Cardinality without enumeration

Athanassios Tzouvaras

Dept. of Mathematics, Univ. of Thessaloniki,
541 24 Thessaloniki, Greece
e-mail:tzouvara@math.auth.gr

Abstract

We show that the notion of cardinality of a set is independent from that of wellordering, and that reasonable total notions of cardinality exist in every model of ZF where the axiom of choice fails. Such notions are either definable in a simple and natural way, or non-definable, produced by forcing. Analogous cardinality notions exist in nonstandard models of arithmetic admitting nontrivial automorphisms. Certain motivating phenomena from quantum mechanics are also discussed in the Appendix.

Keywords: Cardinality notion, choice, wellordering, generic class, separation class, nonstandard model of arithmetic, quantum mechanics.

1 Introduction

In current set theory, in order to assign a *size* to a set, i.e. a cardinal number, we must first be able to *enumerate* its elements along a well-ordered sequence, i.e., assign to it an ordinal number. This is because in ZF sizes are sets (initial ordinals) which carry an inherent well-ordering. Every beginner in set theory knows how tightly the theory of cardinal numbers is interwoven with that of ordinals.

However the two concepts, of size and enumeration[1], seem to be independent – at least one-way: If a set X is enumerable by the elements of a set κ,

[1]By enumeration throughout we mean, basically, wellordering, although in some cases, as in nonstandard models of arithmetic, wellordering is only relative to the structure.

which is used as a measure of size, then indeed X gets a size through the act of enumeration, namely that of κ. But we can imagine situations where one, using e.g. an oracle, or just a device, can *consistently* assure us that the size of X is κ, although one is in principle unable to construct an enumeration of its elements. Some striking examples, concerning finite sets, come from quantum mechanics and are discussed in the Appendix.

Here we shall be concerned only with the formal/mathematical aspects of the question: Is it consistent to assume that in a universe of sets (satisfying the usual axioms except choice) there is a reasonable total and consistent notion of cardinality, and yet some sets have no enumeration, i.e., no well-ordering? Note that the second part of the above statement, i.e. the existence of sets without well-ordering is automatically satisfied if the universe in question is a model of ZF+$\neg$AC. In such universes certain sets admit wellorderings and satisfy choice, while others do not. We shall call the former *well-orderable.*

2 The well-orderable sets

Let $M \models$ ZF. A set $x \in M$ is said to be *well-orderable* in M if there is an ordinal α and a bijection $f : x \to \alpha$. If x is well-orderable, clearly there is a cardinal κ of M and a bijection $f : x \to \kappa$. Let WS^M denote the (definable) class of well-orderable sets of M. For simplicity we often drop the superscript from WS^M, as well as from other classes relativized to M, like Ord^M, etc.

Observe that WS is an ideal of M, i.e., WS contains all finite sets, and is closed under finite unions and subsets. Moreover, if HWS denotes the class of *hereditarily well-orderable* sets, then $Ord \subseteq HWS$ and HWS is an inner model of a good part of ZF, namely the following holds:

Proposition 2.1 *The class HWS is a model of: Extensionality, empty set, pair, infinity, regularity, separation, replacement and choice. It fails to satisfy union and power set.*

Proof. Extensionality, empty set, pair, infinity, regularity and separation are obviously true in HWS. To verify replacement, let $x \in HWS$ and let $F''x = y$, where $F : HWS \to HWS$ is a definable class-function. By assumption there is a $\alpha \in Ord$ and a bijection $f : x \to \alpha$. Define $g : y \to \alpha$

as follows:

$$g(u) = \min\{\beta < \alpha : \exists z \in x(F(z) = u \ \&\ f(z) = \beta)\}.$$

Then clearly g is 1-1. Since $g''y \subseteq \alpha$, $g''y \in HWS$, and hence $y \in HWS$.

Choice: Let $x \in HWS$, and let $\preceq$ be a wellordering of x in M. Clearly $x \times x \in HWS$ and $\preceq \subseteq x \times x$. Therefore $\preceq \in HWS$, hence x has a wellordering in HWS.

Concerning union, it is well-known that there is a model in which the set of reals is the union of a countable set S whose members are countable sets (see e.g. [7], Theorem 10.6). Then $S \in HWS$ and yet $\cup S \notin HWS$.

That power set fails in general follows from the fact that $\omega \in HWS$ but $P(\omega)$, i.e., the reals, in general is not in HSW. $\dashv$

3 Notions of cardinality

We asked earlier if there is a "reasonable total and consistent notion of cardinality". We have to specify what this notion amounts to. Let $M \models$ ZF. Intuitively a notion of cardinality for M will be a mapping C which assigns to every set $x \in M$ a real cardinal number in the sense of M, i.e., $C(x) \in Card^M$, and which satisfies certain obvious compatibility conditions. Namely, if x, y are disjoint then we must have $C(x \cup y) = C(x) + C(y)$, for all x, y $C(x \times y) = C(x) \cdot C(y)$, and the like. The list of reasonable compatibility requirements for C is contained in the following

Definition 3.1 Let M be a model of ZF. A *notion of cardinality* for M is a mapping $C \subset M$ such that:

(1) $dom(C) = M$ and $rng(C) = Card$.

(2) $C(\kappa) = \kappa$, for every $\kappa \in Card$.

(3) For any disjoint sets x, y $C(x \cup y) = C(x) + C(y)$.

(4) For any x, y $C(x \times y) = C(x) \cdot C(y)$.

(5) If $f : x \to y$ is an injective mapping, then $C(x) \leq C(y)$.

A cardinality notion C is said to be *standard* if in addition the converse of (5) holds, i.e., if $C(x) \leq C(y)$ implies that there is an injective f from x into y.

Remark 3.2 (i) It follows from (5) that if f is 1-1 and $x \subseteq dom(f)$, then $C(x) \leq C(f''x)$ and, using f^{-1}, $C(f''x) \leq C(x)$, therefore $C(x) = C(f''x)$. Moreover if $x \subseteq y$, then by (5), for $f = id$, $C(x) \leq C(y)$, that is C is monotonic.

(ii) If C is standard, then $C(x) = C(y)$ implies that there is an injective f from x onto y. So the existence of a standard notion of cardinality implies AC. Moreover in the presence of AC there is a unique notion of cardinality, the standard one. Thus in order to depart from the standard cardinality, we must drop AC.

(iii) If $x \in WS$, then there is a cardinal κ and a bijection $f : x \to \kappa$, hence, by (2) and (5) above, $C(x) = |x| = \kappa$. Therefore the cardinality notion C agrees with the standard cardinality on the sets where the latter is defined.

(iv) If $x \notin WS$ then $C(x) \geq \omega$, since every finite set is by definition well-orderable.

(v) One might want to strengthen clauses (3) and (4) so as to capture infinitary sums and products of sets. However in the absence of AC infinitary operations become problematic. For example even if each x_i is nonempty, we cannot infer that $\Pi_i x_i$ is nonempty without the AC. The same applies to exponentiation. The set x^y cannot be treated without AC.

Recall that in absence of choice, finite sets have at least two non-equivalent definitions.

Definition 3.3 x is said to be *finite* if it is empty or there is an $n \in \omega$, $n \geq 1$ and a bijection $f : x \to n$. x is said to be *D-finite* if there is no injection $g : x \to x$ such that $rng(x) \neq x$.

We shall be interested in the first notion of finiteness. Let Fin be the class of finite sets. Then $Fin \subseteq WS$. Moreover

Proposition 3.4 *Let C be any notion of cardinality. Then*
$x \in Fin$ iff $C(x) = n$ for some $n \in \omega$.

