What is so special with the powerset operation?

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Abstract

The powerset operator, $\mathcal{P}$, is an operator which (1) sends sets to sets,(2) is defined by a positive formula and (3) raises the cardinality of its argument, i.e., $|\mathcal{P}(x)| > |x|$. As a consequence of (3), $\mathcal{P}$ has a proper class as least fixed point (the universe itself). In this paper we address the questions: (a) How does $\mathcal{P}$ contribute to the generation of the class of all positive operators? (b) Are there other operators with the above properties, "independent" of $\mathcal{P}$?

Concerning (a) we show that every positive operator is a combination of the identity, powerset, and almost constant operators. This enables one to define what a $\mathcal{P}$-independent operator is. Concerning (b) we show that every $\mathcal{P}$-independent bounded positive operator is not $\mathcal{P}$-like.

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1 Introduction

The powerset operation is at the same time the strength and the weakness of set theory: The strength because it is the main tool for building the set universe (if there is such a thing), and the weakness because of its relative character. There are a great many different powersets of a given set (depending on the model in which the set is located), with none of them being any better or worse with respect to the others. Specifically many people doubt e.g. a) if there is such a thing as "the universe of sets", b) if there is such a thing as "the set of real numbers", c) if the Continuum Hypothesis (CH) is a definite mathematical statement. For example in Feferman [2], p. 405, we read:

> My own view – as is widely known – is that the Continuum Hypothesis is what I have called an "inherently vague" statement, and that the continuum itself, or equivalently the powerset of the natural numbers, is *not* a definite mathematical object. Rather, it's a conception we have of the totality of "arbitrary" subsets of the set of natural numbers, a conception that is clear enough for us to ascribe many evident properties to that supposed object (such as the impredicative comprehension axiom scheme) but which cannot be sharpened in any way to determine or fix that object itself.

Let $L = \{\in\}$ be the language of set theory augmented with class (i.e. second-order) variables denoted by upper case letters X, Y, S etc. As usually lower case variables range over sets. In this paper we treat the powerset $\mathcal{P}$ as an operator, i.e., as a definable mapping from classes to classes. In general a (unary) *operator with parameters* is produced by a second-order formula $\phi(v, \overline{c}, S)$ of the language of set theory, where S is a class variable, v is a set variable and $\overline{c} = (c_1, \ldots, c_n)$ is a tuple of set parameters. $\phi(v, \overline{c}, S)$ gives rise to the operator $\Gamma_{\phi(\overline{c})}$ defined by

$$\Gamma_{\phi(\overline{c})}(X) = \{x : \phi(x, \overline{c}, X)\}. \tag{1}$$

In general $\Gamma_{\phi(\overline{c})}$ sends classes to classes but, mainly, we shall be interested in those ϕ such that for every set a, $\Gamma_{\phi(\overline{c})}(a)$ is a set. Such an operator will be called *set-theoretic*, or a *set-operator*.

An operator $\Gamma_{\phi(\overline{c})}$ is said to be *monotone* if $X \subseteq Y \Rightarrow \Gamma_{\phi(\overline{c})}(X) \subseteq \Gamma_{\phi(\overline{c})}(Y)$. In order for $\Gamma_{\phi(\overline{c})}$ to be monotone it suffices for ϕ to be *positive in S*. ϕ is positive in S if it is constructed from formulas not containing S and atomic formulas $u \in S$ using only the logical operations $\wedge$, $\vee$, $\exists$ and $\forall$. (See e.g. [3].)

Remark 1.1 Most of the time we shall be dealing with mappings $\Gamma_{\phi(\overline{c})}(a)$ where $\Gamma_{\phi(\overline{c})}$ is a set-theoretic positive operator and a is a set. So why appeal to formulas $\phi(v, \overline{c}, S)$ with a class variable S? The reason is that in the definition of positive formulas we want the only atomic formula containing S to be of the form $u \in S$. If instead of S we had a set variable s, then we should allow positive formulas containing, besides $u \in s$, the atomic formulas $u = s$ and $s \in u$. This on the one hand may contradict the fact that $\Gamma_{\phi(\overline{c})}$ is set theoretic (e.g. if $s \in u$ is allowed) and on the other it makes lemma 1.2 below false.

The following canonical form for positive formulas will be heavily used throughout this paper.

Lemma 1.2 (Moschovakis) *Let $\phi(v, \overline{c}, S)$ be a positive formula of L. Then there is a quantifier-free and S-free formula $\theta(v, \overline{c}, \overline{w}, u)$, where $\overline{w} = (w_1, \ldots, w_m)$, and a string of quantifiers $\overline{Q} = (Q_1, \ldots, Q_m)$ such that, for every x and every class $X \neq V$,*

$$\phi(x, \overline{c}, X) \iff (\overline{Q}\overline{w})(\forall u)(\theta(x, \overline{c}, \overline{w}, u) \vee u \in X).$$

Proof. See [3], pp. 57-58. ⊣

By 1.2 we may assume that every positive formula has the form

$$\phi(v, \overline{c}, S) := (\overline{Q}\overline{w})(\forall u)(\theta(v, \overline{c}, \overline{w}, u) \vee u \in S). \tag{2}$$

We shall refer to (2) as the *canonical form* of ϕ. The string of quantifiers $\overline{Q}$ in the above form measures the *complexity* of ϕ.

The letters Γ, Δ etc. will range over positive operators. Positivity is a syntactic property, hence absolute for the various fragments of ZF. So let us set

$$\mathcal{O} = \{\Gamma : \Gamma \text{ is positive operator of the language of set theory}\}.$$

The main operation in $\mathcal{O}$ is composition, but there are also certain natural finitary operations under which $\mathcal{O}$ is closed. Given $\Gamma_1, \ldots, \Gamma_n$, let $\Gamma_1 \cup \cdots \cup \Gamma_n$, $\Gamma_1 \cap \cdots \cap \Gamma_n$, be the operators defined by

$$(\Gamma_1 \cup \cdots \cup \Gamma_n)(X) = \Gamma_1(X) \cup \cdots \cup \Gamma_n(X),$$

$$(\Gamma_1 \cap \cdots \cap \Gamma_n)(X) = \Gamma_1(X) \cap \cdots \cap \Gamma_n(X).$$

Clearly if $\Gamma_1, \ldots, \Gamma_n$ are positive then so are $\Gamma_1 \cup \cdots \cup \Gamma_n$ and $\Gamma_1 \cap \cdots \cap \Gamma_n$.

Given any family $\mathfrak{G}$ of positive operators, let

$$\langle \mathfrak{G} \rangle$$

denote the smallest class containing the elements of $\mathfrak{G}$ and closed under composition and finite unions and meets.

