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DIGRAPH COMPETITIONS AND COOPERATIVE GAMES

ABSTRACT. Digraph games are cooperative TU-games associated to domination structures which can be modeled by directed graphs. Examples come from sports competitions or from simple majority win digraphs corresponding to preference profiles in social choice theory. The Shapley value, core, marginal vectors and selectope vectors of digraph games are characterized in terms of so-called simple score vectors. A general characterization of the class of (almost positive) TU-games where each selectope vector is a marginal vector is provided in terms of game semi-circuits. Finally, applications to the ranking of teams in sports competitions and of alternatives in social choice theory are discussed.

KEY WORDS: Digraph competition, cooperative TU-game, Shapley value, core, marginal vector, selectope vector, simple score vector

MATHEMATICS SUBJECT CLASSIFICATION: 90D12 (cooperative games), 90D43 (games involving graphs), 90A08 (social choice).

1. INTRODUCTION

A directed graph or *digraph* is a pair (N, D) where N is a finite set of nodes and $D \subset N \times N$ is a binary relation on N. Such a digraph can represent various domination structures. As an example we mention a *sports competition* in which there are teams or players that play matches against one another. In that case the nodes represent the teams that participate in the competition, while $(i, j) \in D$ means that team i has won the match it played against team j. A closely related economic example is a *preference relation*. In this case the nodes represent alternatives between which an agent or a group of agents can choose. On the individual level, $(i, j) \in D$ means that an agent prefers alternative i to alternative j when comparing these alternatives pairwise (see, e.g., Sen (1979)). On the group level, the digraph can represent the *simple majority win* digraph corresponding to a preference profile which consists of one preference relation for every individual in the group. Finally

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we mention that digraphs are also used to represent directed *social networks* in, e.g., Gould (1987) and White and Borgatti (1994).

In the sequel we use the term *digraph competition* for a domination structure modelled by a digraph. The 'strength' of a node in a digraph competition depends on all relations that are present in the digraph. The strength of a team in a sports competition, for example, depends on the results of the matches that it has played itself but also on the results of other matches. Usually we want to rank the teams knowing which team has 'won' the competition, which one is second, and so on. In *social choice theory* we want to know which alternative is the 'most preferred' by a society of individuals. The degree of preference for an alternative in a preference profile depends on the positions of all alternatives in the preference relations of the individuals. In directed social networks we want to know whether a certain position in the network is more powerful than another position.

The purpose of this paper is to evaluate or measure the strength of nodes in a digraph competition by game theoretic means. Laffond, Laslier and LeBreton (1993) evaluate this strength by using non-cooperative zero-sum games. In this paper we apply techniques from cooperative game theory. Our motivation is the following. Measuring the strength of nodes in a digraph competition can be seen as an allocation problem in which one has to distribute a number of points such that the stronger the position of the node the more points it gets. Cooperative game theory provides the tools to analyze such allocation problems. To start, we initially assign one point to each node. Measuring the strength of nodes then boils down to redistributing these initial points. We analyze this redistribution problem by assigning to every digraph competition a *cooperative game with transferable utility* – or simply a TU-game – on the set of nodes N. In the tradition of cooperative game theory we assign to every subset of nodes $E \subset N$ the number of points corresponding to nodes on which there is no claim from nodes outside E, i.e., which are not dominated by any node outside E. We refer to these games as *(conservative) digraph games.* Such digraph games turn out to be almost positive and hence convex.

Solutions for TU-games prescribe reasonable payoff distributions for the worth of the 'grand coalition' N. Therefore, applying solutions to digraph games yields distributions of the total number of points which can be thought of as measuring the strength of the nodes in the underlying digraph competition. In this paper we analyze two important solutions for TU-games, namely the Shapley value and the core. It is shown that the *Shapley value* of an arbitrary digraph game, which by definition is the average of all *marginal vectors* of the game, is also equal to the average of the so-called *simple score vectors* of *simple subdigraphs*. Moreover, we show that the *core* of a digraph game is equal to the convex hull of the set of the simple score vectors. The proof of this result uses the correspondence between these simple score vectors and *selectope vectors* as defined in Hammer, Peled and Sorensen (1977), Vasil'ev (1981) and Derks, Haller and Peters (2000). It is well known that each marginal vector is a selectope vector and hence, in the context of digraph games, a marginal vector is a simple score vector. We show that digraph competitions for which each simple score vector is also a marginal vector of the corresponding digraph game are exactly those that do not contain an *anti-directed semi-circuit*. This result is derived using a new general result on the coincidence between marginal vectors and selectope vectors for almost positive games.

