

Constructive mathematics with the knowledge predicate \mathcal{K} satisfied by every currently known theorem

Apoloniusz Tyszk

Technical Faculty
Hugo Kołłątaj University
Balicka 116B, 30-149 Kraków, Poland
E-mail: rttyszka@cyf-kr.edu.pl

Abstract

\mathcal{K} denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set \mathcal{K} is time-dependent, publicly available, and contains theorems both from formal and constructive mathematics. Any theorem of any mathematician from past or present forever belongs to \mathcal{K} . Mathematical statements with known constructive proofs exist in \mathcal{K} separately and form the set $\mathcal{K}_c \subseteq \mathcal{K}$. We assume that mathematical sets are atemporal entities. They exist formally in ZFC theory although their properties can be time-dependent (when they depend on \mathcal{K}) or informal. Algorithms always terminate. We explain the distinction between algorithms whose existence is provable in ZFC and constructively defined algorithms which are currently known. By using this distinction, we obtain non-trivially true statements on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that belong to constructive and informal mathematics and refer to the current mathematical knowledge on \mathcal{X} .

Key words and phrases: constructive mathematics, constructively defined algorithms, current mathematical knowledge, informal mathematics, known algorithms, time-dependent notion of truth in constructive mathematics.

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1 Introduction

This article is an extended and changed version of [19].

\mathcal{K} denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set \mathcal{K} is time-dependent, publicly available, and contains theorems both from formal and constructive mathematics. Any theorem of any mathematician from past or present forever belongs to \mathcal{K} . Mathematical statements with known constructive proofs exist in \mathcal{K} separately and form the set $\mathcal{K}_c \subseteq \mathcal{K}$.

We assume that mathematical sets are atemporal entities. They exist formally in *ZFC* theory although their properties can be time-dependent (when they depend on \mathcal{K}) or informal. The true statement " \mathcal{K} is non-empty" is outside \mathcal{K} forever because \mathcal{K} is not a formal set.

Paul Cohen proved in 1963 that the equality $2^{\aleph_0} = \aleph_1$ is independent of *ZFC*, see [3]. Before 1963, the statement "*There is a constructively defined integer $n \geq 1$ such that $2^{\aleph_0} = \aleph_n$* " was outside \mathcal{K} . Since 1963, this statement is outside \mathcal{K} forever. The true statement "*There exists a set $\mathcal{X} \subseteq \{1, \dots, 49\}$ such that $\text{card}(\mathcal{X}) = 6$ and \mathcal{X} never occurred as the winning six numbers in the Polish Lotto lottery*" refers to the current non-mathematical knowledge and is outside \mathcal{K} forever.

Algorithms always terminate. Semi-algorithms may not terminate. There is the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in *ZFC*) and *known algorithms* (i.e. algorithms whose definition is constructive and currently known), see [2], [14, p. 9]. By using this distinction, we obtain non-trivially true statements on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that belong to constructive and informal mathematics and refer to the current mathematical knowledge on \mathcal{X} . For every such statement Φ , Observations 1 and 2 hold.

Observation 1. *The truth of Φ concerning the current mathematical knowledge is implied by a true statement of the form:*

(the conjunction of i conditions of the form $\alpha \in \mathcal{K}$) \wedge
 (the conjunction of j conditions of the form $\beta \notin \mathcal{K}$) \wedge
 (the conjunction of k conditions of the form $\gamma \in \mathcal{K}_c$) \wedge
 (the conjunction of l conditions of the form $\delta \notin \mathcal{K}_c$),

where $i, j, k, l \in \mathbb{N}$ and $\alpha, \beta, \gamma, \delta$ are mathematical statements.

Observation 2. *The proof of Φ uses mathematical theorems. For example, the proof of Statement 6 uses the following implication: if*

$$\mathcal{X} = \{n \in \mathbb{N} : \text{the interval } [-1, n] \text{ contains more than}$$

$$29.5 + \frac{11!}{3n+1} \cdot \sin(n) \text{ primes of the form } k! + 1\}$$

then $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, 501893]$.

Observation 3 is known from the beginning of computability theory and shows that the predicate \mathcal{K} increases intuitive mathematics.

Observation 3. *Church's thesis is based on the fact that the currently known computable functions are recursive, where the notion of a computable function is informal.*

In Observation 4, the predicate \mathcal{K} trivially increases constructive mathematics.

Observation 4. *The largest known prime number has the form $2^n - 1$.*

2 Time-dependent notion of truth in constructive mathematics

Below is the English summary of [18] available at the internet address of [18].

