FRAGILITY AND INDESTRUCTIBILITY OF THE TREE PROPERTY

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Abstract. We prove various theorems about the preservation and destruction of the tree property at ω_2. Working in a model of Mitchell [9] where the tree property holds at ω_2, we prove that ω_2 still has the tree property after ccc forcing of size $\aleph_1$ or adding an arbitrary number of Cohen reals. We show that there is a relatively mild forcing in this same model which destroys the tree property. Finally we prove from a supercompact cardinal that the tree property at ω_2 can be indestructible under ω_2-directed closed forcing.

1. Introduction

In this paper we prove some facts about destruction and preservation of the tree property at ω_2. For completeness we recall some definitions.

Definition 1. *Let κ and λ be cardinals with κ regular.*

(1) *A* κ-tree *is a tree of height κ with levels of size less than κ.*
(2) *A tree T has a* κ-branch *if and only if there is a $b \subseteq T$ such that b is linearly ordered by $<_T$ and o.t.$(\langle b, <_T\rangle) = \kappa$*
(3) *A* κ-Aronszajn *tree is a κ-tree with no κ-branch.*
(4) *A λ^+-tree T is* special *if and only if there is a function $f : T \to \lambda$ such that for all $x, y \in T$, if $x <_T y$ then $f(x) \neq f(y)$.*

Remark 1. *A special λ^+-tree is Aronszajn*

Definition 2. *A regular cardinal κ has the* tree property *if and only if every κ-tree has a κ-branch or equivalently there are no κ-Aronszajn trees.*

The tree property is a well studied combinatorial principle. Aronszajn proved that there is a special ω_1-Aronszajn tree [6] and a generalization due to Specker [12] shows that if $\kappa^{<\kappa} = \kappa$, then there is a special κ^+-Aronszajn tree. In particular CH implies that there is a special ω_2-Aronszajn tree. Mitchell [9] showed that the tree property at ω_2 is equiconsistent with the existence of a weakly compact cardinal. The forcing used in this result can easily be modified to give the Tree property at the double successor of any regular cardinal from a weakly compact cardinal. Many questions about the successors of singulars have been answered, but we shall not concern ourselves with them here.

To obtain the tree property at ω_2 in Mitchell's result, we force to make the weakly compact cardinal into ω_2, but with a forcing that will preserve the tree property. The tree property in the extension is a *generic* large cardinal property,

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a remnant of the fact that ω_2 was weakly compact in an inner model. Destruction and preservation of large cardinals is an important topic in set theory. In this paper we study the destruction and preservation of the tree property as a generic large cardinal property. There are immediate differences between large cardinals and generic large cardinal properties in terms of destruction and preservation. Let us focus on weak compactness and the tree property. Levy and Solovay proved that if κ is weak compact, then it remains weakly compact after any forcing of size less than κ. By contrast if ω_2 has the tree property, then forcing with $\mathrm{Coll}(\omega, \omega_1)$ makes ω_2 in to ω_1 and there is always an ω_1-Aronszajn tree. So it is possible for small forcing to destroy the tree property. The aim of this paper is explore how fragile or robust the tree property is as a generic large cardinal property.

Indestructibility of the tree property plays a key role in research of forcing extensions where the tree property holds. For example the work of Abraham[1], and Cummings and Foreman[2] relies on constructing models in which the tree property is indestructible. In [1], one first constructs a model in which ω_2 has the tree property in an indestructible way. One then forces over that model to obtain the tree property at ω_3 and argues that the tree property at ω_2 cannot be destroyed by further forcing.

Fragility of the tree property is also important although it seems less is known about it. Results about the fragility of the tree property constrain the choice of forcing posets in results like [1],[2],[10].

In Section 2 we describe Mitchell's forcing for obtaining the tree property at ω_2 from a weakly compact cardinal and state some of its properties. Also in this section we will state some of the lemmas used in the proof that the tree property holds at ω_2 in the extension. Our results about the indestructibility of the tree property will concern the extension by Mitchell's forcing and another related model. For the remainder of the paper Mitchell's forcing will refer to the forcing from this section and Mitchell's model will refer to the extension by Mitchell's forcing.

In Section 3 we prove that the tree property in Mitchell's model is indestructible under both ccc forcing of size $\aleph_1$ and adding arbitrarily many Cohen reals. To do this we prove Lemma 6 a generalization of a lemma used in the proof that the tree property holds at ω_2 in Mitchell's model. Ideas from this lemma have found application in recent work of Sinapova [11] and the lemma itself is used in recent work of Neeman [10].

In Section 4 we show that in Mitchell's model there is a countably distributive ω_2-cc forcing of size ω_2, which adds a special ω_2-Aronszajn Tree, in fact it adds a $\square_{\omega_1}$-sequence.

In Section 5 we present a modification of Mitchell's forcing which allows us to construct a model in which the tree property at ω_2 is robust. Starting from a supercompact cardinal, we use a variation of the forcing from [1] to obtain a model of the tree property at ω_2 in which the tree property is indestructible under ω_2-directed closed forcing.

In Section 6 we conclude with some remarks and open problems.

2. Mitchell's forcing

In this section we describe Mitchell's forcing and state a few of its properties. This presentation of Mitchell's forcing along with a complete analysis appears in [1].

