, we can assume that at each time moment a company state is determined by the prices for its shares and other securities and by their yield. In the idealized model these specific parameters are commonly known and available to each of the traders, thus determining their choice. In reality, however, in order to make a decision on purchase or sale of shares it is also required to know the degree of risk connected with volatility of securities. However, in the simplest illustrative classical model we will so far limit ourselves to discussing only the deterministic processes. At the same time we can assume that all similar shares are indistinguishable, from this follows the identity of their price at each moment of time. A certain price function can be accepted as the coordinate. Then all the similar shares have the same coordinate, their trajectories are indistinguishable and this allows us considering the whole volume of similar shares as a single point mass.

The choice of price function is not limited by any requirements and transition from one function to another is limited to a simple change of variables in the equations of motion, however it is convenient to choose this function in such a way that the originating state space would be maximally symmetrical. This will eventually determine the simplicity of notation of the equation of motion in the future. Actually, the right choice of the variable which we will consider a coordinate is to ensure the observance of the inertia principle – conservation of motion velocity at absence of external forces. Thus, in the economic model similar to classical mechanics we will choose a coordinate as a price function so as to ensure the homogeneity of the economic space.

In accordance with this requirement it is preferable to accept as a coordinate the logarithmic scale, like for instance in Black-Scholes-Merton model. In this case the coordinate space appears to be homogeneous, as variation of the origin of coordinates $x_i^1 = x_i^0 + \delta x_{01}$ only means changing of the price scale (all prices at the stock-exchange are changed the same number of times) and cannot effect their dynamic properties. In the alternative variant when the price itself is a coordinate, the increase of all prices by a certain similar value results in their relative equalizing, which certainly effects the dynamic properties of the market. Therefore the price scale $x_i = C_i$ is not homogenous. Let us also note that in a homogeneous economic space the velocity of the point masses, proportional to$d(\ln S)/dt$, actually represents the share yield is not changed due to offset of the origin of coordinates.

Besides we should take into account that the price of a single share is in inverse proportion to their amount, which is a conventional parameter and does not depend of the objective properties of a company. Therefore we should choose as a corresponding coordinate the logarithmic function of full symmetry (N_iC_i) of a shares portfolio (not of a single share), which in a certain sense reflects the value of the company's capital stock. Finally we will determine the **coordinate** of the company, considered as a point mass, as

$$\{x_i\} \equiv \ln (N_iC_i) \tag{1}$$

The analog of **distance** between the point masses is the value$x_2 - x_1 = \ln (N_1C_1) - \ln (N_2C_2) = \ln \left(\frac{N_1C_1}{N_2C_2}\right)$. At the same time, assuming the existence of an absolute time scale analogously to classical mechanics (in various reference systems the notion of simultaneity is absolute), we will obtain the general law of velocity composition:

$$v_{21} + v_{32} = v_{31}, \text{ where } v_{i;j} = d\,(x_i - x_j)/dt \tag{2}$$

Like in classical mechanics, we have limited the state of the point mass by setting the coordinate and the velocity. The reason for this, similarly to physics, is the fact that in case of absence of external forces, the coordinate and velocity are responsible for determining the trajectory of free motion. Any deviations from it mean either the occurrence of external factors (forces) effecting the trajectory, or the changes in the internal structure of production (which does not allow considering a company as a point mass any more). Thus, the accepted definition of state is closely connected with the notion of force. Discussing its meaning, R. Feynman writes: “The real content of the Newtonian laws is the following: it is assumed that the force has independent properties in addition to the law$\vec{F} = m\vec{a}$”. Paraphrasing this, we can say that the physicians' assumption is that the velocity of variation of body impulse does not depend on the impulse itself. This is not always true. For instance, the friction force is proportional to the velocity. However, we can still consider the friction force not as a fundamental force, but as an averaged effect of a large number of interactions of the body with the molecules of medium. We will also make similar assumptions while constructing econophysical models, most of which are intrinsically dissipative.

3. The Analog of the First Newton Law

The law of inertia (the first Newtonian law in physics) follows from the homogeneity of the state space, which was ensured by a suitable selection of the variable of state. Therefore our further considerations are more of an illustration of this law in economic systems rather than proving. Besides, by determining the conditions under which the system can be considered closed, we are approaching to the understanding of the notion of force in economics. Let us first give a more rigorous determination of the notion of a free particle in the quantum-mechanical model. In mechanics this means absence of interaction either with other point masses or with force fields. We can assume that in economics this state corresponds to absence of transaction with holdings of shares and increase of their price only at the expense of receiving the surplus value (at unchanged production technology). At the same time, it is important to note that in a homogeneous space the value, which is retained, is the share yield, not the velocity of share price increase, which equals in the accepted approximation to the profitability of the capital stock $d(\ln NC)/dt = d(\ln C)/dt$ and which is the analog of the **velocity** of motion. Such behavior corresponds to the mechanism of extended reproduction, when at a constant amount of shares the received profit is spent for instance for procurement of new machine tools and hiring of new employees at the same technology and production output. We can also consider a production factor the allocation of part of the profit for paying dividends on shares. Constant velocity corresponds to constant percentage of such deductions. Thus, simple extended reproduction corresponds to the **analog of the inertia motion** in classical mechanics. At the same time the analog of the first Newton law can be formulated as:
Company's profitability remains constant at constant at constant surplus value and production technology (including proportions of profit distribution).

