ARONSZAJN TREES AND THE SUCCESSORS OF A SINGULAR CARDINAL

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ABSTRACT. From large cardinals we obtain the consistency of the existence of a singular cardinal κ of cofinality ω at which the Singular Cardinals Hypothesis fails, there is a bad scale at κ and κ^{++} has the tree property. In particular this model has no special κ^+-trees.

1. INTRODUCTION

We prove the following result.

Theorem 1.1. *If κ is supercompact and $\lambda > \kappa$ is weakly compact, then there is a forcing extension in which κ is a singular strong limit cardinal of cofinality ω,* SCH *fails at κ, there is a bad scale at κ and the tree property holds at κ^{++}.*

To begin we recall some basic definitions.

Definition 1.2. *Let κ and λ be cardinals with κ regular.*

1. *A* κ-tree *is a tree of height κ with levels of size less than κ.*
2. *A* cofinal branch *through a tree T is a linearly ordered subset of T whose order type is the height of T.*
3. *A* κ-Aronszajn tree *is a κ-tree with no cofinal branch.*
4. *κ has the* tree property *if and only if there are no κ-Aronszajn trees.*
5. *A λ^+-tree T is special if and only if there is a function $f : T \to \lambda$ such that if x and y are comparable in the tree, then $f(x) \neq f(y)$.*

Definition 1.3. *For a singular cardinal η, the Singular Cardinal Hypothesis (SCH) at η is the assertion that if η is strong limit, then $2^\eta = \eta^+$.*

The tree property is well studied. There are many classical results and there has also been some recent research. We review the classical results. The tree property at $\aleph_0$ is precisely König's Lemma [7]. Aronszajn [9] constructed an $\aleph_1$-Aronszajn tree. Generalizing Aronszajn's construction Specker [17] proved that if $\kappa^{<\kappa} = \kappa$, then there is a special κ^+-tree. In particular CH implies that there is a special ω_2-tree. Mitchell [13] proved that relative to ZFC the tree property at ω_2 is equiconsistent with the existence of a weakly compact cardinal. Variations of the forcing from Mitchell's result play a central role in further forcing results about the tree property.

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An old question asks whether it is consistent that every regular cardinal greater that $\aleph_1$ can have the tree property. One of the main obstacles is arranging the tree property at the successors of a singular cardinal. Shelah [11] proved that if ν is a singular limit of supercompact cardinals, then ν^+ has the tree property. The main result of [11] shows that assuming the existence of a little more than a huge cardinal, it is relatively consistent that $\aleph_{\omega+1}$ has the tree property. Recently Sinapova [16] showed that one can obtain the tree property at $\aleph_{\omega+1}$ from just ω supercompact cardinals using a very different construction from Magidor and Shelah.

For this paper we are motivated by trying to arrange the tree property at both the successor and double successor of a singular cardinal. For a general singular κ there are two relevant partial results. First, Cummings and Foreman [3] have shown from a supercompact cardinal with a weakly compact cardinal above, it is relatively consistent that there is a singular cardinal of cofinality ω whose double successor has the tree property. Second, using a forcing of Gitik and Sharon [5], Neeman [14] proved that starting from ω supercompact cardinals it is consistent that there is a singular cardinal of cofinality ω at which SCH fails and whose successor has the tree property. Making the singular cardinal into a small cardinal like $\aleph_\omega$ or $\aleph_{\omega^2}$ is difficult. Recently, Sinapova [15] was able to define a version of the Gitik-Sharon forcing to obtain the analog of Neeman's result [14] where $\kappa = \aleph_{\omega^2}$. The result for $\kappa = \aleph_\omega$ is open.

Our forcing is a combination of the Gitik-Sharon [5] forcing and the forcing from the result of Cummings and Foreman [3]. The forcing from [3] is a variant of a forcing due to Mitchell [13]. In the model for Theorem 1.1, we prove that there is a bad scale at κ. A bad scale at κ is a PCF theoretic object whose existence implies $\kappa^+ \notin I[\kappa^+]$ which in turn implies the failure of weak square, $\square^*_\kappa$. By a theorem of Jensen [6] weak square is equivalent to the existence of a special Aronszajn tree. So in particular the model for Theorem 1.1 has no special κ^+-trees. For an account of scales and their use in singular cardinal combinatorics we refer the interested reader to [2].

There is a natural model related to the model for Theorem 1.1 which is a candidate for the full tree property at κ^+. We are kept from this further result by difficulties involved in reproducing the argument of Neeman [14]. To illustrate this our presentation of the forcing will take an increasing sequence of regular cardinals $\langle \kappa_n \mid n < \omega \rangle$ as a parameter. If we take $\kappa_n = \kappa^{+n}$ for all $n < \omega$ as in [5], then we obtain the model for Theorem 1.1. If we instead let each κ_n be a supercompact cardinal as in [14], then we obtain a model that is a candidate for the full tree property at κ^+. We will also prove that there are no special κ^+-trees in the model obtained from letting the κ_n's be supercompact. We include this argument, because it is different from the argument given in the proof of Theorem 1.1.

The paper is organized as follows. In Section 2 we formulate a branch lemma, which will be used in the proof of Theorem 1.1 and has independent interest. We also recall another classical branch lemma needed in the proof below. In Section 3 we prove some preliminary lemmas, which allows us to define the main forcing. In Section 4 we define the main forcing and prove some of its properties. In Section 5 we prove that regardless of the choice of the sequence $\langle \kappa_n \mid n < \omega \rangle$, the tree property holds at κ^{++} in the extension. In Section 6 we give the two different models which both have no special κ^+-trees.

2. Branch lemmas

A *branch lemma* is a statement of the form 'Forcing of type X cannot add a branch through a tree of type Y'. Branch lemmas form a key component in our arguments that the tree property holds in the generic extension. Often we will need to do extra forcing to see that a tree has a branch and we will apply branch lemmas to see that the extra forcing could not have added the branch. There are two basic branch lemmas in the literature. One of the lemmas argues on the basis of chain condition and the other on the basis of closure. The original lemma used to show that forcing with good chain condition cannot add a branch through a branchless tree is due to Kunen and Tall [8].

Lemma 2.1. *Suppose that T is a branchless tree of height κ and $\mathbb{P}$ is κ-Knaster, then forcing with $\mathbb{P}$ cannot add a branch through T.*

The proof is very specific to Knasterness and requires that the degree of Knasterness be the same as the height of the tree. The assumption that T be branchless also seems like it should be able to be eliminated. We will generalize Lemma 2.1, but first we need a definition.