There are various infinitary versions of $\cup$ and $\cap$. First we have those induced by formulas with parameters $\phi(c)$. For any specific c, we have the operator $\Gamma_{\phi(c)}$, so they induce the operators $\bigcup_{c \in V} \Gamma_{\phi(c)}$, $\bigcap_{c \in V} \Gamma_{\phi(c)}$ defined by

$$\bigcup_{c \in V} \Gamma_{\phi(c)}(X) = \{x : (\exists c)(x \in \Gamma_{\phi(c)}(X)\},$$

$$\bigcap_{c \in V} \Gamma_{\phi(c)}(X) = \{x : (\forall c)(x \in \Gamma_{\phi(c)}(X)\}.$$

Such infinitary $\bigcup$ and $\bigcap$ are called *uniform*, as they come from a single formula. Given a family $\mathfrak{G}$ let

$$\langle \mathfrak{G} \rangle_{unif}$$

denote the smallest family containing $\mathfrak{G}$ and closed under composition, finitary $\cup, \cap$ and uniform $\bigcup, \bigcap$. Obviously, uniform $\bigcup, \bigcap$ correspond to unbounded quantifiers and

$$\langle \mathfrak{G} \rangle \subseteq \langle \mathfrak{G} \rangle_{unif}$$

In the last section we shall consider also some other kinds of infinitary $\bigcup, \bigcap$.

In the class of all operators the constant or almost constant ones naturally play a significant role.

Definition 1.3 Γ is said to be *almost constant* if there is a set size family of classes $\{A_i : i \leq \mu\}$, μ a cardinal number, such that for every set x, $\Gamma(x) = A_i$, for some $i \leq \mu$. If $\Gamma(x) = A$ for all x, Γ is said to be *constant* and is denoted C_A. The class of all almost constant operators is denoted by $\mathfrak{C}$.

Remark 1.4 Note that in the above definition we require Γ to be almost constant on *sets* only and not on all classes. And the latter is not implied by the former. For example the operator $\Gamma(X) = \{x : (\forall y)(x \in y \Rightarrow y \in X)\}$ is constant on sets, namely, $\Gamma(a) = \emptyset$ for every set a, but not on proper classes. The two notions, however, coincide for "set-continuous" operators (see [1]). Γ is said to be *set-continuous* if for every class X, $\Gamma(X) = \bigcup\{\Gamma(x) : x \subseteq X\}$. Most of the common positive operators, like $\mathcal{P}$, are set-continuous. It is easy to check that if Γ is set-continuous, and almost constant on sets, then it is almost constant on all classes.

We come now to the powerset operator. The main characteristics of $\mathcal{P}$ are the following:

(1) It is set-theoretic (this of course being a consequence of the powerset axiom).

(2) It is positive.

(3) $|x| < |\mathcal{P}(x)|$, for every set x.

An immediate consequence of (3) is

(4) The least fixed point of $\mathcal{P}$ is a proper class (the universe itself, if we work in ZFC[1]).

Of the above properties (3) is obviously the crucial one. We can easily find set-theoretic positive operators, "independent" of $\mathcal{P}$, satisfying (1), (2) and (4). For example the operator of the induction that generates the class of ordinals is of this kind. But (3) seems to be inherently connected with $\mathcal{P}$.

Definition 1.5 An operator $\Gamma_{\phi(\overline{c})} \in \mathcal{O}$ is said to be $\mathcal{P}$*-like* if it satisfies properties (1)-(3) above.

The questions we address in this paper are the following:

(a) What is the role of $\mathcal{P}$ in the creation of the class of all positive operators?

(b) Are there other $\mathcal{P}$-like operators "independent" of $\mathcal{P}$?

In section 2 we answer question (a): Every positive operator is a combination of the identity operator Id, the powerset, and almost

[1] If we work in ZFC minus foundation, the least fixed point of $\mathcal{P}$ is V_{wf}, the class of well-founded sets.

constant operators. This enables one to give a strict definition of $\mathcal{P}$-independent operators. In section 3 we consider and partially answer question (b): Every $\mathcal{P}$-independent bounded operator Γ (i.e., defined by a bounded positive formula), is not $\mathcal{P}$-like.

2 Generating the class of positive operators

In this section we shall prove the following:

Theorem 2.1 *The class of all positive operators is generated from $\mathcal{P}$, Id and almost constant operators by composition and finitary and uniformly infinite unions and intersections. That is*

$$\mathcal{O} = \langle \mathcal{P}, Id, \mathfrak{C} \rangle_{unif}.$$

We consider first the operators $\Gamma_{\phi(\overline{c})}$ for $\phi(v, \overline{c}, S)$ with canonical form $(\overline{Q}\overline{w})(\forall u)(\theta(v, \overline{c}, \overline{w}, u) \vee u \in S)$, where $\overline{Q} = \emptyset$.

Proposition 2.2 *Let $\phi(v, \overline{c}, S) = (\forall u)(\theta(v, \overline{c}, u) \vee u \in S)$, where $\overline{c} = (c_1, \ldots, c_n)$, $n \geq 0$ is a tuple of parameters, and θ is a disjunction of atomic and negated atomic formulas. Then $\Gamma_{\phi(\overline{c})} = C_B \cup \Delta$, where B is a constant class and $\Delta = \mathcal{P}$, or $\Delta = Id$, or $\Delta \in \mathfrak{C}$.*

Proof. The proof is by examining several cases. There are seven main cases which we designate by the letters **A** through **G**. In some of them subcases are also considered, designated by letters (a), (b) etc., and even subcases of them of the form (a1), (a2) etc. Cases **A**-**G** arise from a syntactic analysis of θ above. Case **A** is somewhat trivial. The nontrivial cases **B**-**G** are produced by five properties (of the metalanguage) concerning θ, $P_1 - P_5$ in the following way, which obviously makes **A**-**G** form a partition of truth:

B: P_1,
C: $\neg P_1$ & P_2,
D: $\neg P_1$ & $\neg P_2$ & P_3,
E: $\neg P_1$ & $\neg P_2$ & $\neg P_3$ & P_4,
F: $\neg P_1$ & $\neg P_2$ & $\neg P_3$ & $\neg P_4$ & P_5,
G: $\neg P_1$ & $\neg P_2$ & $\neg P_3$ & $\neg P_4$ & $\neg P_5$.

(A): u does not occur in θ.

Then for every set a, $\Gamma_{\phi(\overline{c})}(a) = \{x : (\forall u)(\theta(x,\overline{c}) \vee u \in a\} = \{x : \theta(x,\overline{c}) \vee (\forall u)(u \in a)\} = \{x : \theta(x,\overline{c})\} = A$, which is a constant class. Hence $\Gamma_{\phi(\overline{c})} = C_A$. So the claim holds. (This case includes the particular cases where $\theta = \top$ or $\bot$.)