4. The Analog of the Second Newton Law

As we are considering the extended reproduction of a certain company as an analog of a free-moving point mass, let us also consider the forces which can originate in this model. In the general case the companies profitability can be associated with its production parameters using the following formula:

$$\frac{d}{dt}(NC) = P \cdot (c_+ - c_-) \tag{3}$$

where P- is the production capacity, measured in $\left[\frac{up}{d}\right]$ - units of products per day; c_+- is the price of unit of product on the market, and c_- - is its cost price, measured in conventional monetary units (which we will further denote as \$) per unit of product $\left[\frac{\$}{up}\right]$. This formula is in fact is the simplified expression of the law of surplus value. Dividing internal and external factors in this expression, we obtain:

$$\frac{NC}{P} \cdot \frac{d(\ln NC)}{dt} = c_+ - c_-, \tag{4}$$

In this representation the first factor $\frac{P}{NC}$ characterizes the properties of the company and the second factor characterizes the velocity of its motion. The prices in the right part of the equation determine the character of economic interaction between the companies, which act as links in the equations of motion. It is natural to associate these factors with mass and impulse of the point mass, respectively. At the same time, economic analog of **mass** can be defined as the quantity of equivalent capital stock ensuring unit productivity:

$$\{m\} \equiv \frac{NC}{P} \tag{5}$$

The profit received from production of a product unit (its surplus value) acts as an **impulse**.

$$\{p\} \equiv c_+ - c_- = \delta c \tag{6}$$

Then we will consider any effect resulting in time variation of this profit (impulse) an analog of the **external force**.

$$\{F\} \equiv \frac{d}{dt}(\delta c) \tag{7}$$

Accordingly, the analog of the second Newton law $ma = F$ is sequent from formula (4) and can be written down as

$$\frac{NC}{P} \cdot \frac{d^2(\ln NC)}{dt^2} = \frac{d}{dt}(\delta c) \tag{8}$$

5. The Standard of Mass as a Basis for Introduction of the Cost of Product

The question of analog of mass in econophysical models is definitely among the most important and tangled ones. The majority of publications mainly relates to utilization of statistical physics methods and studying of statistics and dynamics of probabilistic distribution of economic indicators [10]. At the same time they traditionally use the law of conservation of monetary aggregates during an elementary transaction between subjects of market as a basis for making analogy with physics. Actually, only on the basis of the ratio $m_1 + m_2 = m_1' + m_2'$ and the hypothesis of equal distribution among the permissible microstates, they obtain Boltzmann distribution, introduce econophysical analog of thermostat and its temperature. Other types of distributions, more corresponding to experimental data, are obtained by phenomenologically introducing various special distribution features of possible outcomes of an elementary transaction.

However, with such an approach the notion of mass is not necessary, as we are speaking only about the distribution of an economic analog of energy, which is identified as the monetary aggregates. In this case the law of energy conservation is observed automatically, but this only relates to the procedure of securities exchange and does not cover the production process. Moreover, similarity to the law of energy conservation is of formal nature only, as no features of energy or its type (kinetic, potential) are stipulated. The law of equal distribution at the permissible microstates and the method of discretization of these states are postulated.

The analog of mass introduced by us earlier in the economic model is free from these faults. Let us note that at the proposed definition of mass, it is invariable both at uniform inertial motion (extended reproduction) and at acceleration caused by the effect of external factors: changing of procurement prices for raw materials, equipment and labor force or selling prices for the manufactured products. At such a definition it only depends on the ratio between the production efficiency and the amount of capital stock invested into the production, i.e. the production technology.

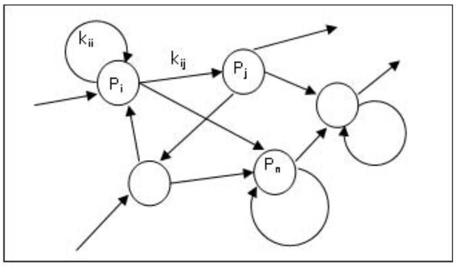

Fig.1. Technological layout of product flows in economic system of interconnected productions (companies).

However due to the fact that various companies produce various types of products, we obtain different measurement units of mass $\left[\frac{\$\cdot d}{up}\right]$.

A similar situation had once occurred relative to the measurement units of real masses (pounds, kilograms, etc.). For elimination of the contradiction we must also introduce a "standard" of economic mass, corresponding to such parameters of production technology and efficiency, which provide production of a unit of nominal product for a nominal unit of capital stock in a unit of time. The ratio between masses of companies is thus connected with the ratio of the number of units of nominal product contained in a unit of real product manufactured by each o the companies. For determination of such ratios we will consider the product flows for the case of a stably developing economy at unvaried technological parameters. A schematic layout of such product flows is shown in Fig.1. The valuek_{ij}, indicated above each pointer shows the number of product units manufactured by the company$i$, required according to the technological parameters for product of a unit of product by the company j (we are discussing a simplified variant of production when one company manufactures only a single type of product). In this case the set of coefficients k_{ij} forms a square matrix, which we will call ***the matrix of technologies***. The valuesP_i form a vector, which we will call ***the vector of productivities***. Let us note that the ratios between the quantities of manufactured products (under condition of complete demand for them) are determined only by the production technologies (i.e. internal properties). At the same time, according to formula (5) the production technology is responsible for determining the mass in economic model, as in case of unvaried technology and extended reproduction the production efficiency increases proportionally to the equivalent of the company's capital stock. Similarly, in classical mechanics the ratio of body masses does not depend on the interaction of these bodies.

