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Greek Geometry and Its Discontents: The Failed Search for Non-Euclidean Geometries in the Greek Philosophical and Mathematical Corpus

Sabetai Unguru

Imre Tóth 2010: *Fragmente und Spuren nichteuklidischer Geometrie bei Aristoteles.* [= Beiträge zur Altertumskunde, 280] Berlin: De Gruyter, geb., 425 S., 119,95 €, ISBN-13: 978-3-11022-415-3.

Reading this book should start with the Appendix (*Anhang*, pp. 395–414), containing the Aristotelian and Platonic Greek sources, and their Latin and German translations, the textual foundations upon which Toth erects his entire interpretive *échafaudage*. Of these nineteen odd pages, the Greek originals occupy a mere seven (pp. 395–401).

The attentive, open-minded reader will not discover any traces whatsoever of non-Euclidean geometry in these sources. Nothing at all. And this should not be surprising: Like Marx, who was no Marxist, Euclid, too, was without party affiliation, he was, of course, no follower of Euclid, he was no Euclidean. Greek geometry was just that, geometry, with no additional qualifiers. Hence, even on the basis of this simple, obvious, semantic consideration, there could not have been any Greek non-Euclidean geometry; but, there is more to it.

Tóth's creation is, largely, godlike, de nihilo. He brings to his sources, of course, like all of us, his biases, but, unlike most historians worth their salt, he does not leave his prejudices at the door before entering the textual world; on the contrary, he invests his sources with all of his preconceived ideas, reading into them all of his biases, a process that enables him to find what he wants and what is clearly not there. It is this claim that the present essay will try to defend.

Introduction

In 1966 the *Archive for History of Exact Sciences* published a book-length article entitled "Das Parallelenproblem im Corpus Aristotelicum," authored by Imre Tóth and communicated by Joseph Ehrenfried Hofmann. This article, containing all the main elements of the book under review, opened a long series of publications on the topic, following year after year, till the death of their author in 2010, in the forms of books, largely repetitive of one another, in a spate of languages (Italian, French, German), articles in learned, and popular journals, chapters in books, separata et cetera, the duplicative character of which is glaring. Thus the bibliography of *Fragmente und Spuren* alone, and it is by no means exhaustive, contains some thirteen (!) items, all of them dealing with the same issue, or rather non-issue, namely, non-Euclidean geometries in Greek Antiquity, making them into a set of monotonous variations, *Variationen über ein eigenes Thema*.

It is well known that the status of Postulate V in the *Elements*, most likely the genial and original creation of Euclid himself, was contested early on, and that many unsuccessful attempts to change its status to that of theorem were undertaken during the centuries, beginning already in Antiquity. It is precisely around the standing of this postulate and the associated issues of parallel lines and the sum of the interior angles in a triangle that the so called *Fragments and Traces*, identified by Tóth in the Aristotelian corpus, and elsewhere, as instances of non-Euclidean geometry, revolve. They represent, allegedly, the remnants of a quite well developed undertaking of geometers at the Platonic Academy, who, recognizing the problematic status of the Euclidean Parallel Postulate, replaced it with another in which parallels meet and the sum of the interior angles in a triangle differs from two rights. In doing so, they proved, before Euclid's time, a number of important theorems belonging to the new geometry. This is the "revolutionary" thesis of Tóth's book. To prove it, he deploys all the paraphernalia at his disposal, and they are an impressive lot. In making his case, he dismisses haughtily, ironically, and sometimes venomously (typically in lengthy footnotes) all those, who disagree with him. The result of his learned, telling, and vigorous effort is impressive and unconvincing.

Practically all of Tóth's sources lend themselves to simpler, more obvious and straightforward, "boring" explanations, less interesting and challenging than those cooked up by Tóth, but, unlike his, consistent with the extant Greek mathematical corpus, without requiring any reliance upon unknown achievements and appeal to nonexistent, alternative geometries to that contained in the *Elements*, something Tóth succeeds in identifying by, inter alia, removing typically his "fragments and traces" from their surrounding context, uprooting them, as it were, from their immediate, natural, and larger environs, a procedure that strikes the reader as ahistorical and, therefore, unacceptable.

Some Examples of Toth's Historical Methodology

Let us now see, how Tóth deals with his "non-Euclidean fragments and traces". Before that, however, it needs pointing out that it is impossible to tackle all instances in which one encounters non sequiturs, misquotations, mistranslations, inaccurate enunciations of theorems, et cetera, in this challenging and wrongheaded book. A few examples should suffice.