Proof. Since $Fin \subseteq WS$, one direction is obvious. Conversely let x be infinite. It suffices to show that for every $n \in \omega$ there is a $u \subseteq x$, $u \in WS$ such that $|u| = n$. By induction on n. For $n = 0$ this is obvious. Suppose there is $u \subseteq x$, $u \in WS$, such that $|u| = n$. If $u = x$, then x would be finite.

Therefore there is $a \in x - u$, $u \cup \{a\} \subseteq x$, $u \cup \{a\}$ belongs to WS and has cardinality $n+1$. $\dashv$

A notion of cardinality C is a *proper class* with respect to M, i.e., $C \subset M$. Yet we are not going to shift to a theory of classes (like GB or KM) in order to accommodate C. We shall keep working in ZF. Usually classes with respect to ZF are identified with predicates $\varphi(x)$ of the language L, i.e., with definable subsets of the models M of ZF. However here we refer to arbitrary subsets of M as classes over M, and to definable subsets of M as definable classes.

As noted in remark 3.2 (iii), for every set $x \in WS$ there is a cardinal κ and a bijection $f : x \to \kappa$. We write then $|x| = \kappa$, and by the definition 3.1, $C(x) = |x|$. Let now $x \notin WS$. For every well-orderable $y \subseteq x$ we have by monotonicity $C(y) \leq C(x)$, or $|y| \leq C(x)$. Therefore for every notion of cardinality C,

$$C(x) \geq \sup\{|y| : y \in WS \ \& \ y \subseteq x\}. \tag{1}$$

So a natural candidate for a notion of cardinality in a model M of ZF is the following:

$$C_0(x) = \sup\{|y| : y \in WS \ \& \ y \subseteq x\}. \tag{2}$$

Remark 3.5 The referee remarked that there have been in the past two other attempts to assign cardinality to non-well-orderable sets. One by using Hartog numbers, $H(x) = \{\alpha \in On : \exists f \ f : \alpha \xrightarrow{1-1} x\}$, and another by A. Lindenbaum and A. Tarski, in [12], by a definition equivalent to $LT(x) = \{\alpha \in On : \exists g \ g : x \xrightarrow{\text{onto}} \alpha\}$. For all x, $C_0(x) \leq H(x) \leq LT(x)$. In particular $H(\aleph_\alpha) = LT(\aleph_\alpha) = \aleph_{\alpha+1}$, while $C_0(\aleph_\alpha) = \aleph_\alpha$. However in some symmetric models (like Feferman-Lévy model, [7], p. 259, where the reals is the countable union of countable sets), $C_0(2^{\aleph_0}) = C_0(\aleph_0) = \aleph_0$, while $H(2^{\aleph}_0) = LT(2^{\aleph}_0) = \aleph_1$. The referee remarked also that it would be more in Cantorian spirit to have a cardinality notion C such that $C(\aleph_0) < C(2^{\aleph_0})$ *in every model of* ZF, and asked if this possible. I do not know the answer, and I just cite it here as an open question.

Despite its naturality, C_0 has a drawback: It may happen that $C_0(x) = C_0(x')$ and yet x, x' do not contain well-orderable sets of same cardinalities. E.g. x can be infinite but contain only *finite* well-orderable subsets, whence $C_0(x) = \omega$, while x' can contain a $y \subseteq x'$, $y \in WS$ such that $|y| = \omega$ and for every other $z \subseteq x'$, $z \in WS$, $|z| \leq \omega$. In some sense, x has "potential" cardinality ω while x' has "actual" cardinality ω. The examples of non-well-orderable sets yielded e.g. in the permutation models of [7] are of the first category. They are infinite sets a for which there is no injective mapping $f : \omega \to a$. On the other hand, if a is of this kind and $y \in WS$ such that $|y| = \omega$ and $y \cap a = \emptyset$, and we set $x = a \cup y$, then clearly $x \notin WS$, $C_0(x) = \omega$ but x is of the second category. To distinguish between them we shall call them weak and strong respectively.

Definition 3.6 A set x is said to be *strong* if there is a $y \subseteq x$ such that $y \in WS$ and $C_0(x) = |y|$. Otherwise x is said to be *weak.*

Note that for every well-orderable set x, $|x|$ is an aleph, and for alephs κ, λ, we have $\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$ without the use of the AC (see [16], p. 51).

Theorem 3.7 *Let* $M \models$ ZF. *Then the mapping* C_0 *defined above is a definable notion of cardinality in* M. *If moreover* $M \models$ AC, *then* C_0 *is the standard notion of cardinality, i.e.,* $C_0(x) = |x|$.

Proof. That C_0 is definable is obvious. Also $C_0(x) = |x|$ if $x \in WS$. If $M \models$ AC, then $M = WS^M$, therefore C_0 is the standard cardinality. Let us now verify that C_0 satisfies conditions (1)-(5) of definition 3.1. (1) is obvious. For (2) just note that every cardinal κ is in WS, hence $C_0(\kappa) = |\kappa| = \kappa$.

(3): Let x, y be disjoint. Suppose first that $x \cup y$ is strong. Then $x \cup y$ contains a well-orderable subset, say z, of maximum cardinality, i.e., $C_0(x \cup y) = |z| = \kappa$. Let $z_1 = z \cap x$ and $z_2 = z \cap y$. Then $z_1, z_2 \in WS$, $z_1 \cap z_2 = \emptyset$, $z = z_1 \cup z_2$, hence $|z| = |z_1| + |z_2|$. Since $|z| = \max(|z_1|, |z_2|)$, it follows that either z_1 or z_2 is a well-orderable subset of $x \cup y$ of maximum cardinality. Without loss of generality suppose this is z_1. Then, a fortiori, z_1 is of maximum cardinality as subset of x, hence $|z_1| = C_0(x)$, and also $|z_1| \geq C_0(y)$. So $C_0(x) + C_0(y) = C_0(x) = \kappa$.

Now suppose that $x \cup y$ is weak. Then necessarily $C_0(x \cup y)$ is a limit cardinal. Then it is easy to see that $C_0(x) + C_0(y) = \max(C_0(x), C_0(y))$ is also limit. Let $\kappa < C_0(x \cup y)$. Then, by the definition of C_0, there is a set $z \in WS$ such that $z \subseteq x \cup y$ $|z| = \kappa$. Let again $z_1 = z \cap x$ and $z_2 = z \cap y$. Then obviously, $|z_1| \leq C_0(x)$ and $|z_2| \leq C_0(y)$. Therefore, $\kappa = |z| = |z_1| + |z_2| \leq C_0(x) + C_0(y)$. Since this holds for every $\kappa < C_0(x \cup y)$, it follows that

$C_0(x \cup y) \le C_0(x) + C_0(y)$. For the converse, assume $C_0(x) \ge C_0(y)$, i.e., $C_0(x) + C_0(y) = C_0(x)$, which is limit as we noted above. Let $\lambda < C_0(x)$ be a cardinal. Then there is $w \in WS$ such that $w \subseteq x$ and $|w| = \lambda$. Consequently, $w \subseteq x \cup y$, therefore $\lambda \le C_0(x \cup y)$. Since this holds for every $\lambda < C_0(x) + C_0(y)$, it follows that $C_0(x) + C_0(y) \le C_0(x \cup y)$.

(4): For the case of $x \times y$, the argument is analogous to the previous one. First suppose $C_0(x \times y)$ is not limit, i.e., again there is a well-orderable $z \subseteq x \times y$ of maximum cardinality, $C_0(x \times y) = C_0(z) = \kappa$. Let $z_1 = dom(z)$ and $z_2 = rng(z)$. Then clearly z_1, z_2 are well-orderable (a wellordering of the pairs of z induces obvious wellorderings for the first and second coordinates of the pairs). Also, $z_1 \subseteq x$, $z_2 \subseteq y$, and $z \subseteq z_1 \times z_2 \subseteq x \times y$. Clearly $z_1 \times z_2 \in WS$. Therefore $|z_1 \times z_2| = |z| = \kappa = \max(|z_1|, |z_2|)$. Let (without loss of generality) $\max(|z_1|, |z_2|) = |z_1|$. Then z_1 is of maximum cardinality for x, i.e., $C_0(x) = |z_1| = \kappa$ and $\kappa \ge C_0(y)$. Therefore $C_0(x) \cdot C_0(y) = C_0(x) = C_0(x \times y)$ and we are done.