Each pointer in the figure indicates the use of products of one company for manufacturing of products of another company. Consequently, we can include here the human, material and energy resources as the required components of the production process. On the other hand, each employee (or a group of employees) can also be considered as system elements consuming one of the resources and producing resources. We can assume, in similar way to the classical mechanics, that the dynamics of an arbitrary complex system can be limited to a multitude of interacting point masses and perfectly rigid bodies (economic units with unvaried technological and material structure). In fact, in case of compliance with this requirement (conservation of proportions of the production efficiency and the coefficients of the matrix of technologies) all point masses of the system travel at the same speed. This corresponds to conservation of pair distances between them – an analog of a perfectly rigid body in mechanics. Such classical model corresponds to the deterministic concepts both in physics and in economics. Its limited nature reflects in the idealization of the notion of point mass. However, while in physics the contradiction arising in this connection can be solved by means of transition a quantum mechanical description; in economic models this idealization is no longer effective in describing a human consciousness as an element of economic systems. In our further publication we will show that the attempt of introducing the consciousness into the description of system state results in the necessity of rejecting deterministic models and allows approximation similar to quantum mechanical formalism.

In the present paper we have limited ourselves to the analysis of deterministic models. Under the conditions of complete consumption of the manufactured products and growth of production of various types of products in proportion to each other, the following ratio must be true:

$$P_i^{(n)} = \sum_j k_{ij} P_j^{(n+1)} \quad , \tag{9}$$

where (n) and $(n+1)$ are the numbers of subsequent production cycles. If ratio (9) is true, all products manufactured during the n-th cycle will be in demand and will determine the vector of productivity of the (n+1)-th cycle. Let us not that for validity of (9) there is still no need for introduction of an analog of mass - though all P_i have different dimensions, the corresponding dimensions of coefficients $[k_{ij}] = \frac{up(i)}{up(j)}$ compensate these differences. For introduction of the standard of mass we will assume that a unit of product manufactured by company (i) contains β_i units of a certain nominal product (NP). Correspondingly, we will call the vector β_i, consisting of these coefficients, ***the vector of utility*** of the manufactured products.

Then we obtain from (9):

$$P_i^{(n)} \beta_i^{(n)} = \sum_j \left(k_{ij} \frac{\beta_i^{(n+1)}}{\beta_j^{(n+1)}} \right) P_j^{(n+1)} \beta_j^{(n+1)}, \text{ or } P_i^{*(n)} = \sum_j k_{ij}^* P_j^{*(n+1)} \tag{10}$$

In the last formula the productivities of companies P_i^* have the same dimension, while the coefficients of technological links between them k_{ij}^* are dimensionless quantities. Now we can compare the number of NP units at the input and at the output of the production process.

Let us note that the expression of product units in NP units is actually an equivalent of their cost, as it allows abstracting from specific utility of various products. It also allows performing equivalent exchange of any products in accordance with their abstract utility. By postulating that a conventional unit is nothing else but a unit of notional product, we can cancel these units in the dimension of economic mass and express it only in time units $[d]$. Thus, this mass is proportional to the manufacturing time of a product, equal in cost to the invested capital, and characterizes its inertia. So we can consider a ***standard of mass*** the mass of such a production, which manufactures products equal in cost to the cost of capital stock per accepted time unit.

6. Dynamics of the Deterministic Model in Economic Systems

Transition from (9) to (10) is yet formal and is possible at any choice of vector β_i. For its concretization we must use additional conditions between production volumes of various products. Another requirements besides (9) is the equal amount of the funds $P_j^{(n)} \beta_j^{(n)}$, received from the sales of products manufactured by the j-the company during the cycle(n), and the funds spent for procurement of resources required for manufacturing of products in the cycle$(n+1)$. This condition means complete demand for the monetary funds received from the sales in each cycle of exchange of goods.

$$P_j^{(n)} \beta_j^{(n)} = P_j^{(n+1)} \sum_i \left(\beta_i^{(n)} k_{ij} \right) \tag{11}$$

Taking into account (9), we can write down the system of equations for calculation of the vector of utility of products (their nominal cost) in each cycle.

$$\left[\frac{\sum_l k_{jl} P_l^{(n+1)}}{P_j^{(n+1)}}\right] \beta_j^{(n)} = \sum_i \left(\beta_i^{(n)} k_{ij}\right) \quad (12)$$

Let us note that, as was expected, the utility is determined by the consumer demand for products, i.e. by the planned production capacity and the matrix of technologies. In turn, this demand (the planned vector of productivities) depends on the matrix of technologies and the current amount of products on the market due to (9). In this conclusion we are using the approximation on the regulatory mechanism of price formation, at which the prices (proportions of product exchange) are changing until all manufactured products are in complete demand. At the same time the price formation process is fully completed during the intervals between the product cycles.