To me there seems to be a contradiction between the claim that geometers at the Platonic Academy proved the truth of a whole chain of non-Euclidean propositions (p. 2) and, yet, rejected that geometry *in toto* (In the original: "zwar abgelehnt und im ontischen Zustand einer konkreten Negativität", p. 1).

It is not strictly the case that the word Η$\iota\tau\eta\sigma\theta\varpi$ introduces just the parallel postulate. It actually precedes all five postulates.

The enunciation of I.29, an equivalent of the fifth postulate, and the first proposition in the *Elements* proven by appeal to it, has nothing to do with co-orthogonality. It reads in Clemens Thaer's rather accurate German translation:

> Beim Schnitt einer geraden Linie mit (zwei) parallelen geraden Linien werden (innere) Wechselwinkel einander gleich, jeder äußere Winkel wird dem innen gegenüberliegenden gleich, und innen auf derselben Seite entstehende Winkel werden zusammen zwei Rechten gleich. (Euklid [1933] 2000: 21)

And now here is Tóth's so-called quotation of Euclid's enunciation (the quotation marks appear in the original): "Wenn zwei Geraden parallel sind, dann sind sie auch koorthogonal." Though this is an immediate consequence of Euclid's theorem, it is not his. Such distortions, more or less innocuous and offensive, appear as quotations throughout Tóth's book.

Finally, though the examples could be multiplied, practically *sine limite*, Tóth goes to extreme lengths to find an Aristotelian Greek text, differing from the generally accepted one of the Loeb Classical Library, say (*De caelo* 281 b3–6), in order to be able to provide a rebarbative interpretation, one of his main textual instances of non-Euclidicity in the entire Aristotelian corpus, to the analysis of which we shall return.

Tóth as Historian, Part I

It is now time to review Tóth's analyses of the alleged historical evidence for non-Euclidean geometry in ancient Greece. This, needless to say, we must also do selectively. Toth's sources include the two *Analytics*, *De sophisticis elenchis,* the *Physica, De caelo, De anima,* the *Problemata, Metaphysica,* the three *Ethics* (*Nicomachea, Eudemia*, and the *Magna Moralia*), as well as Platon's *Kratylos*, by all counts an impressive lot. These sources include: (1) examples entirely unrelated to the issue at hand, which even Tóth has great difficulty reading non-Euclidically, (2) examples out of which Tóth manages to extract with his ahistorical scalpel supportive evidence, and (3) examples which are neither here nor there. In our survey, we shall attempt to illustrate them all.

Analytica priora 64b28, the first presumably corroborative fragment quoted by Tóth is a mere four-words-mutilated torso, saying, in Tredennick's translation: "Begging or assuming the point at issue" A. J. Jenkinson renders it basically in the same fashion. As it stands, it says nothing at all about geometry, and certainly nothing about non-Euclidean geometry. Tóth will use it, however, in connection with the second part of the example, one of many, of what he calls "Die nichteuklidischen Stellen im *Corpus Aristotelicum* und in Platon" (p. 395).

It is, as expected, a discussion of *petitio principii*, which, for the sake of understanding, I shall quote in extenso, including some preceding sentences:

> Now some things are naturally knowable through themselves, and others through something else (for principles are knowable through themselves, while the examples which fall under the principles are knowable through something else); and when any one tries to prove by means of itself that which is not knowable by means of itself, then he is begging the point at issue. This may be done by directly postulating the proposition which is to be proved; but we may also have recourse to some other propositions of a sort which are of their very nature proved by means of our proposition, and prove the point at issue by means of them: *e.g.*, supposing that A is proved by B and B by C, and it is the nature of C to be proved by A; for if anyone argues this way it follows that he is proving A by means of itself. This is exactly what those persons do who think that they are drawing parallel lines; for they do not realize that they are making assumptions which cannot be proved unless the parallel lines exist (H. Tredennick's translation in Aristotle [1938] 1983: 495).