Let now $C_0(x \times y)$ be limit. Consequently, so is $C_0(x) \cdot C_0(y)$, i.e., $\max(C_0(x), C_0(y))$. Let $\kappa < C_0(x \times y)$, and let $z \in WS$, $z \subseteq x \times y$ with $|z| = \kappa$. Let again $z_1 = dom(z) \subseteq x$, $z_2 = rang(z) \subseteq y$, whence $z \subseteq z_1 \times z_2 \subseteq x \times y$. We have $|z_1| \le C_0(x)$, $|z_2| \le C_0(y)$, so $|z| \le |z_1| \cdot |z_2| \le C_0(x) \cdot C_0(y)$. This proves that $C_0(x \times y) \le C_0(x) \cdot C_0(y)$. Conversely, let $C_0(x) = \max(C_0(x), C_0(y))$, so $C_0(x) \cdot C_0(y) = C_0(x)$ and $C_0(x)$ is limit. Let $\lambda < C_0(x)$. Then there is well-orderable $u \subseteq x$ such that $|u| = \lambda$. Choose $b \in y$. Then $u \times \{b\}$ is well-orderable, $u \times \{b\} \subseteq x \times y$ and $|u \times \{b\}| = \lambda$. Therefore $\lambda \le C_0(x \times y)$. This shows that $C_0(x) \cdot C_0(y) \le C_0(x \times y)$.

(5): Let $f : x \to y$ be an injection and let $\kappa \le C_0(x)$ be such that there is $u \in WS$, $u \subseteq x$ with $|u| = \kappa$. Obviously, the set $f''u$ is well-orderable, $f''u \subseteq y$ and $|f''u| = |u| = \kappa$. Therefore $\kappa \le C_0(y)$. Since this holds for every $\kappa \le C_0(x)$, it follows that $C_0(x) \le C_0(y)$. $\dashv$

Is the above class C_0 the only definable notion of cardinality? We' ll see below that the answer is No. Observe that if C is a notion of cardinality in M, then it gives rise to a mapping $F_C : Card \to Card$ defined as follows:

$$F_C(\kappa) = \min\{\lambda : (\exists x \notin WS)(\exists y \in WS)(C(x) = \lambda \ \&\ y \subseteq x \ \&\ |y| = \kappa). \quad (3)$$

In words, $F_C(\kappa)$ is the least C-cardinality of a non-well-orderable set that contains a well-orderable subset of cardinality κ. F_C is well-defined since for every infinite κ there is a $x \notin WS$ and a $y \in WS$ such that $y \subseteq x$ and $|y| = \kappa$. Indeed let $a \in M - WS$. We can always find a set y disjoint from a such that $|y| = \kappa$ (without the help of AC, e.g. if $rank(x) = \alpha$, we can take as y the first κ ordinals beyond V_α). If we set $x = a \cup y$, then $x \notin WS$, hence it is as required.

By definition $F_C(\kappa) \geq \kappa$ for every κ. Moreover F_C is eventually the identity, i.e., there is κ_0 such that $F_C(\kappa) = \kappa$ for all $\kappa \geq \kappa_0$. Indeed, let $a \in M - WS$ and let $C(a) = \kappa_0$. Then for every $\kappa \geq \kappa_0$, take as before $y \in WS$ with $|y| = \kappa$, disjoint from a and let $x = a \cup y$. Then $x \notin WS$ and $C(x) = C(a \cup y) = C(a) + C(y) = \kappa$. Therefore $F_C(\kappa) \leq \kappa$, consequently, $F_C(\kappa) = \kappa$.

Let $M \models \mathrm{ZF} + \neg\mathrm{AC}$. Call a set $a \in M - WS$ *thin* if a is weak and $C_0(a) = \omega$. (As noted earlier, many models of $\mathrm{ZF} + \neg\mathrm{AC}$ contain thin sets, cf. e.g. the symmetric models in [7].)

Lemma 3.8 *(i) Let M contain a thin set. Then $F_{C_0} = id$.*

(ii) For every cardinality notion C,

$$C(x) \geq \sup\{F_C(|y|) : y \in WS \ \&\ y \subseteq x\}.$$

(iii) $F_C(\kappa + \lambda) = F_C(\kappa) + F_C(\lambda)$ and $F_C(\kappa \cdot \lambda) = F_C(\kappa) \cdot F_C(\lambda)$

Proof. (i) Indeed, let a be a thin set. For every infinite κ, let y be disjoint from a and $|y| = \kappa$. If $x = a \cup y$, then $x \notin WS$, x contains a well-orderable set of cardinality κ and $C(x) = C(a) + C(y) = \omega + \kappa = \kappa$.

(ii) By the definition of F_C, for every $x \notin WS$ and every $y \subseteq x$, $F_C(|y|) \leq C(x)$. Therefore

$$C(x) \geq \sup\{F_C(|y|) : y \in WS \ \&\ y \subseteq x\}.$$

(iii) Let $F_C(\kappa + \lambda) = \mu$. Then there is $x \notin WS$ and $y \in WS$ such that $y \subseteq x$, $|y| = \kappa + \lambda$ and $C(x) = \mu$. Then clearly, y can be split into two $y_1, y_2 \in WS$ such that $|y_1| = \kappa$, $|y_2| = \lambda$ and $y_1, y_2 \subseteq x$. Therefore $F_C(\kappa) \leq \mu$ and $F_C(\lambda) \leq \mu$, hence $\mu = F_C(\kappa + \lambda) \geq F_C(\kappa) + F_C(\lambda)$. Conversely, let $x_1, x_2 \notin WS$ such that $x_1 \cap x_2 = \emptyset$ with $C(x_1) = F_C(\kappa)$ and $C(x_2) = F_C(\lambda)$ and $y_1 \subseteq x_1$, $y_2 \subseteq x_2$, with $|y_1| = \kappa$ and $|y_2| = \lambda$. But then $x_1 \cup x_2 \in WS$, $C(x_1 \cup x_2) = F_C(\kappa) + F_C(\lambda)$, $|y_1 \cup y_2| = \kappa + \lambda$ and $y_1 \cup y_2 \subseteq x_1 \cup x_2$. Therefore $F_C(\kappa + \lambda) \leq F_C(\kappa) + F_C(\lambda)$. The preservation of $\cdot$ is shown similarly. $\dashv$

In view of clause (ii) of the above lemma, given a definable mapping $F : Card \to Card$, such that $F(\kappa) \geq \kappa$, $F(\kappa + \lambda) = F(\kappa) + F(\lambda)$ and $F(\kappa \cdot \lambda) = F(\kappa) \cdot F(\lambda)$, one might think to define, in analogy to (2) above, the mapping C_F as follows:

$$C_F(x) = \begin{cases} |y| \text{ if } x \text{ is strong and } y \subseteq x \text{ is of maximum cardinality,} \\ \sup\{F(|y|) : y \in WS \ \& \ y \subseteq x\}, \text{ otherwise.} \end{cases}$$

Theorem 3.9 *Suppose $F : Card \to Card$ is a definable mapping such that $F(\kappa+\lambda) = F(\kappa)+F(\lambda)$ and $F(\kappa\cdot\lambda) = F(\kappa)\cdot F(\lambda)$. Then C_F is a (definable) cardinality notion.*

Proof. Note that if F is as stated, then F is also monotonic, i.e., $\kappa \leq \lambda \Rightarrow F(\kappa) \leq F(\lambda)$ (because if $\kappa \leq \lambda$, then $F(\lambda) = F(\kappa+\lambda) = F(\kappa)+F(\lambda) \geq F(\lambda)$).