Definition 3. *Let θ be a regular cardinal and α be an ordinal. We call the forcing for adding α-many subsets of θ* $\mathrm{Add}(\theta, \alpha)$.

Definition 4. *Let κ be an inaccessible cardinal. Let $\mathbb{P}(\alpha) = \mathrm{Add}(\omega, \alpha)$ for all $\alpha \leq \kappa$, which we think of as finite partial functions from α to 2. $\mathbb{M}(\kappa)$ is the collection of pairs (p, q) such that $p \in \mathbb{P} =_{def} \mathbb{P}(\kappa)$ and q is a function with domain a countable subset of κ and for all $\beta \in dom(q), q(\beta)$ is a $\mathbb{P}(\beta)$-name for a condition in $\mathrm{Add}(\omega_1, 1)_{V^{\mathbb{P}(\beta)}}$. For the ordering, let $(p, q) \leq (p', q')$ if and only if $p' \subseteq p$, $dom(q') \subseteq dom(q)$ and for all $\beta \in dom(q')$, $p \restriction \beta \Vdash^{V}_{\mathbb{P}(\beta)} q(\beta) \leq q'(\beta)$ in $\mathrm{Add}(\omega_1, 1)_{V^{\mathbb{P}(\beta)}}$.*

We can now state the forcing direction of Mitchell's theorem more precisely.

Theorem 1. *If there exists a weakly compact cardinal κ, then the generic extension by $\mathbb{M} =_{def} \mathbb{M}(\kappa)$ satisfies*

(1) $2^{\omega} = \kappa = \omega_2$,
(2) $\omega_1^V = \omega_1$ *and*
(3) ω_2 *has the tree property.*

We state some of the properties of Mitchell's forcing that are used in the proof of the above theorem and the proofs of our theorems below. For completeness we reference the precise numbering of [1].

Definition 5. *A poset $\mathbb{P}$ is κ-*Knaster *if for every sequence $\langle p_\alpha \mid \alpha < \kappa \rangle$, there is $I \subseteq \kappa$ unbounded such that for all $\alpha, \beta \in I$, p_α and p_β are compatible.*

Lemma 1 (Lemma 2.4 of [1])**.** $\mathbb{M}$ *is κ-Knaster.*

Notice that in the definition of the ordering there is a connection between the first and second coordinates. The next lemma shows that at the cost of a little more forcing we can disengage the coordinates.

Lemma 2 (Lemmas 2.5 and 2.8 of [1])**.** *There is a countably closed forcing $\mathbb{R}$ such that $\mathbb{M}$ is the projection of $\mathbb{P} \times \mathbb{R}$.*

For $\alpha < \kappa$, the map from $\mathbb{M}$ to $\mathbb{M}(\alpha)$ given by $(p, q) \mapsto (p \restriction \alpha, q \restriction \alpha)$ is a projection. This means that we can form the quotient $\mathbb{M}/\mathbb{M}(\alpha)$. It turns out that this quotient resembles $\mathbb{M}$ as defined in $V^{\mathbb{M}(\alpha)}$. The following Lemma combines Definition 2.7 and Lemmas 2.8 and 2.12 of [1].

Lemma 3. *In $V^{\mathbb{M}(\alpha)}$ there is an $\aleph_1$-Knaster forcing $\mathbb{P}'$ and a countably closed forcing $\mathbb{R}'$ such that $\mathbb{M}/\mathbb{M}(\alpha)$ is the projection of $\mathbb{P}' \times \mathbb{R}'$.*

This completes the facts that we will need about $\mathbb{M}$. We will now state a few lemmas that are used in the proof of Theorem 1, which we will use also. The following are often called *branch lemmas.*

Lemma 4 (Silver)**.** *Let τ and η be regular cardinals with $2^\tau \geq \eta$, T be an η-tree and $\mathbb{R}$ be τ^+-closed poset. Forcing with $\mathbb{R}$ cannot add a new branch through T, that is every cofinal branch through T in $V^{\mathbb{P}}$ is in V.*

Lemma 5. *κ-Knaster forcing cannot add a branch through a branchless tree of height κ.*

3. Indestructibility of the tree property in $V^{\mathbb{M}}$

Having described $\mathbb{M}$ we can state our indestructibility theorem precisely.

Theorem 2. *Work in $V^{\mathbb{M}}$. Suppose that $\mathbb{Q}$ is either a ccc poset of size $\aleph_1$ or $\mathrm{Add}(\omega, \mu)$ for some cardinal μ, then the tree property holds at ω_2 in the extension by $\mathbb{Q}$.*

In order to prove Theorem 2, we will need to make use of a generalization of Lemma 4. We state and prove the lemma and then return to the proof of Theorem 2.

Lemma 6. *Let τ, η be cardinals with η regular and $2^\tau \geq \eta$. Let $\mathbb{P}$ be τ^+-cc and $\mathbb{R}$ be τ^+-closed. Let $\dot{T}$ be a $\mathbb{P}$-name for an η-tree. Then in $V^{\mathbb{P}}$ forcing with $\mathbb{R}$ cannot add a branch through T.*

We begin with a definition and a lemma given in [8].