The outline of the paper is as follows. Section 2 defines digraph games. Section 3 considers the Shapley value. Section 4 analyzes the structure of the core and the relation between marginal vectors, selectope vectors and simple score vectors. Section 5 discusses applications. Finally, Section 6 considers possible extensions.

2. DIGRAPH GAMES

Since we assume the finite set of nodes N to be fixed, we represent a digraph competition on N by its binary relation $D \subset N \times N$. The collection of all digraph competitions on N is denoted by $\mathcal{D}^N$. For a digraph competition $D \in \mathcal{D}^N$ and $i \in N$ the nodes in $S_D(i) := \{j \in N | (i, j) \in D\}$ are called the *successors* of i in D, and the nodes in $P_D(i) := \{j \in N | (j, i) \in D\}$ are called the *predecessors* of i in D. For $E \subset N$ we define $S_D(E) := \cup_{i \in E} S_D(i)$ and $P_D(E) := \cup_{i \in E} P_D(i)$.

A TU-game on N is a pair (N, v) where the *characteristic function* $v : 2^N \to \mathbb{R}$ is such that $v(\emptyset) = 0$. Since N is fixed we represent each TU-game on N by its characteristic function. We denote the set of all TU-games v on N by $\mathcal{G}^N$. In this section we assign a cooperative game in $\mathcal{G}^N$ to every digraph competition in $\mathcal{D}^N$. So, in particular the set of players in such a game corresponds to the set of nodes N. Since in game theoretic tradition coalitions are assigned the worth they can guarantee themselves in total, we propose to define a digraph game in a way such that to every coalition $E \subset N$ it assigns the number of successors of E that have no predecessors outside E.

DEFINITION 2.1. The *(conservative) digraph game* corresponding to $D \in \mathcal{D}^N$ is the game $v_D : 2^N \to \mathbb{R}$ given by $v_D(E) = \#\{j \in S_D(E) | P_D(j) \subset E\}$ for all $E \subset N$.

Using *unanimity games*[1] u_T, it is easy to see that a digraph game can be expressed as follows.

LEMMA 2.2. *For every* $D \in \mathcal{D}^N$ *it holds that* $v_D = \sum_{i \in S_D(N)} u_{P_D(i)}$.

By Lemma 2.2 each digraph game can be expressed as a positive sum of unanimity games. Since all unanimity games are convex we have the following corollary.

COROLLARY 1. *For every* $D \in \mathcal{D}^N$ *it holds that* v_D *is convex, i.e.,*

$$v_D(E) + v_D(F) \leq v_D(E \cup F) + v_D(E \cap F) \textit{ for all } E, F \subset N.$$

It is well known that convex games have nice properties, in particular with respect to the Shapley value and the core.

3. THE SHAPLEY VALUE OF A DIGRAPH GAME

The shapley value is the function $\Phi : \mathcal{G}^N \to \mathbb{R}^N$ which assigns to each TU-game the average of the marginal vectors of that game, i.e., $\Phi(v) = \frac{1}{(\#N)!} \sum_{\pi \in \Pi(N)} m^\pi(v)$, where $\Pi(N)$ denotes the collection of all bijections $\pi : N \to \{1, \dots, \#N\}$, and for a bijection $\pi \in \Pi(N)$ the *marginal vector* $m^\pi(v) \in \mathbb{R}^N$ is given by:

$$\begin{aligned} m_i^\pi(v) =& v(\{j \in N | \pi(j) \leq \pi(i)\}) \\ &- v(\{j \in N | \pi(j) < \pi(i)\}) \text{ for all } i \in N. \end{aligned}$$

By $\mathcal{M}(v) = \{m^{\pi}(v) | \pi \in \Pi(N)\}$ we denote the set of all marginal vectors in v. By means of the digraph game we have assigned a TU-game to every digraph competition $D \in \mathcal{D}^N$. Since the Shapley value assigns real numbers to all players in a TU-game, the Shapley value of a digraph game can be seen as a function that evaluates the strength of the nodes in a digraph competition. It turns out that the Shapley value of the digraph game corresponding to $D \in \mathcal{D}^N$ coincides with the β-vector[2] $\beta(D)$ which is given by $\beta_i(D) = \sum_{j \in S_D(i)} \frac{1}{\#P_D(j)}$ for all $i \in N$.

