Forcing and antifoundation

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Abstract

It is proved that the forcing apparatus can be built and set to work in ZFCA (=ZFC minus foundation plus the antifoundation axiom AFA). The key tools for this construction are greatest fixed points of continuous operators (a method sometimes referred to as "corecursion"). As an application it is shown that the generic extensions of standard models of ZFCA are models of ZFCA again.

1 Preliminaries

It is well known that the constituents of forcing machinery, including the forcing relation $\Vdash$ itself, are defined by $\in$-recursion. So it is natural to ask what happens when foundation is missing. Let ZFC$^-$ be ZFC without the foundation axiom. We shall show that in ZFC$^-$ plus an antifoundation axiom, we can restore the forcing machinery by the help of greatest fixed points of continuous operators. ZFC$^-$ will be our basic system, although there are other interesting unfounded set theories, like New Foundations (NF). The reason is that after [1], the standard approach to non-well-foundedness has become the one using directed graphs and transitive closures of sets. And transitive closures are available in ZFC$^-$ (and its variants) though not in NF.

In the place of foundation axiom we shall adopt Aczel's antifoundation axiom AFA. To formulate it we need a few definitions. So for the reader's convenience, we shall first recall certain notions and facts from [1].

A *directed graph* is a pair $(g, \to_g)$, where $\to_g$ is a binary relation on g. If there is no danger of confusion we write $a \to b$ instead of $a \to_g b$. An n-path of g, for $n \in \mathbb{N} \cup \{\infty\}$, is always a *directed* path, i.e., a sequence $a_1 \to a_2 \to \cdots$ consisting of n edges. An *accessible pointed graph* (*apg* for short) is a triple $(g, \to, a_0)$, where $(g, \to)$ is a directed graph and a_0 is a distinguished node, the *point* of g, such that every other node is joined with a_0 by a finite path. Again if there is no danger of confusion we write just g instead of $(g, \to, a_0)$. A *decoration* of an apg $(g, \to, a_0)$ is a mapping $d : g \to V$ such that for every node a, $d(a) = \{d(b) : a \to b\}$. In such a case the set $d(a_0)$ assigned to the point is said to be a *picture of* g. Aczel's axiom claims the following:

(AFA) Every apg has a unique decoration.

(Actually, if AFA holds for apg's, it holds also for every directed graph.) It follows from AFA that for every apg $(g, \to, a_0)$, there is a unique set $d(a_0)$ having g as a picture. We denote it by $\sigma(g, \to, a_0)$, or just $\sigma(g)$, that is

$$\sigma(g) = \sigma(g, \to, a_0) = d(a_0).$$

For example, to every set x there corresponds the apg

$$\gamma(x) = (TC(x \cup \{x\}), \to, x),$$

where $TC(x)$ denotes the transitive closure of x, and for every $y, z \in TC(x \cup \{x\})$, $y \to z$ if $z \in y$. $\gamma(x)$ is said to be the *$\in$-graph of x*. Obviously the mapping $d : \gamma(x) \to V$ such that $d(y) = y$ for every $y \in TC(x \cup \{x\})$, is a decoration of $\gamma(x)$. Since this decoration is unique, it follows that

$$\sigma(\gamma(x)) = x.$$

AFA in the above form talks about decorations in general, not *injective* (i.e., 1-1) ones. If the decoration d of the apg (g, a_0) is injective, the set $d(a_0)$ is said to be an *exact* picture of g. For example the $\in$-graph $\gamma(x)$ defined above is an exact picture of x, and in this sense it is unique. However x may have many other nonexact and nonisomorphic pictures (so in general $\gamma(\sigma(g)) \neq g$)). The question "which graphs are exact pictures", or equivalently, "which graphs admit injective decorations" is important and leads to an equivalent reformulation of AFA.

In the sequel we shall often work with large, i.e. class size graphs.

Definition 1.1 The graph $(C, \rightarrow_C)$ is said to be a *system* if C is a class but for every $a \in C$, the class $\{b \in C : a \rightarrow_C b\}$ of the children of a in C is a set.

As usual we often write just C for the system $(C, \rightarrow_C)$. For every system C and every $a \in C$, let us set:

$a_C = \{b \in C : a \rightarrow_C b\}$ (the set of children nodes of a in C),

$C \restriction a =$ the apg with point a and nodes and edges those of C lying on paths starting from a.

Obviously for any $a \in C$, a_C and $C \restriction a$ are sets.

Let C be a system and let $R \subseteq C \times C$ be a relation on C. R is said to be a *bisimulation* on C if for all $a, b \in C$:

$$aRb \;\Rightarrow\; (\forall x \in a_C)(\exists y \in b_C)(xRy) \;\&\; (\forall y \in b_C)(\exists x \in a_C)(xRy).$$

For example the identity $=$ is a bisimulation on every C.

Lemma 1.2 *For every system C there is a greatest bisimulation $\equiv_C$ on it. Specifically, $\equiv_C$ is the union of all small (i.e. set) bisimulations on C.*

Proof. See [1], Theorem 2.4. $\dashv$

C is said to be *$\equiv_C$-extensional* (or *strongly extensional*) if for all $a, b \in C$,

$$a \equiv_C b \;\Rightarrow a = b$$

(i.e., if $=$ is the greatest bisimulation on C).

It is proved ([1], Thm. 2.23) that AFA is equivalently reformulated as follows:

(AFA) A graph g is an exact picture iff it is $\equiv_g$-extensional.

Further, given any two apg's $(g_1, \rightarrow_1, a_1)$, $(g_2, \rightarrow_2, a_2)$, we can always think of them as subgraphs of a larger graph $(g, \rightarrow)$ where $g_1 = g \restriction a_1$ and $g_2 = g \restriction a_2$. So we can drop $\rightarrow_i$ from the notation. We say that (g_1, a_1), (g_2, a_2) are *bisimilar* and write $(g_1, a_1) \equiv (g_2, a_2)$, if $a_1 \equiv_g a_2$. A simple consequence of AFA is the following:

Lemma 1.3 *(g_1, a_1), (g_2, a_2) are pictures of the same set (i.e., $\sigma(g_1) = \sigma(g_2)$) iff $(g_1, a_1) \equiv (g_2, a_2)$.*

We come now to operators and their fixed points. An *operator* Γ is given by a formula $\phi(v, U)$ of the language of set theory, with a set variable v and a class variable U. (No quantifiers binding proper-class variables are allowed in ϕ.) The operator Γ_ϕ induced by ϕ is the mapping $\Gamma_\phi(X) = \{x : \phi(x, X)\}$, sending classes to classes.

An operator Γ is said to be *set continuous*, if for every class X, $\Gamma(X) = \bigcup\{\Gamma(x) : x \subseteq X\}$. It is easy to see that this property is equivalent to the conjunction of the following two ones: (a) $X \subseteq Y \Rightarrow \Gamma(X) \subseteq \Gamma(Y)$ (Γ is *monotone*) and (b) $a \in \Gamma(X) \Rightarrow a \in \Gamma(x)$ for some $x \subseteq X$ (Γ is *set based*). (In the preceding notation, lower case variables x, y denote sets, while upper case X, Y, denote classes.) If Γ is induced by $\phi(v, U)$, in order for Γ to be monotone it suffices for ϕ to be *positive in* U. (ϕ is positive in U if it is constructed by formulas not containing U and atomic formulas $v \in U$ using only the logical operations $\wedge$, $\vee$, $\exists$ and $\forall$. See e.g. [5].) X is said to be the *least* (resp. *greatest*) fixed point of Γ, if for any other fixed point Y, $X \subseteq Y$ (resp. $Y \subseteq X$). Since operators do not involve quantification over proper classes, we feel free to talk about them (informally) in the context of ZFC or ZFC$^-$ (instead of their conservative extensions GBN (Gödel-Bernays-von Neumann) and GBN$^-$, respectively).