From now on we assume that u occurs in θ and $\theta \not\Leftrightarrow \top, \bot$. By assumption $\theta = \sigma_1 \vee \cdots \vee \sigma_k$, where each σ_i is atomic or negated atomic. We refer to σ_i as *literals*. Let $W = \{v, u, c_1, \ldots, c_n\}$ be the set of variables and constants of θ. We let the letters α, β range over elements of W. For any $\alpha, \beta \in W$, let $F(\alpha, \beta)$ be the set of literals containing α, β. Namely

$$F(\alpha, \beta) = \{\alpha \in \beta, \beta \in \alpha, \alpha = \beta, \alpha \notin \beta, \beta \notin \alpha, \alpha \neq \beta\}.$$

Each σ_i belongs to some $F(\alpha, \beta)$. Without loss of generality we may assume that no σ_i is in some $F(\alpha, \alpha)$. This is because the formulas of $F(\alpha, \alpha)$ are either valid in ZF ($\alpha = \alpha$, $\alpha \notin \alpha$) or contradictory ($\alpha \neq \alpha$, $\alpha \in \alpha$). If some σ_i is valid then $\theta \Leftrightarrow \top$ and we are reduced to case **A** above. If some σ_i is contradictory, we may drop it. If all σ_i are contradictory, then $\theta \Leftrightarrow \bot$, and we go back to case **A** again.

So let $\sigma_i \in F(\alpha, \beta)$ for $\alpha \neq \beta$. Further let $\theta = \theta_1 \vee \theta_2$ where θ_1 is the disjunction of literals containing u and θ_2 the disjunction of the rest. Then

$$\Gamma_{\phi(\overline{c})}(a) = \{x : \theta_2(x, \overline{c}) \vee (\forall u)(\theta_1(x, \overline{c}, u) \vee u \in a)\} =$$

$$\{x : \theta_2(x, \overline{c})\} \cup \{x : (\forall u)(\theta_1(x, \overline{c}, u) \vee u \in a)\} =$$

$$B \cup \{x : (\forall u)(\theta_1(x, \overline{c}, u) \vee u \in a)\},$$

where B is a constant class depending on $\overline{c}$. So $\Gamma_{\phi(\overline{c})} = C_B \cup \Delta$, where

$$\Delta(X) = \{x : (\forall u)(\theta_1(x, \overline{c}, u) \vee u \in X)\}$$

and u occurs in every literal of θ_1. So from now on without loss of generality we may assume that $\theta = \theta_1$ and $\Delta = \Gamma_{\phi(\overline{c})}$.

Since u occurs in every literal of θ, each σ_i has one of the forms: $u \in \alpha$, $\alpha \in u$, $u = \alpha$, $u \notin \alpha$, $\alpha \notin u$, $u \neq \alpha$, where $\alpha \in \{v, c_1, \ldots, c_n\}$. Define for every set a, the classes

$$-a = \{u : u \notin a\}, \ \hat{a} = \{u : a \in u\}, \ -\hat{a} = \{u : a \notin u\}.$$

Then the formulas

$$u \notin \alpha,\ \ \alpha \in u,\ \ \alpha \notin u,\ \ u = \alpha,\ \ u \neq \alpha$$

above are written, respectively,

$$u \in -\alpha,\ \ u \in \hat{\alpha},\ \ u \in -\hat{\alpha},\ \ u \in \{\alpha\},\ \ u \in -\{\alpha\}.$$

Let $\Theta_1(v, u)$ be the set of literals of θ containing only v and u and $\Theta_2(u, \overline{c})$ be the set of the rest. Then $\theta = \theta_1 \vee \theta_2$ where $\theta_1 = \bigvee \Theta_1$ and $\theta_2 = \bigvee \Theta_2$, and, in view of the above translation,

$$\Theta_1 \subseteq \{u \in v, u \in -v, u \in \hat{v}, u \in -\hat{v}, u \in \{v\}, u \in -\{v\}\}. \tag{3}$$

To write out Θ_2 let us use distinct letters for constants appearing in distinct types of literals. Namely the letters $b, d, e, f, g, h,$ with subscripts, are used for constants occurring in the following formulas respectively:

$$u \in b,\ \ u \in -d,\ \ u \in \hat{e},\ \ u \in -\hat{f},\ \ u \in \{g\},\ \ u \in -\{h\}.$$

Then

$$\Theta_2 \subseteq \{u \in b_i,\ u \in -d_j,\ u \in \hat{e_k},\ u \in -\hat{f_l},\ u \in \{g_m\},\ u \in -\{h_p\}\}, \tag{4}$$

where i, j, k, l, m, p range over certain finite sets. Using just classes instead of formulas, (3) and (4) are written

$$\Theta_1'(v) \subseteq \{v, -v, \hat{v}, -\hat{v}, \{v\}, -\{v\}\}, \tag{5}$$

$$\Theta_2'(\overline{b}, \overline{d}, \overline{e}, \overline{f}, \overline{g}, \overline{h}) \subseteq \{b_i,\ -d_j,\ \hat{e_k},\ -\hat{f_l},\ \{g_m\},\ -\{h_p\}\}. \tag{6}$$

In this notation we easily see that

$$\Gamma_{\phi(\overline{c})}(X) = \{x : \bigcup \Theta_1'(x) \cup \bigcup \Theta_2'(\overline{b}, \overline{d}, \overline{e}, \overline{f}, \overline{g}, \overline{h}) \cup X = V\},$$

or, in more detail,

$$\Gamma_{\phi(\overline{c})}(X) = \{x : \bigcup \Theta_1'(x) \cup (\bigcup_i b_i) \cup (\bigcup_j -d_j) \cup (\bigcup_k \hat{e_k}) \cup (\bigcup_l -\hat{f_l}) \cup$$

$$(\bigcup_m \{g_m\}) \cup (\bigcup_p -\{h_p\}) \cup X = V\}. \tag{7}$$

The finite sequences $\overline{b}$, $\overline{d}$, $\overline{e}$, $\overline{f}$, $\overline{g}$, $\overline{h}$ need not be all disjoint and some of them may be empty. However, due to the fact that $\theta \not\Leftrightarrow \top$, we do have

$$\overline{b} \neq \overline{d},\ \ \overline{e} \neq \overline{f},\ \ \overline{g} \neq \overline{h},\ \ \overline{d} \neq \overline{f}. \tag{8}$$

($\overline{d} \neq \overline{f}$, because for every d, $-d \cup -\hat{d} = V$.)

If we set $A = \bigcup \Theta_2'(\overline{b}, \overline{d}, \overline{e}, \overline{f}, \overline{g}, \overline{h})$, A is a constant class independent of X (and depending only on $\overline{c}$). So

$$\Gamma_{\phi(\overline{c})}(X) = \{x : \bigcup \Theta_1'(x) \cup A \cup X = V\}. \tag{9}$$

Now we have to examine various cases concerning the forms of $\Theta_1'(x)$ and $\Theta_2'(\overline{b}, \overline{d}, \overline{e}, \overline{f}, \overline{g}, \overline{h})$. Since θ is not a tautology, Θ_1' does not contain complementary classes, e.g. $\hat{v}$ and $-\hat{v}$.