Moreover, it can be found as the average of the score vectors of all simple subdigraphs of D. The *score vector* of digraph D is the vector $\sigma(D) \in \mathbb{R}^N$ given by $\sigma_i(D) = \#S_D(i)$ for all $i \in N$. A digraph A is a *simple subdigraph* of $D \in \mathcal{D}^N$ if $A \subset D$ and $\#P_A(i) = 1$ for all $i \in S_D(N)$. The collection of all simple subdigraphs of $D \in \mathcal{D}^N$ is denoted by $Sim(D)$. Score vectors corresponding to simple subdigraphs of $D \in \mathcal{D}^N$ are called *simple score vectors* in D.

THEOREM 3.1. *For every $D \in \mathcal{D}^N$ it holds that*

$$\Phi(v_D) = \frac{1}{\#Sim(D)} \sum_{A \in Sim(D)} \sigma(A).$$

Proof. Let $D \in \mathcal{D}^N$. Since $v_D = \sum_{j \in S_D(N)} u_{P_D(j)}$, additivity of the Shapley value[3] implies that $\Phi_i(v_D) = \sum_{j \in S_D(N)} \Phi_i(u_{P_D(j)}) = \sum_{j \in S_D(i)} \frac{1}{\#P_D(j)}$.

Since $\#Sim(D) = \prod_{h \in S_D(N)} \#P_D(h)$ and $\#\{A \in Sim(D) \mid j \in S_A(i)\} = \prod_{h \in S_D(N) \setminus \{j\}} \#P_D(h)$ for every $(i, j) \in D$, it follows for every $i \in N$ that

$$\Phi_i(v_D) = \sum_{j \in S_D(i)} \frac{1}{\#P_D(j)} = \frac{\sum_{j \in S_D(i)} \left(\prod_{h \in S_D(N) \setminus \{j\}} \#P_D(h) \right)}{\prod_{h \in S_D(N)} \#P_D(h)}$$

$$= \frac{1}{\#Sim(D)} \sum_{j \in S_D(i)} \#\{A \in Sim(D) \mid j \in S_A(i)\}$$

$$= \frac{1}{\#Sim(D)} \left(\sum_{A \in Sim(D)} \#S_A(i) \right)$$

$$= \frac{1}{\#Sim(D)} \sum_{A \in Sim(D)} \sigma_i(A) . \qquad \square$$

EXAMPLE 3.2. *Consider the digraph* $D = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 3)\}$ *on* $N = \{1, 2, 3\}$. *The corresponding digraph game* v_D *is given by* $v_D = u_{\{1\}} + u_{\{1,2\}} + u_{\{1,2,3\}}$. *For this digraph* $\Phi(v_D) = (1\frac{5}{6}, \frac{5}{6}, \frac{1}{3})$. *This digraph has six simple subdigraphs which are given by* $A_1 = \{(1, 1), (1, 2), (1, 3)\}$, $A_2 = \{(1, 1), (1, 2), (2, 3)\}$, $A_3 = \{(1, 1), (1, 2), (3, 3)\}$, $A_4 = \{(1, 1), (2, 2), (1, 3)\}$, $A_5 = \{(1, 1), (2, 2), (2, 3)\}$, $A_6 = \{(1, 1), (2, 2), (3, 3)\}$, *with simple score vectors* $\sigma(A_1) = (3, 0, 0)$, $\sigma(A_2) = \sigma(A_4) = (2, 1, 0)$, $\sigma(A_3) = (2, 0, 1)$, $\sigma(A_5) = (1, 2, 0)$, $\sigma(A_6) = (1, 1, 1)$. *Taking the average of these simple score vectors also gives* $\Phi(v_D) = (1\frac{5}{6}, \frac{5}{6}, \frac{1}{3})$. $\square$

4. THE CORE AND SELECTOPE OF A DIGRAPH GAME

The *core* of an arbitrary TU-game $v \in \mathcal{G}^N$ is given by $Core(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N)$ and $\sum_{i \in E} x_i \geq v(E)$ for all $E \subset N\}$. As stated in Section 2 each digraph game is convex. In Shapley (1971) and Ichiishi (1981) it is shown that the core of a convex game v equals the convex hull of the marginal vectors[4], i.e., $Core(v) = Conv(\mathcal{M}(v))$. It turns out that the core of a digraph game v_D also coincides with the convex hull of all simple score vectors in D. In the proof of this result we use the correspondence between the convex hull of the simple score vectors in D and selectope vectors of v_D.