Lemma 1.4 *In* ZFC$^-$ *every set continuous operator* Γ *has a least fixed point (l.f.p.) and a greatest fixed point (g.f.p.) denoted* Γ_∞ *and* Γ^∞ *respectively. Specifically,*

$$\Gamma_\infty = \bigcap\{X : \Gamma(X) \subseteq X\}, \qquad \Gamma^\infty = \bigcup\{x : x \subseteq \Gamma(x)\}.$$

Proof. See [1], Theorems 6.4 and 6.5. See also [2], §15. $\dashv$

When working in ZFC, in many cases $\Gamma_\infty = \Gamma^\infty$. But in ZFC$^-$ we very often have $\Gamma_\infty \neq \Gamma^\infty$. Now it is well known that every recursion makes use of the least fixed point of some monotone operator. This is why the method of defining notions as g.f.p.'s is often referred to as "corecursion" (cf. [2] for a further discussion). And indeed, if one works in ZFC$^-$ and has to choose between Γ_∞ and Γ^∞ as the proper definition of a notion given in terms of Γ, then one often chooses Γ^∞, because it contains *all* objects scoped by the definition, (unless of course Γ^∞ contains also "undesirable" elements).

The explicit description of Γ^∞ as $\bigcup\{x : x \subseteq \Gamma(x)\}$ implies that Γ^∞ is definable. In addition it offers a method for showing that a particular a

belongs to Γ^∞. Namely the following fact will be repeatedly used in the next sections:

Lemma 1.5 *Let Γ be any set continuous operator. Then $a \in \Gamma^\infty$ iff there is a set x such that $a \in x$ and $x \subseteq \Gamma(x)$.*

As a useful application of g.f.p.'s observe that the definition of bisimulation is inductive. Specifically, given a system C, consider the operator Γ_C defined as follows: For every $X \subseteq C \times C$:
$\Gamma_C(X) =$

$$\{(a,b) : (\forall x \in a_C)(\exists y \in b_C)((x,y) \in X) \ \& \ (\forall y \in b_C)(\exists x \in a_C)((x,y) \in X)\}.$$

It is easy to see that Γ_C is set continuous. Then, using Lemmas 1.2 and 1.4, one can easily verify that:

Lemma 1.6 *i) R is a bisimulation on C iff $R \subseteq \Gamma_C(R)$.*
ii) $\equiv_C$ is the g.f.p. of Γ_C.

In this paper we use g.f.p.'s to define forcing in ZFCA. In section 2 we define names and the generic extension $M[G]$ of a model M of ZFCA. In section 3 we define the forcing relation and prove its basic properties. The main result of the paper is Theorem 3.10. In section 4 we show that every generic extension of a standard model of ZFCA is a model of ZFCA.

2 Names

Let $\mathbb{P} = (\mathbb{P}, <, \mathbf{1})$ be a partial ordering with greatest element $\mathbf{1}$. p, q range over elements of $\mathbb{P}$. $p \leq q$ means that p extends q. A $\mathbb{P}$-*name* is a set x whose elements are pairs (y,p) such that y is a $\mathbb{P}$-name and $p \in \mathbb{P}$. In well-founded set theories this circular definition is formally given as a definition by $\in$-recursion and leads to the class of names $V^{\mathbb{P}}$. In general, it follows from the definition that X is a class of $\mathbb{P}$-names if $X \subseteq \mathcal{P}(X \times \mathbb{P})$, where $\mathcal{P}$ is the powerset operator. So X can be taken to be a fixed point of the operator

$$\Gamma_1(X) = \mathcal{P}(X \times \mathbb{P}). \tag{1}$$

If we call "names" only the objects of the least fixed point $\Gamma_{1\infty}$, then the non-well-founded sets will be nameless. To be concrete, suppose $x = \{(x,p)\}$, for $p \in \mathbb{P}$, is in our universe. Then x fulfils the requirements for being a name "in the wide sense", i.e., the property of x's being a name is compatible with the property required for its elements. (If x is supposed to be a name, then its element must consist of a pair whose first component, x, is a name and the second component is an element of $\mathbb{P}$. But this is the case, so no contradiction arises.) But obviously $x \notin \Gamma_{1\infty}$. However $x \in \Gamma_1^\infty$. To see this it suffices to show that $\{x\} \subseteq \Gamma_1^\infty$, or that $\{x\} \subseteq \Gamma_1(\{x\})$, i.e., $\{x\} \subseteq \mathcal{P}(\{x\} \times \mathbb{P})$, which is indeed the case.

Corollary 2.1 *The operator* Γ_1 *above has a g.f.p.*

Proof. Just check that Γ_1 is set continuous. $\dashv$

The class Γ_1^∞ will be called the *class of* $\mathbb{P}$*-names* and will be denoted by $V^{\mathbb{P}}$. Note that by its definition, the class Γ_1^∞ is definable. The letters t, s with subscripts range over names.

So far we have been working in ZFC$^-$. However in order to build the ingredients of forcing mechanism, ZFC$^-$ does not suffice. So from now on we shall be working in ZFCA=ZFC$^-$+AFA. Also instead of V, the real world, we shall be working in a transitive model M of ZFCA. For a notion of forcing $\mathbb{P} \in M$, $M^{\mathbb{P}}$ will denote the class of $\mathbb{P}$-names of M.

Let us first inspect the $\in$-graphs of names. Due to the fact that each name is a set of pairs and each pair (x, y) is defined to be the set $\{\{x\}, \{x, y\}\}$, the $\in$-graph $\gamma(t)$ of a name t, contains, for each $(s, p) \in t$ the subgraph consisting of the edges $t \to \{s\} \to s$, $t \to \{s, p\} \to s$, and $t \to \{s, p\} \to p$, i.e., the graph of figure 1.

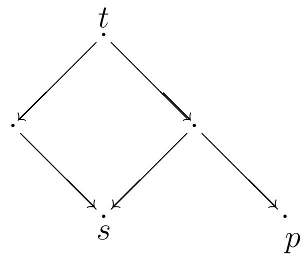

Figure 1

Now it is convenient to abbreviate the graph of figure 1 (including the descendants of p) by the single *weighted* edge $t \xrightarrow{p} s$. Intuitively s is a child of t with weight $p \in \mathbb{P}$. In general a $\mathbb{P}$-*weighted apg* (or just *weighted apg*) is an apg such that every edge is labelled by some $p \in \mathbb{P}$. (Formally, a weighted graph can be defined as a set of nodes and a set of triples (a, b, p), where p is the weight of the edge (a, b).)

Decorations of weighted graphs are to be slightly different from those of ordinary graphs.