(B): $\Theta_1'(v)$ contains $-\{v\}$.

Observe that the following hold for every set x:
(i) $-\{x\} \cup x = -\{x\}$.
(ii) $-\{x\} \cup -x = V$
(iii) $-\{x\} \cup \hat{x} = -\{x\}$.
(iv) $-\{x\} \cup -\hat{x} = V$.
We consider the subcases (a) and (b) below.

(a) Suppose $\Theta_1'(v)$ contains also $-v$ or $-\hat{v}$. Then, by the relations (ii) and (vi) above, (9) yields for every $X \neq V$

$$\Gamma_{\phi(\overline{c})}(X) \supseteq \{x : -\{x\} \cup -x \cup A \cup X = V\} = V$$

or

$$\Gamma_{\phi(\overline{c})}(X) \supseteq \{x : -\{x\} \cup -\hat{x} \cup A \cup X = V\} = V.$$

Therefore $\Gamma_{\phi(\overline{c})} = C_V$.

(b) Suppose $\Theta_1'(v)$ contains neither $-v$ nor $-\hat{v}$. So it may contain only v and $\hat{v}$. By (i) and (iii) above, (9) becomes

$$\Gamma_{\phi(\overline{c})}(X) = \{x : -\{x\} \cup A \cup X = V\} = \{x : x \in A \cup X\} = A \cup X,$$

hence $\Gamma_{\phi(\overline{c})} = C_A \cup Id$. In particular, if ϕ contains no parameters, then $\Gamma_{\phi(\overline{c})} = Id$.

(C): $\Theta_1'(v)$ does not contain $-\{v\}$ and Θ_2' contains a class $-\{h_p\}$.

Then $\Theta_1'(v) \subseteq \{v, -v, \hat{v}, -\hat{v}, \{v\}\}$ and $\overline{h} \neq \emptyset$. We shall show that $\Gamma_{\phi(\overline{c})}$ is almost constant.

If Θ_2' contains at least two distinct such classes, $-\{h_p\}$, $-\{h_q\}$, then $\Gamma_{\phi(\overline{c})}(\emptyset) \supseteq \{x : -\{h_p\} \cup -\{h_q\} = V\} = V$. The same happens if $\Theta_1'(v)$ contains both v and $-v$ or both $\hat{v}$ and $-\hat{v}$. We consider three subcases (a), (b) and (c) which present no essential difference in their treatment.

(a) Suppose $\Theta_1'(v)$ contains $-v$. Observe that for every x,

(v) $-x \cup \hat{x} = -x$,

(vi) $-x \cup \{x\} = -x$.

So we may assume that $\Theta_1'(v) \subseteq \{-v, -\hat{v}\}$. From the following proof it will be clear that nothing is lost if we assume that $\Theta_1'(v) = \{-v, -\hat{v}\}$. Then

$$\Gamma_{\phi(\overline{c})}(X) = \{x : -x \cup -\hat{x} \cup A \cup -\{h_p\} \cup X = V\},$$

where A is a constant class defined by the other possible parameters of ϕ. Thus

$$\Gamma_{\phi(\overline{c})}(X) = \{x : h_p \in A \cup X \cup -x \cup -\hat{x}\}.$$

If $h_p \in A \cup X$, $\Gamma_{\phi(\overline{c})}(X) = V$, otherwise $\Gamma_{\phi(\overline{c})}(X) = \{x : h_p \in -x \cup -\hat{x}\} = -\hat{h_p} \cup -h_p$. It follows that for every $X \neq V$, $\Gamma_{\phi(\overline{c})}(X) = V$, or $\Gamma_{\phi(\overline{c})}(X) = \hat{h_p} \cup -h_p$, and therefore $\Gamma_{\phi(\overline{c})}$ is almost constant.

(b) Suppose $\Theta_1'(v)$ contains v. Suppose without loss of generality that $\Theta_1'(v) = \{v, \hat{v}, \{v\}\}$ (the case $\Theta_1'(v) = \{v, -\hat{v}, \{v\}\}$ is similar). Then

$$\Gamma_{\phi(\overline{c})}(X) = \{x : x \cup \hat{x} \cup \{x\} \cup A \cup -\{h_p\} \cup X = V\} =$$

$$\{x : h_p \in A \cup X \cup x \cup \hat{x} \cup \{x\}\}.$$

Again if $h_p \in A \cup X$, then $\Gamma_{\phi(\overline{c})}(X) = V$. Otherwise $\Gamma_{\phi(\overline{c})}(X) = h_p \cup \hat{h_p} \cup \{h_p\}$, i.e., $\Gamma_{\phi(\overline{c})}$ is almost constant.

(c) Suppose $\Theta_1'(v)$ contains neither $-v$ nor v. Then $\Theta_1'(v) \subseteq \{\hat{v}, -\hat{v}, \{v\}\}$. The treatment of all these possible subcases is as of the previous ones and leads to the conclusion that $\Gamma_{\phi(\overline{c})}$ is almost constant.

<u>**(D)**: $\Theta_1'(v)$ does not contain $-\{v\}$ and Θ_2' does not contain any class $-\{h_p\}$ and $\Theta_1'(v)$ contains $-v$.</u>

It follows from (v) and (vi) that we may assume that $\Theta_1'(v) \subseteq \{-v, -\hat{v}\}$.

(a) Suppose $\Theta'_1(v) = \{-v\}$. Then

$$\Gamma_{\phi(\overline{c})}(X) = \{x : -x \cup A \cup X = V\} = \{x : x \subseteq A \cup X\} = \mathcal{P}(A \cup X).$$

Therefore $\Gamma_{\phi(\overline{c})} = \mathcal{P}(C_A \cup Id)$. In particular, for $A = \emptyset$, $\Gamma_{\phi(\overline{c})} = \mathcal{P}$.

(b) Suppose $\Theta'_1(v) = \{-\hat{v}\}$. Then

$$\Gamma_{\phi(\overline{c})}(X) \supseteq \{x : -\hat{x} \cup A \cup X = V\} = \{x : \hat{x} \subseteq A \cup X\}.$$

We have to examine the form of A. In general we have

$$\Gamma_{\phi(\overline{c})}(X) = \{x : \hat{x} \subseteq (\bigcup_i b_i) \cup (\bigcup_j -d_j) \cup (\bigcup_k \hat{e_k}) \cup (\bigcup_l -\hat{f_l}) \cup (\bigcup_m \{g_m\}) \cup X\}. \tag{10}$$

(b1) Suppose $\overline{d} \neq \emptyset$. Observe that for all x and d_i,

(vii) $\hat{x} \subseteq -d_1 \cup \cdots \cup -d_j \iff d_1 \cap \cdots \cap d_j \subseteq -\hat{x} \iff x \notin \bigcup(d_1 \cap \cdots \cap d_j)$.