The basic philosophical idea of intuitionism is that mathematical entities exist only as mental constructions and that the notion of truth of a proposition should be equated with its verification or the existence of proof. However different intuitionists explained the existence of a proof in fundamentally different ways. There seem to be two main alternatives: the actual and potential existence of a proof. The second proposal is also understood in two alternative ways: as knowledge of a method of construction of a proof or as knowledge-independent and tenseless existence of a proof. This paper is a presentation and analysis of these alternatives.

In constructive mathematics ([13]) and the traditional Brouwerian intuitionism ([10, p. 135]), the truth of a mathematical statement means that we know a constructive proof. Therefore, the truth of a mathematical statement depends on time, where the statement is formally stated in the classical mathematics without the predicate \mathcal{K} .

In this article, mathematical statements on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ refer to time because they refer to the current mathematical knowledge on \mathcal{X} . They cannot be formally stated in the classical mathematics without the predicate \mathcal{K} and their logical values may change in time.

In Martin-Löf's terminology ([9, p. 142]), every currently known theorem is actually true whereas every theorem (known or unknown) is potentially true. Actual truth is knowledge dependent and tensed. Potential truth is knowledge independent and tenseless.

3 Basic definitions and examples

Definition 1 applies to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven.

Definition 1. We say that a non-negative integer k is a known element of \mathcal{X} , if $k \in \mathcal{X}$ and we know an algebraic expression that defines k and consists of the following signs: 1 (one), + (addition), - (subtraction), \cdot (multiplication), $^$ (exponentiation with exponent in \mathbb{N}), ! (factorial of a non-negative integer), ((left parenthesis),) (right parenthesis).

The set of known elements of \mathcal{X} is finite and time-dependent, so cannot be defined in the formal language of classical mathematics. Let t denote the largest twin prime that is smaller than (((((((9!)!)!)!)!)!)!)!. The number t is an unknown element of the set of twin primes.

Definition 2. Conditions (1)-(5) concern sets $\mathcal{X} \subseteq \mathbb{N}$.

- (1) A known algorithm with no input returns an integer n satisfying $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.
- (2) A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.
- (3) For every known algorithm \mathcal{A} with no input, the statement " \mathcal{A} returns the logical value of the statement $\text{card}(\mathcal{X}) = \omega$ " is outside \mathcal{K} .
- (4) There are many elements of \mathcal{X} and it is conjectured, though so far unproven, that \mathcal{X} is infinite.
- (5) \mathcal{X} is naturally defined. The infiniteness of \mathcal{X} is false or unproven. \mathcal{X} has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.

Condition (3) implies that no known proof shows the finiteness/infiniteness of \mathcal{X} . No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies Conditions (1)–(4) and is widely known in number theory or naturally defined, where this term has only informal meaning.

Let $[\cdot]$ denote the integer part function.

Example 1. *The set*

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } \left[\frac{(((((((9!)!)!)!)!)!)!)!}{\pi} \right] \text{ is odd} \\ \emptyset, & \text{otherwise} \end{cases}$$

does not satisfy Condition (3) because we know an algorithm with no input that computes $\left[\frac{(((((((9!)!)!)!)!)!)!)!}{\pi} \right]$. The set of known elements of \mathcal{X} is empty. Hence, Condition (5) fails for \mathcal{X} .

Example 2. *([2], [14, p. 9]). The function*

$$\mathbb{N} \ni n \xrightarrow{h} \begin{cases} 1, & \text{if the decimal expansion of } \pi \text{ contains } n \text{ consecutive zeros} \\ 0, & \text{otherwise} \end{cases}$$

is computable because $h = \mathbb{N} \times \{1\}$ or there exists $k \in \mathbb{N}$ such that

$$h = (\{0, \dots, k\} \times \{1\}) \cup (\{k+1, k+2, k+3, \dots\} \times \{0\})$$

No known algorithm computes the function h .

Example 3. *The set*

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if the continuum hypothesis holds} \\ \emptyset, & \text{otherwise} \end{cases}$$

is decidable. This \mathcal{X} satisfies Conditions (1) and (3) and does not satisfy Conditions (2), (4), and (5). These facts will hold forever.