The passage is clear. To grasp it, there is no need to appeal to non-Euclidean geometry, of which there is no trace in the entire Greek mathematical corpus, a detail making Tóth's entire enterprise extremely doubtful. For an exhaustive, balanced discussion of the passage, I recommend to the reader Thomas L. Heath (whom Tóth dismisses cavalierly), *Mathematics in Aristotle* (Heath 1970: 27–30). Before we listen to Tóth, we must follow him, however, with the next passage he tackles in connection with it, namely *Analytica priora* 66a 14–15 (for which Tóth's reference is wrong, and which is not contained in Heath's book). Here is what it says, in the same translation:

> since presumably it is by no means incongruous that the same fallacy should follow from several hypotheses, *e.g.*, that the impossible conclusion ‚parallel lines meet' should follow both on the hypothesis that the interior is greater than the exterior

> angle and on the hypothesis that the sum of the angles of a triangle is greater than two right angles. (Aristotle 1938 [1983]: 495)

This too is clear. Absurd conclusions follow from absurd premises, and, moreover, the same absurdity may follow from different absurd hypotheses. That's it. Now, here is Tóth:

> die falsche Hypothese eines stumpfen Parallelwinkels […], den zwei Geraden mit einer gemeinsamen Sekante bilden, führt zur ebenfalls falschen Konklusion: "die Parallelen schneiden sich"; es handelt sich offensichtlich um Gerolamo Saccheris nichteuklidishe *Hypothese des stumpfen Winkels* im Falle der unendlichen Konfiguration von parallelen Geraden: $\Pi(AB) > R$. (p. 4)

> […] aus der falschen Hypothese, die Winkelsumme des Dreiecks sei größer als zwei Rechte, 2R, folgt dieselbe falsche Konklusion: *die Parallelen schneiden sich* […] – dies ist eine Variante der Hypothese des stumpfen Winkels, ausgesagt für die finite Konfiguration eines Dreiecks; symbolisch: $\Delta > 2R$. (Ibid., emphases in the original)

If this strikes the reader as absurd, indeed it is, in full conformity with Aristotle's statement above. How, then, does Tóth justify this absurd interpretation? He does not. And yet he devotes an entire thoroughly ahistorical chapter, full of its own oddities, though ingenious and learned, in the second part of his book (pp. 117–158) to this so-called justification, which is not. In a nutshell, it makes, inter alia, the following assertions, all of which follow from his reading anachronistically into the Aristotelian text what it clearly does not, and cannot, say. They amount to solemn declarations, containing Tóth's profession of faith, and not to a real, historical substantiation of his beliefs:

Comparing *Elements*, I.16 with I.32, he points out their non-equivalence, because the former is "a theorem of Bolyai's *absolute* geometry," while the latter is Euclidean. "They lie," therefore, "in two theoretically utterly different planes" (p. 124). Euclid would not have grasped the difference.

Tóth reads the Aristotelian passage as a clear-cut and consistent adherence to the "universal" (p. 126) principles of non-Euclidean geometry, where the first part covers the status of nonexistent "parallels," when the corresponding external and internal angles, created by a secant cutting two "parallel" lines, are unequal, while the second part treats a triangle whose internal angles exceed together two rights:

> Unter diesen Umständen ist aber die erste Hypothese der *priora* […] mit der zweiten […] insofern *geometrisch äquivalent*, dass sie beide nichteuklidische Aussagen sind. Da die zweite Hypothese […] der *priora* die Hypothese des stumpfen Winkels für die Figur des *Dreiecks*, $\Delta > 2R$, darstellt, enthält auch die erste Hypothese […] dieselbe Hypothese des stumpfen Winkels. Sie ist aber anstatt für das Dreieck, für die Konfiguration von *parallelen* Geraden, $\Pi(AB) > R$ formuliert. (p. 127)

Tóth goes on, flexing his logico-mathematical muscles, to point out, against all historical evidence, that Aristotle and the geometers of the Academy were fully aware of the necessary, and simultaneous, universality of the Euclidean and non-Euclidean theorems concerning the sum of the internal angles of the triangle:

> Das heißt aber, dass *die formale Negation* der affirmative-universalen euklidischen Aussage *Elem.* I 32: $\Delta = 2R$ – *alle* Dreiecke haben eine Winkelsumme *gleich* mit 2R, wieder zu universalen – und zwar sogar zu zwei ebenfalls affirmative-universalen Aussagen führt: $\Delta < 2R$ – *alle* Dreiecke haben eine Winkelsumme, die *kleiner* ist als 2R, und $\Delta > 2R$ – *alle* Dreiecke haben eine Winkelsumme, die *grosser* ist als 2R. (p. 129)