Definition 2.2. *Let κ be a regular cardinal. $\mathbb{P}$ has the* κ-approximation property *if for every ordinal μ and for every $\mathbb{P}$-name $\dot{x}$ for a subset of μ, if for every $z \in \mathcal{P}_\kappa(\mu)_V$, $\Vdash_{\mathbb{P}} \dot{x} \cap z \in V$, then $\Vdash_{\mathbb{P}} \dot{x} \in V$.*

Note that if $\mathbb{P}$ has the κ-approximation property, then $\mathbb{P}$ cannot add a branch through a tree of height κ. Suppose T is a tree of height κ. We may assume that the underlying set of T is an ordinal μ. Any name for a cofinal branch $\dot{b}$ is a name for a subset of μ. We claim that all of the less than κ sized pieces of $\dot{b}$ are in V. Suppose $z \in \mathcal{P}_\kappa(\mu)_V$. Then since κ is regular and z has size less than κ, there is a level of the tree α so that all nodes in z are below level α. We choose a condition in $\mathbb{P}$, which decides the value of $\dot{b}$ at level α. This determines $\dot{b} \cap z$, so it must be in V. Therefore by the κ-approximation property, $\Vdash_{\mathbb{P}} \dot{b} \in V$. Before we prove our generalization, we need the following proposition.

Proposition 2.3. *Suppose that $\mathbb{P}$ is a poset and $\dot{x}$ is a $\mathbb{P}$-name for a subset of some cardinal μ. Assume that for all $z \in \mathcal{P}_\kappa(\mu)$, $\Vdash_{\mathbb{P}} \dot{x} \cap z \in V$, but $\Vdash_{\mathbb{P}} \dot{x} \notin V$. Then for all $p \in \mathbb{P}$ and all $y \in \mathcal{P}_\kappa(\mu)$, there are $p_1, p_2 \leq p$ and $z \supseteq y$ such that p_1, p_2 decide the value of $\dot{x} \cap z$ and they decide different values.*

Proof. Suppose that the conclusion fails. Then we have p and y so that for any two extensions of p and any $z \supseteq y$ of size less than κ if these extensions decide the value of $\dot{x} \cap z$, then they give the same value. It follows that p forces $\dot{x} \in V$. □

The following was obtained by reflecting on arguments of Mitchell [13].

Lemma 2.4. *Let κ be a regular cardinal. Suppose that $\mathbb{P}$ is a poset and $\mathbb{P} \times \mathbb{P}$ is κ-cc, then $\mathbb{P}$ has the κ-approximation property.*

Proof. Suppose that the lemma is false. Then we have a poset $\mathbb{P}$ and a name $\dot{x}$, which fails to be approximated. We work by recursion to construct an antichain of size κ in $\mathbb{P} \times \mathbb{P}$. In particular, we construct $\langle (p^0_\alpha, p^1_\alpha) \mid \alpha < \kappa \rangle$ and a function $f : \kappa \to \mathcal{P}_\kappa(\mu)$. Assume that we have constructed (p^0_α, p^1_α) for $\alpha < \beta$ and $f \restriction \beta$ for some $\beta < \kappa$. Let $y = \bigcup f``\beta$ which is in $\mathcal{P}_\kappa(\mu)$. Choose $p_\beta \in \mathbb{P}$ which decides the value of $\dot{x} \cap y$ to be x_β. Apply Proposition 2.3 to p_β and y to obtain conditions

p^0_β, p^1_β and $f(\beta) \in \mathcal{P}_\kappa(\mu)$ such that p^0_β, p^1_β decide different values for $\dot{x} \cap f(\beta)$. Record the values that each condition decides as x^0_β, x^1_β. This completes the construction.

We claim that $\{(p^0_\alpha, p^1_\alpha) \mid \alpha < \kappa\}$ is an antichain of size κ. Suppose that we had $\alpha < \beta$ such that $(p^0_\alpha, p^1_\alpha), (p^0_\beta, p^1_\beta)$ are compatible. Then $x^k_\beta \cap f(\alpha) = x^k_\alpha$ for $k = 0, 1$. Note that $x^0_\beta \cap f(\alpha) = x^1_\beta \cap f(\alpha) = x_\beta \cap f(\alpha)$ by the choice of x_β. This implies that $x^0_\alpha = x^1_\alpha$ a contradiction. □

We will need this stronger lemma to prove that the tree property holds in our model. We will also need the following lemma which is used in [3] and [1]. We refer the interested reader to either paper for a proof.

Lemma 2.5 (Silver)**.** *Suppose that τ, η are cardinals with $2^\tau \geq \eta$. If $\mathbb{Q}$ is τ^+-closed, then forcing with $\mathbb{Q}$ cannot add a branch through an η-tree.*

3. Preliminaries to the main forcing

We give some definitions and results which allow us to define the main forcing. For the remainder of the paper we work in a ground model V of GCH where κ is a supercompact cardinal which is indestructible under κ-directed closed forcing [10]. For ease of argument we are going to assume that $\lambda > \kappa$ is measurable with U^* a normal measure on λ. Weakening the result to use only weak compactness is straightforward. Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of regular cardinals less than λ with $\kappa = \kappa_0$. Let ν be the supremum of the κ_n's. Since ν will be collapsed and ν^+ will be preserved, we let $\mu = \nu^+$.

Let $\mathbb{A} = Add(\kappa, \lambda)$. In $V^{\mathbb{A}}$, κ is still supercompact. We let U be a supercompactness measure on $\mathcal{P}_\kappa(\mu)$ and for each $n < \omega$ let U_n be the projection of U on to $\mathcal{P}_\kappa(\kappa_n)$. The measures U_n concentrate on the sets X_n of $x \in \mathcal{P}_\kappa(\kappa_n)$ such that $x \cap \kappa$ is an inaccessible cardinal. We define $\mathbb{P}$ the diagonal Prikry forcing in the model $V^{\mathbb{A}}$ using the measures U_n.

Definition 3.1. *A condition in $\mathbb{P}$ is a sequence*

$$p = \langle x_0, x_1, \dots x_{n-1}, A_n, A_{n+1}, \dots \rangle$$

where

1. *for all $i < n$, $x_i \in X_i$,*
2. *for all $i < n-1$, $x_i \subseteq x_{i+1} \cap \kappa_i$ and $\mathrm{o.t.}(x_i) < \kappa \cap x_{i+1}$ and*
3. *for all $i \geq n$, $A_i \in U_i$ and $A_i \subseteq X_i$.*

We call n the length of p and denote it $\ell(p)$. Given another condition

$$q = \langle y_0, \dots y_{m-1}, B_m, B_{m+1}, \dots \rangle$$

we define $p \leq q$ if and only if $n \geq m$, for all $i < m$, $y_i = x_i$, for all i with $m \leq i < n$, $x_i \in B_i$, and for all $i \geq n$ $A_i \subseteq B_i$.