Definition 6. *Let $\mathbb{R}$ be a poset and T be a tree. Assume that $\dot{b}$ is a $\mathbb{R}$-name for a branch through T. We say that r_1, r_2 force contradictory information about $\dot{b}$ at level γ if for every pair $r_1', r_2' \in \mathbb{R}$ with $r_1' \leq r_1$ and $r_2' \leq r_2$ if r_1', r_2' each decide the value of $\dot{b} \cap \mathrm{Lev}_\gamma T$, then they decide different values.*

The following lemma shows that conditions forcing contradictory information act as one might expect.

Lemma 7. *Let T be a tree, $\mathbb{R}$ be a poset and $\dot{b}$ be a $\mathbb{R}$-name for a branch through T. Suppose that r_1, r_2 force contradictory information about $\dot{b}$ at some level γ, then*

1. *for all $\gamma' > \gamma$, r_1, r_2 force contradictory information about $\dot{b}$ at level γ' and*
2. *for all $r_1' \leq r_1$ and $r_2' \leq r_2$, r_1', r_2' force contradictory information about $\dot{b}$ at level γ.*

For a proof see [8]. We prove another basic fact [8] which says that if a branch is forced to be new then we can always extend to find conditions that force contradictory information about the branch at some level.

Proposition 1. *Let T be a tree and $\mathbb{R}$ a poset. Suppose that $\dot{b}$ is an $\mathbb{R}$-name for a branch through T such that $\Vdash_{\mathbb{R}} \dot{b} \notin V$. Then for every $r_1, r_2 \in \mathbb{R}$ there are r_1', r_2' and γ such that $r_1' \leq r_1$, $r_2' \leq r_2$ and r_1', r_2' force contradictory information about $\dot{b}$ at level γ.*

Proof. Assume that the proposition is false. So there are r_1, r_2 such that for any $r_1' \leq r_1, r_2' \leq r_2$ and γ, r_1', r_2' do not force contradictory information about $\dot{b}$ on level γ. We work through the meaning of 'do not force contradictory information.' For any r_1', r_2' and γ as above there are $r_1'' \leq r_1'$ and $r_2'' \leq r_2'$ such that r_1'' and r_2'' decide the same value for $\dot{b} \cap \mathrm{Lev}_\gamma T$. In particular we have a dense set below $(r_1, r_2) \in \mathbb{R} \times \mathbb{R}$ that forces interpretation of $\dot{b}$ by the left generic to be equal to that of the right generic. This contradicts the fact that $\Vdash_{\mathbb{R}} \dot{b} \notin V$. □

We are now ready to begin the proof of Lemma 6

Proof. In $V^{\mathbb{P}}$ let $\dot{b}$ be an $\mathbb{R}$-name for a cofinal branch through T, the interpretation of $\dot{T}$. Assume for a contradiction that $\Vdash_{\mathbb{R}} \dot{b} \notin V^{\mathbb{P}}$.

Claim. *For all $r_1, r_2 \in \mathbb{R}$, there is a dense set $D_{r_1,r_2} \subseteq \mathbb{P}$ such that for all $p \in D_{r_1,r_2}$, we have*

(1) *$p \Vdash$ There are $r_1', r_2' \in \mathbb{R}$ and $\gamma < \eta$ such that $r_1' \leq r_1$, $r_2' \leq r_2$ and r_1', r_2' force contradictory information about $\dot{b}$ on level γ and*
(2) *p decides the value of γ, r_1', r_2'.*

We apply Proposition 1 in $V^{\mathbb{P}}$ with $\mathbb{R}, T, \dot{b}$ as in the lemma to see that in fact every condition in $\mathbb{P}$ forces (1). We can then extend a given condition to obtain (2), which shows that the set of conditions satisfying (1) and (2) is dense. This completes the claim.

Next given $r \in \mathbb{R}$ we construct a maximal antichain A in $\mathbb{P}$ and conditions r_1^*, r_2^* and $\gamma^* < \eta$ such that $r_1^*, r_2^* \leq r$ and for all $p \in A$, $p \Vdash r_1^*, r_2^*$ force contradictory information about $\dot{b}$ at level γ^*. The above claim might give us different conditions in $\mathbb{R}$ for incompatible extensions in $\mathbb{P}$. We now show that we can find a pair of conditions in $\mathbb{R}$ that works for each element in a maximal antichain in $\mathbb{P}$.

To construct $A, r_1^*, r_2^*, \gamma^*$ as above we construct an increasing sequence of length less than τ^+ of antichains $\langle A_\alpha \rangle$ in $\mathbb{P}$, decreasing sequences $\langle r_i^\alpha \rangle$ for $i = 1, 2$ and a sequence of ordinals $\langle \gamma_\alpha \rangle$ as follows. For definiteness, we fix a well order of $\mathbb{P}$.

To begin the construction we fix $r \in \mathbb{R}$ and let $p_0 \in D_{r,r}$. Let r_1^0, r_2^0, γ_0 witness this, that is they satisfy (1) of the claim. Let $A_0 = \{p_0\}$.