4 A consequence of the physical limits of computation

Statement 1. *No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy Conditions (1)–(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less.*

Proof. The proof goes by contradiction. We fix an integer n that satisfies Condition (1). Since Conditions (1)–(3) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n+1 \notin \mathcal{X}, n+2 \notin \mathcal{X}, n+3 \notin \mathcal{X}, \dots \quad (\text{T})$$

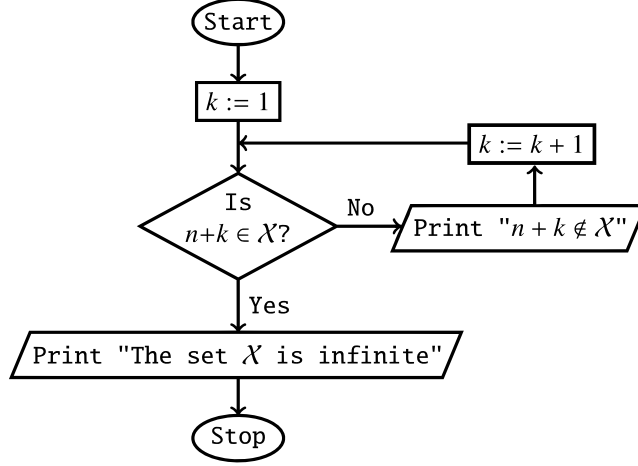


Figure 1 Semi-algorithm that terminates if and only if \mathcal{X} is infinite

The sentences from the sequence (T) and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set \mathcal{X} is finite, contrary to the conjecture in Condition (4). \square

The physical limits of computation ([8]) disprove the assumption of Statement 1.

5 Statements which refer to Conditions (1) - (5)

Edmund Landau's conjecture states that the set \mathcal{P}_{n^2+1} of primes of the form $n^2 + 1$ is infinite, see [16], [17], [22].

Statement 2. Condition (1) remains unproven for $\mathcal{X} = \mathcal{P}_{n^2+1}$.

Proof. For every set $\mathcal{X} \subseteq \mathbb{N}$, there exists an algorithm $\text{Alg}(\mathcal{X})$ with no input that returns

$$n = \begin{cases} 0, & \text{if } \text{card}(\mathcal{X}) \in \{0, \omega\} \\ \max(\mathcal{X}), & \text{otherwise} \end{cases}$$

This n satisfies the implication in Condition (1), but the algorithm $\text{Alg}(\mathcal{P}_{n^2+1})$ is unknown because its definition is ineffective. \square

Statement 3. The statement

$$\exists n \in \mathbb{N} (\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, n+3])$$

remains unproven in ZFC and classical logic without the law of excluded middle.

Statement 4. The set

$$\mathcal{X} = \{k \in \mathbb{N} : \text{card}(\mathcal{P}_{n^2+1} \cap [-1, k]) > 10^{10^{10}}\} \cup \{k \in \mathcal{P}_{n^2+1} : \text{card}(\mathcal{P}_{n^2+1} \cap [-1, k]) \leq 10^{10^{10}}\}$$

satisfies Conditions (2) - (4). Condition (1) fails for \mathcal{X} , $\text{card}(\mathcal{X}) < \omega \Rightarrow \text{card}(\mathcal{X}) \leq 10^{10^{10}}$.

Proof. Since $\text{card}(\mathcal{P}_{n^2+1} \cap [2, 10^{28})) = 2199894223892$ ([17]) and the inequality $\text{card}(\mathcal{P}_{n^2+1}) \geq 10^{10^{10}}$ remains unproven, Conditions (3) and (4) hold. \square

For a non-negative integer n , let $\theta(n)$ denote the largest integer divisor of $10^{10^{10}}$ smaller than n . Let $\kappa : \mathbb{N} \rightarrow \mathbb{N}$ be defined by setting $\kappa(n)$ to be the exponent of 2 in the prime factorization of $n + 1$.

Statement 5. ([20, p. 250]). *The set*

$$\mathcal{X} = \{n \in \mathbb{N} : (\theta(n) + \kappa(n))^2 + 1 \text{ is prime}\}$$

satisfies Conditions (1)-(5) except the requirement that \mathcal{X} is naturally defined. Condition (1) holds with $n = 10^{10^{10}}$.

Let $\mathcal{P}_{n!+1}$ denote the set of primes of the form $n! + 1$.

Conjecture 1. ([1, p. 443], [5]). *The set $\mathcal{P}_{n!+1}$ is infinite.*

For a non-negative integer n , let $\rho(n)$ denote $29.5 + \frac{11!}{3n+1} \cdot \sin(n)$.