The verification is similar to that of the previous theorem. Again conditions (1) and (2) of definition 3.1 are obvious.

(3): Let $x \cap y = \emptyset$. Assume first that $x \cup y$ is strong, hence $C_F(x \cup y) = |z|$ for some $z \subseteq x \cup y$. Let $z_1 = z \cap x$, $z_2 = z \cap y$ and some of the z_i, say z_1 is of maximum cardinality. Then $C_F(x) = F(|z_1|)$ and $|z_1| \geq |u|$ for every $u \subseteq y$, $u \in WS$. Therefore, by monotonicity, $F(|z_1|) \geq F(|u|)$, which means that $C_F(x) \geq C_F(y)$. Also $F(|z|) = F(|z_1| + |z_2|) = F(|z_1|) + F(|z_2|) = \max(F(|z_1|), F(|z_2|)$. Therefore $C_F(x \cup y) = F(|z|) = \max(F(|z_1|), F(|z_2|) = F(|z_1|) = C_F(x) = C_F(x) + C_F(y)$.

Now assume that $x \cup y$ is weak. Then $C_F(x \cup y)$ is limit. Using the properties of F it is easy to show that $\max(C_F(x), C_F(y))$ is limit too. Let $F(|z|) < C_F(x \cup y)$ for some $z \subseteq x \cup y$, $z \in WS$. Taking again z_1, z_2 as above, $F(|z_1|) \leq C_F(x)$, $F(|z_2|) \leq C_F(y)$, and $F(|z|) = F(|z_1|) + F(|z_2|) \leq C_F(x) + C_F(y$. It follows that $C_F(x \cup y) \leq C_F(x) + C_F(y$. For the converse, notice that if, say, $C_F(x) \geq C_F(y)$, then $C_F(x)$ is limit. Let $F(|u|) < C_F(x)$, for some $u \subseteq x$. Then $u \subseteq x \cup y$ and $F(|u|) \leq C_F(x \cup y)$. This shows that $C_F(x) + C_F(y) \leq C_F(x \cup y)$.

Clause (4) is proved similarly using the fact that F preserves $\cdot$.

(5) Let $f : x \to y$ be an injection and $F(|z|) \leq C_F(x)$ for some $z \subseteq x$, $z \in WS$. Then, $F(|z|) = F(|f''z|) \leq C_F(y)$. Hence $C_F(x) \leq C_F(y)$. $\dashv$

Lemma 3.10 *(i) For every $F : Card \to Card$, $F \le F_{C_F}$.*
(ii) For every notion of cardinality C, $C_{F_C} \le C$.

Proof. (i) Let $\kappa \in Card$, and let $F_{C_F}(\kappa) = \lambda$. Then there are $x_0 \notin WS$, $y_0 \in WS$, $y_0 \subseteq x_0$, $|y_0| = \kappa$ and $C_F(x_0) = \lambda$. But $C_F(x_0) = \sup\{F(|y|) : y \in WS, y \subseteq x_0\}$. Therefore $F(\kappa) = F(|y_0|) \le C_F(x_0) = \lambda = F_{C_F}(\kappa)$.

(ii) By definition, if x is strong and $y \subseteq x$ is of maximum cardinality, $C_{F_C}(x) = |y| \le F_C(|y|) \le C(x)$, by the definition of F_C. Otherwise, $C_{F_C}(x) = \sup\{F_C(|y|) : y \subseteq x, x \in WS\} \le C(x)$. ⊣

To see that in general the inequalities above are proper, consider a C defined as follows. Let a be a thin set and κ be an arbitrary infinite cardinal. Set:

$$C(x) = \begin{cases} |x| \text{ if } x \subseteq a \text{ and } x \text{ is finite,} \\ \kappa \text{ if } x \subseteq a \text{ and } x \text{ is infinite,} \\ C(x \cap a) + C_0(x - a) \text{ otherwise.} \end{cases}$$

It is easy to verify that C is a notion of cardinality. Then $C_{F_C}(a) = \sup\{F_C(|y|) : y \subseteq a, y \in WS\}$. But every $y \subseteq a$ such that $y \in WS$, is finite, and, by the definition of C, $F_C(n) = \omega$ for all $n \in \omega$. Hence $C_{F_C}(a) = \sup\{F_C(n) : n \in \omega\} = \omega$. On the other hand $C(a) = \kappa$. That is, $C_{F_C} \ne C$.

Thin sets generalize naturally as follows: A set x is *κ-thin* if x is weak and $C_0(x) = \kappa$. That is, x contains well-orderable subsets of size λ for every $\lambda < \kappa$, but no well-orderable subset of size κ.

Theorem 3.11 *For every infinite cardinal κ, there is a $M \models \mathrm{ZF} + \neg\mathrm{AC}$ containing κ-thin sets.*

Proof. (Sketch) For simplicity we consider permutation rather than symmetric models, i.e., we work in ZFC with urelements (see [7], p. 251, for details). Let N be such a model and let $A \in N$ be a set of urelements such that $N \models |A| \ge \kappa$. Let G be the group of permutations of A (in the real world) and for every set $x \in N$ let $G_x = \{\pi \in G : \pi(x) = x\}$ be the subgroup fixing x. Let $\mathcal{F}$ be the filter of subgroups generated by G_B, $B \subseteq A$ and $|B| < \kappa$. Clearly $\mathcal{F}$ is a normal filter, i.e., for every $H \in \mathcal{F}$ and every $g \in G$, $g^{-1}Hg \in \mathcal{F}$. x is called *symmetric* if $G_x \in \mathcal{F}$. Let $M \subseteq N$ be the set of hereditarily symmetric sets of N. As usual $M \models$ ZF and $A \in M$. Moreover A is κ-thin. Indeed, for every $B \subseteq A$ with $|B| = \lambda < \kappa$, $B \in M$, since $G_B \in \mathcal{F}$. On the other hand there is no injection $f \in M$ such that $f : \kappa \to A$. Assume not, and let f be such a mapping. Then $|f''\kappa| = \kappa$ and there is $B \subseteq A$ such that $|B| < \kappa$ and for every π such that $\pi{\restriction}B = id$, $\pi(f) = f$. Choose $a \in (f''\kappa) - B$. Then there is π such that $\pi{\restriction}B = id$ and $\pi(a) \neq a$. But $a = f(\alpha)$ for some $\alpha < \kappa$. Hence $\pi(a) = \pi(f)(\pi(\alpha)) = f(\alpha) = a$, a contradiction. Therefore no well-orderable subset of A of cardinality κ exists in M, i.e., $C_0^M(A) = \kappa$ and A is weak. $\dashv$

4 Forcing cardinality notions

Besides the definable notions of cardinality considered in the previous section, certain non-definable ones can be constructed by forcing.

Definition 4.1 Given a standard countable $M \models$ ZF. Let P^M_{card}, or just P_{card}, be the (definable) class of M whose elements p satisfy the following conditions (which are, roughly, the conditions of definition 3.1):

(i) p is a function such that $rng(p) \subseteq Card$.

(ii) If $\kappa \in dom(p)$, then $p(\kappa) = \kappa$.

(iii) If $x \in dom(p)$, then $p(x) \geq C_0(x) = \sup\{|y| : y \subseteq x \ \& \ y \in WS\}$.

(iv) If $x, y, x \cup y \in dom(p)$ and $x \cap y = \emptyset$, then $p(x \cup y)) = p(x) + p(y)$.