Assuming we have constructed, r_1^α, r_2^α, A_α, γ_α, we define $r_1^{\alpha+1}$, $r_2^{\alpha+1}$, $A_{\alpha+1}$, $\gamma_{\alpha+1}$ or halt the construction. If A_α is a maximal antichain then terminate the construction and set $A = A_\alpha$, $r_i^* = r_i^\alpha$ for $i = 1, 2$ and $\gamma^* = \gamma_\alpha$. Otherwise we choose the least p in the well order of $\mathbb{P}$ such that p is incompatible with everything in A_α. Choose $p' \leq p$ with $p' \in D_{r_1^\alpha, r_2^\alpha}$ and let $r_1^{\alpha+1} \leq r_1^\alpha$, $r_2^{\alpha+1} \leq r_2^\alpha$ and $\gamma_{\alpha+1} > \gamma_\alpha$ witness that $p' \in D_{r_1^\alpha, r_2^\alpha}$. Define $A_{\alpha+1} = A_\alpha \cup \{p'\}$.

For the limit step assume that α is a limit ordinal and that for all $\beta < \alpha$ we have constructed, $r_1^\beta, r_2^\beta, A_\beta, \gamma_\beta$. By the construction so far $\langle r_1^\beta \mid \beta < \alpha \rangle$ and $\langle r_2^\beta \mid \beta < \alpha \rangle$ are each decreasing sequences and $\langle A_\beta \mid \beta < \alpha \rangle$ is an increasing sequence of antichains. Let $A_\alpha = \bigcup_{\beta < \alpha} A_\beta$, r_1^α, r_2^α be lower bounds for the appropriate sequences and $\gamma_\alpha = (\sup_{\beta < \alpha} \gamma_\beta) + 1$.

Easy induction shows that for each relevant α, A_α is an antichain. Combining the τ^+-cc of $\mathbb{P}$ and the τ^+-closure of $\mathbb{R}$, we can continue the construction at limit stages. Again by the τ^+-cc of $\mathbb{P}$, the construction must terminate at some stage less than τ^+.

We claim that $r_1^*, r_2^*, A, \gamma^*$ are as required. Let $p \in A$, then by construction we chose p at some stage $\alpha + 1$ and $p \in D_{r_1^\alpha, r_2^\alpha}$ with witnesses $r_1^{\alpha+1}, r_2^{\alpha+1}, \gamma_{\alpha+1}$. Now for $i = 1, 2$ we have $r \geq r_i^\alpha \geq r_i^{\alpha+1} \geq r_i^*$. $p \Vdash$ "$r_1^{\alpha+1}, r_2^{\alpha+1}$ force contradictory information at level $\gamma_{\alpha+1}$."

Passing to stronger conditions and moving up in levels, we get $p \Vdash$ " r_1^*, r_2^* force contradictory information at level γ." This is what we wanted. Note that we can modify this construction to make γ^* above any ordinal less than η that we fixed in advance.

Next we construct a binary tree of conditions. Let χ be least such that $2^\chi \geq \eta$. Then $\chi \leq \tau$. Using the above construction repeatedly, we get conditions r_s for $s \in {}^{<\chi}2$ such that $r_{s \frown 0}, r_{s \frown 1} \leq r_s$ and there is a maximal antichain A_s in $\mathbb{P}$ such that for all $p \in A_s$, $p \Vdash r_{s \frown 0}, r_{s \frown 1}$ force contradictory information about $\dot{b}$ at some level. If $\ell(s)$ is a limit ordinal then $r_s \leq r_{s \restriction \alpha}$ for all $\alpha < \ell(s)$.

Inductively we can ensure that for all $\alpha < \chi$ there is a level ζ_α such that for all $s, t \in {}^{<\chi}2$ with $\ell(s) = \ell(t) = \alpha$ r_s and r_t force contradictory information about $\dot{b}$ at level ζ_α. This is possible since the set of $s \in {}^{<\chi}2$ with a given length α is smaller than η. Let $\zeta = \sup_{\alpha<\chi} \zeta_\alpha$ and note that $\zeta < \eta$.

Since we are working in V and $\mathbb{R}$ is τ^+-closed in V, for each $s \in {}^{\chi}2$ we can find a lower bound r_s for the decreasing sequence $\langle r_{s\restriction\alpha} : \alpha < \chi\rangle$. Let G be $\mathbb{P}$-generic over V. Then $G \cap A_s \neq \emptyset$ for all $s \in {}^{<\chi}2$.

Work in $V[G]$. We claim that for all $s \neq t \in {}^{\chi}2$, r_s, r_t force contradictory information about $\dot{b}$ on level ζ. To see this look at the largest $\alpha < \chi$ where the s and t agree. Since $G \cap A_{s\restriction\alpha} \neq \emptyset$, it follows that $r_{(s\restriction\alpha)^\frown 0}, r_{(s\restriction\alpha)^\frown 1}$ force contradictory information about $\dot{b}$ at some level less than ζ. Moving up in levels we have that r_s, r_t force contradictory information about $\dot{b}$ on level ζ. It follows that level ζ of T has $2^\chi \geq \eta$ many nodes, a contradiction. □

We are now ready to prove Theorem 2.

Proof. For ease of argument we assume that we started with a measurable cardinal, but note that the weakening to a weakly compact cardinal poses no problem. Let κ be a measurable cardinal. Let $\mathbb{M}$ be Mitchell's poset as described above. Let $j : V \to M$ witness that κ is measurable where we assume that ${}^{\kappa}M \subseteq M$.