The dynamics of the production process itself in the discussed model is determined from (9) on the basis of the initial value of the vector of productivity and the matrix of technologies:

$$P_i^{(n)} = [k_{ij}]^{-n} P_j^{(0)} \quad (13)$$

In case of stable proportional economic growth the proportions both between the productivities of various companies and between the mass coefficients of the manufactured products in case of production expansion are retained. Then we can write down for productivity in subsequent cycles$P_i^{(n+1)} = \lambda P_i^{(n)}$, and for finding the vector $P_i^{(n)}$ from (15) we obtain the system

$$\lambda^{-1} P_i^{(n)} = \sum_j k_{ij} P_j^{(n)} \quad , \quad (14)$$

which uniquely defines the vector of productivity. It can be considered as a task of finding the proper values (λ^{-1}), and the proper vectors of the linear operator$\hat{K} \equiv [k_{ij}]$. Each of the obtained solutions sets the ratios between the productivities of various companies, which provide stable extended reproduction of the economic system in general with the exponential parameter$\lambda > 1$.

By substituting (14) into (12), we obtain the equation for determining the vector of utility in this case

$$\lambda^{-1} \beta_j^{(n)} = \sum_i \beta_i^{(n)} k_{ij} \quad (15)$$

Thus, in the trivial case of dynamics with the initial value of the vector of productivities corresponding to one of the proper vectors of the matrix of technologies, the corresponding proper value determines the exponential indicator of the extended reproduction, while the proper vector of utility corresponding to the same proper value determines the nominal utility (cost) of the manufactured products. Let us note that the module of the vector of productivities increases exponentially in time, while the vector of utility of the products exponentially decreases, respectively. It is a natural result with account of the fact that the volume of all monetary funds remains the same, but in case if it increases proportionally to the gross product, the prices in such an extended reproduction remain unchanged.

In a more generalized case for an enclosed economic system the solution at arbitrary initial values of the vector of productivities can be found by standard methods, i.e. basis expansion of the proper vectors or directly according to the formulas (12, 13).

7. Passage to the Limit to the Continuous Description of the Dynamics

In order to obtain differential equations of dynamics the passage to the limit to continuous description of the enclosed economic system is required. For this purpose we will assume that the production and the exchange of products take place continuously and the division into production cycles is nominal. Then from the matrix of technologies $[k_{ij}]_{\Delta t}$, corresponding to the time period Δt, we can proceed to the description using cycles with the time δt, which is sufficiently small to consider minor the increments of the dynamic values (vectors of productivity and utility). At the same time the new matrix is interconnected with the previous one by the ratio:

$$[k_{ij}]_{\Delta t}^{=} [k_{ij}]_{\delta t}^{n}\,, \text{ where } n = \Delta t/\delta t >> 1. \tag{16}$$

In this case the reverse matrix will be generally similar to the identity matrix and can be approximately represented in the following form:

$$[k_{ij}]_{\delta t}^{-1} \approx I + \frac{\delta t}{\Delta t}\left([k_{ij}]_{\Delta t}^{-1} - I\right) \tag{17}$$

Then for the minor increment of the values of the vector of productivity during the time period dt we can obtain, with account of (13):

$$\frac{dP_i}{dt} \approx \frac{1}{\Delta t}\left([k_{ij}]_{\Delta t}^{-1} - I_{ij}\right) P_j^{(t)} \tag{18}$$

The obtained matrix $[\alpha_{ij}] \equiv \frac{1}{\Delta t}\left([k_{ij}]_{\Delta t}^{-1} - I_{ij}\right)$ in the model of continuous dynamics actually determines the interconnection of the velocities of point masses. Therefore we will further refer to it as the matrix of velocities. In case of non-interconnected productions when the matrix of technologies is diagonal, we obtain the trivial dynamics of motion of the point masses with constant velocities:

$$v_i = \frac{d}{dt}\ln P_i = \frac{[k_{ii}]_{\Delta t}^{-1} - 1}{\Delta t} \tag{19}$$

as the mass conserves at unvaried internal properties of the company and we have:

$$\frac{d}{dt}\ln P_i = \frac{d}{dt}\ln\left(\frac{P_i}{NC_i}NC_i\right) = \frac{d}{dt}\left(\ln m_i + \ln NC_i\right) = v_i \tag{20}$$

However in a generalized case the productivities of various companies turn out to be interconnected and their dynamics is determined not only by the matrix of productivity, but also by the initial state of the system. In this case it is impossible to obtain a simple expression for the velocities, such as (20). Therefore we will further discuss an illustrative example of the dynamics of interconnected productions in the framework of the “island model” widely used in various branches of economics.

8. The Island Model of the Dynamics of an Enclosed Economic System

The simplest illustration for the aforesaid ratios can be already obtained for a system comprising two interacting productions. This model is equivalent i.e. to the island model, in which an island is inhabited by two tribes of ”shepherds” and “farmers”, one of which breeds (produces) goats and the other grows grains. The matrix of technologies determines which part of the manufactured products is consumed by the tribe itself, and which part is exchanged for the products of the other tribe.

Not explaining in details the analysis of solutions of the system (18) in this case, let us note that they are determined only by the coefficients of the matrix of technologies and by the initial value of the vector of productivities. The trajectories of such system in the coordinates $(P_1; P_2)$ are determined by the position of the initial point, they do not cross and have asymptotes corresponding to the proper vectors of the matrix of technologies.