Using the measurability of λ we are going to show that there are many places where the measure U (and hence each U_n) reflects. For $\alpha < \lambda$ let $\mathbb{A}_\alpha$ be $Add(\kappa, \alpha)$. Let $\dot{U}$ be an $\mathbb{A}$-name for U. Following the set up of [3] we have the following lemma.

Lemma 3.2. *There is a set $B \subseteq \lambda$ of Mahlo cardinals with $B \in V$ such that*

1. *if g is $\mathbb{A}$-generic over V, then for all $\alpha \in B$, $i_g(\dot{U}) \cap V[g \restriction \alpha] \in V[g \restriction \alpha]$ and*
2. *$B \in U^*$.*

Proof. Let $\beta < \lambda$. For each canonical $\mathbb{A}_\beta$-name $\dot{X}$ for a subset of $\mathcal{P}_\kappa(\mu)$ choose a maximal antichain of conditions in $\mathbb{A}$ deciding the statement "$\dot{X} \in \dot{U}$". By the κ^+-cc of $\mathbb{A}$ and the inaccessibility of λ, the supremum of the domains of conditions in $\mathbb{A}$ appearing in any of the above antichains is less than λ. Let $F(\beta) < \lambda$ be greater than this supremum. The set of limit points of F is club. Let B be the set of Mahlo limit points of F. B is as required for the lemma. □

Let g be $\mathbb{A}$-generic over V. For each $\alpha \in B$, let $U^\alpha =_{def} i_g(\dot{U}) \cap V[g \restriction \alpha]$. It is clear that U^α is a supercompactness measure on $\mathcal{P}_\kappa(\mu)$ in $V[g \restriction \alpha]$. For $\alpha \in B$ we define $\mathbb{P}_\alpha$ in $V[g \restriction \alpha]$ to be the Diagonal Prikry forcing obtained from U^α in the same way we defined $\mathbb{P}$ from U. We call the associated measures U^α_n. Next we note that a Prikry sequence for $\mathbb{P}$ gives a Prikry sequence for $\mathbb{P}_\alpha$. This follows from a characterization of genericity for Prikry forcing due to Mathias [12]. The version needed here is that $\vec{x}$ is $\mathbb{P}$-generic if and only if for all sequences of measure one sets $\langle A(n) \mid n < \omega \rangle$, $\vec{x}(n) \in A(n)$ for all sufficiently large n. From this it is easy to see that an $\mathbb{A} * \mathbb{P}$-generic object induces a $\mathbb{A}_\alpha * \mathbb{P}_\alpha$-generic object for each $\alpha \in B$. In particular, we just restrict the $\mathbb{A}$-generic object and use the same Prikry generic sequence. It follows that $\mathrm{RO}(\mathbb{A}_\alpha * \mathbb{P}_\alpha)$ is isomorphic to a complete subalgebra of $\mathrm{RO}(\mathbb{A} * \mathbb{P})$ where $\mathrm{RO}(-)$ denotes the regular open algebra. Our work with the above posets will rely on the notion of a *projection*.

Definition 3.3. *Let $\mathbb{P}$ and $\mathbb{Q}$ be posets. A map $\pi : \mathbb{P} \to \mathbb{Q}$ is a* projection *if*

(1) $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$,
(2) *for all $p, p' \in \mathbb{P}$, $p' \leq p$ implies that $\pi(p') \leq \pi(p)$ and*
(3) *for all $p \in \mathbb{P}$ and $q \leq \pi(p)$, there is $p' \leq p$ such that $\pi(p') \leq q$.*

Definition 3.4. *Suppose that $\pi : \mathbb{P} \to \mathbb{Q}$ is a projection. Then in $V^{\mathbb{Q}}$ define $\mathbb{P}/\mathbb{Q} = \{p \in \mathbb{P} \mid \pi(p) \in \dot{G}_{\mathbb{Q}}\}$ ordered as a suborder of $\mathbb{P}$. If G is $\mathbb{Q}$-generic, then we may write $\mathbb{P}/G$ for $\mathbb{P}/\mathbb{Q}$ as computed in $V[G]$.*

Fact 3.5. *In the context of Definition 3.4, $\mathbb{P}$ is isomorphic to a dense subset of $\mathbb{Q} * \mathbb{P}/\mathbb{Q}$.*

We now continue with facts about $\mathbb{A} * \mathbb{P}$ and related posets.

Lemma 3.6. *For all $\alpha \in B$ there is a projection $\pi_\alpha : \mathbb{A} * \mathbb{P} \to \mathrm{RO}(\mathbb{A}_\alpha * \mathbb{P}_\alpha)$.*

Proof. This follows from general considerations about the regular open algebras of posets. First we use the map that takes $\mathbb{A} * \mathbb{P}$ densely into its regular open algebra. Then viewing $\mathrm{RO}(\mathbb{A}_\alpha * \mathbb{P}_\alpha)$ as its isomorphic copy inside $\mathrm{RO}(\mathbb{A} * \mathbb{P})$, we take the meet over all conditions in $\mathrm{RO}(\mathbb{A}_\alpha * \mathbb{P}_\alpha)$ that are above a given condition in the range of the first map. It is easy to see that the composition of the above two maps gives a projection. □

Note that the projection we get here did not rely on special properties of λ so by a similar argument we have the following lemma.

Lemma 3.7. *For every $\alpha, \beta \in B$ with $\alpha < \beta$, there is a projection $\pi_{\alpha,\beta} : \mathbb{A}_\beta * \mathbb{P}_\beta \to \mathrm{RO}(\mathbb{A}_\alpha * \mathbb{P}_\alpha)$*

Remark 3.8. *In the previous two lemmas we used the fact that there are projections from $\mathrm{RO}(\mathbb{A} * \mathbb{P})$ to $\mathrm{RO}(\mathbb{A}_\beta * \mathbb{P}_\beta)$ and from $\mathrm{RO}(\mathbb{A}_\beta * \mathbb{P}_\beta)$ to $\mathrm{RO}(\mathbb{A}_\alpha * \mathbb{P}_\alpha)$ which we denote σ_β and $\sigma_{\alpha,\beta}$ respectively. We also note that $\sigma_{\alpha,\beta} \circ \pi_\beta = \pi_{\alpha,\beta}$.*

Though we will eventually need something stronger, we have the following.