In the first part of the proof we let $\mathbb{Q}$ be ccc of size $\aleph_1$. Let H be V-generic for $j(\mathbb{M})$ and let G be the induced V-generic filter over $\mathbb{M}$. Then we have $j``G \subseteq H$ since each condition in $\mathbb{M}$ is in V_κ. It follows that in $V[H]$ there is a generic elementary embedding $j : V[G] \to M[H]$ with critical point ω_2. We can assume that $\mathbb{Q} \in H_{\omega_2}$ and it follows that $j(\mathbb{Q}) = \mathbb{Q}$ and $j \restriction \mathbb{Q} = id_{\mathbb{Q}}$. Thus we can lift the generic embedding further to $j : V[G][x] \to M[H][x]$ where x is $V[H]$-generic for $\mathbb{Q}$.

Assume for a contradiction that $\dot{T} \in V[G]$ is an $\mathbb{Q}$-name for an ω_2-Aronszajn tree. Since $V \models {}^{\kappa}M \subseteq M$ and G is generic for κ-cc forcing, $V[G] \models {}^{\kappa}M[G] \subseteq M[G]$. It follows that $\dot{T} \in M[G]$ and thus $T \in M[G][x]$. We argue that T has acquired a branch in $M[H][x]$. Since the critical point of j is κ, $j(T) \restriction \kappa = T$. In $M[H][x]$, $j(T)$ is a tree of height $j(\kappa)$. So if we fix a point on level κ of $j(T)$, this determines a branch through T. So we have a branch through T in $M[H][x]$ and we would like to show that the forcing to get from $M[G][x]$ up to $M[H][x]$ could not have added this branch, a contradiction. To show this we look at a particular outer model of $M[H][x]$. By Lemma 3 applied to $j(\mathbb{M})/\mathbb{M}$, $M[H]$ can be viewed a submodel of $M[G][H_1][H_2]$ where H_1 is generic for Cohen forcing and H_2 is generic for a countably closed term forcing from $M[G]$. By Easton's Lemma and the fact that x is generic over $V[H] \supseteq V[G][H_1]$, x, H_1, H_2 are mutually generic and it follows that $M[H][x] \subseteq M[G][x][H_2][H_1]$.

To obtain a contradiction it suffices to show that H_2, H_1 could not have added the branch. We apply Lemma 6 with $M[G]$ as the ground model, $\mathbb{Q}$ as the *ccc* forcing and the forcing which adds H_2 as the countably closed forcing. It follows that T is still branchless in $M[G][x][H_2]$. By a standard argument, the ω_2 of $M[G][x]$ is collapsed to an ordinal of cofinality ω_1 by the addition of H_2. In $M[G][x][H_2]$ the height of T is no longer a cardinal, so we replace T by its restriction to a cofinal ω_1 sequence of levels. Note that the resulting tree need not be an ω_1-tree, since T is an ω_2 tree in $M[G][x]$ and thus can have levels of size ω_1. Now since Cohen forcing is $\aleph_1$-Knaster, we have that our restricted tree is branchless in $M[G][x][H_2][H_1]$ by Lemma 5. This is a contradiction.

We move on to the case where $\mathbb{Q} = \mathrm{Add}(\omega, \mu)$ for some cardinal μ. We are thinking of elements of$\mathrm{Add}(\omega, \mu)$ as finite partial functions from μ to 2. We can assume that $\mu \geq \omega_2$, since otherwise we are in the previous case. Again let $\dot{T}$ be a $\mathbb{Q}$-name for an ω_2-Aronszajn tree. We view $\dot{T}$ as a name for a subset of ω_2 which codes an ω_2-tree in a fixed way. Hence for each $\alpha < \omega_2$, there is a maximal antichain A_α so that each $q \in A_\alpha$ decides the statement $\alpha \in \dot{T}$. Let $B = \cup\{\mathrm{dom}\, q \mid q \in A_\alpha$ for some $\alpha\}$. Then $|B| = \omega_2$ and $\mathbb{Q} \upharpoonright B \simeq \mathrm{Add}(\omega, \omega_2)$. We replace $\mathbb{Q}$ with its isomorphic copy where $\dot{T}$ is forced to be equal to a $\mathbb{Q} \upharpoonright \omega_2$-name.