Statement 6. *The set*

$$\mathcal{X} = \{n \in \mathbb{N} : \text{the interval } [-1, n] \text{ contains more than } \rho(n) \text{ elements of } \mathcal{P}_{n!+1}\}$$

satisfies Conditions (1)-(5) except the requirement that \mathcal{X} is naturally defined. $501893 \in \mathcal{X}$. Condition (1) holds with $n = 501893$. $\text{card}(\mathcal{X} \cap [0, 501893]) = 159827$. $\mathcal{X} \cap [501894, \infty) = \{n \in \mathbb{N} : \text{the interval } [-1, n] \text{ contains at least 30 elements of } \mathcal{P}_{n!+1}\}$.

Proof. For every integer $n \geq 11!$, 30 is the smallest integer greater than $\rho(n)$. By this, if $n \in \mathcal{X} \cap [11!, \infty)$, then $n + 1, n + 2, n + 3, \dots \in \mathcal{X}$. Hence, Condition (1) holds with $n = 11! - 1$. Since the inequality $\text{card}(\mathcal{P}_{n!+1}) \geq 30$ remains unproven, Condition (3) holds. The interval $[-1, 11! - 1]$ contains exactly three primes of the form $k! + 1$: $1! + 1, 2! + 1, 3! + 1$. For every integer $n > 503000$, the inequality $\rho(n) > 3$ holds. Therefore, the execution of the following *MuPAD* code

```
m:=0:
for n from 0.0 to 503000.0 do
if n<1!+1 then r:=0 end_if:
if n>=1!+1 and n<2!+1 then r:=1 end_if:
if n>=2!+1 and n<3!+1 then r:=2 end_if:
if n>=3!+1 then r:=3 end_if:
if r>29.5+(11!/(3*n+1))*sin(n) then
m:=m+1:
print([n,m]):
end_if:
end_for:
```

displays the all known elements of \mathcal{X} . The output ends with the line $[501893.0, 159827]$, which proves Condition (1) with $n = 501893$ and Condition (4) with $\text{card}(\mathcal{X}) \geq 159827$. \square

T. Nagell proved in [11] (cf. [15, p. 104]) that the equation $x^2 - 17 = y^3$ has exactly 16 integer solutions, namely $(\pm 3, -2), (\pm 4, -1), (\pm 5, 2), (\pm 9, 4), (\pm 23, 8), (\pm 282, 43), (\pm 375, 52), (\pm 378661, 5234)$. The set

$$\bigcup_{\substack{(x,y) \in \mathbb{Z} \times \mathbb{Z} \\ (x^2 - y^3 - 17) \cdot (y^2 - x^3 - 17) = 0}} \{(x+8)^8\}$$

has exactly 23 elements. Among them, there are 14 integers from the interval $[1, 2199894223892]$. Let \mathcal{W} denote the set

$$\bigcup_{\substack{(x,y) \in \mathbb{Z} \times \mathbb{Z} \\ (x^2 - y^3 - 17) \cdot (y^2 - x^3 - 17) = 0}} \{k \in \mathbb{N} : k \text{ is the } (x+8)^8 - \text{th element of } \mathcal{P}_{n^2+1}\}$$

From [17], it is known that $\text{card}(\mathcal{P}_{n^2+1} \cap [2, 10^{28})) = 2199894223892$. Hence, $\text{card}(\mathcal{W} \cap [2, 10^{28})) = 14$ and 14 elements of \mathcal{W} can be practically computed. The inequality $\text{card}(\mathcal{P}(n^2+1)) \geq (378661+8)^8$ remains unproven. The last two sentences and Statement 6 imply the following corollary.

Corollary 1. *If we add \mathcal{W} to \mathcal{X} , then the following statements hold:*

Condition (1) fails for \mathcal{X} ,

$$159827 + 14 \leq \text{card}(\mathcal{X}),$$

the above lower bound is currently the best known,

$$\text{card}(\mathcal{X}) < \omega \Rightarrow \text{card}(\mathcal{X}) \leq 159827 + 23,$$

the above upper bound is currently the best known,

\mathcal{X} satisfies Conditions (2) - (5) except the requirement that \mathcal{X} is naturally defined.

Corollary 2. *Since the inequality $\text{card}(\mathcal{P}_{n^2+1}) > 9^{999}$ remains unproven and $10^{953} < 9^{999} < 10^{954}$, analogical statements hold when we add to \mathcal{X} the set*

$$\bigcup_{i \in \mathbb{N}} \left\{ k \in \mathbb{N} : k - 501894 \text{ is the } \left(\left\lceil \frac{9^{999}}{10^i} \right\rceil + 1 \right) - \text{th element of } \mathcal{P}_{n^2+1} \right\}$$

which has at most 955 elements.

For a non-negative integer i , let $d(i)$ denote the smallest prime divisor of $\left\lceil 31 + \frac{10^6}{i+1} \right\rceil$.