Lemma 3.9. *$\mathbb{A} * \mathbb{P}$ is μ-cc.*

Proof. $\mathbb{A}$ is κ^+-cc and $\Vdash_{\mathbb{A}} \mathbb{P}$ is μ-cc, so it follows that $\mathbb{A} * \mathbb{P}$ is μ-cc. □

It is easy to see that this proof applies to $\mathbb{A}_\alpha * \mathbb{P}_\alpha$ for $\alpha \in B$. Before moving on to the definition of the main forcing we record some facts about the extension by $\mathbb{A} * \mathbb{P}$. The proofs of properties of $\mathbb{P}$ which lead to the following lemma are easy adaptations of the proofs in [5].

Lemma 3.10. *$V^{\mathbb{A}*\mathbb{P}}$ satisfies*

(1) *κ is singular strong limit of cofinality ω,*
(2) *$\kappa^+ = (\nu^+)^V = \mu$ and*
(3) *$2^\kappa = \lambda$.*

4. The main forcing

We are now ready to define the final poset, which we call $\mathbb{R}$. Conditions are triples (a, p, f) such that (a, p) is a condition in $\mathbb{A} * \mathbb{P}$ and f is a function with the following properties:

(1) $\mathrm{dom}(f) \subseteq B$ and $|\mathrm{dom}(f)| \leq \nu$.
(2) For all $\alpha \in \mathrm{dom}(f)$, $f(\alpha)$ is an $\mathbb{A}_\alpha * \mathbb{P}_\alpha$-name for a condition in the forcing $\mathrm{Add}(\mu, 1)_{V^{\mathbb{A}_\alpha * \mathbb{P}_\alpha}}$.

The ordering is defined by $(a, p, f) \leq (a_1, p_1, f_1)$ if and only if

(1) $(a, p) \leq (a_1, p_1)$ in $\mathbb{A} * \mathbb{P}$,
(2) $\mathrm{dom}(f_1) \subseteq \mathrm{dom}(f)$ and
(3) for all $\alpha \in \mathrm{dom}(f_1)$, $\pi_\alpha(a, p) \Vdash f(\alpha) \leq f_1(\alpha)$ in $\mathrm{Add}(\mu, 1)_{V^{\mathbb{A}_\alpha * \mathbb{P}_\alpha}}$.

We also define a certain restriction of $\mathbb{R}$.

Definition 4.1. *For $\beta \in B$, let $\mathbb{R} \restriction \beta$ be the poset defined as follows. Conditions are triples (a, p, f) such that $(a, p) \in \mathbb{A}_\beta * \mathbb{P}_\beta$ and f is a function as before, but its domain is a subset of $B \cap \beta$. Let $(a, p, f) \leq (a_1, p_1, f_1)$ if and only if $(a, p) \leq (a_1, p_1)$, $dom(f) \supseteq dom(f_1)$, and for all $\alpha \in dom(f_1)$, $\pi_{\alpha,\beta}(a, p) \Vdash f(\alpha) \leq f_1(\alpha)$.*

It is easy to see that a generic for $\mathbb{R}$ induces a generic for $\mathbb{R} \restriction \beta$. Our forcing has many of the same properties the forcing from [3].

4.1. **Basic Properties of $\mathbb{R}$.**

Lemma 4.2. *$\mathbb{R}$ is λ-Knaster and for all $\beta \in B$, $\mathbb{R} \restriction \beta$ is β-Knaster.*

Proof. This is a standard Δ-system argument combined with the fact that any two Prikry conditions with the same stem are compatible. □

We have the standard projections as in [1] or [3].

Lemma 4.3. *There are projections*

$$\mathbb{R} \to \mathbb{A} * \mathbb{P} \text{ and}$$
$$\mathbb{R} \to \mathrm{RO}(\mathbb{A}_\alpha * \mathbb{P}_\alpha) * \mathrm{Add}(\mu, 1)_{V^{\mathbb{A}_\alpha * \mathbb{P}_\alpha}}$$

for $\alpha \in B$ given by

$$(a, p, f) \mapsto (a, p) \text{ and}$$
$$(a, p, f) \mapsto (\pi_\alpha(a, p), f(\alpha))$$

respectively.

Using the first projection we see that $2^\kappa \geq \lambda$ and using the second projection we see that each $\alpha \in B$ is collapsed to have size μ in the extension by $\mathbb{R}$. As in the Cummings-Foreman paper, we have that the extension by $\mathbb{R}$ is contained in an extension by $(\mathbb{A} * \mathbb{P}) \times \mathbb{Q}$ where $\mathbb{Q}$ is μ-closed.

Definition 4.4. *Let $\mathbb{Q}$ be the set of third coordinates from $\mathbb{R}$ together with the ordering $f_1 \leq f_2$ if and only if $\mathrm{dom}(f_1) \supseteq \mathrm{dom}(f_2)$ and for all $\alpha \in \mathrm{dom}(f_2)$, $\Vdash_{\mathbb{A}_\alpha * \mathbb{P}_\alpha} f_1(\alpha) \leq f_2(\alpha)$.*

Lemma 4.5. *$\mathbb{Q}$ is μ-closed and the identity map is a projection from $(\mathbb{A} * \mathbb{P}) \times \mathbb{Q}$ to $\mathbb{R}$.*

The proof is a straightforward adaptation of Lemma 2.8 of [1].

For suitable choice of generics we have $V^{\mathbb{A}*\mathbb{P}} \subseteq V^{\mathbb{R}} \subseteq V^{(\mathbb{A}*\mathbb{P})\times\mathbb{Q}}$. Using these facts we can prove the following lemma.

Lemma 4.6. *$V^{\mathbb{R}}$ satisfies*

(1) *κ is singular strong limit of cofinality ω,*
(2) *ν is collapsed to have size κ and μ is preserved and*
(3) *$2^\kappa = \kappa^{++} = \lambda$.*

Proof. By Lemma 4.5, every $< \mu$ sequence from $V^{\mathbb{R}}$ is in $V^{\mathbb{A}*\mathbb{P}}$. It follows that κ is singular strong limit of cofinality ω in $V^{\mathbb{R}}$. It also follows that μ is preserved since if it were collapsed then it would have been collapsed by $\mathbb{A} * \mathbb{P}$. Since $\mathbb{R}$ projects on to $\mathbb{A} * \mathbb{P}$, we have that ν is collapsed to have size κ and $2^\kappa \geq \lambda$. We have that $2^\kappa = \lambda$, since $2^\kappa = \lambda$ in $V^{\mathbb{A}*\mathbb{P}}$ and every κ sequence from $V^{\mathbb{R}}$ is in this model. Finally, each $\beta \in B$ is collapsed to have cofinality μ by Lemma 4.3 and λ is preserved by Lemma 4.2. □

4.2. **Complex Properties of $\mathbb{R}$.** In this subsection we prove that in $V^{\mathbb{R}\restriction\beta}$ the forcing $\mathbb{R}/\mathbb{R} \restriction \beta$ is equivalent to a forcing with a definition similar to $\mathbb{R}$. This will provide a key component in our proof that the tree property holds at κ^{++} in the extension.