Definition 2.2 Let g be a weighted graph. A mapping $d : g \to M$ is said to be a *decoration of* g if for every $a \in g$,

$$d(a) = \{(d(b), p) : a \xrightarrow{p} b \in g\}.$$

Lemma 2.3 *Let* $M \models \mathrm{ZFCA}$.

i) Every $\mathbb{P}$*-name* t *gives rise to a* $\mathbb{P}$*-weighted graph.*

ii) Every $\mathbb{P}$*-weighted graph* g *has a unique decoration* $d : g \to M$*. Moreover for every* $a \in g$*,* $d(a)$ *is a* $\mathbb{P}$*-name.*

Proof. i) This is obvious from the discussion that preceded definition 2.2.

ii) Let g be a weighted graph. The existence (and uniqueness) of a decoration for g follows from the Solution Lemma (see [1], p.13). Indeed we have to find a unique d such that for every $a \in g$

$$d(a) = \{(d(b), p) : a \xrightarrow{p} b \in g\}.$$

Consider the system of equations

$$x_a = \{(x_b, p) : a \xrightarrow{p} b \in g\},\ a, b \in g,$$

where x_a, $a \in g$, are variables (x_a and p are treated here as urelements. For details cf. [1] and [2]). By the Solution Lemma (which is essentially equivalent to AFA), the above system of equations has a unique solution. (ξ_a), $a \in g$. Setting $d(a) = \xi_a$ we obtain the required decoration of g.

To see that each $d(a)$ is a name, let $u = d''g$. Since $d(a) = \{(d(b), p) : a \xrightarrow{p} b \in g\}$, it follows that $d(a) \subseteq u \times \mathbb{P}$, or $d(a) \in \mathcal{P}(u \times \mathbb{P})$. Therefore $u \subseteq \mathcal{P}(u \times \mathbb{P}) = \Gamma_1(u)$ and so $u \subseteq \Gamma_1^\infty = M^{\mathbb{P}}$. This proves the claim. $\dashv$

Henceforth, for any name t, $\gamma(t)$ will denote the weighted graph corresponding to it.

Lemma 2.4 (ZFCA) *Let $M \models$ ZFCA be a standard transitive model, $\mathbb{P} \in M$ and G be a generic subset of $\mathbb{P}$. Then there is a G-interpretation of names $I_G : M^{\mathbb{P}} \to M$ such that for every $t \in M^{\mathbb{P}}$, $I_G(t) = \{I_G(s) : (\exists p \in G)((s, p) \in t)\}$.*

Proof. Let t be a name and let $\gamma(t)$ be its weighted graph. We transform $\gamma(t)$ into the graph $\gamma_G(t)$ as follows:

Call a path of $\gamma(t)$ *principal* if it is of the form:

$$t \xrightarrow{p_1} s_1 \xrightarrow{p_2} s_2 \xrightarrow{p_3} s_3 \cdots,$$

(finite or infinite) where all p_i are in G.

The algorithm for constructing $\gamma_G(t)$ is as follows:

Step 1. Erase all nodes which do not belong to some principal path.

Step 2. Delete the weights from the edges of the principal paths.

This completes the construction of $\gamma_G(t)$. Clearly $\gamma_G(t)$ is an apg with point, say, a_0. By AFA of the real world, there is a unique decoration d of $\gamma_G(t)$ and hence $\sigma(\gamma_G(t)) = d(a_0)$. So we can set

$$I_G(t) = \sigma(\gamma_G(t)).$$

This completes the definition of the G-interpretation I_G. It is easy to check that for every t, $I_G(t) = \{I_G(s) : (\exists p \in G)((s, p) \in t)\}$. $\dashv$

We come now to standard names. A name is *standard* if it consists of elements of the form $(s, \mathbf{1})$ where s is a standard name. So the class of standard names $\widehat{M}$ can be defined again as the g.f.p. of the operator

$$\Gamma_2(X) = \mathcal{P}(X \times \{\mathbf{1}\}).$$

Since for every X, $\Gamma_2(X) \subseteq \Gamma_1(X)$, it follows that $\Gamma_2^\infty \subseteq \Gamma_1^\infty$, i.e., $\widehat{M} \subseteq M^{\mathbb{P}}$. However this does not guarantee that for each $x \in M$ there is a standard name $\widehat{x}$ such that $I_G(\widehat{x}) = x$.

Lemma 2.5 *Let $M \models$ ZFCA and G be a generic set for M. Then for every $x \in M$ there is a set $\widehat{x} \in M$ such that:*
i) $I_G(\widehat{x}) = x$.
ii) $\widehat{x} = \{(\widehat{y}, \mathbf{1}) : y \in x\}$,
ii) $\widehat{x} \in \widehat{M}$,

Proof. Let $x \in M$ and consider the $\in$-graph $\gamma(x)$ of x. Replace every edge $a \to b$ of $\gamma(x)$, by the weighted edge $a \xrightarrow{\mathbf{1}} b$. This transforms $\gamma(x)$ into the weighted graph $\overline{\gamma(x)}$ with point x. By lemma 2.3, there is a unique decoration d of $\overline{\gamma(x)}$, and let $\widehat{x} = \sigma(\overline{\gamma(x)}) = d(x)$.

i) By lemma 2.3, $d(x)$, i.e. $\widehat{x}$, is a name and its weighted graph is just $\overline{\gamma(x)}$, i.e., $\gamma(\widehat{x}) = \overline{\gamma(x)}$. The procedure that leads from $\gamma(x)$ to $\overline{\gamma(x)}$ is, roughly, the converse of that leading from the weighted graph $\gamma(t)$ of a name to $\gamma_G(t)$. Therefore $\gamma_G(\widehat{x}) = \gamma(x)$ (or, which amounts to the same thing, $\gamma_G(\widehat{x}) \cong \gamma(x)$). Hence $I_G(\widehat{x}) = \sigma(\gamma_G(\widehat{x})) = \sigma(\gamma(x)) = x$.

ii) That $\widehat{x} = \{(\widehat{y}, \mathbf{1}) : y \in x\}$ follows immediately from the inspection of the graph of $\widehat{x}$.

iii) Since $\widehat{M} = \Gamma_2^\infty = \bigcup\{x : x \subseteq \Gamma_2(x)\}$, in order to prove that $\widehat{x} \in \widehat{M}$, we have to show that there is a set a such that $\widehat{x} \in a$ and $a \subseteq \Gamma_2(a)$. For any $\widehat{x}$, let $dom(\widehat{x}) = \{\widehat{y} : (\widehat{y}, \mathbf{1}) \in \widehat{x}\}$. Given $\widehat{x}$, define inductively the sequence (a_n), $n \in \mathbb{N}$, as follows: $a_0 = \{\widehat{x}\}$ and $a_{n+1} = \cup\{dom(\widehat{y}) : \widehat{y} \in a_n\}$. Let $a = \cup_n a_n$. Then $\widehat{x} \in a_0 \subseteq a$. On the other hand, for every $\widehat{y} \in a$, clearly $\widehat{y} \subseteq a \times \{\mathbf{1}\}$, so $\widehat{y} \in \mathcal{P}(a \times \{\mathbf{1}\}) = \Gamma_2(a)$. Therefore $a \subseteq \Gamma_2(a)$ and we are done. $\dashv$

Given a generic $G \subseteq \mathbb{P}$, let $M[G] = \{I_G(t) : t \in M^{\mathbb{P}}\}$.