Statement 7. *The set*

$$\mathcal{X} = \bigcup_{i \in \mathbb{N}} \left\{ k^i : k \text{ is the } d(i) - \text{th element of } \mathcal{P}_{n^2+1} \right\}$$

satisfies Conditions (2) - (5) except the requirement that \mathcal{X} is naturally defined. Condition (1) fails for \mathcal{X} . $\text{card}(\mathcal{X}) \geq 946732$. If $\text{card}(\mathcal{P}_{n^2+1}) \leq 28$, then $\text{card}(\mathcal{X}) = 946732$. If $29 \leq \text{card}(\mathcal{P}_{n^2+1}) \leq 30$, then $\text{card}(\mathcal{X}) = 946745$. If $\text{card}(\mathcal{P}_{n^2+1}) \geq 31$, then $\text{card}(\mathcal{X}) = \omega$.

Proof. The inequality $\text{card}(\mathcal{P}_{n!+1}) \geq 23$ holds, see [4]. The inequality $\text{card}(\mathcal{P}_{n!+1}) \geq 29$ remains unproven. The execution of the following *MuPAD* code

```
[m,n]:=[0,0]:
for i from 0 to 10^6-1 do
A:=numlib::primedivisors(floor(31+(10^6/(i+1)))):
if A[1]<=23 then m:=m+1 end_if:
if A[1]<=29 then n:=n+1 end_if:
end_for:
print([m,n]):
```

displays [946732, 946745]. The last claim of Statement 7 holds because $d\left(\left[31 + \frac{10^6}{i+1}\right]\right) = 31$ for every integer $i \geq 10^6$. Condition (1) fails for \mathcal{X} because we cannot rule out the possibility that $29 \leq \text{card}(\mathcal{P}_{n!+1}) \leq 30$. \square

6 Satisfiable conjunctions which consist of Conditions (1)–(5) and their negations

Open Problem 1. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ which satisfies Conditions (1)–(5) ?*

Open Problem 1 asks about the existence of a year $t \geq 2024$ in which the conjunction

$$(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge (\text{Condition 3}) \wedge (\text{Condition 4}) \wedge (\text{Condition 5})$$

will hold for some $\mathcal{X} \subseteq \mathbb{N}$. For every year $t \geq 2024$ and for every $i \in \{1, 2, 3\}$, a positive solution to Open Problem i in the year t may change in the future. Currently, the answers to Open Problems 1–5 are negative.

The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge (\text{Condition 3}) \wedge (\text{Condition 4}) \wedge (\text{Condition 5})$$

The set $\mathcal{X} = \{0, \dots, 10^6\} \cup \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge (\text{Condition 3}) \wedge (\text{Condition 4}) \wedge \neg(\text{Condition 5})$$

Let $f(1) = 10^6$, and let $f(n+1) = f(n)^{f(n)}$ for every positive integer n . The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } 2^{2^{f(9^9)}} + 1 \text{ is composite} \\ \{0, \dots, 10^6\}, & \text{otherwise} \end{cases}$$

satisfies the conjunction

$$(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge \neg(\text{Condition 3}) \wedge (\text{Condition 4}) \wedge \neg(\text{Condition 5})$$

Open Problem 2. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge \neg(\text{Condition 3}) \wedge (\text{Condition 4}) \wedge (\text{Condition 5})?$$

The numbers $2^{2^k} + 1$ are prime for $k \in \{0, 1, 2, 3, 4\}$. It is open whether or not there are infinitely many primes of the form $2^{2^k} + 1$, see [7, p. 158] and [12, p. 74]. It is open whether or not there are infinitely many composite numbers of the form $2^{2^k} + 1$, see [7, p. 159] and [12, p. 74]. Most mathematicians believe that $2^{2^k} + 1$ is composite for every integer $k \geq 5$, see [6, p. 23]. The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } 2^{2^{f(9^9)}} + 1 \text{ is composite} \\ \{0, \dots, 10^6\} \cup \\ \{n \in \mathbb{N} : n \text{ is the sixth prime number of the form } 2^{2^k} + 1\}, & \text{otherwise} \end{cases}$$

satisfies the conjunction

$$\neg(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge \neg(\text{Condition 3}) \wedge (\text{Condition 4}) \wedge \neg(\text{Condition 5})$$

Open Problem 3. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$\neg(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge \neg(\text{Condition 3}) \wedge (\text{Condition 4}) \wedge (\text{Condition 5})?$$

It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \mathcal{P}_{n^2+1}$ will solve Open Problem 2. The same is true for Open Problem 3. It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \{k \in \mathbb{N} : 2^{2^k} + 1 \text{ is composite}\}$ will solve Open Problem 1. The same is true for Open Problems 2 and 3.