He ascribes, incredibly, to Aristotle the knowledge of hyperbolic, non-Euclidean geometry (p. 137), in a learned, ingenious, inappropriate discussion (pp. 133–139), full of gratuitous assumptions, imprecise enunciations, and unsupported conclusions. He claims that the non-Euclidean hypothesis is contained implicitly in the *Elements* (!) (p. 140), that Ptolemy formulated explicitly, for the first time in history, the two non-Euclidean hypotheses, in a work he never wrote (*Über die Inzidenz der Nicht-Koorthogonalen*) (the title given by Proclus being, *That Lines Produced from Angles Less than Two Right Angles Meet One Another*); that the Euclidean enunciation of the fifth postulate is redundant (!) (p.141); that the *Elements* contain no postulate stating the indefinite, limitless extension of the straight line (!) (what about Postulate 2?), and so on.

As if this weren't enough, Tóth ends his discussion of the passage in question with an excursus (pp. 150–158), which is a striking and blatant example of an ahistorical discourse in which faithfulness to the sources is relinquished and replaced by alternative modern enunciations and timeless logical analysis. A couple of examples should suffice:

> Die pythagoreische Definition des Begriffs *logos* lautet: "der *logos* ist ein geordnetes Paar von zwei natürlichen Zahlen". (p. 151)

Definition V.5 of the *Elements* is stated as follows:

> [Z]wei *logoi* sind miteinander gleich, wenn die zwei infiniten Dyaden (Dedekind-Schnitte), die sie jeweils im Universum der *pythagoreischen logoi* induzieren, miteinander gleich sind. (p. 152)

Finally, proposition XIII.6, which in Euclid reads: "If a rational straight line be cut in extreme and mean ratio, each of the segments is the irrational straight line called apotome," [Euclid [1926] 1956: 449] is rendered by Tóth as follows:

> [D]er Satz behauptet, dass der *logos,* der Extreme und der Mitte […] *alogos estin*, d.h. dass der *eudoxische logos* der Extremen und der Mitte eine [sic] *pythagoreischer alogos* ist. (p. 153, my Latin transcription of the Greek-written terms)

The "Excursus" ends on a note combining mystical nonsense with "historical writing" of the worst kind, making it, deservingly, quotable in extenso:

> Die oben analysierte Stelle der *priora* […] weist aber darauf hin, dass Saccheris und Legendres Satz bereits den Geometern der Akademie bekannt gewesen sein dürfte [!]. Aber ebenso wie Legendre von Saccheri, hat auch Saccheri keine Kenntnis von dieser Stelle der *priora* gehabt. Alle drei haben aber demselben Raum westlichen Denkens angehört. Keine materielle Spuren irgendeiner Verbindung zwischen den drei Ereignissen ist nachweisbar und höchstwahrscheinlich ist eine derartige Verbindung auch inexistent. Es gehört aber zu den faszinierendsten, sowohl mysteriösen als auch immanenten Zügen des Geistes, dass in seiner immateriellen Substanz auch echte, quasi hydrodynamische Strömungen vorhanden sind, die Ideen tragen und sie auch wahrnehmbar weitertragen (p. 158).

Ipse dixit.

Tóth as Historian, Part II

We can now tackle only a couple of additional so called non-Euclidean Aristotelian passages dealt with by Tóth in his book, but what shall be said about those applies equally, *mutatis mutandis*, to all the rest, and also have a look at the fragment in Plato's *Cratylus*, in which Tóth sees a crucial testimonial for the existence of non-Euclidean geometry among the members of the Academy.

In the *Posterior Analytics* 85b28–86a4, Aristotle treats the procedure for identifying the final cause, and says, inter alia:

> Again, we cease our inquiry for the reason and assume we know it when we reach a fact whose existence does not depend upon any other fact […]. If, then, the same principle applies to all causes and reasoned facts, and if our knowledge of all final causes is most complete under the conditions which we have just described, then in all other cases too our knowledge is most complete when we reach a fact which does not depend further upon any other fact. *So when we recognize that the sum of the exterior angles of a figure is equal to four right angles, because the figure is isosceles, there still remains the reason why the figure is isosceles, viz., that it is a triangle, and this because it is a right-lined figure. If this reason depends upon nothing else, our knowledge is now complete. Moreover our knowledge is now universal; and therefore universal knowledge is superior.* [Translation by Tredennick in Aristotle [1960] 1997: 143, 145; my emphasis]