(v) If $x, y, x \times y \in dom(p)$, then $p(x \times y) = p(x) \cdot p(y)$.

(vi) If $x, y \in dom(p)$ and there is an injective $f : x \to y$, then $p(x) \leq p(y)$.

P_{card} is the class of forcing conditions used below. As usual p *extends* q, $p \preceq q$, if $p \supseteq q$. So the greatest element of P, denoted by 1_P, is $\emptyset$. p, q with no common extension are said to be *incompatible*. We denote this by $p|q$. A $G \subseteq P$ is said to be *generic* if (a) $p \in G$ and $p \preceq q$ imply $q \in G$, (b) if $p, q \in G$ then there is $r \in G$ such that $r \preceq p$ and $r \preceq q$ and (c) $G \cap D \neq \emptyset$ for every definable class D which is dense in $(P, \preceq)$. Existence of generic sets follows from the countability of M.

Lemma 4.2 *(a) If G is P_{card}-generic, and $C_G = \bigcup\{p : p \in G\}$, then C_G is a class function satisfying conditions (1)-(5) of definition 3.1, i.e., G is a notion of cardinality in M. (We refer to C_G as a* generic cardinality notion.*)*

(b) Let $x \notin WS$ and κ be a cardinal number such that $\kappa \geq \{|y| : y \subseteq x \;\&\; y \in WS\}$. Then there is a generic notion of cardinality C such that $C(x) = \kappa$.

Proof. (a) We have to verify that C_G fulfils conditions (1)-(5) of definition 3.1. For this it suffices to check that for any two sets x, y, and any cardinal κ the sets $D_{x,y} = \{p : x, y \in dom(p)\}$, $D_x = \{p : x \in dom(p)\}$, and $E_\kappa = \{p : \kappa \in rng(p)\}$ are dense in P_{card}. The proof is easy and left to the reader.

(b) Given $x \notin WS$ and κ such that $\kappa \geq \sup\{|y| : y \subseteq x \;\& y \in WS\}$, clearly the set $p_0 = \{(x, \kappa)\}$ is an element of P_{card}. If (D_n), $n \geq 1$, is the sequence of dense subsets of P_{card} in M, let $p_0 \succeq p_1 \succeq p_2 \succeq \cdots$, with $p_n \in D_n$. The set G generated by (p_n) is generic and $p_0 \in G$. If we set $C = C_G$, then $C(x) = \kappa$. $\dashv$

Remark 4.3 It would be of interest to stress that the above proof goes through only because condition (iii) is included in the definition 4.1. Without (iii), the sets $D_{x,y}$ and D_x cannot be dense in P_{card}. In other words, given a set $x \notin WS$, we cannot force an arbitrary infinite cardinality κ onto it, in a consistent way, unless $\kappa \geq C_0(x)$. Had we dropped condition (iii) and $\kappa < C_0(x)$, then $\{(x, \kappa)\}$ would be an element of P_{card}, but if $y \subseteq x$, such that $y \in WS$ and $|y| > \kappa$, then the set D_y would have no element below $\{(x, \kappa)\}$.

Of course if $M \models$ AC, things become trivial because in this case the elements of P_{card} are functions p such that $p(x) = |x|$ for every $x \in dom(p)$. Therefore P_{card} contains no incompatible elements, and the only generic G is P itself. So $C_G = Card$, the cardinality function of M. However the following holds:

Lemma 4.4 *If $M \models \neg$AC, then every generic $G \subseteq P$, as well as C_G, is a non-definable subset of M.*

Proof. If $M \models \neg$AC, then $M - WS$ is a proper class. Given $p \in P_{card}$, there are q_1, q_2 such that $q_1 \preceq p$, $q_1 \preceq p$ and $q_1 | q_2$. Indeed just take a set $x \notin dom(p)$ such that $x \notin WS$ and let $\kappa, \lambda \geq \sup\{C_0(x), p(x) : x \in dom(x)\}$. Then $q_1 = p \cup \{(x, \kappa)\}$ and $q_2 = p \cup \{(x, \lambda)\}$ are incompatible extensions of p.

Let $D = \{p \in P : \forall q \in G(q \not\preceq p)\}$. D is dense in P. Assume not. Then there is p_0 such that $(\forall p \preceq p_0)(\exists q \in G)(q \preceq p_0)$. By the above, there are $p_1, p_2 \preceq p_0$ such that $p_1|p_2$. If $q_1 \preceq p_1$, $q_2 \preceq p_2$, with $q_1, q_2 \in G$, then $q_1|q_2$ which contradicts genericity. Now if we assume G is definable, so is D. Hence, by genericity, $G \cap D \neq \emptyset$, a contradiction.

To show that C_G is non-definable too, just observe that G can be recovered from C_G as follows: $G = \{C_G{\restriction}x : x \in M\}$, i.e., G is the class of the restrictions of the mapping C_G on sets. $\dashv$

It follows that for every generic $G \subseteq P_{card}$ in a model $M \models \text{ZF} + \neg\text{AC}$, C_G is a non-definable notion of cardinality for M and that there is a large variety of such notions which agree only on the well-orderable sets. The problem, however, is that such notions seem to be "external" to the inhabitants of M. Rather G would be familiar to the inhabitants of $M[G]$, the generic extension of M – if such a model could exist (see the remark below).

Remark 4.5 We didn't say anything about the structure $M[G]$ produced (if possible) by a P_{card}-generic class G. Forcing with proper classes, instead of sets, causes certain difficulties. In fact for every p.o. class $(P, \preceq)$ and every P-generic G, $M[G]$ can be defined in pretty same way as if P were a set. Moreover $M[G]$ is a standard transitive set which satisfies extensionality, empty set, pairing, infinity, regularity and even separation. These are proved without employing any forcing relation. In fact the main difficulty with proper-class forcing is to define the forcing relation $\Vdash$ *inside* the model, and then prove the fundamental lemmas (definability, truth and extension lemmas). Even if $\Vdash$ is definable, this does not guarantee the validity of the lemmas, unless extra conditions are imposed on P and G. For example only under these strong genericity conditions, one can prove replacement in $M[G]$. Our specific notion P_{card} and its generic subsets does not seem to satisfy the extra conditions required, and $M[G]$ cannot be shown to satisfy replacement. The interested reader is referred to [1] and [19].

5 Non-definable Separation and Replacement classes

By the end of the last section, just before remark 4.5, we said that non-definable cardinality notions look “external”. Let us make precise what we mean by an “internal” class $X \subseteq M$. We mean that if $a \in M$, then every $b \subseteq a$ defined in M by the help of X, is an element of M (e.g. for internal X, all pieces $X \cap a$, $a \in M$, belong to M). What is required from X of this kind is to satisfy a separation axiom scheme. We shall avoid the term “internal” since it is used with other meaning in set theory, and we shall use the term “separation class” or “Sep-class”. Stronger or weaker notions of “internal” are possible (see next definition).

For the precise definition we have, formally, to expand our language $L = \{\in\}$ to $L_S = \{\in, S\}$ by adding a unary predicate $S(\cdot)$. We write $x \in S$ instead of $S(x)$. Given a language $L' \supseteq L = \{\in\}$, let **Sep** denote the separation scheme:

$$\exists x \forall y (y \in x \iff y \in a \,\&\, \varphi(x)),$$

for every $\varphi \in L'$. If $(M, \in, \cdots)$ is an L'-structure, $M \models$ **Sep** means that M satisfies **Sep** of the corresponding language. Similarly **Rep** denotes the replacement scheme:

$$\forall x \in a \exists! y\ \phi(x,y) \Rightarrow \exists z \forall x \in a \exists y \in z\ \phi(x,y).$$

Definition 5.1 Let M be a standard transitive model of $M \models \mathrm{ZF}$ and let $X \subseteq M$. X is said to be a *Sep-class* of M, if $(M, X) \models$ **Sep**. X is said to be a *Rep-class* of M, if $(M, X) \models$ **Rep**. Finally X is said to be *graded* if $X \cap M_\alpha \in M$ for every α, where M_α are the levels of M.