Theorem 2.6 (ZFCA) *Let $M \models$ ZFCA, $\mathbb{P} \in M$ and $G \subseteq \mathbb{P}$ be generic. Then $M[G]$ exists, $G \in M[G]$ and $M \subseteq M[G]$.*

Proof. The existence of $M[G]$ follows from the existence of $I_G(t)$ for each name t, which is due to AFA of the metatheory. Also by the previous lemma $M \subseteq M[G]$. Given G, set $\underline{G} = \{(\widehat{p}, p) : p \in \mathbb{P}\}$. We show $\underline{G}$ is a name. Clearly $\underline{G} \subseteq M^{\mathbb{P}} \times \mathbb{P}$, and since $\underline{G}$ is a set, $\underline{G} \in \mathcal{P}(M^{\mathbb{P}} \times \mathbb{P}) = \Gamma_1(M^{\mathbb{P}})$. But $M^{\mathbb{P}} = \Gamma_1^{\infty}$, hence $\Gamma_1(M^{\mathbb{P}}) = M^{\mathbb{P}}$. So $\underline{G} \in M^{\mathbb{P}}$. Now if G is generic, $I_G(\underline{G}) = \{I_G(\widehat{p}) : p \in G\} = \{p : p \in G\} = G$, whence $G \in M[G]$. $M \subseteq M[G]$ follows from lemma 2.5 (i). $\dashv$

It is well known that every forcing poset $\mathbb{P}$ is densely embedded in a unique complete Boolean algebra denoted $B(\mathbb{P})$ (see e.g. [3]). Then the generic set G generates a generic ultrafilter on $B(\mathbb{P})$. Sometimes it is more convenient to work with $B(\mathbb{P})$ rather than $\mathbb{P}$. In the next sections we feel free to switch from $\mathbb{P}$ to $B(\mathbb{P})$ and from the generic set G to the generic ultrafilter it generates.

3 The forcing relation

In this section we intend to define the forcing relation $p \Vdash \phi$ for $p \in \mathbb{P}$ and $\phi \in L(M^{\mathbb{P}})$, to the effect that the following properties hold:

(a) $p \Vdash \phi$ is definable in M (Definability Lemma),

(b) $p \Vdash \phi \ \&\ q \le p \ \Rightarrow\ q \Vdash \phi$ (Extension Lemma) and

(c) $M[G] \models \phi(I_G(t_1), \ldots, I_G(t_n)) \Leftrightarrow (\exists p \in G)(p \Vdash \phi(t_1, \ldots, t_n))$ (Truth Lemma).

Of course as usual the crux of the matter is the definition of $p \Vdash t \in s$ and $p \Vdash t = s$. Further, $p \Vdash t \in s$ can be easily defined in terms of $p \Vdash t = s$ by setting

$$p \Vdash t \in s := (\exists t_1)(\exists q \ge p)((t_1, q) \in s \ \&\ p \Vdash t = t_1).$$

So our main task is to define $p \Vdash t = s$ so as to fulfill Lemmas (a)-(c) and especially the Truth Lemma. The Truth Lemma for $t = s$ amounts to the equivalence $M[G] \models I_G(t) = I_G(s) \iff (\exists p \in G)(p \Vdash t = s)$, or, since $M[G]$ is a transitive structure, to

$$I_G(t) = I_G(s) \iff (\exists p \in G)(p \Vdash t = s).$$

So first let us characterize $I_G(t) = I_G(s)$ in the context of ZFCA. For every $p \in \mathbb{P}$ and generic $G \subseteq \mathbb{P}$, let us write

$t \in_p s := (\exists q \geq p)((t,q) \in s)$,
$t \in_G s := (\exists p \in G)(t \in_p s)$.
Clearly $t \in_p s \ \&\ q \leq p \ \Rightarrow t \in_q s$. $M^{\mathbb{P}}$ endowed with $\in_G$ is a directed system.

Let $\equiv_G$ be the greatest bisimulation of $(M^{\mathbb{P}}, \in_G)$. It follows from lemma 1.6 that $\equiv_G$ is the g.f.p. of the operator Γ_G defined by:

$$\begin{aligned}\Gamma_G(X) = \{(t,s) : & (\forall t_1 \in_G t)(\exists s_1 \in_G s)((t_1,s_1) \in X)\ \& \\ & (\forall s_1 \in_G s)(\exists t_1 \in_G t)((t_1,s_1) \in X)\}.\end{aligned}$$

The required characterization of $I_G(t) = I_G(s)$ is the following:

Lemma 3.1 *For any $t, s \in M^{\mathbb{P}}$, $I_G(t) = I_G(s)$ iff $t \equiv_G s$ iff $(t,s) \in \Gamma_G^{\infty}$.*

Proof. By definition, $I_G(t) = \sigma((\gamma_G(t))$. Therefore $I_G(t) = I_G(s)$ iff $\sigma(\gamma_G(t)) = \sigma((\gamma_G(s))$. By 1.3, the latter holds if $\gamma_G(t) \equiv \gamma_G(s)$. Now clearly $\gamma_G(t)$ is the $\in_G$-graph of t and by the definition of the graph bisimilarity, $\gamma_G(t) \equiv \gamma_G(s)$ iff $t \equiv_G s$. $\dashv$

Therefore we have to define $p \Vdash t = s$ so as to satisfy

$$t \equiv_G s \iff (\exists p \in G)(p \Vdash t = s). \tag{2}$$

For that purpose we shall employ Kunen's definition of $p \Vdash^* t = s$ (see [4], p. 195) which is as follows:

$$p \Vdash^* t = s \iff \forall (t_1, p_1) \in t$$

$$\{q \leq p : q \leq p_1 \Rightarrow \exists (s_1, q_1) \in s (q \leq q_1 \ \&\ q \Vdash^* t_1 = s_1)\}$$

is dense below p, and $\forall (s_1, q_1) \in s$

$$\{q \leq p : q \leq q_1 \Rightarrow \exists (t_1, p_1) \in t (q \leq p_1 \ \&\ q \Vdash^* t_1 = s_1)\}$$

is dense below p.

Of course we have also Shoenfield's [6] more standard simultaneous definitions of $p \Vdash^* t = s$ and $p \Vdash^* t \in s$, and the two definitions are proved equivalent over a well-founded set universe (at least for $p \Vdash^* t = s$ and $p \Vdash^* t \in s$ with $p \in G$). But when foundation is missing their equivalence is open. And of the two, the most suitable one turns out to be Kunen's (see Remark 3.11 below).

When foundation is missing the above relation $p \Vdash^* t = s$ can be realized as the g.f.p. of the following operator Φ:

Definition 3.2 For every $X \subseteq \mathbb{P} \times M^{\mathbb{P}} \times M^{\mathbb{P}}$, $(p,t,s) \in \Phi(X)$ iff $\forall (t_1, p_1) \in t$

$$\{q \leq p : q \leq p_1 \Rightarrow \exists (s_1, q_1) \in s \ (q \leq q_1 \ \& \ (q, t_1, s_1) \in X)\}$$

is dense below p, and $\forall (s_1, q_1) \in s$

$$\{q \leq p : q \leq q_1 \Rightarrow \exists (t_1, p_1) \in t \ (q \leq p_1 \ \& \ (q, t_1, s_1) \in X)\}$$

is dense below p.