Table 1 shows satisfiable conjunctions of the form

$$\#(\text{Condition 1}) \wedge (\text{Condition 2}) \wedge \#(\text{Condition 3}) \wedge (\text{Condition 4}) \wedge \#(\text{Condition 5})$$

where $\#$ denotes the negation \neg or the absence of any symbol.

	$(\text{Cond. 2}) \wedge (\text{Cond. 3}) \wedge (\text{Cond. 4})$	$(\text{Cond. 2}) \wedge \neg(\text{Cond. 3}) \wedge (\text{Cond. 4})$
$(\text{Cond. 1}) \wedge (\text{Cond. 5})$	Open Problem 1	Open Problem 2
$(\text{Cond. 1}) \wedge \neg(\text{Cond. 5})$	$\mathcal{X} = \{n \in \mathbb{N} : \text{the interval } [-1, n] \text{ contains more than } 29.5 + \frac{11!}{3n+1} \cdot \sin(n) \text{ primes of the form } k! + 1\}$	$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } 2^{2^{f(9^9)}} + 1 \text{ is composite} \\ \{0, \dots, 10^6\}, & \text{otherwise} \end{cases}$
$\neg(\text{Cond. 1}) \wedge (\text{Cond. 5})$	$\mathcal{X} = \mathcal{P}_{n^2+1}$	Open Problem 3
$\neg(\text{Cond. 1}) \wedge \neg(\text{Cond. 5})$	$\mathcal{X} = \{0, \dots, 10^6\} \cup \mathcal{P}_{n^2+1}$	$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } 2^{2^{f(9^9)}} + 1 \text{ is composite} \\ \{0, \dots, 10^6\} \cup \{n \in \mathbb{N} : n \text{ is the sixth prime number of the form } 2^{2^k} + 1\}, & \text{otherwise} \end{cases}$

Table 1 Five satisfiable conjunctions

7 Statements which refer to Conditions (1a)-(5a) and (6)-(11)

Definition 3. Conditions (1a)-(5a) concern sets $\mathcal{X} \subseteq \mathbb{N}$.

(1a) A known algorithm with no input returns a positive integer n satisfying $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

(2a) A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.

(3a) For every known algorithm \mathcal{A} with no input, the statement " \mathcal{A} returns the logical value of the statement $\text{card}(\mathcal{X}) < \omega$ " is outside \mathcal{K} .

(4a) There are many elements of \mathcal{X} and it is conjectured, though so far unproven, that \mathcal{X} is finite.

(5a) \mathcal{X} is naturally defined. The finiteness of \mathcal{X} is false or unproven. \mathcal{X} has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.

Statement 8. The set

$$\mathcal{X} = \left\{ n \in \mathbb{N} : \text{the interval } [-1, n] \text{ contains more than } 6.5 + \frac{10^6}{3n+1} \cdot \sin(n) \text{ squares of the form } k! + 1 \right\}$$

satisfies Conditions (1a)-(5a) except the requirement that \mathcal{X} is naturally defined. $95151 \in \mathcal{X}$. Condition (1a) holds with $n = 95151$. $\text{card}(\mathcal{X} \cap [0, 95151]) = 30311$. $\mathcal{X} \cap [95152, \infty) = \{n \in \mathbb{N} : \text{the interval } [-1, n] \text{ contains at least 7 squares of the form } k! + 1\}$.

Proof. For every integer $n > 10^6$, 7 is the smallest integer greater than $6.5 + \frac{10^6}{3n+1} \cdot \sin(n)$. By this, if $n \in \mathcal{X} \cap (10^6, \infty)$, then $n+1, n+2, n+3, \dots \in \mathcal{X}$. Hence, Condition (1a) holds with $n = 10^6$. It is conjectured that $k! + 1$ is a square only for $k \in \{4, 5, 7\}$, see [21, p. 297]. Hence, the inequality $\text{card}(\{k \in \mathbb{N} \setminus \{0\} : k! + 1 \text{ is a square}\}) > 3$ remains unproven. Since $3 < 7$, Condition (3a) holds. The interval $[-1, 10^6]$ contains exactly three squares of the form $k! + 1$: $4! + 1, 5! + 1, 7! + 1$. Therefore, the execution of the following MuPAD code

```
m:=0:
for n from 0.0 to 1000000.0 do
if n<25 then r:=0 end_if:
if n>=25 and n<121 then r:=1 end_if:
if n>=121 and n<5041 then r:=2 end_if:
if n>=5041 then r:=3 end_if:
if r>6.5+(1000000/(3*n+1))*sin(n) then
m:=m+1:
print([n,m]):
end_if:
end_for:
```

displays the all known elements of \mathcal{X} . The output ends with the line [95151.0, 30311], which proves Condition (1a) with $n = 95151$ and Condition (4a) with $\text{card}(\mathcal{X}) \geq 30311$. \square