Therefore, by (vii),

$$\Gamma_{\phi(\overline{c})}(\emptyset) \supseteq \{x : \hat{x} \subseteq \bigcup_j -d_j\} = \{x : x \notin \bigcup(d_1 \cap \cdots \cap d_j)\} =$$

$$-\bigcup(d_1 \cap \cdots \cap d_j).$$

Now $\bigcup(d_1 \cap \cdots \cap d_j)$ is a set, and for every $X \neq V$, $\Gamma_{\phi(\overline{c})}(X) \supseteq \Gamma_{\phi(\overline{c})}(\emptyset)$, or $-\Gamma_{\phi(\overline{c})}(X) \subseteq \bigcup(d_1 \cap \cdots \cap d_j)$. So the family $\{\Gamma_{\phi(\overline{c})}(X) : X \subseteq V\}$ is set size, hence $\Gamma_{\phi(\overline{c})}$ is almost constant.

(b2) Suppose $\overline{d} = \emptyset$. Then (10) becomes

$$\Gamma_{\phi(\overline{c})}(X) = \{x : \hat{x} \subseteq (\bigcup_i b_i) \cup (\bigcup_k \hat{e_k}) \cup (\bigcup_l -\hat{f_l}) \cup (\bigcup_m \{g_m\}) \cup X\}. \tag{11}$$

Claim 1. For every set a, $\hat{x} \subseteq (\bigcup_i b_i) \cup (\bigcup_k \hat{e_k}) \cup (\bigcup_l -\hat{f_l}) \cup (\bigcup_m \{g_m\}) \cup a \iff x \in \{e_1, \ldots, e_k\}$.

Proof. The direction "$\Leftarrow$" is obvious. For the converse, suppose $x \notin \{e_1, \ldots, e_k\}$. Then pick a set y such that

(i) $x \in y$, (ii) $\{e_1, \ldots, e_k\} \cap y = \emptyset$. (iii) $\{f_1, \ldots, f_l\} \subseteq y$, and (iv) $y \notin (\bigcup_i b_i) \cup (\bigcup_m \{g_m\}) \cup a$. Note that (ii) and (iii) are compatible because of the assumption (8). Then by (i), $y \in \hat{x}$, by (ii), $y \notin \bigcup_k \hat{e_k}$ and by (iii) $y \notin \bigcup_l -\hat{f_l}$. Hence $y \notin (\bigcup_i b_i) \cup (\bigcup_k \hat{e_k}) \cup (\bigcup_l -\hat{f_l}) \cup (\bigcup_m \{g_m\}) \cup a$. Therefore $\hat{x} \not\subseteq (\bigcup_i b_i) \cup (\bigcup_k \hat{e_k}) \cup (\bigcup_l -\hat{f_l}) \cup (\bigcup_m \{g_m\}) \cup a$. This proves the claim.

It follows from Claim 1 that for every set a,

$$\Gamma_{\phi(\overline{c})}(a) = \{x : \hat{x} \subseteq (\bigcup_i b_i) \cup (\bigcup_k \hat{e_k}) \cup (\bigcup_l -\hat{f_l}) \cup (\bigcup_m \{g_m\}) \cup a\} =$$

$$\{x : x \in \{e_1, \ldots, e_k\}\} = \{e_1, \ldots, e_k\}.$$

Hence $\Gamma_{\phi(\overline{c})}$ is constant.

(c) Finally suppose $\Theta'_1(v) = \{-v, -\hat{v}\}$. Then

$$\Gamma_{\phi(\overline{c})}(X) \supseteq \{x : -x \cup -\hat{x} = V\} = V,$$

hence $\Gamma_{\phi(\overline{c})} = C_V$, i.e., $\Gamma_{\phi(\overline{c})}$ is constant.

<u>**(E)** : $\Theta'_1(v)$ does not contain $-\{v\}$ and Θ'_2 does not contain any class $-\{h_p\}$ and $\Theta'_1(v)$ does not contain $-v$ and $\Theta'_1(v)$ contains $-\hat{v}$.</u>

Then $\Theta'_1(v) \subseteq \{v, -\hat{v}, \{v\}\}$. Observe that

(vii) $-\hat{x} \cup \{x\} = -\hat{x}$ and

(viii) $-\hat{x} \cup x = -\hat{x}$.

By (vii) and (viii) we may assume that $\Theta'_1(v) = \{-\hat{v}\}$.

But this is a minor modification of case **C** (b).

<u>**(F)**: $\Theta'_1(v)$ does not contain $-\{v\}$ and Θ'_2 does not contain any class $-\{h_p\}$ and $\Theta'_1(v)$ does not contain $-v$ and $\Theta'_1(v)$ does not contain $-\hat{v}$ and $\overline{d} = \emptyset$.</u>

Then $\Theta'_1(v) \subseteq \{v, \hat{v}, \{v\}\}$.

(a) Suppose $\Theta'_1(v)$ contains $\hat{v}$. Then

$$\Gamma_{\phi(\overline{c})}(X) = \{x : x \cup \{x\} \cup \hat{x} \cup (\bigcup_i b_i) \cup (\bigcup_k \hat{e_k}) \cup (\bigcup_l -\hat{f_l}) \cup (\bigcup_m \{g_m\}) \cup X = V\}.$$

<u>Claim 2</u>. For every set a, $\Gamma_{\phi(\overline{c})}(a) = \{f_1, \ldots, f_l\}$. Hence $\Gamma_{\phi(\overline{c})}$ is constant.

Proof. Obviously $\{f_1, \ldots, f_l\} \subseteq \Gamma_{\phi(\overline{c})}(a)$. For the converse let $y \notin \{f_1, \ldots, f_l\}$. Towards reaching a contradiction, assume $y \in \Gamma_{\phi(\overline{c})}(a)$. By the above expression of $\Gamma_{\phi(\overline{c})}(a)$, clearly, $\hat{y} \cup (\bigcup_k \hat{e_k}) \cup (\bigcup_l -\hat{f_l})$ would be a coset, or equivalently, $-\hat{y} \cap (\bigcap_k -\hat{e_k}) \cap (\bigcap_l \hat{f_l})$ would be a set. But

$$u \in -\hat{y} \cap (\bigcap_k -\hat{e_k}) \iff \{y, e_1, \ldots, e_k\} \cap u = \emptyset$$

and

$$u \in \bigcap_l \hat{f_l} \iff \{f_1, \ldots, f_l\} \subseteq u.$$

By (8) and the fact that $y \notin \{f_1, \ldots, f_l\}$, $\{y, e_1, \ldots, e_k\} \cap \{f_1, \ldots, f_l\} = \emptyset$. So there are class many u in $-\hat{y} \cap (\bigcup_k -\hat{e_k}) \cap (\bigcap_l \hat{f_l})$, a contradiction.