In order to further illustrate the possibilities of constructing the economic theory analogously to classical mechanics we must find an analog of the potential energy, write down the Lagrangian of the system and set the law of its motion in the form of a variation principle. However in the system of linear differential equations of the first order (18) the accelerations and forces are proportional to the velocity and do not constitute a total differential of any function of coordinates. This prevents us from introducing an analog of the potential energy. The reason for this is the simplicity if the analyzed model. On the example of the mechanical models we can see that the oscillating nature of motion originates only if a third body (spring) is available, which is actually the potential energy carrier. In case of elastic collision of two bodies the role of such an additional body is fulfilled by their parts moving with different velocities. Therefore in the economic model we must also consider a more complex model comprising at least three bodies for creating an analog of the potential energy.

We can consider as such a link i.e. an intermediary, which purchases the products of both tribes and then sells it at once or in some time. In an idealized model these services can be provided by a single person, who does not manufacture any products and practically does not require expenses for his upkeep. Due to (5) this corresponds to a negligible mass of such a body, as it requires practically no capital investments. On the other hand the input and the output of such an economic unit are not bound by any technological process. Similarly to a flexible spring, the ends of which can have arbitrary positions in the space, formally our intermediary can also provide any output productivity irrespective of the input parameters at the expense of the accumulated reserve of products. In fact, such an intermediate link can no longer be described as a point mass by a set of coefficients in the matrix of technologies. Accumulation of resources in this economic object can occur without capital expenditures at the expense of the effect of delay of product transfer from the manufacturer to the consumer. Similarly to the mechanism of elastic collision the potential energy of an idealized spring in determined not by its mass (which is negligible), but by the internal energy (relative motion of the components). Increasing of the spring force corresponds to the decrease of the delay time in the process of transfer of interaction, and to the corresponding decrease of the accumulated potential energy. In this case the intermediate element of the economic system turns into a simple kinematic link.

Detailed analysis of such systems is beyond the scope of the present paper. We are planning to cover the analysis of conditions of occurrence of oscillating motions in enclosed economic systems in one of our further publications. However, even now we can assume that the possibility of storing part of the manufactured products results in differential equations with delay, which have well-studied oscillating solutions in the theory of evolution (Voltairre).

Discussing of mechanical systems usually dose not require the knowledge of mechanisms of occurrence of the force function. It can be set on the basis of additional assumptions, i.e. the Hooke law or the law of universal gravitation. Discussing the dynamics of open economic systems, we will also postulate a certain force depending on the coordinates of the system's point masses, leaving for the future the analysis of mechanisms of its occurrence. We will further demonstrate one of such possibilities by introducing into the discussed island model the possibility of storing surplus products in a warehouse.

9. The Economic Analog of the Elastic Interaction Near the Point of Equilibrium

During the exchange process the prices or products, which determine the surplus value, depend both on the demand and on the supply. Let us analyze the interaction of two productions in the island model. In equilibrium case when the production volumes change proportionally all the manufactured products are completely consumed, the price for the products remains unchanged and no interaction forces according to (7) occur. Let us now assume that due to certain factors the production efficiency of one (or both productions) has increased. It results in an imbalance and lack of demand for a part of the products, which will now start to accumulate at the manufacturer's warehouses. As it tends to accumulate, the supply will exceed the demand more and more and the product price will be eventually decreasing. This corresponds to the occurrence of the restoring (balancing) force $F_{12} = {}^{d}\!/_{dt} c_{12} = -F_{21}$. Let us note that the action force F_{12} turn out to be equal in terms of module to the counteraction force F_{21} due to the fact that the prices are reduced to the nominal unit of product and thus the losses of one of the tribes during the exchange are strictly equal to the profit of the other tribe. Thus, the analog of the ***third Newton law*** is observed. The new distance between the point masses corresponding to the companies now equals

$$x_2 - x_1 = \ln(NC_2) - \ln(NC_1) = \ln\left(\frac{NC_2}{P_2}\right) - \ln\left(\frac{NC_1}{P_1}\right) - \ln\left(\frac{P_1}{P_2}\right) = m_2 - m_1 - \ln\left(\frac{P_1}{P_2}\right) = (x_2 - x_1)_0 + \ln\left(\frac{P_1}{P_2}\right)_0 - \ln\left(\frac{P_1}{P_2}\right) \tag{21}$$

and its measurement (at constant masses)

$$\delta x_{12} = \ln\left(\frac{P_1}{P_2}\right)_0 - \ln\left(\frac{P_1}{P_2}\right) \approx \frac{P_2 - P_{20}}{P_{20}} - \frac{P_1 - P_{10}}{P_{10}} \tag{22}$$

(with small deviations from the equilibrium).