First we make note of a slightly different, but equivalent (in the sense of forcing) definition of $\mathbb{R}$ and its restrictions. Note that instead of the condition $(a, p) \in \mathbb{A} * \mathbb{P}$ in the first two coordinates of $\mathbb{R}$, we could have just taken a condition in $\mathrm{RO}(\mathbb{A} * \mathbb{P})$. The projections needed in the definition of the ordering are just the projections σ_β from Remark 3.8. If we call this new poset $\mathbb{R}'$, it is easy to see that $\mathbb{R}$ is a dense subset of $\mathbb{R}'$. Similarly we define $\mathbb{R}' \restriction \beta$ as above by replacing $\mathbb{A}_\beta * \mathbb{P}_\beta$ with $\mathrm{RO}(\mathbb{A}_\beta * \mathbb{P}_\beta)$ in the definition of $\mathbb{R} \restriction \beta$.

Having defined these auxiliary posets it is easy to see that the map from $\mathbb{R}$ to $\mathbb{R}' \restriction \beta$ given by $(a, p, f) \mapsto (\pi_\beta(a, p), f \restriction \beta)$ is a projection. In $V^{\mathbb{R}\restriction\beta}$ we will define a poset $\mathbb{R}^*$ which is equivalent to the poset $\mathbb{R}/\mathbb{R}' \restriction \beta$ and which resembles $\mathbb{R}$. To define the ordering we will need suitable projections from $(\mathbb{A} * \mathbb{P})/(\mathbb{A}_\beta * \mathbb{P}_\beta)$ to $\mathrm{RO}(\mathbb{A}_\gamma * \mathbb{P}_\gamma)/(\mathbb{A}_\beta * \mathbb{P}_\beta)$ for $\gamma > \beta$ in B. These projections will be given by the following proposition. (The posets $\mathbb{P}, \mathbb{Q}$ and $\mathbb{R}$ in the following proposition have no relation to the posets we've defined above.)

Proposition 4.7. *Let $\mathbb{P}, \mathbb{Q}$ and $\mathbb{R}$ be posets and assume that there are projections $\pi : \mathbb{P} \to \mathbb{Q}$ and $\sigma : \mathbb{Q} \to \mathbb{R}$. If G is $\mathbb{R}$-generic, then in $V[G]$ $\pi \restriction \mathbb{P}/G$ is a projection from $\mathbb{P}/G$ to $\mathbb{Q}/G$.*

Proof. Clearly the restriction of π is order preserving and sends the top element of $\mathbb{P}/G$ to the top element of $\mathbb{Q}/G$. For the moment we work in V. Let $p \in \mathbb{P}$ and $q \leq \pi(p)$. Let D be the set of $r \in \mathbb{R}$ such that there is $p' \in \mathbb{P}$ with

(1) $p' \leq p$,
(2) $\pi(p') \leq q$ and
(3) $\sigma(\pi(p')) = r$.

We claim that D is dense in $\mathbb{R}$ below $\sigma(q)$. Suppose that $r \leq \sigma(q)$ since σ is a projection there is $q' \leq q$ such that $\sigma(q') \leq r$. Since π is a projection there is a $p' \leq p$ such that $\pi(p') \leq q'$. Clearly $\sigma(\pi(p')) \in D$. Suppose that $p \in \mathbb{P}/G$ and $q \leq \pi(p)$ is in $\mathbb{Q}/G$. Then since $\sigma(q) \in G$, $D \cap G \neq \emptyset$ where D is defined as above. Let p' witness that some $r \in D \cap G$. Then $p' \in \mathbb{P}/G$, $p' \leq p$ and $\pi(p') \leq q$ as required. $\square$

Working in $V^{\mathbb{R}\restriction\beta}$, we define the forcing $\mathbb{R}^*$ as follows. We let $(a, p, f) \in \mathbb{R}^*$ if and only if $(a, p) \in (\mathbb{A} * \mathbb{P})/(\mathbb{A}_\beta * \mathbb{P}_\beta)$ and f is a partial function with domain a subset of B of size $< \mu$ such that for each $\gamma \in B$, $f(\gamma)$ is an $(\mathbb{A}_\gamma * \mathbb{P}_\gamma)/(\mathbb{A}_\beta * \mathbb{P}_\beta)$-name for a condition in $\mathrm{Add}(\mu, 1)$. The ordering is defined in a similar way to that for $\mathbb{R}$, but for each γ using the restriction of π_γ to $(\mathbb{A} * \mathbb{P})/(\mathbb{A}_\beta * \mathbb{P}_\beta)$. These restrictions are projections by the previous proposition applied with π_γ in place of π and $\sigma_{\beta,\gamma}$ (from Remark 3.8) in place of σ.

We can now state the main technical lemma of this section.

Lemma 4.8. *There is a map i from $\mathbb{R}$ to $\mathbb{R}' \restriction \beta * \dot{\mathbb{R}}^*$ such that $x \leq y$ if and only if $i(x) \leq i(y)$ and the range of i is dense.*

The proof is very similar to the proof of Lemma 2.12 in [1]. As with $\mathbb{R}$ we can show that $\mathbb{R}^*$ is the projection of a product (see Definition 4.4 and Lemma 4.5).

Lemma 4.9. *In $V^{\mathbb{R}\restriction\beta}$, there is a μ-closed forcing $\mathbb{Q}^*$ such that the identity map is a projection from $(\mathbb{A} * \mathbb{P})/(\mathbb{A}_\beta * \mathbb{P}_\beta) \times \mathbb{Q}^*$ to $\mathbb{R}^*$.*

The proof is straightforward.