As before we have a lifted embedding $j : V[G] \to M[H]$ in $V[H]$. We lift this embedding to the extension $V[G][x]$ where x is $V[G]$-generic over $\mathbb{Q}$. We let x^* be $j(\mathbb{Q})$-generic over $M[H]$ with $j''x \subseteq x^*$. Again using the lifted elementary embedding there is a branch through T in $M[H][x^*]$. We let x_0 be the initial segment of x which is generic for $\mathrm{Add}(\omega, \omega_2)$. Then by the argument of the preceding paragraph, $T \in M[G][x_0]$ and our assumption for a contradiction implies it is an Aronszajn tree in this model. Since $\mathrm{crit}(j) = \omega_2$, we can write $x^* = x_0 \times x_1$, where x_1 is generic for the quotient $j(\mathbb{Q})/(\mathbb{Q} \upharpoonright \omega_2)$. Note that this quotient is just Cohen forcing. Using mutual genericity and Lemma 3, we write $M[H][x^*] \subseteq M[G][x_0][H_2][H_1][x_1]$ where H_1, H_2 are generic for the forcings $\mathbb{P}', \mathbb{R}'$ from Lemma 3. Again we apply Lemma 6 to show that T is still branchless in $M[G][x_0][H_2]$. In $M[G][x_0][H_2]$, $(\omega_2)^{M[G][x_0]}$ has been collapsed to an ordinal of cofinality ω_1. Again we look at a restriction of the tree of a cofinal set of levels. To finish the proof we note that $H_1 \times x_1$ is generic for Cohen forcing which has the ω_1-Knaster property and thus cannot add a branch through a branchless tree by Lemma 5. This is a contradiction, since T has a branch in $M[G][x_0][H_2][H_1][x_1]$. □

4. Adding an ω_2-Aronszajn tree in $V^{\mathbb{M}}$

In this section we describe a forcing for adding $\square_{\omega_1}$ with countable conditions. The forcing below is due to Foreman and Magidor [4].

Definition 7. *Let λ be an inaccessible cardinal. Let $c \in \mathbb{C}$ if and only if c is a function with* $\mathrm{dom}(c) \subseteq \mathrm{Lim}(\lambda)$, $|\mathrm{dom}(c)| = \aleph_0$ *and for all* $\alpha \in \mathrm{dom}(c)$,

1. *$c(\alpha)$ is a countable closed subset of α,*
2. *if $\mathrm{cf}(\alpha) = \omega$, then $c(\alpha)$ is unbounded in α,*
3. *if $\mathrm{cf}(\alpha) > \omega$, then $\max(c(\alpha))$ is greater than all $\beta < \alpha$ with $\beta \in \mathrm{dom}(c)$ and*
4. *for all $\beta \in \mathrm{Lim}(c(\alpha))$, $\beta \in \mathrm{dom}(c)$ and $c(\alpha) \cap \beta = c(\beta)$.*

Let $c_2 \leq c_1$ if and only if $\mathrm{dom}(c_2) \supseteq \mathrm{dom}(c_1)$ *and for all $\alpha \in \mathrm{dom}(c_1)$, $c_2(\alpha)$ end extends $c_1(\alpha)$.*

The following lemmas are easy.

Lemma 8. $\mathbb{C}$ *is countably closed and λ-cc*

Lemma 9. *If X is V-generic for $\mathbb{C}$ and $C_\alpha = \bigcup\{c(\alpha) \mid c \in X$ and $\alpha \in \mathrm{dom}(c)\}$, then for all $\alpha \in \mathrm{Lim}(\lambda) \cap \mathrm{cof}(> \omega)^V$, C_α is a club of order type ω_1 in α.*

It follows that in a generic extension by $\mathbb{C}$, λ is collapsed to ω_2 and $\langle C_\alpha \mid \alpha \in \mathrm{Lim}(\lambda)\rangle$ is a $\square_{\omega_1}$-sequence. Having shown the basic properties of $\mathbb{C}$, we return to the setting of Mitchell's forcing. Recall κ is a measurable cardinal in V and G be V-generic for $\mathbb{M}$. Let $\mathbb{C} \in V$ be the forcing described above with $\lambda = \kappa$.

Lemma 10. *In $V[G]$, $\mathbb{C}$ is countably distributive and ω_2-cc*

Proof. We prove the latter property first. Let $\dot{A}$ be an $\mathbb{M}$ name for an antichain in $\mathbb{C}$ of size ω_2. We can view $\dot{A}$ as a κ-sequence of names each of which names a member of $\mathbb{C}$. For $\alpha < \lambda$ let $m_\alpha \in \mathbb{M}$ decide the value of the α^{th} member of $\dot{A}$ to be c_α. Since $\mathbb{M}$ is κ-Knaster, we can find $I \subseteq \kappa$ unbounded such that for all $\alpha, \beta \in I$, m_α is compatible with m_β. It follows that $\{c_\alpha \mid \alpha \in I\}$ is an antichain of size κ in $\mathbb{C}$, a contradiction.

We use Easton's Lemma (Lemma 15.19 of [5]) to show that in $V[G]$, $\mathbb{C}$ is countably distributive. Let X be $\mathbb{C}$-generic over $V[G]$. In $V[G][X]$ we force with the quotient $(\mathbb{P} \times \mathbb{R})/\mathbb{M}$ to obtain $V[G][X][G']$. This quotient forcing exists by Lemma 2. So X and G' are mutually generic and we have $V[G][X][G'] = V[G][G'][X]$. By the general theory of projections we have that $\mathbb{M} * ((\mathbb{P} \times \mathbb{R})/\mathbb{M})$ is isomorphic to $\mathbb{P} \times \mathbb{R}$. We use this isomorphism to read off generics H_1, H_2 for $\mathbb{P}, \mathbb{R}$ respectively such that $V[G][G'][X] = V[H_1][H_2][X]$. Now $H_2 \times X$ is generic for $\mathbb{R} \times \mathbb{C}$, which is countably closed in V. Since H_1 is generic for ccc forcing, we can apply Easton's lemma to see that every ω-sequence of ordinals from $V[H_1][H_2][X]$ is in $V[H_1]$. It follows that forcing with $\mathbb{C}$ over $V[G]$ did not add any ω-sequences, since $V[H_1] \subseteq V[G][X] \subseteq V[G][X][G'] = V[H_1][H_2][X]$. $\square$

So in $V[G]$ forcing with $\mathbb{C}$ preserves cardinals and adds a $\square_{\omega_1}$-sequence. It is a well known fact due to Jensen[3] that $\square_{\omega_1}$ implies that there is a special ω_2-Aronszajn tree.