In accordance with the most generalized assumptions the relative price for the products of the two companies is determined by the volumes of demand and supply. In the equilibrium state when the proportions between the productivities of the two companies are retained the demand is equal to the supply and the manufactured products is in complete demand. In case of violation of equilibrium (productivity of the first company has increased) the surplus of its products is created, which can, on the one hand, decrease the price of exchange (and shift the proportion between the productivities), and on the other hand, allows to store the non-demanded products at a warehouse. At the same time the decision on the proportion of surplus products to be exchanged or to be stored can depend both on objective and subjective factors. In the present paper we will discuss for illustration purposes only one of the many possible variants of such a strategy. By analogy with mechanics we can use the case of collision of two balls with a spring between them as an analog of this model. Lack of accumulation of stored products at the warehouse means that the relative price of the products changes immediately in accordance with the new productivities. It means that the interconnection between the companies' productivities is absolutely rigid and the corresponding analog of force tends to infinity. This specific case of immediate price changes until complete demand for the products is above described by the equations (18). If all the surplus products are kept at the warehouse, the exchange is performed in unvaried proportions. In this case the relative price of products also remains unchanged and the force does not occur.

Let us assume that the relative price of the exchanged products varies as long as a part of one of them remains non-demanded. In intermediate cases the nature of interaction is determined by the velocity of price changing, which in our assumption depends on the share of non-demanded products in the amount of exchanged products. Taking into account that this velocity has zero value in the point of equilibrium, with small deviations from it in the linear approximation for a quasi-continuous process we can write down:

$$\frac{d}{dt}c_{12} = -\xi \frac{P_{1pe3}}{k_{12}P_2}, \tag{23}$$

where the coefficient ξ characterizes the measure set by the management – the analog of the ***rigidity of connection***, and P_{1pe3} is the part of the manufactured products reserved at the warehouses of the first company. With account of the ratios set by the matrix of technologies,

$$P_i^{(n)} - P_{i\,pe3}^{(n)} = \sum_j k_{ij} P_j^{(n+1)} \tag{24}$$

and proceeding to the description of the continuous process we obtain the analog of the ***Hooke law***

$$F_{12} = \frac{d}{dt} c_{12} - \eta_{12} \delta x_{12} \tag{25}$$

in which the analog of the ***coefficient of elasticity*** η_{12} takes on the value

$$\eta_{12} = \xi \left(1 + \frac{k_{21} P_{10}^2}{k_{12} P_{20}^2} \right) \tag{26}$$

In the accepted assumption it depends both on the production management determining the coefficientξ, and on the matrix of technologies. Prior to analyzing the analog of potential energy, let us note that the ratio of amount of the stored products to the products exchanged per time unit is proportional to the impulse of force, which follows from the ratio:

$$\delta c_{12} = \int F_{12} dt \approx -\xi \frac{\int P_{1pe3} dt}{k_{12} P_{20}} \tag{27}$$

This impulse of force equals the variation of the impulse of body, which is per se the integral form of the second Newton law. The economic interpretation of this ratio in our model means that the variation of the economic analog of the impulse of object – the surplus value of manufacturing of a unit of products, occurs at invariable matrix of technologies at the expense of the impulse of force proportional to the amount of non-demanded products accumulated at the warehouse.

10. The Work of Force in the Model of Economic System

In classical mechanics the work of force is defined as $dA = Fdx$. With account of the second Newton law we can derive from it:

$$dA = Fdx = \frac{d\,(mv)}{dt} dx = v \cdot d\,(mv) = d\left(\frac{mv^2}{2} \right) = dK \tag{28}$$

δx_1 Actually it means that the work of all forces is equal to the variation of the kinetic energy of the point mass. This ratio is often considered the definition of the kinetic energy - the work, which a body is able to execute until complete stoppage. In order to find out the meaning of the economic analog of the kinetic energy let us note that the "aim" of work execution can be nominally considered the motion of a body against a counteracting force. Then, the larger is the value of motion or this force, the larger is the value of the work performed.

In the economic models one can often encounter a different situation, when a specific work needs to be done in order to "boost" a company – impart to it certain profitability (velocity). It can be provided by external forces – decreasing of product cost price or

increasing of purchasing prices (it results in the increase of the surplus value - body impulse in our interpretation).
Rewriting (28) using economic definitions, with account of (8) we obtain:

$$\frac{d\left(\delta c\right)}{dt}d\left(\ln NC\right)=\frac{NC}{P}\cdot d\left[\frac{d\left(\ln NC\right)}{dt}\right]^{2} \tag{29}$$

Analogously to physics one of the interpretations of this formula allows calculating the work, which a body is able to execute until complete stoppage. In the economic interpretation it means that at the expense of the available kinetic energy connected with the positive profitability (velocity), under the effect of a constant negative force (decrease of surplus value) a company can pass a certain distance before stoppage – to increase its capital corresponding number of times, after that its profitability becomes zero. And vice versa, by performing a work – moving it at a certain distance (increasing the capital several times) under the effect of a positive force (increase of surplus value), a certain velocity (positive profitability) and associated kinetic energy can be imparted to the company.