5. The tree property at κ^{++}

The proof in this section is somewhat different from the proof in the paper of Cummings and Foreman [3]. We rely on the same analysis in terms of projections, but the forcings involved are no longer as nice. Our task is further complicated by a mistake on the very last page the Cummings and Foreman paper. In particular they attempt to prove that the quotient forcing in their paper corresponding to $(\mathbb{A} * \mathbb{P})/(\mathbb{A}_\beta * \mathbb{P}_\beta)$ in our paper has the Knaster property. The key point in the argument is to show that conditions which witness the compatibility of conditions in $\mathbb{A} * \mathbb{P}$ are forced in to the quotient. A careful read of the paper shows that Cummings and Foreman have not done enough work to show that such conditions are forced in to the quotient. To fix this problem we provide a further analysis of the quotient forcing. Our task is made a little easier, since in light of Lemma 2.4 we only need to show that the quotient squared has chain condition. The proof given below adapts easily to give the proof the analogous fact about the forcing in the Cummings and Foreman paper.

Lemma 5.1. *The tree property holds at κ^{++} in $V^{\mathbb{R}}$.*

Let $j : V \to M$ be the ultrapower by the normal measure U^*. We want to lift j to an elementary embedding $j : V[G] \to M[H]$ where G is V-generic for $\mathbb{R}$ and H is V-generic for $j(\mathbb{R})$. For all $r \in \mathbb{R}$, $j(r) = r$. Recall that B is in the normal measure on λ, so $\lambda \in j(B)$. So we have $j(\mathbb{R}) \restriction \lambda = \mathbb{R}$. We choose H which is V-generic for $j(\mathbb{R})$ and let G be the induced generic object for $\mathbb{R}$. It follows that in $V[H]$ we may lift the embedding. For a contradiction we assume that T is a κ^{++}-Aronszajn tree in $V[G]$.

Lemma 5.2. *There is a $\beta \in B$ such that $V[G \restriction \beta]$ models that $T \restriction \beta$ is an β-Aronszajn tree*

Proof. Let j be as above. By hypothesis $V[G]$ models T is an Aronszajn tree. We may assume that the underlying set of T is λ. It follows that $j(T) \restriction \lambda = T$. T has an $\mathbb{R}$-name which can be coded as a subset of λ and hence $T \in M[G]$. Since $M[G]$ is a submodel of $V[G]$, T is an Aronszajn tree in $M[G]$. Recall that $M[G] = M[H \restriction \lambda]$. Putting all of this together we have $M[H] \models$ "There is a $\beta \in \pi(B)$ such that $M[H \restriction \beta] \models j(T) \restriction \beta$ is a β-Aronszajn tree." So by elementarity we have the lemma. □

Clearly there is a branch through $T \restriction \beta$ in $V^{\mathbb{R}}$ as T has height κ^{++}. We are going to show that the quotient forcing $\mathbb{R}/\mathbb{R} \restriction \beta$ could not have added the branch, a contradiction. By Lemmas 4.8 and 4.9 it will suffice to show that forcing with $\mathbb{Q}^* \times (\mathbb{A} * \mathbb{P})/(\mathbb{A}_\beta * \mathbb{P}_\beta)$ over the model $V[G \restriction \beta]$ could not have added a branch through $T \restriction \beta$. Note that the hypotheses of Lemma 2.5 hold in $V[G \restriction \beta]$ about $T \restriction \beta$ and $\mathbb{Q}^*$. It follows that $T \restriction \beta$ is still branchless in $V[G \restriction \beta]^{\mathbb{Q}^*}$. Moreover in this model β has been collapsed to have cofinality $\mu = \kappa^+$. We replace $T \restriction \beta$ with its restriction to a cofinal set of levels of order type μ. To finish the proof we would like to show that $((\mathbb{A} * \mathbb{P})/(\mathbb{A}_\beta * \mathbb{P}_\beta))^2$ is μ-cc in $V[G \restriction \beta]^{\mathbb{Q}^*}$ in order to apply Lemma 2.4. It will be enough to show the following.

Lemma 5.3. *In $V^{\mathbb{A}_\beta * \mathbb{P}_\beta}$, $((\mathbb{A} * \mathbb{P})/(\mathbb{A}_\beta * \mathbb{P}_\beta))^2$ has the μ-cc*

Corollary 5.4. *$((\mathbb{A} * \mathbb{P})/(\mathbb{A}_\beta * \mathbb{P}_\beta))^2$ has the μ-cc in $V[G \restriction \beta]^{\mathbb{Q}^*}$*

Proof of Corollary 5.4 from Lemma 5.3. For ease of notation let $\mathbb{T}$ denote the quotient. By Easton's Lemma it is enough to show that $\mathbb{T}^2$ has μ-cc in $V^{\mathbb{R} \restriction \beta}$. By Lemma 5.3 we have that $\mathbb{A}_\beta * \mathbb{P}_\beta * \mathbb{T}^2$ is μ-cc in V. By Easton's Lemma we have that $\mathbb{A}_\beta * \mathbb{P}_\beta * \mathbb{T}^2$ has μ-cc in $V^{\mathbb{Q} \restriction \beta}$. It follows that $\mathbb{T}^2$ has μ-cc in $V^{\mathbb{Q} \restriction \beta \times (\mathbb{A}_\beta * \mathbb{P}_\beta)}$ which finishes the proof. □

Before proceeding with the proof of the Lemma 5.3 we prove preliminary facts. First we need a proposition which is an adaptation of a claim in the proof of Lemma 7.1 of [3].

Proposition 5.5. *Let $(b, (t, \dot{B})) \in \mathbb{A}_\beta * \mathbb{P}_\beta$ and $(a, (s, \dot{A})) \in \mathbb{A} * \mathbb{P}$. $(b, (t, \dot{B})) \Vdash (a, (s, \dot{A})) \notin (\mathbb{A} * \mathbb{P})/(\mathbb{A}_\beta * \mathbb{P}_\beta)$ if and only if one of the following holds*

1. *$a \restriction \beta, b$ are incompatible in $\mathbb{A}_\beta$.*
2. *$a \restriction \beta, b$ are compatible, $s \not\leq t$ and $t \not\leq s$.*
3. *$a \restriction \beta, b$ are compatible, $s \leq t$ and $a \cup b \Vdash$ there is n such that $\ell(s) \leq n < \ell(t)$ and $t(n) \notin \dot{A}(n)$.*
4. *$a \restriction \beta, b$ are compatible, $t \leq s$ and $a \restriction \beta \cup b \Vdash$ there is n such that $\ell(t) \leq n < \ell(s)$ and $s(n) \notin \dot{B}(n)$.*

The proof is exactly the same as in [3]. So we have characterized when a condition in $\mathbb{A}_\beta * \mathbb{P}_\beta$ forces a condition in $\mathbb{A} * \mathbb{P}$ out of the quotient. We will also need sufficient conditions for something to forced in to the quotient.