Lemma 3.3 *i) Φ is set continuous; hence it has a g.f.p.*
ii) $(p,t,s) \in \Phi(X)$ & $q \leq p \ \Rightarrow \ (q,t,s) \in \Phi(X)$.

Proof. i) Left to the reader. (Lemma 3.8 below provides a detailed analysis of the conditions such as $(p,t,s) \in \Phi(x)$. This analysis proves also this claim.)

ii) Immediate from the definition. $\dashv$

For every $p \in \mathbb{P}$, let $(p] = \{q : q \leq p\}$. Below x, y are elements of M.

Lemma 3.4 *For every $x \subseteq M^{\mathbb{P}} \times M^{\mathbb{P}}$ and $t, s \in M^{\mathbb{P}}$,*

$$(t,s) \in \Gamma_G(x) \ \Rightarrow \ (\exists p \in G)(p,t,s) \in \Phi((p] \times x).$$

Proof. We argue by contradiction. Assume the contrary and fix $x \subseteq M^{\mathbb{P}} \times M^{\mathbb{P}}$ and t, s such that $(t,s) \in \Gamma_G(x)$ and $(\forall p \in G)(p,t,s) \notin \Phi((p] \times x)$. The first of them yields

$$(\forall t_1 \in_G t)(\exists s_1 \in_G s)((t_1, s_1) \in x) \ \& \ (\forall s_1 \in_G s)(\exists t_1 \in_G t)((t_1, s_1) \in x). \quad (3)$$

Also, by definition 3.2, the formula $(p,t,s) \notin \Phi((p] \times x)$ is written as follows:

$$\exists (t_1, p_1) \in t \ (\exists q \leq p)(\forall r \leq q)$$

$$[(r \le p_1 \ \& \ \forall (s_1, q_1) \in s \ (r \le q_1 \Rightarrow (r, t_1, s_1) \notin (p] \times x))] \vee$$
$$\exists (s_1, q_1) \in s \ (\exists q \le p)(\forall r \le q)$$
$$[(r \le q_1 \ \& \ \forall (t_1, p_1) \in t \ (r \le p_1 \Rightarrow (r, t_1, s_1) \notin (p] \times x))].$$

The last formula is made of two symmetric disjuncts. To make it more readable first observe that for $r \le q$, $r \in (p]$, hence $(r, t_1, s_1) \notin (p] \times x)$ iff $(t_1, s_1) \notin x$. Also we can interchange the initial existential quantifiers, in each disjunct. Let us set

$$A(q, t, s, x) := \exists (t_1, p_1) \in t \ (\forall r \le q)$$
$$[(r \le p_1 \ \& \ \forall (s_1, q_1) \in s(r \le q_1 \Rightarrow (t_1, s_1) \notin x)] \tag{4}$$

Then the formula $(p, t, s) \notin \Phi((p] \times x)$ is written

$$(\exists q \le p)[A(q, t, s, x) \ \vee \ A(q, s, t, x)].$$

And thus our assumption $(\forall p \in G)(p, t, s) \notin \Phi((p] \times x)$ is the formula:

$$(\forall p \in G)(\exists q \le p)[A(q, t, s, x) \ \vee \ A(q, s, t, x)] \tag{5}$$

Define $f : \mathbb{P} \to B(\mathbb{P})$ as follows:

$$f(p) = \bigvee\{q \le p : A(q, t, s, x)\}, \text{ if } (\exists q \le p) A(q, t, s, x),$$
$$\bigvee\{q \le p : A(q, s, t, x)\} \text{ otherwise.}$$

Obviously (a) f is definable, (b) $f(p) \le p$ for all $p \in \mathbb{P}$ and, by (5), (c) for all $p \in G$, $f(p) \le p$ and $f(p) \ne 0$.

Claim. For all $p \in \mathbb{P}$, $f(p) \notin G$.

Proof. For $p \notin G$ this is obvious since $f(p) \le p$. Let $p_0 \in G$. Suppose $f(p_0) \in G$. We shall reach a contradiction. By (5), $f(p_0) = \bigvee\{q \le p_0 : A(q, t, s, x)\}$ or $f(p_0) = \bigvee\{q \le p_0 : A(q, s, t, x)\}$. Assume the first, the other case being similar. By the genericity of G, there is $q_0 \le p_0$ such that $q_0 \in G$ and $A(q_0, t, s, x)$ holds, i.e.,

$$\exists (t_1, p_1) \in t(\forall r \le q_0)[(r \le p_1 \ \& \ (\forall (s_1, q_1) \in s)(r \le q_1 \Rightarrow (t_1, s_1) \notin x)]. \tag{6}$$

Fix a pair $(t_1, p_1) \in t$ having the above property. Then by (6), $q_0 \le p_1$ and since $q_0 \in G$, it follows $p_1 \in G$ and $t_1 \in_G t$. From the latter and assumption (3), there is s_1 and $q_1 \in G$ such that $(s_1, q_1) \in s$ and $(t_1, s_1) \in x$. Let $q_2 = q_0 \cdot q_1$. Then $q_2 \le q_0, q_1$, $(s_1, q_1) \in s$ and $(t_1, s_1) \in x$. But this contradicts (6). This proves the claim.

By the Claim, for all $p \in \mathbb{P}$, $-f(p) \in G$. By the genericity of G, $\bigwedge_{p \in \mathbb{P}} -f(p) = r \in G$. But then $f(r) \le r \le -f(r)$, hence $f(r) = 0$ which is impossible since $r \in G$. This proves the lemma. $\dashv$.

In fact the previous lemma can be strengthened as follows.

Lemma 3.5

$$y \subseteq \Gamma_G(x) \;\Rightarrow\; (\exists p \in G)(\forall (t,s) \in y)(p,t,s) \in \Phi((p] \times x).$$

Proof. The proof is analogous to that of 3.4 so we only sketch it. We argue again by contradiction and let $y \subseteq \Gamma_G(x)$ while

$$(\forall p \in G)(\exists (t,s) \in y)(p,t,s) \notin \Phi((p] \times x).$$

Using the formula $A(q,t,s,x)$ of the previous proof, the last formula is written

$$(\forall p \in G)(\exists q \le p)(\exists (t,s) \in y)[A(q,t,s,x) \;\vee\; A(q,s,t,x)] \tag{7}$$

Define $f : \mathbb{P} \to B(\mathbb{P})$ by:

$$f(p) = \begin{array}{l} \bigvee\{q \le p : (\exists (t,s) \in y)A(q,t,s,x)\}, \text{ if } (\exists q \le p)(\exists (t,s) \in y)A(q,t,s,x), \\ \bigvee\{q \le p : (t,s) \in y)A(q,s,t,x)\} \text{ otherwise.} \end{array}$$

It suffices to prove $f(p) \notin G$ for all $p \in G$. Assume $f(p_0) \in G$. Then there is $q_0 \in G$ such that $(\exists (t,s) \in y)A(q_0,t,s,x)$. So for some $(t_0,s_0) \in y$, $A(q_0,t_0,s_0,x)$. Since $(t_0,s_0) \in \Gamma_G(x)$, we get as before a contradiction $\dashv$

Corollary 3.6 *For all names t, s,*

$$t \equiv_G s \;\Rightarrow (\exists p \in G)((p,t,s) \in \Phi^\infty).$$

Proof. Let $t \equiv_G s$, i.e. $(t,s) \in \Gamma_G^\infty$. Since Γ_G is set continuous there is $x \subseteq M^{\mathbb{P}} \times M^{\mathbb{P}}$ such that $(t,s) \in x \subseteq \Gamma_G(x)$. By lemma 3.5, there is $p_0 \in G$ such that $(\forall (t,s) \in x)(p_0,t,s) \in \Phi((p_0] \times x)$. Now it is easy to see that

$$(p_0] \times x \subseteq \Phi((p_0] \times x).$$

Indeed let $(q,t,s) \in (p_0] \times x$, i.e., $q \leq p_0$ and $(t,s) \in x$. Then $(p_0,t,s) \in \Phi((p_0] \times x)$, and hence, by lemma 3.3, $(q,t,s) \in \Phi((p] \times x)$. So if we set $z = (p_0] \times x$, then $(p_0,t,s) \in z \subseteq \Phi(z)$. By lemma 1.5, this means that $(p_0,t,s) \in \Phi^\infty$. $\dashv$

We now prove the converse of 3.4.