Obviously every definable $X \subseteq M$ is a Rep-class; every Rep-class is a Sep-class; and every Sep-class is graded. None of these implications can be reversed. The following are open:

Problems. Given a transitive model $M \models \mathrm{ZF}$,

1) Is every P_{card}-generic class G of M, a Sep-class or a Rep-class of M?

2) More generally, does there exist a non-definable notion of cardinality $C \subseteq M$ which is a Sep-class or a Rep-class for M?

We could only prove the following:

Proposition 5.2 *Let $M \models$ ZF and let G be P^M_{card}-generic. Then C_G is graded.*

Proof. We have to show that $C_G \cap M_a$ is a set for every α. Clearly it suffices to show that $C_G \cap M_{\alpha+2}$ is an element of M, for every α. Now for every $x \in M$, there is a $p \in G$ such that $C_G \restriction x = p$ (and $x = dom(p)$). (Indeed, for every x, the set $D_x = \{p : x \subseteq dom(p)\}$ is dense in P, hence there is $q \in G$ such that $x \subseteq dom(q)$. If $p = q \restriction x$, then $p \succeq q$, so $p \in G$, therefore, $C_G \restriction x = p$.) Let $\alpha \in Ord$. Since $M_\alpha \in M$, there is $p \in G$ such that $C_G \restriction M_\alpha = p$, where $dom(p) = M_\alpha$. Let $p^{-1''} M_\alpha = a$. If $f = p \restriction a$, then it is easy to verify that $C_G \cap M_{\alpha+2} = f$. Indeed, let $(x, \kappa) \in f$. Then $x, \kappa \in M_\alpha$, hence $(x, \kappa) \in M_{\alpha+2}$. Since $(x, \kappa) \in p \subseteq C_G$, it follows $(x, \kappa) \in C_G \cap M_{\alpha+2}$. Therefore $f \subseteq C_G \cap M_{\alpha+2}$. Conversely, let $(x, \kappa) \in C_G \cap M_{\alpha+2}$. Then $x, \kappa \in M_\alpha$ and $C_G(x) = p(x) = \kappa$. Moreover $x \in a = dom(f)$, i.e., $(x, \kappa) \in f$, so $C_G \cap M_{\alpha+2} \subseteq f \in M$. Since $f \in M$, we are done. $\dashv$

Although the above problems are open, we have something to say about. Namely a characterization of Rep-classes. The following definition comes from the analog of Reflection Theorem (see e.g. [7] or [11]) for the structure (M, X).

Definition 5.3 Let $M \models$ ZF. A class $X \subseteq M$ is said to be *reflective* if the Reflection Theorem holds in the structure (M, X), i.e., if for every formula $\varphi(\overline{x}, \overline{b}, S)$ of L_S, with free variables $\overline{x}$ and parameters $\overline{b}$, and every ordinal β, there is an ordinal $\alpha \geq \beta$, such that for all $\overline{x} \in M_\alpha$,

$$(M, X) \models \varphi(\overline{x}, \overline{b}, S) \iff (M_\alpha, X \cap M_\alpha) \models \varphi(\overline{x}, \overline{b}, S). \tag{4}$$

Proposition 5.4 *X is a Rep-class iff X is reflective and graded.*

Proof. Let X be a Rep-class. Trivially X is graded. Also (M, X) satisfies Replacement. Now by a straightforward generalization of the proof of the standard Reflection, we can show the Reflection Theorem for (M, X). Indeed, the standard proof of Reflection is by induction on the length of ϕ, and the main step is the one for $\exists$. This is proved by the help of the Replacement axiom. In exactly the same way the corresponding step for formulas of L_S is proved by the help of the Replacement axiom in (M, X). Thus X is reflective.

The proof of the other direction is rather folklore, but we include it for the sake of completeness. Suppose X is reflective and graded and let $\phi(x, y, \overline{c}, S)$ be a formula ϕ of L_S, $\overline{c}$ parameters, and a set $a \in M$ such that

$$(M, X) \models \forall x \in a \exists ! y \phi(x, y, \overline{c}, S).$$

It suffices to show that there is $b \in M$ such that

$$(M, X) \models \forall x \in a \exists y \in b\ \phi(x, y, \overline{c}, S).$$

Let β be such that $\overline{c}, a \in M_\beta$. Let

$$\psi(u, v, \overline{c}, S) := \phi(u, v, \overline{c}, S)\ \&\ \forall x \in a \exists ! y \phi(x, y, \overline{c}, S).$$

By the assumption there is $\alpha > \beta$ such that for all $u, v \in M_\alpha$,

$$(M, X) \models \psi(u, v, \overline{c}, S) \iff (M_\alpha, X \cap M_\alpha) \models \psi(u, v, \overline{c}, S). \tag{5}$$

Since X is graded, $X \cap M_\alpha \in M$. So

$$\exists x \in a[(M_\alpha, X \cap M_\alpha) \models \phi(x, y, \overline{c}, S)]$$

is a formula of L. It follows by Separation in M that

$$b = \{y \in M_\alpha : \exists x \in a[(M_\alpha, X \cap M_\alpha) \models \phi(x, y, \overline{c}, S)]\}$$

is a subset of M_α belonging to M. Therefore

$$\forall x \in a \exists y \in b[(M_\alpha, X \cap M_\alpha) \models \phi(x, y, \overline{c}, S)]. \tag{6}$$

But $a, b \subseteq M_\alpha$, so $x \in a$ and $y \in b$ implies $x, y \in M_\alpha$, and so by (5)

$$(M, X) \models \phi(x, y, \overline{c}, S) \iff (M_\alpha, X \cap M_\alpha) \models \phi(x, y, \overline{c}, S).$$

Therefore by (6) and the last relation

$$(M, X) \models \forall x \in a \exists y \in b\ \phi(x, y, \overline{c}, S).$$

$\dashv$

6 Cardinality of finite sets

In the preceding sections we showed the existence of several cardinality notions in models of ZF + $\neg$AC. But as follows from proposition 3.4, all of them coincide on finite sets. So if we are to deviate from standard *finite* cardinality, a different treatment and/or context is needed.

It is well known that the theory of finite sets is, essentially, Peano Arithmetic. So one approach to nonstandard cardinality notions could be through nonstandard models of PA. Given $N \models$ PA, the "internal" subsets of N are the *coded* ones, i.e., those representable by a single element of N. These are exactly the definable and bounded ones. Such a set $X \subseteq N$ is denoted D_a, for some $a \in N$, if a is a code for X (a is a code for X if $x \in X \iff N \models p_x | a$, where p_x is the x-th prime of N). Let $S(N)$ be the collection of coded subsets of N. For each $D_a \in S(N)$, $D_a \neq \emptyset$, there is $b \in N$, $b > 0$, and a coded bijection $c : D_a \to [0, b-1]$, where $[0, b-1]$ is the initial segment of the first b elements of N. Then we write $C_0(D_a) = |D_a| = b$ and say that b is the (internal) cardinality of D_a. In fact C_0 provides a definable enumeration of the elements of D_a, by means of the numbers $0, \ldots, b-1$.