Remark 2. *We note that the forcing $\mathbb{P} \times \mathbb{R}/\mathbb{M}$ is also countably distributive and ω_2-cc, and it adds a special ω_2-Aronszajn tree. The proofs of distributivity and chain condition are routine. To see that it adds a special ω_2-Aronszajn tree, we note that $V^{\mathbb{R}}$ is an inner model of $V^{\mathbb{P}\times\mathbb{R}}$ with the same ω_2 and moreover $V^{\mathbb{R}} \models \mathrm{CH}$. It follows that there is a special ω_2-tree in $V^{\mathbb{R}}$ and it is still special in $V^{\mathbb{P}\times\mathbb{R}}$.*

5. Making the tree property indestructible under closed forcing

In this section we assume that κ is supercompact and we construct a model in which the tree property is indestructible under ω_2-directed closed forcing. It is known that in models of PFA the tree property is indestructible under ω_2-closed forcing [7]. However, the result of this section provides a more flexible proof for a weaker conclusion. In particular the forcing from this section generalizes easily to higher cardinals.

The definition of the forcing that we present is not the most general, but it is enough for the application. For a complete analysis of this style of forcing in a different and more general context, we refer the reader to [1] or [2]. Our account of the forcing comes from [2]. Before defining the forcing, we give the definition of *Laver function.*

Definition 8. *A function $f : \kappa \to V_\kappa$ is a* Laver function *if for every λ and every $x \in H_{\lambda^+}$, there is $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, ${}^\lambda M \subseteq M$ and $j(f)(\kappa) = x$.*

Definition 9. *Let $\mathbb{P} =_{def} \mathrm{Add}(\omega, \kappa)$ and for all $\alpha < \kappa$ let $\mathbb{P}(\alpha) =_{def} \mathrm{Add}(\omega, \alpha)$. Let F be a Laver function from κ to V_κ. We define a forcing $\mathbb{M}^*$ by induction on $\beta \leq \kappa$. Let $(p, q, f) \in \mathbb{M}^*(\beta)$ if and only if*

(1) $p \in \mathbb{P}(\beta)$,
(2) q *is a partial function on* β *with* $|\operatorname{dom}(q)| = \aleph_0$, $\operatorname{dom}(q)$ *is a set of successor ordinals and if* $\alpha \in \operatorname{dom}(q)$, *then* $f(\alpha)$ *is a* $\mathbb{P}(\alpha)$*-name for a condition in* $\operatorname{Add}(\omega_1, 1)_{V^{\mathbb{P}(\alpha)}}$ *and*
(3) f *is a partial function on* β *with* $|\operatorname{dom}(f)| = \aleph_0$, $\operatorname{dom}(f)$ *is a set of limit ordinals such that for all* $\alpha \in \operatorname{dom}(f)$, $\Vdash_{\mathbb{M}^*(\alpha)}$ *"*$F(\alpha)$ *is* α*-directed closed forcing and* $f(\alpha) \in F(\alpha)$*."*

Define the ordering on $\mathbb{M}^*(\beta)$ *by* $(p, f, r) \leq (p', f', r')$ *if and only if*

(1) $p \leq p'$ *in* $\mathbb{P}(\beta)$,
(2) $\operatorname{dom}(q) \supseteq \operatorname{dom}(q')$,
(3) *for all* $\alpha \in \operatorname{dom}(q')$, $p \restriction \alpha \Vdash_{\mathbb{P}(\alpha)} q(\alpha) \leq q'(\alpha)$,
(4) $\operatorname{dom}(f) \supseteq \operatorname{dom}(f')$ *and*
(5) *for all* $\alpha \in \operatorname{dom}(f')$, $(p, q, f) \restriction \alpha \Vdash_{\mathbb{M}^*(\alpha)} f(\alpha) \leq f'(\alpha)$

We set $\mathbb{M}^* =_{def} \mathbb{M}^*(\kappa)$.

We state without proof the lemmas needed to show the following theorem. The proofs of the lemmas are routine modifications of the analogous lemmas in [2].

Theorem 3. *Let* G *be* V*-generic over* $\mathbb{M}^*$. $V[G]$ *satisfies*

(1) $2^\omega = \omega_2 = \kappa$,
(2) $\omega_1 = \omega_1^V$ *and*
(3) *the tree property at* ω_2.

Lemma 11. $\mathbb{M}^*$ *is* κ*-Knaster.*

Lemma 12. $\mathbb{M}^*$ *projects onto the posets* $\mathbb{P}$, $\mathbb{M}^*(\alpha)$ *and* $\mathbb{P}(\alpha) * \operatorname{Add}(\omega_1, 1)_{V^{\mathbb{P}(\alpha)}}$ *for any* $\alpha < \kappa$.