Lemma 3.7 *Let $y \subseteq \mathbb{P} \times M^{\mathbb{P}} \times M^{\mathbb{P}}$. Then*

$$(\exists p \in G)(p,t,s) \in \Phi(y) \;\Rightarrow\; (t,s) \in \Gamma_G(pr(y)),$$

where $pr(y) = \{(t,s) : (\exists p)(p,t,s) \in y\}$.

Proof. Let $(p,t,s) \in \Phi(y)$ for some $p \in G$, and let $t_1 \in_{p_1} t$ for some $p_1 \in G$. We shall find s_1 and $q_1 \in G$ such that $s_1 \in_{q_1} \in s$ and $(t_1,s_1) \in pr(y)$. Let $p_2 = p \cdot p_1$. Then $p_2 \in G$ and, by the assumption, the set

$$D = \{q \leq p_2 : q \leq p_1 \Rightarrow \exists (s_1,q_1) \in s(q \leq q_1 \;\&\; (q,t_1,s_1) \in y)\}$$

is dense below p_2. Since $p_2 \in G$, $D \cap G \neq \emptyset$. Let $r \in D \cap G$. Then there is $(s_1,q_1) \in s$ with $r \leq q_1$ and $(t_1,s_1) \in pr(y)$. So $s_1 \in_G s$ and $(t_1,s_1) \in pr(y)$. We showed that $(\forall t_1 \in_G t)(\exists s_1 \in_G s)(t_1,s_1) \in pr(y)$. Similarly we see that $(\forall s_1 \in_G s)(\exists t_1 \in_G t)(t_1,s_1) \in pr(y)$. Therefore $(t,s) \in \Gamma_G(pr(y))$. $\dashv$

Lemma 3.8 *Let $p_0 \in G$, and t_0, s_0, y such that $(p_0,t_0,s_0) \in y \subseteq \Phi(y)$. Then there is $u \subseteq y$ such that $(p_0,t_0,s_0) \in u \subseteq \Phi(u)$ and $\forall (t,s) \in pr(u)(\exists q \in G)((q,t,s) \in u)$.*

Proof. Let (p,t,s) be given. For every $(t_1,p_1) \in t$ and $(s_1,q_1) \in s$ let

$$\Phi^{-1}_{(t_1,p_1)}(p,t,s) = \{(q,t_1,s_1) : q \leq p \cdot p_1 \;\Rightarrow\; (\exists q_1)((s_1,q_1) \in s \;\&\; q \leq q_1)\},$$

$$\Phi^{-1}_{(s_1,q_1)}(p,t,s) = \{(q,t_1,s_1) : q \leq p \cdot q_1 \;\Rightarrow\; (\exists p_1)((t_1,p_1) \in t \;\&\; q \leq p_1)\}.$$

It is easy to check that for every (p,t,s) end every y,

$$(p,t,s) \in \Phi(y) \iff$$

a) For all $(t_1,p_1) \in t$, $\Phi^{-1}_{(t_1,p_1)}(p,t,s) \subseteq y$ and

$$\{q : (\exists s_1)(q,t_1,s_1) \in \Phi^{-1}_{(t_1,p_1)}(p,t,s)\}$$

is dense below p, and

b) For all $(s_1,q_1) \in s$, $\Phi^{-1}_{(s_1,q_1)}(p,t,s) \subseteq y$ and

$$\{q : (\exists t_1)(q,t_1,s_1) \in \Phi^{-1}_{(s_1,q_1)}(p,t,s)\}$$

is dense below p.

Further, let us set for every (p,t,s) and for every set w,

$$\Phi^{-1}(p,t,s) = \bigcup\{\Phi^{-1}_{(t_1,p_1)}(p,t,s), \Phi^{-1}_{(s_1,q_1)}(p,t,s) : (t_1,p_1) \in t, (s_1,q_1) \in s\},$$

$$\Phi^{-1}(w) = \bigcup\{\Phi^{-1}(p,t,s) : (p,t,s) \in w\}.$$

Let now $(p_0,t_0,s_0) \in y \subseteq \Phi(y)$ with $p_0 \in G$. Define the sets $(u_n), n \in \mathbb{N}$, and u as follows: $u_0 = \{(p_0,t_0,s_0)\}$, $u_{n+1} = \Phi^{-1}(u_n)$ and $u = \bigcup_n u_n$.

Claim.
(i) $(p_0,t_0,s_0) \in u$.
(ii) $u_n \subseteq y$ for every n.
(iii) $u_n \subseteq \Phi(u_{n+1})$ for every n. Therefore $u \subseteq \Phi(u)$.
(vi) For every $(t,s) \in pr(u)$ there is $q \in G$ such that $(q,t,s) \in u$.

Proof. (i) is obvious. (ii) By induction on n. Obviously $u_0 \subseteq y$. Now since $y \subseteq \Phi(y)$, clearly $\Phi^{-1}(y) \subseteq y$. So suppose $u_n \subseteq y$. Then

$$u_{n+1} = \Phi^{-1}(u_n) \subseteq \Phi^{-1}(y) \subseteq y.$$

(iii) Let $(q,t_1,s_1) \in u_n$. Then $\Phi^{-1}(q,t_1,s_1) \subseteq \Phi^{-1}(u_n) = u_{n+1}$. Now since $(q,t_1,s_1) \in u_n \subseteq y \subseteq \Phi(y)$, the density conditions (a), (b) mentioned above are satisfied, therefore $(q,t_1,s_1) \in \Phi(u_{n+1})$. This shows that $u_n \subseteq \Phi(u_{n+1})$.

(iv) Let $(t_1,s_1) \in pr(u)$. We have to show that there is $q \in G$ such that $(q,t_1,s_1) \in u$. This is proved by induction on n for the elements of the sets

$pr(u_n)$. To illustrate the idea let us prove it for the elements of $pr(u_1)$. Then the induction step is easily grasped. Take some $(t_1, s_1) \in pr(u_1)$. Then for some q, $(q, t_1, s_1) \in u_1 = \Phi^{-1}(p_0, t_0, s_0)$. It follows that (q, t_1, s_1) belongs either to $\Phi^{-1}_{(t_1,p_1)}(p_0, t_0, s_0)$ for some p_1 such that $(t_1, p_1) \in t_0$, or to $\Phi^{-1}_{(s_1,q_1)}(p_0, t_0, s_0)$ for some q_1 such that $(s_1, q_1) \in s_0$.