(b) Suppose $\Theta'_1(v)$ does not contain $\hat{v}$. Then $\Theta'_1(v) \subseteq \{v, \{v\}\}$. So

$$\Gamma_{\phi(\overline{c})}(X) = \{x : x \cup \{x\} \cup (\bigcup_i b_i) \cup (\bigcup_k \hat{e_k}) \cup (\bigcup_l -\hat{f_l}) \cup (\bigcup_m \{g_m\}) \cup X = V\}.$$

If we set

$$A = (\bigcup_i b_i) \cup (\bigcup_k \hat{e_k}) \cup (\bigcup_l -\hat{f_l}) \cup (\bigcup_m \{g_m\}),$$

then clearly A is not a coset. Therefore for every set a, $\Gamma_{\phi(\overline{c})}(a) = \{x : x \cup \{x\} \cup A \cup a = V\} = \emptyset$. Hence $\Gamma_{\phi(\overline{c})}$ is constant.

<u>**(G)**: $\Theta'_1(v)$ does not contain $-\{v\}$ and Θ'_2 does not contain any class $-\{h_p\}$ and $\Theta'_1(v)$ does not contain $-v$ and $\Theta'_1(v)$ does not contain $-\hat{v}$ and $\overline{d} \neq \emptyset$.</u>

(a) Suppose first that $\Theta'_1(v)$ contains $\hat{v}$. Assume for simplicity that $\overline{d}$ consists of a single parameter d (the case of many d's is quite similar). Then for every set a,

$$\Gamma_{\phi(\overline{c})}(a) = \{x : x \cup \hat{x} \cup \{x\} \cup B \cup -d \cup a = V\},$$

where B is the constant class defined from the rest of the parameters. The above is written

$$\Gamma_{\phi(\overline{c})}(a) = \{x : d \cap -B \cap -a \subseteq x \cup \hat{x} \cup \{x\}\}.$$

If $d \cap -B \cap -a = \emptyset$, then clearly $\Gamma_{\phi(\overline{c})}(a) = V$. So assume $d \cap -B \cap -a \neq \emptyset$. Now observe that

$$X \subseteq x \cup \hat{x} \cup \{x\} \iff (\forall y)(y \in X \rightarrow y \in x \vee x \in y \vee x = y) \iff$$

$$(\forall y)(y \in X \wedge y \notin x \wedge y \neq x \rightarrow x \in y) \iff x \in \bigcap(X - (x \cup \{x\})).$$

Therefore

$$\Gamma_{\phi(\overline{c})}(a) = \{x : x \in \bigcap(d \cap -B \cap -a - (x \cup \{x\}))\} = \bigcap(d - (B \cup a \cup x \cup \{x\})). \tag{12}$$

We have the following subcases:

(a1) $d \subseteq B \cup a$. Then for every x, $d - (B \cup a \cup x \cup \{x\}) = \emptyset$ hence $\bigcap(d - (B \cup a \cup x \cup \{x\})) = \bigcap \emptyset = V$. Therefore, by (12), $\Gamma_{\phi(\overline{c})}(a) = \{x : x \in V\} = V$.

(a2) $d \not\subseteq B \cup a$. Then, for $x \notin d \cup \mathcal{P}(d)$, $d - (B \cup a \cup x \cup \{x\}) \neq \emptyset$, hence for such x, $\bigcap(d-(B\cup a\cup x\cup\{x\})) \subseteq \bigcup d$. By (12), $\Gamma_{\phi(\overline{c})}(a) \subseteq \bigcup d$. If for some x, $d \subseteq B\cup a\cup x\cup\{x\}$, then clearly $x \in d\cup\mathcal{P}(d)$. Therefore in any case $\Gamma_{\phi(\overline{c})}(a) \subseteq (\bigcup d) \cup d \cup \mathcal{P}(d)$. Thus $\Gamma_{\phi(\overline{c})}$ is almost constant.

(b) Now suppose that $\Theta_1'(v)$ does not contain $\hat{v}$, i.e., $\Theta_1'(v) \subseteq \{v, \{v\}\}$. Then as in the previous case, for every set a,

$$\Gamma_{\phi(\overline{c})}(a) = \{x : x\cup\{x\}\cup B\cup -d\cup a = V\} = \{x : d-(B\cup a) \subseteq x\cup\{x\}\}.$$

Recall that d is a constant set and B a constant class. So when a ranges over all sets, $d - (B \cup a)$ ranges over some subsets of d. Therefore for $a \in V$, the classes $\{x : d - (B \cup a) \subseteq x \cup \{x\}\}$ are set many, hence $\Gamma_{\phi(\overline{c})}$ is almost constant.

We have exhausted all possible cases, so the proof is complete. $\dashv$

Proposition 2.3 *Let $\phi(v, \overline{c}, S) \Leftrightarrow (\forall u)(\theta(v, \overline{c}, u) \vee u \in S)$. Then $\Gamma_{\phi(\overline{c})} = \Gamma_1 \cap \cdots \cap \Gamma_k$, where for each i, $\Gamma_i = \mathcal{P}$, or $\Gamma_i = Id$, or Γ_i is almost constant. Therefore $\Gamma_{\phi(\overline{c})} \in \langle \mathcal{P}, Id, \mathfrak{C}\rangle$.*

Proof. Let $\theta = \bigwedge_{i\leq k} \theta_i$ be the conjunctive normal form of θ. Then each θ_i is a disjunction of atomic or negated atomic formulas. Moreover we have

$$\phi(v, \overline{c}, S) \Leftrightarrow (\forall u)(\theta \vee u \in S) \Leftrightarrow (\forall u)(\bigwedge_{i\leq k} \theta_i \vee u \in S) \Leftrightarrow$$

$$(\forall u) \bigwedge_{i\leq k}(\theta_i \vee u \in S) \Leftrightarrow \bigwedge_{i\leq k}[(\forall u)(\theta_i \vee u \in S)].$$

Therefore, if $\phi_i(v, \overline{c}, S) := (\forall u)(\theta_i(v, \overline{c}, u) \vee u \in S)$, then for all $X \neq V$, $\Gamma_{\phi(\overline{c})}(X) = \bigcap_{i\leq k} \Gamma_{\phi_i}(X)$. By proposition 2.2, each Γ_{ϕ_i} is the join of a constant with either $\mathcal{P}$, or Id, or an almost constant. So the claim follows. $\dashv$