The statement, quoted more fully than Tóth does it, to show its context, is clear, unremarkable, unproblematic. It has nothing to do with alternative geometries, nothing at all. The English translation is accurate and corresponds fully to Iulius Pacius's Latin translation (p. 404). However, Tóth mistranslates part of the last two emphasized sentences above, as follows: "Wenn aber dies nicht mehr besteht, weil anderes gilt […][!], dann haben wir den höchsten Grad des Wissens erreicht" (pp. 320 f.). And he comments:

> Die Summe der *n* Außenwinkel ist in jedem geradlinigen Polygon mit vier Rechten, 4R, gleich. Wenn aber dies nicht der Fall ist, weil die Summe der Außenwinkel eine andere ist, dann besitzen wir ein Wissen, das seinen höchsten Grad erreicht hat (p. 4).

Why does he do this? Presumably to be able to find a connection, even if a weak one, between this innocent statement and its non-Euclidean counterpart (pp. 314–322):

> Der Text der *posteriora* 85b38-86a3 enthält aber eine etwas obskure aber ebenso auffallende Bemerkung, welche die Vermutung erlaubt, dass den Geometern der Akademie auch der Fall bekannt gewesen sein dürfte, wenn die Summe der Außenwinkel des geradlinigen Polygons nicht mit 4R gleich ist. (p. 320)

Tóth's generous use of the subjunctive in his reconstructions and creative interpretations of historical texts speaks volumes. Without his mistranslation of the passage, even Tóth could not have done this. I shall not repeat his quasi-arguments, which I find, needless to say, eminently unconvincing and blatantly anachronistic.

There is a passage in *De caelo*, which provides extremely important fodder for Tóth's obsession with non-Euclidean geometry, and to which he dedicates pages upon numerous pages of his fascinatingly wayward, perverse book. It is the following passage, in Latin alphabet transcription: *lego d', oion to trigōnon adunaton duo orthas echein, ei tade, kai he diametros symmetros, ei tade* (281b5-6). What this means in Guthrie's Loeb translation is:

> say it is impossible for a triangle to contain two right angles *if such-and-such conditions are fulfilled,* or, the diagonal is commensurate with the sides *if such-and-such conditions are fulfilled.* (Aristotle [1939] 2000: 113)

Tóth's Greek text eliminates the second *ei tade*, and his German translation, punctuated differently than the Greek, reads:

> denn ich behaupte zum Beispiel: es ist unmöglich für das Dreieck eine Winkelsumme zu haben, die gleich mit zwei Rechten ist; wenn dies [der Fall ist] ist auch die Diagonale [des Quadrates] kommensurabel. (p. 371)

Proceeding in this manner enables Tóth to read the quoted text as an important theorem of non-Euclidean geometry, one of the two he has managed to identify, and to engage in singing Hosanna in the Highest to the achievements of Greek non-Euclidean geometers. What can one say to this?

The context of the passage is clear. It deals with the different meanings and cognitive status of the concepts possible, impossible, true, false, and it is in this context that the two impossible examples, of the triangle, whose internal angles are not together equal to 2R, and the diagonal of the square, which is commensurable with the side, are brought. Again, impossible hypotheses bring in their wake impossible consequences. C'est tout.

Our last Aristotelian example, gleaned from Tóth's chrestomathy of alleged non-Euclidean instances, will be drawn from the ethical works, because of the great significance Tóth attaches to them. We can deal with only one example, but this suffices, as what shall be said about it is transferable, *mutatis mutandis*, to the rest. Tóth's lengthy, insightful, highly readable discussion of the ethical passages, predicated as it is on the assumption that they refer necessarily to non-Euclidean geometry, is contained in Chapter 5 of the Second Part of his book, titled: "*Geometria more ethico.* Strukturanalogie der ethisch-politischen Praxis und der axiomatischen Grundlegung der Geometrie: *Ethica Nicomachea […]*; *Magna moralia […]*; *Ethica Eudemia […]*; *Problemata* […] Die Alternative: "euklidisch-nichteuklidisch" und die Inkorruptibilität ihrer Entscheidung." This detailed title says it all. Tóth attaches, rightly, great significance to the fact that man's singularity in the natural realm resides in his freedom of choice, in his capacity of making freely his decisions, and rejoices in the bare fact that the only example of voluntary, free choice given by Aristotle is geometrical, not ethical. His conclusion, unwarranted to my mind by the textual evidence, is that in geometry, like in ethics, freedom of decision and choice reigns supreme that is, the Greek geometers were fully aware that it is up to them to choose which geometry to follow and develop, Euclidean or non-Euclidean, and that choice was completely, absolutely, free, and independent of any geometrical considerations!