Case 1. Suppose that there is p_1 such $(t_1, p_1) \in t_0$ and $p_1 \notin G$. Then obviously $p_0 \not\leq p_1$, and hence the implication in the defining condition of $\Phi^{-1}_{(t_1,p_1)}(p_0, t_0, s_0)$ is vacuously true. Hence $(p_0, t_1, s_1) \in \Phi^{-1}_{(t_1,p_1)}(p_0, t_0, s_0)$ for every $s_1 \in dom(s)$. Since $p_0 \in G$, the claim holds.

Case 2. Suppose that there is q_1 such $(s_1, q_1) \in s_0$ and $q_1 \notin G$. As before $(p_0, t_1, s_1) \in \Phi^{-1}_{(s_1,q_1)}(p_0, t_0, s_0)$ for every $t_1 \in dom(t)$ and the claim holds.

Case 3. Suppose that cases 1 and 2 are false, i.e., $(t_1, p_1) \in t_0 \Rightarrow p_1 \in G$ and $(s_1, q_1) \in s_0 \Rightarrow q_1 \in G$. Then take any p_1 such that $(t_1, p_1) \in t_0$ and any q_1 such that $(s_1, q_1) \in s_0$. Then $p_0, p_1, q_1 \in G$. Take $q = p_0 \cdot p_1 \cdot q_1$. Then $q \in G$ and (q, t_1, s_1) satisfies the defining condition of $\Phi^{-1}_{(t_1,p_1)}(p_0, t_0, s_0)$. Therefore $(q, t_1, s_1) \in \Phi^{-1}_{(t_1,p_1)}(p_0, t_0, s_0)$ and the condition holds.

This completes the proof of the claim and the lemma. $\dashv$

Corollary 3.9 *For all names t, s,*

$$(\exists p \in G)((p, t, s) \in \Phi^\infty) \;\Rightarrow\; t \equiv_G s.$$

Proof. Let $(p_0, t_0, s_0) \in \Phi^\infty$ for some $p_0 \in G$. We have to show that $(t_0, s_0) \in \Gamma^\infty_G$. By the set continuity of Φ there is y such that $(p_0, t_0, s_0) \in y \subseteq \Phi(y)$. By lemma 3.8 we may assume that for all $(t, s) \in pr(y)$ there is $q \in G$ such that $(q, t, s) \in y$. By lemma 3.7 $(p_0, t_0, s_0) \in \Phi^\infty$ and $p_0 \in G$ entail that $(t_0, s_0) \in pr(y)$. We claim that

$$pr(y) \subseteq \Gamma_G(pr(y)).$$

Indeed let $(t, s) \in pr(y)$. By the condition for y, there is $q \in G$ such that $(q, t, s) \in y \subseteq \Phi(y)$. So by 3.4, $(t, s) \in \Gamma_G(pr(y)$. So we found $z = pr(y)$ such that $(t_0, s_0) \in z \subseteq \Gamma_G(z)$. Therefore $(t_0, s_0) \in \Gamma^\infty_G$. $\dashv$

Theorem 3.10 *For all names t, s,*

$$t \equiv_G s \;\Leftrightarrow (\exists p \in_G)((p, t, s) \in \Phi^\infty).$$

Proof. Immediate from corollaries 3.6 and 3.9. $\dashv$

Remark 3.11 If instead of Kunen's definition of $p \Vdash^* t = s$ one employs Shoenfield's definition cited in [6], p.375, one is led to the following "circular" formula:

$$p \Vdash^* t = s \Leftrightarrow (\forall q \leq p)[(\forall t_1 \in_q t)(\exists q_1 \leq q)(\exists s_1 \in_{q_1} s)(q_1 \Vdash^* t_1 = s_1) \;\&\; (\forall s_1 \in_q s)(\exists q_1 \leq q)(\exists t_1 \in_{q_1})(q_1 \Vdash^* t_1 = s_1)].$$

If Φ_0 is the operator induced by the last formula, then one can prove lemma 3.4 for Φ_0, but lemma 3.7 remains open. Hence Φ_0 is weaker than Φ. A natural strengthening of Φ_0 is obtained if we take the "uniform" variant of the above formula with respect to the quantifier $\exists q_1 \leq q$, i.e., if we define $p \Vdash^* t = s$ by:

$$p \Vdash^* t = s \Leftrightarrow (\forall q \leq p)(\exists q_1 \leq q)[(\forall t_1 \in_q t)(\exists s_1 \in_{q_1} s)(q_1 \Vdash^* t_1 = s_1) \;\&\; (\forall s_1 \in_q s)(\exists t_1 \in_{q_1})(q_1 \Vdash^* t_1 = s_1)].$$

Now if Φ_1 is the operator induced by the last formula, then we can prove lemma 3.7 for Φ_1, but 3.4 remains open. That is, Φ_1 is stronger than Φ. In general we have for every X, $\Phi_1(X) \subseteq \Phi(X) \subseteq \Phi_0(X)$. But I do not know if these inclusions are proper, neither whether theorem 3.10 holds for Φ_0 and Φ_1.

Definition 3.12 For each particular ϕ, the relation $p \Vdash \phi$ is defined inductively as follows:
(a) $p \Vdash t = s$ if $(p, t, s) \in \Phi^\infty$.
(b) $p \Vdash t \in s$ if $(\exists s_1 \in_p s)(p \Vdash s_1 = t)$.
(c) $p \Vdash \phi \;\&\; \psi$ if $p \Vdash \phi$ and $p \Vdash \psi$.
(d) $p \Vdash \neg\phi$ if $(\forall q \leq p)(\neg(p \Vdash \phi))$.
(e) $p \Vdash (\forall v)\phi(v)$ if $(\forall t \in M^{\mathbb{P}})(p \Vdash \phi(t))$.

Lemma 3.13 (Definability Lemma) *The relation "$p \Vdash \phi$" is definable in M.*

Proof. Immediate from definition 3.12 and the fact that Φ^∞ is definable. $\dashv$

Lemma 3.14 (Extension Lemma) *If $p \Vdash \phi$ and $q \leq p$, then $q \Vdash \phi$.*

Proof. By lemma 3.3, we easily infer that $(p, t, s) \in \Phi^\infty$ and $q \leq p$ imply $(q, t, s) \in \Phi^\infty$. Therefore the claim holds for $t = s$. Also $t \in_p s$ and $q \leq p$ imply $t \in_q s$. So it holds also for $t \in s$. The other steps are trivial. $\dashv$

Lemma 3.15 (Truth Lemma) (ZFCA) *Let $M \models$ ZFCA, $\mathbb{P} \in M$ a forcing notion and $G \subseteq \mathbb{P}$ generic. If $\phi(v_1, \ldots, v_n)$ is any formula of L with free variables among v_i, and $t_1, \ldots, t_n \in M^{\mathbb{P}}$, then*

$$M[G] \models \phi(I_G(t_1), \ldots, I_G(t_n)) \iff (\exists p \in G)(p \Vdash \phi(t_1, \ldots, t_n)).$$

Proof. By induction on the length of ϕ.