11. The Analog of Potential Energy in the Models of Economic Systems

The possibility of introducing the potential energy in mechanics is connected with the invariance of action of forces relative to the selection of motion trajectory. This allows writing down the law of mechanical energy conservation for motion in the field of such forces. As the forces in the analyzed economic model have been defined as the velocity of variation of the surplus value, which can depend on the selected management strategy, in the general case the field of such forces is not surely potential. However, in the above-discussed example the condition of potentiality is observed due to the assumption on the fact that the velocity of price variation (analog of the interaction force) depends only in the ratio of productivities. Then

$$\int F_{12}dx_{12}=\int\frac{d\left(\delta c_{12}\right)}{dt}d\left(\frac{P_1}{P_2}\right)=\int f\left(\frac{P_1}{P_2}\right)d\left(\frac{P_1}{P_2}\right)=U\left({}^{P_1}/_{P_2}\right)_2-U\left({}^{P_1}/_{P_2}\right)_1=U(\delta x_2)-U(\delta x_1) \tag{30}$$

Near the point of equilibrium in the linear approximation we obtain the analog of the ***potential energy of the harmonic oscillator:***

$$U\left({}^{P_1}/_{P_2}\right)=\eta_{12}\cdot\left(\frac{P_2}{P_{20}}-\frac{P_1}{P_{10}}\right)^{2} \tag{31}$$

Now we can also formulate the analog of the law of conservation of mechanical energy for the analyzed system, which connects the variations of the relative profitability (velocity) of the interacting companies with the deviation from the position of equilibrium of their relative productivity (distance).

$$\sum_i\left[\frac{NC}{P}\right]_i\cdot\left(\frac{d\left(\ln P_i\right)}{dt}\right)^{2}+\sum_{i;j}\eta_{ij}\cdot\left(\frac{P_i}{P_{i0}}-\frac{P_j}{P_{j0}}\right)^{2}=const \tag{32}$$

In case of absence of external effects the motion of such system of two companies will be of oscillating nature with frequency depending on the coefficient of elasticity and the equivalent mass of the system.

$$\nu_{12} = \sqrt{\frac{\eta_{12}}{m_{12}}}, \text{ where } m_{12} = \frac{m_1 m_2}{m_1 + m_2} \tag{33}$$

In case of increase of number of point masses in the system we will accordingly obtain the spectrum of oscillations. Returning to the general case of dynamics, not limiting ourselves to minor oscillations near the equilibrium position, the trajectories of motion will include both the oscillating and the asymptotic component. The latter corresponds to the inertia motion with constant velocity in the space of generalized coordinates, corresponding to the decomposition of the vector of productivity in terms of own vectors of the matrix of technologies.

12. The Internal Energy in the Models of Economic Systems

In a certain sense the conservation of form of interacting bodies means conservation of distances between their components. In case of the economic system it means that the internal structure of companies remains unchanged. Otherwise, "dissipation" of the kinetic energy occurs. Reverse process is also possible, when as a result of interaction the structure of one or both companies is arranged in order. This arrangement means equalizing of profitability of separate components of the companies (or production components). At the same time their velocities on the scale of price coordinates align, and, accordingly, part of the internal energy becomes kinetic. Thus, in the economic models we can also consider the internal energy (for complex systems) and temperature along with the notion of kinetic energy. Moreover, as the laws of statistical physics do not require any additional assumptions, they can be completely transferred to the economic systems, provided that the laws of conservation for elementary interactions are observed. Nevertheless, an essential difference between the discussion of complex physical and economic systems can lay in the fact that the property of identity of particles (gas molecules, for instance) is not observed here. Therefore the statistical ratios become more complex, also taking account of the distribution of parameters of the system's components.

13. The Variational Formulation of the Laws of Motion Of the Economic System

Analogously to classical mechanics we can formulate the laws of dynamics of the analyzed economic system in the variational form. In case of monogenic external forces and holonomic kinematic links of the system this can always be done in the framework of the Hamilton variational principle. For this purpose we must introduce the Lagrangian function, which in our model takes the form corresponding to the case of small oscillations near the point of equilibrium.

$$L(P;\dot{P}) = \sum_i \left[\frac{NC}{P}\right]_i \cdot \left(\frac{d\,(\ln P_i)}{dt}\right)^2 - \sum_{i;j} \eta_{ij} \cdot \left(\frac{P_i}{P_{i0}} - \frac{P_j}{P_{j0}}\right)^2 \tag{34}$$

Let us note that the Lagrangian function depends on the generalized coordinates and velocities of the system, which are unambiguously connected in time with the vector of productivity and its derivative with the help of (5) and the assumption on the constancy of masses. Actually, the introduction of such function allows writing down the equations of motion (8) both in the local form of Lagrange equations and in the form of variational principle.

$$\delta S = \delta \int_{t_1}^{t_2} L(P;\dot{P})dt = 0 \tag{35}$$

Formally this equation is equivalent to the above-written analog of the Newton law; however it enables us to see the "motive force" regulating the development of the economic system in a different way. As expected, it depends on the accepted management strategy and the matrix of technologies of the system, which determine the specific form of the Lagrangian function. The presence of subjective element (management strategy) in this function allows considering the connection of the analog of the ***principle of least action*** with the variational principles used in the economic theory (principle of minimization of arbitration, maximization of the function of utility, etc.). In our further publications we are planning to discuss in details the connection between the mechanical and economic formulations of these principles.