In general let us define:

Definition 6.1 Let N be a model of PA. A *notion of cardinality* for N is a mapping $C : S(N) \to N$ such that:
(1) $rng(C) = N$.
(2) $C(\emptyset) = 0$.
(3) $C(D_a \cup \{x\}) = C(D_a) + 1$, for $x \notin D_a$.
(4) For any disjoint sets D_a, D_b, $C(D_a \cup D_b) = C(D_a) + C(D_b)$.
(5) For any D_a, D_b, $C(D_a \times D_b) = C(D_b) \cdot C(D_b)$.
(6) If there is a coded injective mapping $e : D_a \to D_b$, then $C(D_a) \leq C(D_b)$.

If C is a cardinality notion for $N \models$ PA, it follows easily from clauses (2) and (3) of the above definition that for every standard n (i.e., $n \in \omega$), $C(D_n) = |D_n|$ (by induction on ω).

Let $N \models$ PA be nonstandard and admitting nontrivial automorphisms (for basic information see [8]). Given a nontrivial automorphism $f : N \to N$, let us define $C_f : S(N) \to N$ as follows:

$$C_f(D_a) = f(|D_a|) = |f''D_a| = |D_{f(a)}|. \tag{7}$$

The standard C_0 is just C_f for $f = id$.

Proposition 6.2 *For every automorphism f, C_f is a cardinality notion.*

Proof. (1) Let $b \in N$ and let $f^{-1}(b) = c$. Clearly there is $a \in N$ such that $|D_a| = c$. Then $C_f(D_a) = f(|D_a|) = f(c) = b$, therefore $rng(C_f) = N$.

(2) A code for $\emptyset$ is 1, i.e., $D_1 = \emptyset$. ω is an initial segment of N and every automorphism f fixes the elements of ω. So $C_f(\emptyset) = f(D_1) = |D_{f(1)}| = |D_1| = 0$.

(3) Let $x \notin D_a$. Then $C_f(D_a \cup \{x\}) = |f''(D_a \cup \{x\})| = |f''(D_a) \cup \{f(x)\}| = |f''(D_a)| + 1 = C_f(D_a) + 1$.

(4) Let $D_a \cap D_b = \emptyset$. Then clearly $D_a \cup D_b$ is coded and, $f''D_a \cap f''D_b = \emptyset$, so

$$C_f(D_a \cup D_b) = |f''(D_a \cup D_b)| = |f''D_a \cup f''D_b| =$$
$$|f''D_a| + |f''D_b| = C_f(D_a) + C_f(D_b).$$

(5) Similar to (4).

(6) Let $e : D_a \to D_b$ be a definable injection. Then $f(e) : D_{f(a)} \to D_{f(b)}$ is a definable injection too. Therefore $|D_{f(a)}| \leq |D_{f(b)}|$, or $C_f(D_a) \leq C_f(D_b)$. $\dashv$

Proposition 6.3 *If f is nontrivial, $C_f \neq C_0$.*

Proof. Since f is non-trivial, there is a nonstandard $b \in N$ such that $f(b) \neq b$. Let a be such that $|D_a| = b$. Then

$$C_f(D_a) = f(|D_a|) = f(b) \neq b = |D_a| = C_0(D_a).$$

$\dashv$

The converse of 6.2 is also true.

Proposition 6.4 *If $C : S(N) \to N$ is a cardinality notion, then $C = C_f$ for some automorphism f of N.*

Proof. Let $C : S(N) \to N$ be a cardinality notion. For every $a \in N$, $a > 0$, clearly the segment $[0, a-1]$ is coded. Define the mapping $f : N \to N$ as follows:

$$f(a) = b \iff C([0, a-1]) = b. \tag{8}$$

To show that f is an automorphism, it suffices to check that f is 1-1, onto, and preserves $+$, $\cdot$ and successors.

f is 1-1: Let $a_1 < a_2$, and let $a_2 - a_1 = c$. Then $[0, a_2 - 1] = [0, a_1 - 1] \cup [a_1, a_2 - 1]$. Moreover, there is a coded bijection $d : [0, c-1] \to [a_1, a_2 - 1]$. Therefore, by the properties of C,

$$f(a_2) = C([0, a_2 - 1]) = C([0, a_1 - 1] \cup [a_1, a_2 - 1]) =$$

$$C([0, a_1 - 1]) + C([a_1, a_2 - 1]) = C([0, a_1 - 1]) + C([0, c-1]) = f(a_1) + f(c).$$

Since, clearly, $f(c) \neq 0$, $f(a_1) \neq f(a_2)$.

f is onto: Let $b \in N$. Since C is onto N, there is D_a such that $C(D_a) = b$. Then clearly there is $d \in N$ and a coded bijection $e : [0, d-1] \to D_a$. Hence $C([0, d-1]) = C(D_a) = b$. Therefore $f(d) = b$.

f preserves $+$:

$$f(a+b) = C([0, a+b-1]) = C([0, a-1]) + C([a, a+b-1]) =$$

$$C([0, a-1]) + C([0, b-1]) = f(a) + f(b).$$

f preserves $\cdot$:

$$f(a \cdot b) = C([0, a \cdot b - 1]) = C([0, a-1] \times [0, b-1]) =$$

$$C([0, a-1]) \cdot C([0, b-1]) = f(a) \cdot f(b).$$

f preserves successors: $f(a+1) = C([0, a]) = C([0, a-1]) + 1 = f(a) + 1$.

Hence f is an automorphism. Now let D_a be a coded set, and let $|D_a| = b$. Then there is a coded bijection $c : D_a \to [0, b-1]$. So

$$C(D_a) = C([0, b-1]) = f(b) = f(|D_a|) = C_f(D_a).$$

Therefore $C = C_f$. $\dashv$

However for non-trivial f, C_f is *not* internal to the model N, in a way analogous to a Sep-class. An analogous notion for PA is the following. If $N \models \mathrm{PA}$ and $X \subseteq N$, X is said to be *inductive* (or "substitutable") if (M, X) satisfies the induction axiom for the expanded language; see e.g. [17], where construction of these sets by forcing techniques is given. Moreover, a model N of PA is recursively saturated iff it has an inductive satisfaction class, see e.g. [9].

Proposition 6.5 *For every non-trivial automorphism f, C_f is not inductive.*

Proof. By proposition 6.4 and relation (8), f is definable from C_f, so it suffices to show that f is not inductive in N. Suppose it is. Let $X = \{x \in N : f(x) = x\}$. Then $0 \in X$ and if $x \in X$ then $x + 1 \in X$. But X is inductive, since it is definable from f. Therefore $X = N$, i.e., $f = id$, contrary to our assumption. $\dashv$

Remark 6.6 (1) The above method of constructing nonstandard cardinality notions by the use of automorphisms could in principle be applied also to models of set theory. However even if a model $M \models$ ZF admits nontrivial automorphisms, not all of them are suitable for our purpose. There must exist an automorphism that moves a cardinal number, while the automorphisms provided e.g. in [2] fix all ordinals. In general, automorphisms of models of ZF are much more complicated constructions than those of models of PA.

(2) The cardinality notions C_f defined above all agree on sets with standard code, i.e., on the (coded) subsets of the standard segment ω of N. It doesn't seem possible to define cardinality of such sets, by the above methods, without enumerating them. The only approach that might help to this direction would be to reconsider the meaning of "existence (of an enumeration)", i.e., to interpret $\exists$ *constructively.* We shall not elaborate further here on this point. Some hints are given in the Appendix.