(a) Let $\phi := t = s$, From lemma 3.1, theorem 3.10 and clause (a) of definition 3.12 we have

$$M[G] \models I_G(t) = I_G(s) \iff t \equiv_G s \iff$$

$$(\exists p \in G)((p, t, s) \in \Phi^\infty) \iff (\exists p \in G)((p \Vdash t = s).$$

(b) Let $\phi := t \in s$. Suppose $(\exists p \in G)(p \Vdash t \in s)$. Then for some $p \in G$, $(\exists s_1 \in_p s)(p \Vdash s_1 = t)$. So $s_1 \in_G s$ and, by (a) above, $I_G(s_1) = I_G(t)$. But $s_1 \in_G s$ implies $I_G(s_1) \in I_G(s)$, hence $I_G(t) \in I_G(s)$. Conversely, let $M \models I_G(t) \in I_G(s)$. Then there is $s_1 \in_G s$ such that $I_G(s_1) = I_G(t)$. Thus, by (a) above, there is $p \in G$ such that $p \Vdash s_1 = t$ and there is $p_1 \in G$ such that $s_1 \in_{p_1} s$. Taking a common extension $q \leq p, p_1$ such that $q \in G$, and using the extension lemma, we have $s_1 \in_q s$ and $q \Vdash s_1 = t$. Therefore $(\exists q \in G)(\exists s_1 \in_q s)(q \Vdash s_1 = t)$, or $(\exists q \in G)(q \Vdash t \in s)$.

The proofs of the inductive steps (c) (d) and (e) are standard. $\dashv$

4 Application

Theorem 4.1 (ZFCA) *Let M be a standard model of* ZFCA. *Then for every generic set G, $M[G] \models$ ZFCA.*

Proof. The condition of standardness is needed only to ensure that every directed graph in the sense of M is a directed graph in the real world. If for instance ω^M is not standard, then M contains graphs having paths of nonstandard length, and these are obviously different from graphs whose paths are standard.

The proof that $M[G] \models \mathrm{ZFC}^-$ is standard using the clauses of Truth Lemma 3.15. So it suffices to show that $M[G] \models$ AFA, i.e., that every directed graph of $M[G]$ has a unique decoration in $M[G]$. Remember that our metatheory is ZFCA because AFA is needed for the existence of $M[G]$ (cf. theorem 2.6). This together with the standardness of M makes uniqueness trivial. Indeed if $g \in M[G]$ is a graph in the sense of M, and d_1, d_2 are two decorations of g in $M[G]$, then g is also a graph and d_1, d_2 are decorations of g in the real world, so by AFA of the metatheory, $d_1 = d_2$. We come to prove existence.

Let $(g, \to_g)$ be a directed graph in $M[G]$. To make things simpler let us assume that $g = g_n \cup g_e$, where g_n, g_e are the disjoint sets of nodes and edges of g respectively. So we write $(a, b) \in g_e$ instead of $a \to_g b$. We must prove that there is a mapping $e : g \to M[G]$ such that for all $a \in g_n$,

$$e(a) = \{e(b) : (a, b) \in g_e\}.$$

Let us fix names $\underline{g_n}$, $\underline{g_e}$ for g_n and g_e, as well as a name $\underline{a}$ for each node $a \in g_n$. Let $a, b \in g$. By the Truth Lemma we have

$$(a, b) \in g_e \iff M[G] \models (a, b) \in g_e \iff (\exists p \in G)(p \Vdash (\underline{a}, \underline{b}) \in \underline{g_e}). \tag{8}$$

Define the weighted graph g' as follows: The nodes of g' are names of nodes of g and for every two such nodes t, s

$$t \xrightarrow{p} s \in g' \iff p \Vdash (t, s) \in \underline{g_e}. \tag{9}$$

From (8) and (9) we get

$$(a, b) \in g_e \iff (\exists p \in G)(\underline{a} \xrightarrow{p} \underline{b} \in g'). \tag{10}$$

Clearly $g' \in M$, so by $M \models$ AFA and lemma 2.3, there is a decoration d for g'. It means that for every $t \in g'$, We have

$$d(t) = \{(d(s), p) : t \xrightarrow{p} s \in g'\},$$

or

$$(d(s), p) \in d(t) \iff t \xrightarrow{p} s \in g'. \tag{11}$$

Therefore all $d(t)$, for $t \in g'$, are names in M. Consider the mapping $e : g \to M[G]$, such that

$$e(a) = I_G(d(\underline{a})).$$

e is the required decoration of g. First note that $e \in M[G]$, since $M[G] \models$ ZFC$^-$. Further from (10), (11) and the definition of I_G we have for all a, b:

$$I_G(d(\underline{a})) = \{I_G(d(\underline{b})) : (\exists p \in G)((d(\underline{b}), p) \in d(\underline{a}))\} =$$

$$\{I_G(d(\underline{b})) : (\exists p \in G)(\underline{a} \xrightarrow{p} \underline{b} \in g')\} = \{I_G(d(\underline{b})) : (a, b) \in g_e\}.$$

Therefore, $I_G(d(\underline{a})) = \{I_G(d(\underline{b})) : (a, b) \in g_e\}$ and hence

$$e(a) = \{e(b) : (a, b) \in g_e\}.$$

This completes the proof. $\dashv$

Aczel has proved that for every $M \models$ ZFC there is a unique $N \supseteq M$ such that $N \models$ ZFCA and M is the class of well-founded sets of N (see [1], Theorem 3.10). Given $M \models$ ZFC, let $afa(M)$ denote this unique AFA-extension of M. Also given $N \models$ ZFCA, let N_{wf} denote the well-founded part of N. Then clearly $afa(N_{wf}) = N$ since both N and $afa(N_{wf})$ are AFA-extensions of N_{wf}.

Corollary 4.2 (ZFCA)

(i) Let $M \models$ ZFC and let $G \subseteq M$ be generic. Then $afa(M[G]) = (afa(M))[G]$.

(ii) In particular, if $N \models$ ZFCA and $G \subseteq N_{wf}$, then $N[G] = afa(N_{wf}[G])$. That is, for such particular G, the generic extension of N is reduced to the ordinary generic extension of its well-founded part.

Proof. (i) By theorem 4.1, $(afa(M))[G]$ is a model of ZFCA. So the claim follows from the fact that both $afa(M[G])$ and $(afa(M))[G]$ are AFA-extensions of the well-founded universe $M[G]$.

(ii) Similarly $N[G]$ and $afa(N_{wf}[G])$ are AFA-extensions of $N_{wf}[G]$). $\dashv$

Remark 4.3 1) Note that the condition $G \subseteq N_{wf}$ in 4.2 (ii) above is not an essential restriction. Given $N \models$ ZFCA and a poset $\mathbb{P} \in N$ one can always choose a poset $\mathbb{P}' \in N_{wf}$ isomorphic to $\mathbb{P}$, and thus we can take $G \subseteq N_{wf}$ without any loss of generality.

2) Given $N \models$ ZFCA and $G \subseteq N_{wf}$, the structure $N\langle G\rangle := afa(N_{wf}[G])$ can be defined independently of the machinery developed in this paper. It is easy to check that $N\langle G\rangle$ is the smallest model of ZFCA extending N

and containing G. However this does not guarantee that $N\langle G\rangle$ is a *forcing extension* of N. From a certain point of you, the main theorem 4.1 proves exactly this: That $N\langle G\rangle = N[G]$, where the latter is a genuine forcing extension.

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