Statement 9. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : \text{card}([-1, k] \cap \mathcal{P}_{n^2+1}) < 10^{10000}\}$$

satisfies the conjunction

$$\neg(\text{Condition 1a}) \wedge (\text{Condition 2a}) \wedge (\text{Condition 3a}) \wedge (\text{Condition 4a}) \wedge (\text{Condition 5a})$$

Statement 10. *There exists a naturally defined set $\mathcal{C} \subseteq \mathbb{N}$ which satisfies the following Conditions (6)-(11).*

- (6) *A known and simple algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{C}$.*
- (7) *For every known algorithm \mathcal{A} with no input, the statement " \mathcal{A} returns the logical value of the statement $\text{card}(\mathcal{C}) = \omega$ " is outside \mathcal{K} .*
- (8) *For every known algorithm \mathcal{A} with no input, the statement " \mathcal{A} returns the logical value of the statement $\text{card}(\mathbb{N} \setminus \mathcal{C}) = \omega$ " is outside \mathcal{K} .*
- (9) *It is conjectured, though so far unproven, that \mathcal{C} is infinite.*
- (10) *For every known algorithm \mathcal{A} with no input, the statement " \mathcal{A} returns an integer n satisfying $\text{card}(\mathcal{C}) < \omega \Rightarrow \mathcal{C} \subseteq (-\infty, n]$ " is outside \mathcal{K} .*
- (11) *For every known algorithm \mathcal{A} with no input, the statement " \mathcal{A} returns an integer m satisfying $\text{card}(\mathbb{N} \setminus \mathcal{C}) < \omega \Rightarrow \mathbb{N} \setminus \mathcal{C} \subseteq (-\infty, m]$ " is outside \mathcal{K} .*

Proof. Conditions (6)-(11) hold for $\mathcal{C} = \{k \in \mathbb{N} : 2^{2^k} + 1 \text{ is composite}\}$. It follows from the following three observations. It is an open problem whether or not there are infinitely many composite numbers of the form $2^{2^k} + 1$, see [7, p. 159] and [12, p. 74]. It is an open problem whether or not there are infinitely many prime numbers of the form $2^{2^k} + 1$, see [7, p. 158] and [12, p. 74]. Most mathematicians believe that $2^{2^k} + 1$ is composite for every integer $k \geq 5$, see [6, p. 23]. \square

8 Subsets of \mathbb{N} and their threshold numbers

Definition 4. *We say that an integer n is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.*

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of \mathcal{X} . If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of \mathcal{X} form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

Definition 5. *We say that a non-negative integer n is a weak threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if $\text{card}(\mathcal{X}) < \omega \Rightarrow \text{card}(\mathcal{X}) \leq n$.*

If a set $\mathcal{X} \subseteq \mathbb{N}$ is infinite, then any non-negative integer n is a weak threshold number of \mathcal{X} . If a set $\mathcal{X} \subseteq \mathbb{N}$ is finite, then the all weak threshold numbers of \mathcal{X} form the set $[\text{card}(\mathcal{X}), \infty) \cap \mathbb{N}$.

Let $\mathcal{X} = \{k \in \mathbb{N} : \text{any proof in ZFC of length } k \text{ or less does not show that } 0 = 1\}$.

Lemma 1. *If $n \in \mathbb{N}$ and $\text{card}(\mathcal{X}) \leq n$, then $\mathcal{X} \subseteq (-\infty, n - 1]$.*

Theorem 1. *For every explicitly given $n \in \mathbb{Z}$, if ZFC proves that n is a threshold number of \mathcal{X} , then ZFC is inconsistent. For every explicitly given $n \in \mathbb{N}$, if ZFC proves that n is a weak threshold number of \mathcal{X} , then ZFC is inconsistent.*

Proof. It follows from Lemma 1 and the second Gödel incompleteness theorem. □

Open Problem 4. *Is there a known (weak) threshold number of \mathcal{P}_{n^2+1} ?*

Open Problem 5. *Is there a known (weak) threshold number of $\mathcal{P}_{n!+1}$?*