Lemma 13. *There is a countably closed forcing* $\mathbb{U}$ *such that* $\mathbb{M}^*$ *is the projection of* $\mathbb{P} \times \mathbb{U}$.

Lemma 14. *For all* $\alpha < \kappa$, *in* $V^{\mathbb{M}^*(\alpha)}$ *there is a* $\aleph_1$*-Knaster forcing* $\mathbb{P}'$ *and a countably closed forcing* $\mathbb{U}'$ *such that* $\mathbb{M}^*/\mathbb{M}^*(\alpha)$ *is the projection of* $\mathbb{P}' \times \mathbb{U}'$.

We are now ready to prove the theorem for this section.

Theorem 4. *Let* G *be* V*-generic for* $\mathbb{M}^*$. *In* $V[G]$ *the tree property still holds at* ω_2 *after any* ω_2*-directed closed forcing.*

Proof. Let $\mathbb{Q} \in V[G]$ be ω_2-directed closed and X be $V[G]$-generic for $\mathbb{Q}$. By the property of our Laver function F, there is an embedding $j : V \to M$ witnessing that κ is $(2^{|\mathbb{Q}|})^+$-supercompact with $j(F)(\kappa) = \dot{Q}$ a canonical name for $\mathbb{Q}$. We work to lift this embedding.

Note that for all $(p, f, r) \in \mathbb{M}^*$, $(p, f, r) \in V_\kappa$. It follows that $j \restriction G = id_G$. So in order to lift j to the extension by G we need only to choose a $j(\mathbb{M}^*)/\mathbb{M}^*$ generic object. Using the elementarity of j and Lemma 12, we have that $j(\mathbb{M}^*)$ projects on to $j(\mathbb{M}*)(\kappa+1)$. It is easy to see from the definition of the forcing that $j(\mathbb{M}^*)(\kappa+1)$ is equivalent to $\mathbb{M}^* * j(F)(\kappa)$. So it is enough to choose a $j(F)(\kappa) = \mathbb{Q}$ generic object and a $j(\mathbb{M}^*)/(\mathbb{M}^* * \mathbb{Q})$ generic object. Let X be $\mathbb{Q}$-generic over $V[G]$ and G' be $j(\mathbb{M}^*)/(\mathbb{M}^* * \mathbb{Q})$-generic over $V[G][X]$.

In $V[G][X][G']$ we can lift the embedding j to $j : V[G] \to M[H]$ where H is the generic object obtained from G, X, G' for $j(\mathbb{M}^*)$. We would like to lift further. First note that $X \in M[H]$. Since j witnesses that κ is $(2^{|\mathbb{Q}|})^+$-supercompact in

V and $G, H \in M[H]$, we have that $j \restriction \mathbb{Q} \in M[H]$. It follows that $j``X \in M[H]$ and $M[H] \models j``X$ is a directed subset of $j(\mathbb{Q})$ of cardinality $|\mathbb{Q}|$. $M[H] \models j(\mathbb{Q})$ is $j(\kappa)$-directed closed and hence we can find a lower bound for $j``X$ in $j(\mathbb{Q})$. Force below this lower bound to obtain X' which is $j(\mathbb{Q})$-generic over $V[G][X][G']$. It follows that we can lift again to obtain an embedding $j : V[G][X] \to M[H][X']$.

Assume for a contradiction that T in $V[G][X]$ is an ω_2-Aronszajn tree. By standard arguments $T \in M[G][X]$ and has acquired a branch b in $M[H][X']$. We work to show that the forcing to get from $M[G][X]$ to $M[H][X']$ could not have added the branch. To start we observe that $b \in M[H]$, since X' is $M[H]$-generic for $j(\kappa)$-closed forcing. From above we have $M[H] = M[G][X][G']$ and by Lemma 14, there are generics H_1, H_2 for $\aleph_1$-Knaster forcing and countably closed forcing respectively such that $M[G][X][G'] \subseteq M[G][X][H_2][H_1]$. By the usual arguments using branch lemmas, we have that T is still branchless in $M[G][X][H_2][H_1]$, a contradiction. □

Remark 3. *The results from Sections 3 and 4 hold in the extension by* $\mathbb{M}^*$*.*

6. Conclusion

Our work with Mitchell's model leaves a gap between forcings which are known to preserve the tree property and forcings which can destroy it. We have shown that there is some relatively mild forcing in the Mitchell Model which destroys the tree property. Is there a ccc forcing in the Mitchell model which adds an ω_2-Aronszajn tree? Our results imply that such a forcing must have size at least ω_2 and that the forcing cannot be Cohen forcing. We have only worked with Mitchell's model and other closely related models. Can the tree property at ω_2 be even more fragile? In particular, is there a model where the tree property holds at ω_2, but is destroyed by adding a Cohen real? In Section 5 using a supercompact cardinal we showed that the tree property at ω_2 can be made indestructible under ω_2-directed closed forcing. We suspect that an reasonable attempt to construct a model as in Section 5 will require at least a strongly compact cardinal and that such a result could be proved using a theorem of Viale and Weiss [13].

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