Proposition 5.6. *If $(b,(t,\dot{B})) \in \mathbb{A}_\beta * \mathbb{P}_\beta$ and $(a,(s,\dot{A})) \in \mathbb{A} * \mathbb{P}$ and*

1. *t extends s,*
2. *$b \leq a \restriction \beta$ and*
3. *$a \cup b \Vdash$ "For all n with $\ell(s) \leq n < \ell(t)$, $t(n) \in \dot{A}(n)$",*

then there is an $\mathbb{A}_\beta$-name for a sequence of measure one sets $\dot{C}$ such that $(b,(t,\dot{B} \cap \dot{C})) \Vdash (a,(s,\dot{A})) \in (\mathbb{A} * \mathbb{P})/(\mathbb{A}_\beta * \mathbb{P}_\beta)$.

Note that here $\dot{B} \cap \dot{C}$ is a compact way of writing the canonical $\mathbb{A}_\beta$-name for the pointwise intersection of the sequences of measure one sets $\dot{B}$ and $\dot{C}$.

Proof. We start with a claim that will establish the existence of certain measure one sets.

Claim. *For each $k < \omega$, b forces that the set $\{\vec{x} \mid \ell(\vec{x}) = k, s \frown \vec{x}$ is a stem and $a \nVdash^{V^{\mathbb{A}_\beta}}_{\mathbb{A}/\mathbb{A}_\beta}$ there is $n < k$, $\vec{x}(n) \notin \dot{A}(\ell(s)+n)\}$ contains a set of the form $\prod_{n<k} C_n$ where for all $n < k$, $C_n \in U^\beta_{\ell(s)+n}$*

Suppose that the claim fails. Then by a version of Rowbottom's theorem there are a $k < \omega$ and a $b' \leq b$ which forces that the complement of the above set contains a product of measure one sets C_n for $n < k$. If we force with $\mathbb{A}$ below $b' \cup a$, then we see that $\prod_{n<k} C_n \cap \prod_{k<n} A_{\ell(s)+n}$ is empty, which is impossible since each of the sets involved is measure one for one of the measures U_i.

Claim. *If $\dot{C}$ is an $\mathbb{A}_\beta$-name for a sequence of measure one sets which witnesses the previous claim for all $k < \omega$, then $\dot{C}$ is as required for the proposition.*

Suppose that the claim fails. There is $(b',(t',\dot{B}')) \leq (b,(t,\dot{B} \cap \dot{C}))$ such that $(b',(t',\dot{B}'))$ forces that $(a,(s,\dot{A})) \notin (\mathbb{A} * \mathbb{P})/(\mathbb{A}_\beta * \mathbb{P}_\beta)$. Working through the clauses of Proposition 5.5 we see that (1),(2) and (4) all fail. So by clause (3), we must have that $b' \cup a$ forces that there is an i such that $t'(i) \notin \dot{A}(i)$. By assumption (3) of the current proposition, we must have that such i is greater than or equal to $\ell(t)$. We set $\vec{x}$ to be the unique sequence satisfying $t \frown \vec{x} = t'$. By the previous claim we have that b' forces that a does not force that there is an n such that $\vec{x}(n) \notin \dot{A}_{\ell(s)+n}$. It follows that $a \cup b'$ does not force that there is an n such that $\vec{x}(n) \notin \dot{A}(\ell(s)+n)$. This contradicts clause (3) of Proposition 5.5. $\square$

Proposition 5.7. *If $(b,(t,\dot{B})) \in \mathbb{A}_\beta * \mathbb{P}_\beta$ and $\dot{r}$ is an $\mathbb{A}_\beta * \mathbb{P}_\beta$-name for a condition in $(\mathbb{A} * \mathbb{P})/(\mathbb{A}_\beta * \mathbb{P}_\beta)$, then we can extend $(b,(t,\dot{B}))$ to $(b',(t',\dot{B}'))$ deciding the value of $\dot{r} = (a,(s,\dot{A}))$ so that there is $a' \leq a$ such that $(b',(t',\dot{B}'))$ and $(a',(s,\dot{A}))$ satisfy the hypotheses of Proposition 5.6.*

Proof. Extending to $(b,(t,\dot{B}))$ to $(b',(t',\dot{B}'))$ which decides the value of $\dot{r}$, we may assume that the stem t' extends the stem s and $b' \leq a \restriction \beta$. Now by Proposition 5.5, we must have that clause (3) fails and so there is $a' \leq b' \cup a$ such that a' forces for all n with $\ell(s) \leq n < \ell(t')$, $t'(n) \in \dot{A}(n)$. Extending b' if necessary we may assume that $b' \leq a' \restriction \beta$. $\square$

Remark 5.8. *It is not hard to see that we could have obtained the conclusion of the previous proposition simultaneously for two conditions $\dot{r}_0$ and $\dot{r}_1$ using the same condition $(b', (t', \dot{B}'))$.*

Our final preliminary fact is about the chain condition of $\mathbb{A} * \mathbb{P}$ in V.

Proposition 5.9. *In V, $(\mathbb{A} * \mathbb{P})^2 \times (\mathbb{A}_\beta * \mathbb{P}_\beta)$ is μ-cc*

The proof is straight forward. Conditions with the same stem are compatible and κ^+ of the $\mathbb{A}$ parts can be formed in to a Δ-system. We are now ready to prove that the quotient squared has chain condition.

Proof of Lemma 5.3. Suppose that $(\dot{r}^0_\alpha, \dot{r}^1_\alpha)$ for $\alpha < \mu$ is an $\mathbb{A}_\beta * \mathbb{P}_\beta$-name for an antichain in $((\mathbb{A} * \mathbb{P})/(\mathbb{A}_\beta * \mathbb{P}_\beta))^2$. Let $(b, (t, \dot{B})) \in \mathbb{A}_\beta * \mathbb{P}_\beta$. By Remark 5.8, for each $\alpha < \mu$ we choose $(b_\alpha, (t_\alpha, \dot{B}_\alpha))$ deciding the value of $\dot{r}^0_\alpha, \dot{r}^1_\alpha$to be $(a^0_\alpha, (s^0_\alpha, \dot{A}^0_\alpha))$ and $(a^1_\alpha, (s^1_\alpha, \dot{A}^1_\alpha))$ with extensions $\bar{a}^i_\alpha \leq a^i_\alpha$ for $i \in 2$. By the proof of Proposition 5.9 we may assume that there are $\alpha < \alpha' < \mu$ such that $s^i_\alpha = s^i_{\alpha'}$ for $i \in 2$ and each of $b_\alpha \cup b_{\alpha'}$ and $\bar{a}^i_\alpha \cup \bar{a}^i_{\alpha'}$ for $i \in 2$ are conditions in $\mathbb{A}$. For $i \in 2$ we now have the hypotheses of Proposition 5.6 for the conditions $(b_\alpha \cup b_{\alpha'}, (t, \dot{B}_\alpha \cap \dot{B}_{\alpha'}))$ and $(\bar{a}^i_\alpha \cup \bar{a}^i_{\alpha'}, (s^i, \dot{A}^i_\alpha \cap \dot{A}^i_{\alpha'}))$ where $t = t_\alpha = t_{\alpha'}$ and $s^i = s^i_\alpha = s^i_{\alpha'}$. It follows that there is a direct extension of $(b_\alpha \cup b_{\alpha'}, (t, \dot{B}_\alpha \cap \dot{B}_{\alpha'}))$, which forces the compatibility of the conditions with index α and α'. $\square$