Let us take the passage from the *Eudemian Ethics*, 1222b15–42, and see what it really says. The passage is quite long and we cannot quote it in full (there is really no need for it), but we shall glean out of it primarily the geometrical example and its immediate context, to appreciate its nature and the kind of enterprise Tóth is engaged in:

> [O]bviously man alone among animals initiates certain conduct, for we should not ascribe conduct to any of the others. And the first principles of that sort, which are the first source of motions, are called first principles in the strict sense, and most rightly those that have necessary results; […] But the strict sense of ‚first principle' is not found in first principles incapable of change, for example those of mathematics, although the term is indeed used of them by analogy, for in mathematics if the first principle were changed virtually all the things proved from it would change, though they do not change owing to themselves, one being destroyed by the other, except by destroying the assumption and thereby establishing a proof. But man is a first principle of a certain motion, for action is motion. And since as in other matters the first principle is a cause of the things that exist or come into existence because of it, we must think as we do in the case of demonstrations. For example, if as the angles of a triangle are together equal to two right angles the angles of a quadrilateral are necessarily equal to four right angles, that the angles of a triangle are equal to two right angles is clearly the cause of that fact; and supposing a triangle were to change, a quadrilateral would necessarily change too—for example if the angles of a triangle became equal to three right angles, the angles of a quadrilateral would become equal to six right angles, or if four, eight; also if a triangle does not change but is as described, a quadrilateral too must of necessity be as described. The necessity of what we are arguing is clear from the *Analytics*; […] Supposing there were no further cause of the triangle's having the property stated, then the triangle would be a sort of first principle or cause of the later stages. Hence

> if in fact there are among existing things some that admit of the opposite state, their first principles also must necessarily have the same quality. (Aristotle [1935] 1996: 263, 265)

It is clear from just reading the above fragmentary quotation attentively, as it both demands and deserves, that Tóth overinterprets it, bringing to it, man being a free agent, his previous commitment to the existence of non-Euclidean geometry in ancient Greece. The passage itself is innocent of Tóth's biases, it contains nothing about non-Euclidean geometry and the alleged awareness of Greek geometers at the Academy of their geometrical freedom to choose between, equally viable, alternative geometries.

The theme of the passage is the relation between *archai*, basic principles, and their consequences. When the *archai* change, their consequences change too and changed consequences lead necessarily to changed *archai*. Since the sum of the angles in a triangle is 2R, the necessary consequence is that the sum of internal angles in a quadrangle, composed of two triangles, is 4R. Should the first sum change to 3R, the latter too would change to 6R. The sum of the interior angles of a triangle is itself a consequence of other principles (*archai*), primarily of the Fifth Postulate. Were this sum independent of anterior principles, which it is not, then $\Delta = 2R$ would itself play the role of an *arche*. That is all.

Concerning the alleged freedom of the geometers to choose between a Euclidean and non-Euclidean geometry, there is nothing as such in this passage. (This applies equally to all the other passages on which Tóth draws.) As to Toth's repeated assertion that one can find in Aristotle the statement that the Euclidean and non-Euclidean worlds are each complete, consistent, and independent universes (though he doesn't put it exactly this way), it is itself independent of the texts, on which it allegedly depends. To take just one example, *Metaphysica* 1052a4–7, this is what Aristotle actually says:

> And it is obvious that with regard to immovable things also, if one assumes that there are immovable things, there is no deception in respect of time. *E.g.*, if we suppose that the triangle is immutable, we shall not suppose that it sometimes contains two right angles and sometimes does not, for this would imply that it changes…

There is, needless to state it, nothing here about Euclidicity and non-Euclidicity and about their being equally justified universes, each complete and independent of the other. Tóth, however, incredibly considers it "ein Korollar von Saccheris Theorem der absoluten Universalität der Winkelsumme" (p. 254). What he says about this passage on pages 254–255 is blatantly anachronistic, wild in its implications, and totally gratuitous.