Proof of Theorem 2.1. Obviously $\langle \mathcal{P}, Id, \mathfrak{C}\rangle_{unif} \subseteq \mathcal{O}$. For the converse observe that if $\phi(v, \overline{c}, S) = (\exists w)\psi(v, \overline{c}, w, S)$, then

$$\Gamma_{\phi(\overline{c})}(X) = \{x : \phi(x, \overline{c}, X)\} = \{x : (\exists d)\psi(x, \overline{c}, d, X)\} =$$

$$\{x : (\exists d)[x \in \Gamma_{\psi(\overline{c},d)}(X)]\} = \bigcup_{d\in V} \Gamma_{\psi(\overline{c},d)}(X),$$

that is, $\Gamma_{\phi(\overline{c})} = \bigcup_{d\in V} \Gamma_{\psi(\overline{c},d)}$. Similarly if $\phi(v,\overline{c},S) = (\forall w)\psi(v,\overline{c},w,S)$, then $\Gamma_{\phi(\overline{c})} = \bigcap_d \Gamma_{\psi(\overline{c},d)}$. Let $\Gamma_{\phi(\overline{c})} \in \mathcal{O}$ and let the canonical form of $\phi(v,\overline{c},S)$ be

$$(Q_1 w_1)\cdots(Q_m w_m)(\forall u)(\theta(v,\overline{c},\overline{w},u) \vee u \in S).$$

It follows that if $\psi(\overline{c},\overline{d},S) = (\forall u)(\theta(v,\overline{c},\overline{d},u) \vee u \in S)$ then

$$\Gamma_{\phi(\overline{c})} = U^1_{d_1\in V}\cdots U^m_{d_m\in V}\Gamma_{\psi(\overline{c},\overline{d})},$$

where $U^i = \bigcup$ if $Q_i = \exists$ and $U^i = \bigcap$ if $Q_i = \forall$. By proposition 2.3, $\Gamma_{\psi(\overline{c},\overline{d})} \in \langle \mathcal{P}, Id, \mathfrak{C}\rangle$, therefore $\Gamma_{\phi(\overline{c})} \in \langle \mathcal{P}, Id, \mathfrak{C}\rangle_{unif}$. $\dashv$

3 $\mathcal{P}$-independence and $\mathcal{P}$-likeness

Theorem 2.1 enables us to give a strict definition of "$\mathcal{P}$-independent" operator. Since every positive operator is in $\langle \mathcal{P}, Id, \mathfrak{C}\rangle_{unif}$, the following definition is natural:

Definition 3.1 A positive operator Γ *is said to be $\mathcal{P}$-independent* if

$$\Gamma \in \langle Id, \mathfrak{C}\rangle_{unif}.$$

In this section we shall address the question: Are there $\mathcal{P}$-independent $\mathcal{P}$-like positive operators? The question, in its full generality, is still open to us but we strongly guess that the answer is negative. So we state it in the form of a conjecture.

CONJECTURE Every $\mathcal{P}$-independent positive operator is not $\mathcal{P}$-like.

The main difficulty in handling the above question is the fact that if $\Gamma_{\phi(c)}, c \in V$, is a uniform family and each $\Gamma_{\phi(c)}$ is almost constant, then $\bigcap_{c\in V} \Gamma_{\phi(c)}$ and $\bigcup_{c\in V} \Gamma_{\phi(c)}$ need not be almost constant (see the example after lemma 3.4). In short one does not have control on the behavior of the infinitary operations $\bigcap$ and $\bigcup$. So instead of the general forms of the latter we shall consider their *bounded* versions, which correspond to bounded quantifiers, i.e., to bounded formulas of set theory.

Definition 3.2 An operator Γ is said to be *bounded* if it is defined by a bounded formula, i.e., a formula in which every quantifier is of the form $\forall x \in y$, $\exists x \in y$.

Given a formula $\phi(c)$ and a set a, the operators $\bigcup_{c\in a}\Gamma_{\phi(c)}$ and $\bigcap_{c\in a}\Gamma_{\phi(c)}$ are said to be produced by *bounded uniform* $\bigcup$ and $\bigcap$.

Let $\mathcal{O}_b$ denote the class of bounded positive operators. Given a class of operators $\mathfrak{G}$, let

$$\langle\mathfrak{G}\rangle^b_{unif}$$

denote the smallest class of operators containing $\mathfrak{G}$, closed under composition, finitary $\cup$, $\cap$ and bounded uniform $\bigcup$ and $\bigcap$.

The following is a "bounded version" of Moschovakis' lemma 1.2.

Lemma 3.3 *Let $\phi(v,\overline{c},S)$ be a positive bounded formula of L. Then there is a quantifier-free and S-free formula $\theta(v,\overline{c},\overline{w},u)$, where $\overline{w}=(w_1,\dots,w_m)$, and a string of bounded quantifiers $\overline{Q}=(Q_1,\dots,Q_m)$ such that, for every x and every class $X\neq V$,*

$$\phi(x,\overline{c},X) \iff (\overline{Q}\overline{w})(\forall u)(\theta(x,\overline{c},\overline{w},u)\vee u\in X).$$

Proof. The proof follows by inspection of the proof of the general result 1.2. $\dashv$

An immediate consequence of 3.3 and the proof of theorem 2.1 is the following:

Lemma 3.4 $\mathcal{O}_b=\langle\mathcal{P},Id,\mathfrak{C}\rangle^b_{unif}$.

Note that simple positive operators, like the union operator $\cup$, do not belong to $\mathcal{O}_b$. For instance $\cup X=\{x:(\exists y)(y\in X\wedge x\in y)\}=\bigcup_{c\in V}C_c(X)$, where for every set c, C_c is the almost constant operator defined by: $C_c(X)=c$ if $c\in X$ and $C_c(X)=\emptyset$ otherwise. This is because, despite its intuitive simplicity, $\cup$ has a defining formula with canonical form in which the prefix $\overline{Q}$ is non-empty, namely $\overline{Q}=\exists$.

Finally, given a class of operators $\mathfrak{G}$, let

$$\langle\mathfrak{G}\rangle_{set}$$

denote the smallest class containing $\mathfrak{G}$, and closed under composition, and the following condition: If I is a set and Γ_i, $i\in I$ is a subfamily of $\mathfrak{G}$ indexed by I, then $\bigcup_{i\in I}\Gamma_i$, $\bigcap_{i\in I}\Gamma_i$ are in $\langle\mathfrak{G}\rangle_{set}$. Obviously for every $\mathfrak{G}$,

$$\langle\mathfrak{G}\rangle\subseteq\langle\mathfrak{G}\rangle^b_{unif}\subseteq\langle\mathfrak{G}\rangle_{set}.$$

Lemma 3.5 *If* $\Gamma \in \langle Id, \mathfrak{C} \rangle_{set}$*, then* Γ *can be expressed in the following disjunctive normal form,*

$$\Gamma = \bigcup_{i \in I} \bigcap_{j \in J_i} \Gamma_{ij},$$

where I *and* J_i *for each* $i \in I$ *are sets, and* $\Gamma_{ij} = Id$ *or* $\Gamma_{ij} \in \mathfrak{C}$*.*

Proof. By induction on the length of the words of the algebraic structure $\langle Id, \mathfrak{C} \rangle_{set}$.