References

- [1] C. K. Caldwell and Y. Gallot, *On the primality of $n! \pm 1$ and $2 \times 3 \times 5 \times \cdots \times p \pm 1$* , Math. Comp. 71 (2002), no. 237, 441–448, <http://doi.org/10.1090/S0025-5718-01-01315-1>.
- [2] J. Case and M. Ralston, *Beyond Rogers' non-constructively computable function*, in: *The nature of computation*, Lecture Notes in Comput. Sci., 7921, 45–54, Springer, Heidelberg, 2013, http://link.springer.com/chapter/10.1007/978-3-642-39053-1_6.
- [3] *Continuum hypothesis*, http://en.wikipedia.org/wiki/Continuum_hypothesis.
- [4] *Factorial prime*, http://en.wikipedia.org/wiki/Factorial_prime.
- [5] *Is $n! + 1$ often a prime?*, <http://math.stackexchange.com/questions/853085/is-n-1-often-a-prime>.
- [6] J.-M. De Koninck and F. Luca, *Analytic number theory: Exploring the anatomy of integers*, American Mathematical Society, Providence, RI, 2012.
- [7] M. Křížek, F. Luca, L. Somer, *17 lectures on Fermat numbers: from number theory to geometry*, Springer, New York, 2001, <http://doi.org/10.1007/978-0-387-21850-2>.
- [8] S. Lloyd, *Ultimate physical limits to computation*, Nature 406 (2000), 1047–1054, <http://doi.org/10.1038/35023282>.
- [9] P. Martin-Löf, *A path from logic to metaphysics*, in: G. Corsi and G. Sambin (eds.), *Atti del Congresso "Nuovi problemi della logica e della filosofia della scienza"*, Viareggio, 8–13 gennaio 1990, vol. II, CLUEB, Bologna, 1991, 141–149, <http://constable.blog/wp-content/uploads/A-path-from-logic-to-metaphysics-1991.pdf>.

- [10] E. Martino, *Intuitionistic proof versus classical truth: The role of Brouwer's creative subject in intuitionistic mathematics*, Springer, 2018, <http://link.springer.com/book/10.1007/978-3-319-74357-8>.
- [11] T. Nagell, *Einige Gleichungen von der Form $ay^2 + by + c = dx^3$* , Norske Vid. Akad. Skrifter, Oslo I (1930), no. 7.
- [12] P. Ribenboim, *The little book of bigger primes*, 2nd ed., Springer, New York, 2004.
- [13] F. Richman, *Interview with a constructive mathematician*, Modern Logic 6 (1996), no. 3, 247–271, <http://projecteuclid.org/journals/modern-logic/volume-6/issue-3/Interview-with-a-constructive-mathematician/rml/1204835729.full>.
- [14] H. Rogers, Jr., *Theory of recursive functions and effective computability*, 2nd ed., MIT Press, Cambridge, MA, 1987.
- [15] W. Sierpiński, *Elementary theory of numbers*, 2nd ed. (ed. A. Schinzel), PWN – Polish Scientific Publishers and North-Holland, Warsaw-Amsterdam, 1987.
- [16] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, A002496, Primes of the form $n^2 + 1$, <http://oeis.org/A002496>.
- [17] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, A083844, Number of primes of the form $x^2 + 1 < 10^n$, <http://oeis.org/A083844>.
- [18] Z. Tworak, *O pojęciu prawdy w intuicjonizmie matematycznym (in Polish) (On the notion of truth in mathematical intuitionism)*, Filozofia Nauki (Philosophy of Science) 18 (2010), no. 4, 49–76, <http://www.ceeol.com/search/article-detail?id=269240>.
- [19] A. Tyszka, *In constructive and informal mathematics, in contradistinction to any empirical science, the predicate of the current knowledge in the subject is necessary*, Asian Research Journal of Mathematics 19 (2023), no. 12, 69–79, <http://doi.org/10.9734/arjom/2023/v19i12773>.
- [20] A. Tyszka, *Statements and open problems on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that contain informal notions and refer to the current knowledge on \mathcal{X}* , Creat. Math. Inform. 32 (2023), no. 2, 247–253, http://semnul.com/creative-mathematics/wp-content/uploads/2023/07/creative_2023_32_2_247_253.pdf.
- [21] E. W. Weisstein, *CRC Concise Encyclopedia of Mathematics*, 2nd ed., Chapman & Hall/CRC, Boca Raton, FL, 2002.
- [22] Wolfram MathWorld, *Landau's Problems*, <http://mathworld.wolfram.com/LandausProblems.html>.