This finishes the proof of Lemma 5.3 and with it the proof of Lemma 5.1

6. No special κ^+-trees

In this section we give two proofs that there are no special κ^+-trees. The first proof applies to the version of $\mathbb{R}$ where we follow Gitik and Sharon and take $\kappa_n = \kappa^{+n}$. The second proof applies to the version of $\mathbb{R}$ where we follow Neeman and let the κ_n's be an increasing sequence of supercompact cardinals.

6.1. $\mathbb{R}$ with Gitik-Sharon.

For this section we assume that $\kappa_n = \kappa^{+n}$ and so $\nu = \kappa^{+\omega}$. We show the following:

Theorem 6.1. *In $V^{\mathbb{R}}$ there is a bad scale on κ^+.*

Proof. By arguments from [4] there is a bad scale $\vec{f}$ at κ in $V^{\mathbb{A}*\mathbb{P}}$ which is witnessed by a stationary set $S \subseteq \kappa^{+\omega+1}$ from V. Using Lemma 4.5 and Easton's Lemma, we see that $V^{\mathbb{A}*\mathbb{P}}$ and $V^{\mathbb{R}}$ have the same $< \kappa$-sequences. It follows that $\vec{f}$ is still a scale in $V^{\mathbb{R}}$ and every bad point of $\vec{f}$ in $V^{\mathbb{A}*\mathbb{P}}$ remains bad in $V^{\mathbb{R}}$. It remains to see that the set S is still stationary in $V^{\mathbb{R}}$. It is enough to show that it is still stationary in the outer model $V^{\mathbb{Q}\times(\mathbb{A}*\mathbb{P})}$. Since $\mathbb{Q}$ is $\kappa^{+\omega+1}$-closed in V, S is stationary in $V^{\mathbb{Q}}$. By Easton's Lemma, $\mathbb{A} * \mathbb{P}$ is $\kappa^{+\omega+1}$-cc in $V^{\mathbb{Q}}$. The result follows. $\square$

Remark 6.2. *It should be noted that with a some extra work this argument applies for other choices of the κ_n's. A referee pointed out that arguments from [5] obtain the failure of approachability at κ for all choices of the κ_n's with no extra work required.*

6.2. $\mathbb{R}$ **with Neeman.** For this section we assume that $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals. Further we assume that each κ_n is indestructible under κ_n-directed closed forcing. Under the above assumptions we prove the following theorem.

Lemma 6.3. *In the extension $V^{\mathbb{R}}$ there are no special κ^+-trees*

Proof. It will be enough to show that the tree property holds at κ^+ in an extension by $\mathbb{A} * \mathbb{P} \times \mathbb{Q}$ and that μ is preserved in this extension. If we had a special κ^+-tree in $V^{\mathbb{R}}$, then it would still be a special κ^+-tree in $V^{(\mathbb{A}*\mathbb{P})\times\mathbb{Q}}$, which is impossible. Recall that in the extension we have collapsed cardinals and μ has become κ^+. By Lemma 4.5, we know that $\mu = \kappa^+$ is preserved in $V^{(\mathbb{A}*\mathbb{P})\times\mathbb{Q}}$. It remains to show that the tree property holds in this model.

Recall that the term forcing $\mathbb{Q}$ was defined in the ground model and is μ-closed. So we consider the extension in question as an extension by $\mathbb{Q}$ and then by $\mathbb{A} * \mathbb{P}$. We want to show that in $V^{\mathbb{Q}}$, $\mathbb{A} * \mathbb{P}$ is Neeman's forcing for some choice of measures and each of the κ_n's is still supercompact.

Lemma 6.4. *$\mathbb{Q}$ is μ-directed closed.*

Proof. Let $\{f_\alpha : \alpha < \eta\}$ be a set of conditions in $\mathbb{Q}$ for some $\eta < \mu$ such that for any pair $\alpha_0, \alpha_1 < \eta$ there is a $\gamma < \eta$ such that $f_\gamma \leq f_{\alpha_0}, f_{\alpha_1}$. Define f to be a function such that $\mathrm{dom}(f) = \bigcup_{\alpha<\eta} \mathrm{dom}(f_\alpha)$. The domain of f has size less than μ, because μ is regular. For each $\beta \in \mathrm{dom}(f)$, let $f(\beta)$ be a name for the union of $f_\alpha(\beta)$ over all $\alpha < \eta$. We claim that for each $\beta \in \mathrm{dom}(f)$, $f(\beta)$ names a condition in $\mathrm{Add}(\mu, 1)_{V^{\mathbb{A}_\beta * \mathbb{P}_\beta}}$. Suppose that the claim fails. Then there is a $\beta \in \mathrm{dom}(f)$, $\rho < \mu$, and condition $(a, p) \in \mathbb{A}_\beta * \mathbb{P}_\beta$ such that $(a, p) \Vdash (\rho, 0), (\rho, 1) \in f(\beta)$. By the definition of f, there are $\alpha_0, \alpha_1 < \eta$, such that $(a, p) \Vdash f_{\alpha_0}(\beta)(\rho) = 0$ and $f_{\alpha_1}(\beta)(\rho) = 1$. But this is impossible, because f_{α_0} and f_{α_1} are compatible in the ordering of $\mathbb{Q}$. $\square$

It follows that each $\dot{U}_n$ is still an $\mathbb{A}$-name for an appropriate measure and that each of the κ_n's is still supercompact since we made each of the κ_n's indestructible under κ_n-directed closed forcing. This finishes the proof since in $V^{\mathbb{Q}}$ we have all of the conditions that we need to work the argument from [14]. $\square$

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