(a) For $\Gamma = Id$ or $\Gamma \in \mathfrak{C}$ this is obvious.

(b) Let $\Gamma = \Gamma_1 \circ \Gamma_2$ and suppose the claim is true for Γ_1, Γ_2. Let $\Gamma_1 = \bigcup_{i \in I} \bigcap_{j \in J_i} \Gamma_{ij}$. Then $\Gamma = \Gamma_1 \circ \Gamma_2 = \bigcup_{i \in I} \bigcap_{j \in J_i} (\Gamma_{ij} \circ \Gamma_2)$. If $\Gamma_{ij} \in \mathfrak{C}$, then clearly $\Gamma_{ij} \circ \Gamma_2 \in \mathfrak{C}$. If $\Gamma_{ij} = Id$, then $\Gamma_{ij} \circ \Gamma_2 = \Gamma_2$, which has also a disjunctive normal form, hence, by the usual distributive laws for $\bigcup$ and $\bigcap$, $\Gamma_1 \circ \Gamma_2$ can be expressed in the above normal form.

(c) If $\Gamma = \bigcup_{i \in I} \Gamma_i$ or $\Gamma = \bigcap_{i \in I} \Gamma_i$, where Γ_i have normal forms, then Γ can easily be written in normal form (using the distributive law in the case of $\bigcap$). This completes the proof. $\dashv$

Lemma 3.6 *Let* $\Gamma \in \langle Id, \mathfrak{C} \rangle_{set}$*. Then either* Γ *is non-set-theoretic, or for some* x*,* $|\Gamma(x)| \leq |x|$*.*

Proof. Let $\Gamma \in \langle Id, \mathfrak{C} \rangle_{set}$. By lemma 3.5, $\Gamma = \bigcup_{i \in I} \bigcap_{j \in J_i} \Gamma_{ij}$, where $\Gamma_{ij} = Id$ or $\Gamma_{ij} \in \mathfrak{C}$. For every $i \in I$, let $\Gamma_i = \bigcap_{j \in J_i} \Gamma_{ij}$. So $\Gamma = \bigcup_{i \in I} \Gamma_i$ and we call Γ_i *clauses.* Let $\Gamma_i^* = \bigcap_{j \in J_i, \Gamma_{ij} \neq Id} \Gamma_{ij}$. Obviously, $\Gamma_i^* = \Gamma_i$ if the clause Γ_i is Id-free, i.e., if for every $j \in J_i$, $\Gamma_{ij} \neq Id$.

We distinguish the following cases:

(a) For some i_0, Γ_{i_0} is Id-free and non-set-theoretic. Then $\Gamma_{i_0}^* = \Gamma_{i_0}$ and $\Gamma_{i_0}^*(x)$ is a proper class X, for some x. Thus $\Gamma(x) = \bigcup_i \Gamma_i(x) \supseteq \Gamma_{i_0}(x) = X$; hence Γ is non-set-theoretic.

(b) Let (a) be false, i.e, for every i, Γ_i either contains Id or is set-theoretic. Let $I' = \{i \in I : \Gamma_i \text{ is } Id\text{-free}\}$ and let $\Delta = \bigcup_{i \in I'} \Gamma_i$. By assumption for every $i \in I'$, Γ_i is set-theoretic and $\Gamma_i \in \mathfrak{C}$; therefore $\Delta \in \mathfrak{C}$, and is set-theoretic. So there is a set s such that $\Delta(x) \subseteq s$ for every set x. Moreover $\Gamma = (\bigcup_{i \in I - I'} \Gamma_i) \cup \Delta$, where for every $i \in I - I'$, $\Gamma_i(x) = \Gamma_i^*(x) \cap x$. Thus for every x

$$\Gamma(x) = (\bigcup_{i \in I - I'} (\Gamma_i^*(x) \cap x)) \cup \Delta(x).$$

Since $|\Gamma_i^*(x) \cap x| \leq |x|$, it follows from the last equation that for every x, $|\Gamma(x)| \leq |x| + |s|$. So taking an infinite x such that $|x| \geq |s|$, we get $|\Gamma(x)| \leq |x|$. Hence Γ is non-cardinality-increasing. This completes the proof. $\dashv$

Theorem 3.7 *Every $\mathcal{P}$-independent positive bounded operator is not $\mathcal{P}$-like.*

Proof. Let Γ be positive, bounded and $\mathcal{P}$-independent. Then by lemma 3.4, $\Gamma \in \langle Id, \mathfrak{C}\rangle_{unif}^b$. Hence $\Gamma \in \langle Id, \mathfrak{C}\rangle_{set}$. By lemma 3.6, either Γ is non-set-theoretic, or for some x, $|\Gamma(x)| \leq |x|$. Therefore Γ is not $\mathcal{P}$-like. $\dashv$

By the preceding theorem, $\mathcal{P}$ is, essentially, the only bounded operator satisfying properties (1)-(3). This looks a little bit strange, and it might reasonably suggest that $\mathcal{P}$ doesn't satisfy all (1)-(3) either. But of the properties (1)-(3) only (1) is disputable (the others being simple mathematical facts), since it is just an axiom of ZF independent from the other ones. Thus $\mathcal{P}$ might reasonably be non-set-theoretic, which means that the powerset axiom could be false.

Let $\mathrm{ZF} - \mathrm{P}$ be the theory ZF minus the powerset axiom. Working in $\mathrm{ZF} - \mathrm{P}$ provides another natural notion of $\mathcal{P}$-independence. Namely if Γ is a set-operator *in the sense of* $\mathrm{ZF} - \mathrm{P}$, i.e., $\mathrm{ZF} - \mathrm{P} \vdash \forall x \exists y (\Gamma(x) = y))$, then one feels that Γ cannot be related to $\mathcal{P}$. The question is: Are the two notions of $\mathcal{P}$-independence related? What we know is the following straightforward consequence of lemma 3.5:

Proposition 3.8 *If Γ is positive and set-theoretic then our first notion of $\mathcal{P}$-independence implies the last one, i.e.,*

$$\Gamma \in \langle Id, \mathfrak{C}\rangle_{unif}^b \;\Rightarrow\; \mathrm{ZF} - \mathrm{P} \vdash \forall x \exists y (\Gamma(x) = y).$$

Open Problems. 1) Is 3.8 true with $\langle Id, \mathfrak{C}\rangle_{unif}$ in place of $\langle Id, \mathfrak{C}\rangle_{unif}^b$?
2) Is the converse of 3.8, as well as of Problem 1, true?

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