14. Hamilton Equations for the Model of Economic System

Presentation of dynamic equations using generalized coordinates and impulses has a number of advantages for describing general problems of mechanics. In the proposed economic system such transition is appropriate due to the fact that the impulse and the velocity of the point mass have transparent economic meaning. It is the surplus value of a unit of product $\delta c = c_+ - c_-$ and the company's profitability $\frac{d/dt(NC)}{NC}$, respectively. Accordingly, the Hamilton equations $\dot{x}_i = \frac{dH}{dp_i}; \ddot{p}_i = -\frac{dH}{dx_i}$ determine the system's dynamic at the Hamiltonian set as a function of these variables. By substituting the economic parameters of the model instead of the notations of coordinates and impulse we obtain the system of equations

$$\left[\begin{array}{l} \frac{d(\ln NC_i)}{dt} = \frac{dH}{d(\delta c_i)} \\ \frac{d(\delta c_i)}{dt} = -\frac{dH}{d(\ln NC_i)} \end{array}\right. \tag{36}$$

Using the equation obtained for the complete energy of the system and substituting the economic variables with account of their links with the coordinates and impulses we obtain the expression for the Hamiltonian in explicit form:

$$H(\ln NC_i; \delta c_i) = \sum_i K_i + \sum_{i;j} U_{ij} + \sum_i U_i \tag{37}$$

where $K_i = \sum_i \left[\frac{NC}{P}\right]_i^{-1} \cdot (\delta c_i)^2$; $U_{i;j} = \sum_{i;j} \eta_{ij} \cdot (\ln(NC_i) - \ln(NC_j))^2$; $U_i = U_i(\ln(NC_i))$.

This expression includes three terms. The first term corresponds to the kinetic energy of each of the companies and depends only on their profitability (velocity) and inertia (mass). The second term characterizes the system's internal potential energy of interaction of the companies and depends both on the ratio of their equivalent capitals (analog of distance between them) and on the coefficient of elasticity of links, which is calculated according to the formula (26). The latter is determined both by the matrix of technologies and the strategy of management, which is set in the simplest case by the coefficientξ. The third term characterizes the potential energy of interaction of the companies with the environment. It depends on the parameters of this interaction and the coordinates of the point masses of the system. As the boundary between the open system and the environment is conventional (analogously to physics), we can either add part of the elements of the environment into the system, or vice versa, transfer certain terms from the first two sums to the third one. Observing certain requirements, they can be represented as a potential field for the remaining system elements. Thus, setting the system's Hamiltonian as a function of its coordinates and impulses completely defines the classical dynamics of the economic system and with account if the coupling equation (24) is equivalent to the previously-obtained equations (8, 23).

Conclusion

In the present paper we have shown that the natural-science methodology of constructing economic models allows formulating the laws of their dynamics without involving any phenomenological assumptions. Parallels between the economic parameters of a model and their physical analogs (outlined in the text) covers practically all the mathematical apparatus of the classical mechanics. Nevertheless, they originate not as a result of transfer of physical properties into the economics, but as a natural consequence of the approach common for both theories. For deriving the equations of motion we have used only two assumptions, with no relation to physics. It is the law of surplus value (8) and the mechanism of price formation (23) depending on the accepted management strategy. Let us note that the analyzed model is one of the simplest ones, as the aim of this paper is not to obtain the complete description of their properties adequate to the real economic systems, but to illustrate the possibility of consecutive performance of physical methodology for construction of a rigorous economic model analogous to models of the classical mechanics. We have shown that the logical construction of economic theory (in the natural scientific sense) can also be based on the scheme of "symmetry + variational principle". From them it is possible to obtain the laws of motion and the laws of conservation for the elementary interaction in economic models (transaction of exchange in the island model). The rest of the variety of existing economic manifestations can be described on the basis of statistical approach with account of specific types of symmetry. It can be considered as a result of a complex "multi-particle" interaction of a huge number of economic subjects under the effect of various external factors.

Let us also note the fundamental differences between the physical and economic theories becoming apparent at similar approaches. In physics we don't know whether the symmetry is primary or it is a logical consequence of more fundamental natural-science principles. However, in the sphere of economy these properties follow from logical assumptions on the equality of rights of the market participants, rules of exchange of products, etc. Then we can assume that the principally unavoidable influence of the subjective element on the properties of economic systems requires analysis of the symmetries and these subjective "motive forces" of the economy.

Basically we should note that two possibilities currently exist: either construction of a harmonious fundamental natural-science economic theory, based on fundamental principles and pure mathematics, or quasi-fundamental (phenomenological) local description of economic manifestations. It is currently being realized in the form of various separate economic theories with various degrees of detailing. The second possibility corresponds to pre-Newtonian state of physics. It can not claim to be fundamental, but already has real practical applications and results. The first possibility, on the other hand, does not appear to be practical, as in order to obtain practical results one should pass all the stages of constructing a theory analogously to physics: single-particle problems in simple and complex potentials, two-particle problems, multi-particle problems, medium field problems, quantum-mechanical generalizations, statistical, thermodynamic, hydrodynamic, solid–state analogs, etc. At the same time it is clear that besides apparatus help in the form of the laws of economic evolution (analogous to Lagrange and Schrödinger equations) it is necessary to use the modeling skills. However, such approach, despite its inconvenience and laboriousness, is consistent. Besides, it provides clear understanding of the sphere of applicability of various economic models on the basis of the accepted principles of appropriateness, utility, etc., varying in their intrinsic principles of symmetry. On the grounds of the aforesaid, we assume that the fundamental approach to construction of economic theory based on the first principles is perspective and deserves close attention.

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