APPENDIX

Let me cite here the quantum phenomenon that has been one of the motivations of the present article (see e.g. [18], or [15] and other articles of the same volume). Consider a pair of two particles p_1 and p_2 of the same sort (say photons), isolated in a closed box, each of which can be in one of two equiprobable states, say "left" (L) and "right" (R). What is the probability that, at any given moment, one of the particles is in state L and the other in R? According to classical statistics there are four possible and equiprobable cases for the pair, namely

$$R(p_1)R(p_2),\ R(p_1)L(p_2),\ L(p_1)R(p_2),\ L(p_1)L(p_2), \tag{9}$$

each with probability 1/4. So the probability to have particles in different states must be the sum of the probabilities of the cases $R(p_1)L(p_2)$ and $L(p_1)R(p_2)$, i.e. 1/2. Yet the actual measurements (note that the above is a real, not a theoretical, experiment) show that the required probability is 1/3! That only means that the system behaves as if there were only three cases: Both particles in L, both particles in R, and particles in different states. That in turn means that the system behaves as if the particles lacked identities (which are here indicated by the subscripts 1 and 2) and were indistinguishable. Indeed, if we drop the subscripts/identities from p_1, p_2, then the four cases of (9) above become

$$R(p)R(p),\ R(p)L(p),\ L(p)R(p),\ L(p)L(p), \tag{10}$$

which actually amount to three

$$R(p)R(p),\ L(p)R(p),\ L(p)L(p), \tag{11}$$

since the second and third case of (10) coincide. The cases of (11) being equiprobable, we get probability 1/3 for the fact $L(p)R(p)$.

What we are interested in here is the following mathematical issue raised by the above situation: The objects p_1, p_2 behave as indistinguishable copies of each other. But then how can we talk of *two* particles, instead of one? More generally, how is it possible to attribute cardinality to sets without being able to "identify" their elements? Concerning identification, we have already made a hint about what we believe it amounts to by writing the particles in the form p_1, p_2. We can agree that identifying the elements of a set X amounts to assigning them a *numerical identity*. In the simplest case of a finite X, this means that we are able to construct a bijection $f : \{1, \ldots, n\} \to X$, for some $n \in \mathbb{N}$. In such a case if $x \in X$ and $x = f(k)$, k is supposed to be the identity of x[2]. We conclude that identification of the elements of X means the existence of a bijection $f : \{1, \ldots, n\} \to X$, for some n, and this is equivalent to the existence of an *enumeration* of X in the form $X = \{x_1, \ldots, x_n\}$, where

[2]This method of assigning identity is used throughout industrial mass production in order to distinguish between millions of identical copies (e.g. cars) outgoing the production line. The method of distinction consists in engraving a serial number on each particular copy that constitutes its identity (e.g. the number engraved on the frame of a car is the only characteristic that helps to identify it). This is precisely a constructive bijection $f : \{1, \ldots, n\} \to X$, between the set X of copies and the number n of their multitude.

$x_k = f(k)$. Consequently, to say that the elements of the collection X are not identifiable, is to say that X is not orderable, while X possesses a cardinal number. This is a situation entirely different from the one we are accustomed to, i.e., the Cantorian reality. G.T. di Francia [4], p.27, puts it as follows:

> What is an electron (or any other elementary particle)? From an *extensional* point of view, an electron is an element of the set of particles we call electron. But here we stumble on a serious problem: can one deal with *sets* of identical particles in any classical sense? No, because a collection of identical particles has only a *cardinal* number; but its elements cannot be ordered and cannot be named individually.

Guided by the idea of a set theory admitting indistinguishable and yet non-identical objects, M. Dalla Chiara, R. Giuntini, and D. Krause [3] proposed a system of axioms that might accommodate such entities.

Even simpler examples exist of situations where size of collections is attainable but no enumeration is possible. Suppose we are dealing with finite sets of elementary particles whose presence is detected only indirectly, via an instrument, e.g. a Geiger counter or a cloud chamber. Then we can safely announce that there are, say, 6 particles inside the chamber as soon as we see 6 distinct trajectories on the chamber's screen. This is a correct measurement of the size of the set of particles, and yet no reasonable way to enumerate the particles seems to exist. The particles are indistinguishable and lack identity with respect to the observer.

The above examples concern micro objects. In the field of macroscopic objects, suppose we have a hermetically closed box containing a number of balls of same substance, size and shape that move freely to all directions. For any observer, the balls cannot be distinguished from each other in any reasonable way, so they lack identity. Yet we may identify their number in several cases. E.g. if we know the weight of each ball and the weight of the box, then we can get the number by just weighing out the box. Or if we send a beam of X-rays through the box.

In all the preceding examples the notion of *constructiveness* occurs time and again. The point is not if an enumeration exists in an absolute way, but rather if a general observer is able to construct it. This however affects the logical meaning of $\exists$, and leads us to the realm of constructive mathematics à la Bishop. It is well-known that there are links between quantum mechanics and constructive mathematics (see e.g. [5]). Some systems of constructive set theory have also appeared, e.g. [13], or [14] (where a constructive quantifier $\exists_I$ is employed), but are beyond the scope of the present article.

Acknowledgements. I would like to express my gratitude to the referee for his thoughtful review of the paper. He (she) detected a serious flaw in an earlier draft of the paper. He suggested the implication $\Leftarrow$ of proposition 5.4. He offered also valuable remarks and suggestions, even some of historical interest.

References

[1] R. Chuaqui, Forcing for the impredicative theory of classes, *J. Symb. Logic* **37**, Number 1 (1972), 1-18.

[2] P. Cohen, Automorphisms of set theory, *Proc. Symp. Pure Mathematics*, volume 25, AMS Providence 1974, 325-330.

[3] M.L. Dalla Chiara, R. Giuntini, and D. Krause, Quasiset theory for microobjects, in: Elena Castelani (Ed.), *Interpreting Bodies, Classical and Quantum Objects in Modern Physics*, Princeton U.P. 1998, 142-152.

[4] Giuliano Toraldo di Francia, A world of individual objects?, in: Elena Castelani (Ed.), *Interpreting Bodies, Classical and Quantum Objects in Modern Physics*, Princeton U.P. 1998, 21-29.

[5] G. Hellman, Constructive mathematics and quantum mechanics: Unbounded operators and the spectral theorem, *J. Phil. Logic* **22** (1993), 221-248.

[6] T. Jech, *The axiom of choice*, North Holland 1973.

[7] T. Jech, *Set Theory*, Springer Verlag 2002.

[8] R. Kay, *Models of Peano Arithmetic*, Oxford Logic Guides, 1991.

[9] L. Kirby, K. McAloon, and R. Murawski, Indicators, recursive saturation and expandability, *Fund. Math.* **CXIV** (1981), 127-139.

[10] D. Krause, On a quasi-set theory, *Notre Dame Journal of Formal Logic* **33** (1992), 402-411.

[11] K. Kunen, *Set theory, an introduction to independence proofs*, North Holland, 1980.

[12] A. Lindenbaum and A. Tarski, *Communications sur les recherces de la Théorie des Ensebles,* in: Alfred Tarski Collected Papers, Volume I, Birkhauser 1986, Steven R. Givant and Ralph N. McKenzie (Eds), pp. 171-204.

[13] J. Myhill, Constructive set theory, *J. Symb. Logic* **40** (1975), 347-382.

[14] J. Myhill, Intensional set theory, in: *Intensional mathematics*, S. Shapiro (Ed.), North Holland 1985, pp. 47-61.

[15] Hans Reichenbach, The genidentity of quantum particles, in: Elena Castelani (Ed.), *Interpreting Bodies, Classical and Quantum Objects in Modern Physics*, Princeton U.P. 1998, 61-72.

[16] H. Rubin and J. Rubin, *Equivalents of the Axiom of Choice*, North Holland, 1963.

[17] J. Schmerl, Peano models with many generic classes, Pacific J. Math. **46** (1973), No 2, 523-536.

[18] Paul Teller, Quantum Mechanics and Haecceities, in: Elena Castelani (Ed.), *Interpreting Bodies, Classical and Quantum Objects in Modern Physics*, Princeton U.P. 1998, 114-141.

[19] A. Zarach, Forcing with proper classes, *Fund. Math.* **LXXXI** (1973), 1-27.