Now to the *Cratylus* passages (436AE and 438CE). They represent, perhaps, the longest, and weakest, of Tóth's fragments, discovered by Vittorio Hösle, who studied under Tóth, and wrote the eulogizing, non-critical preface

(*Vorwort*) to the book under review. Against Tóth's unfounded claim, there is absolutely nothing in the dialogue even mildly reminiscent of the cognitive status of non-Euclidean geometry, specifically, if its internal consistency necessarily means that it is also true. Nothing. The crucial parts deal, indeed, with the distinction between consistency and truth, pointing out that a single small mistake in the course of a geometrical proof, may lead, if faultlessly pursued, to false, though correct, necessary consequences. Socrates speaks, in Benjamin Jowett's translation:

> But that, friend Cratylus, is no answer. For if he did begin in error, he may have forced the remainder into agreement with their original error and with himself; there would be nothing strange in this, any more than in geometric diagrams [*diagrammatōn*, i.e., in geometrical proofs], which have often a slight and invisible flaw in the first part of the process, and are consistently mistaken in the long deductions which follow. And this is the reason why every man should expend his chief thought and attention on the consideration of his first principles – are they or are they not rightly laid down? And when he has duly sifted them, all the rest will follow. (Crat. 364 c–e in Plato [1963] 1980: 437)

Now, pray tell, where is there even the slightest mention here of the status of non-Euclidean geometry? There is nothing, except in the minds of Imre Tóth and Vittorio Hösle!

Conclusion

In the fourth and final act of Beaumarchais' *Le mariage de Figaro*, in his monologue, Figaro says: "Sans la liberté de blamer, il n'est point d'éloge flatteur." It is in this spirit that this entire essay was conceived.

Imre Tóth has written a curious book. Judged as a compendium of *varia et curiosa*, on the margin of ancient Greek philosophy and mathematics, an anthology of personal thoughts, maxims and reflections on geometry in the course of history, it is a challenging, well written (though its German is, at times, rather idiosyncratic, and its proofreading faulty), readable, intelligent, and learned book. I learned from it quite a bit. Judged, however, by its declared theme and by its main claims, it is, I am afraid, as I tried to show in this essay, a failure.

Still, a reader desiring ardently to save Tóth from himself could, of course, overlook all that has been said above and somewhat arbitrarily extract from the book a defensible thesis. Such a blindly sympathetic reader would argue that the book is just an attempt to discuss the foundations of pre-Euclidean geometry, a discipline characterized by faulty attempts to prove $\Delta = 2R$ (I.32), in the absence of the Fifth Postulate, and by various attempts to lay the ground for a proper theory of parallels by finding a proper order for propositions I.27–I.32. This pre-Euclidean "foundational crisis" found its solution when Euclid finally formulated the Parallel Postulate. There is nothing wrong with such a speculative thesis, which is neither revolutionary nor new. It does not, however, represent faithfully the gist of the book, and it discards its main content and spirit.

Tóth's fundamental methodological historical principle is a sui generis version of Ockham's razor, in which *necessitate* has become *modernitate*: "Entia non sunt multiplicanda praeter *modernitate*." It is, therefore, possible to say that, on the one hand, Tóth has erected an impressive structure, a multilayered mansion. On the other hand, however, as a historical study, his book is a house built on sand.

His Aristotle has undergone baptism, coming out of the sacrament as ArisTóthle. (It was Jean-Claude Pont who, in his book on the history of non-Euclidean geometry of 1986, referred to Tóth's Aristotle as ArisToth (Pont 1986: quoted on p. 311 of Toth's book). It is not the Stagirite we know, but one of his far off relatives.

Like me, Imre Tóth (originally Roth) was born a Romanian Jew. In January 1975, he sent me one of his studies, *Die nicht-euklidische Geometrie in der Phänomenologie des Geistes* (Frankfurt, 1972), a weird and erudite book, indeed. It displays all the main ingredients of his many other works on the topic, together with his *nulli secundus* philosophico-historical Weltanschauung; its text is one third of its content, the other two thirds being taken by amazingly lengthy and learned notes. The book was accompanied by a friendly letter in Rumanian, dated 23 January, in which he said, inter alia: "Since I assume that, at least in part, if not in toto, your opinions differ from mine, I have nothing against your publishing a critical review of my work, be it as ruthlessly formulated as possible."

I see this essay as a belated, and friendly, fulfillment of Tóth's friendly old request. May he rest in peace.

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Sabetai Unguru
Cohn Institute for the History and Philosophy
of Science and Ideas, University of Tel-Aviv
Ramat Aviv
69978 Tel Aviv
Israel
E-Mail: usabetai@post.tau.ac.il