It is well known that a TU-game $v \in \mathcal{G}^N$ can be expressed in a unique way as a linear combination $v = \sum_{T \subset N} \Delta_v(T) u_T$ of the unanimity games, where the *dividends* $\Delta_v(T)$ are given by $\Delta_v(T) = v(T)$ if $\#T = 1$, and $\Delta_v(T) = v(T) - \sum_{\substack{F \subset T \\ F \neq T}} \Delta_v(F)$ if $\#T \geq 2$ (see Harsanyi (1959)). A TU-game $v \in \mathcal{G}^N$ is *almost positive* if $\Delta_v(T) \geq 0$ for all $T \subset N$ with $\#T \geq 2$. We denote the class of almost positive games by $\mathcal{G}_+^N$. Note that almost positive games are convex. A *selector* on N is a function $\alpha\colon 2^N \backslash \{\emptyset\} \to N$ that assigns to every non-empty coalition in N one of the players in that coalition as a representative for that coalition. Let $\mathcal{A}(N)$ be the set of all selectors on N. Given a selector $\alpha \in \mathcal{A}(N)$ with $\alpha(T) \in T$ for all $T \subset N$ the corresponding *selectope vector* is the vector $s^\alpha \in \mathbb{R}^N$ given by $s_i^\alpha(v) = \sum_{\substack{T \subset N \\ i \in T, \alpha(T)=i}} \Delta_v(T)$, i.e., it assigns the dividends of coalitions fully to their representative players.

By $\mathcal{S}(v) = \{s^\alpha(v) \mid \alpha \in \mathcal{A}(N)\}$ we denote the set of all selectope vectors of v. The *selectope* is defined by $Sel(v) = Conv(\mathcal{S}(v))$, i.e., the convex hull of the selectope vectors. Denoting the set of simple score vectors in D by $\Sigma(D) = \{\sigma(A) \in \mathbb{R}^N \mid A \in Sim(D)\}$, it turns out that the set of selectope vectors is a subset of $\Sigma(D)$. Moreover, their convex hulls are equal.

LEMMA 4.1. *For every $D \in \mathcal{D}^N$ it holds that (i) $\mathcal{S}(v_D) \subset \Sigma(D)$, and (ii) $Conv(\mathcal{S}(v_D)) = Conv(\Sigma(D))$.*

Proof. To prove (i), take $s^\alpha(v_D) \in \mathcal{S}(v_D)$ for some selector $\alpha \in \mathcal{A}(N)$. Defining $A \in Sim(D)$ by $P_A(i) = \alpha(P_D(i))$ for all $i \in S_D(N)$ (, and $P_A(i) = \emptyset$ for $i \in N \setminus S_D(N)$), yields that $s_i^\alpha(v_D) = \#\{j \in N \mid i \in P_A(j)\} = \#S_A(i) = \sigma_i(A)$ for all $i \in N$, and thus $s^\alpha(v_D) \in \Sigma(D)$. This proves (i).

To prove (ii) we distinguish two cases.

First, suppose there are no two nodes $h, j \in N$, $h \neq j$, with $P_D(h) = P_D(j)$ and $\#P_D(j) \geq 2$. For every $\sigma(A) \in \Sigma(D)$ we can then define a selector $\alpha \in \mathcal{A}(N)$ by $\alpha(E) = P_A(i)$ if $E = P_D(i)$ for some $i \in S_D(N)$, and for all other coalitions take $\alpha(E) \in E$ to be an arbitrary player in E. Then $\sigma_i(A) = \#S_A(i) = \#\{j \in N \mid i \in P_A(j)\} = s_i^\alpha(v_D)$ for all $i \in N$, and thus $\sigma(A) \in \mathcal{S}(v_D)$. So, $\Sigma(D) \subset \mathcal{S}(v_D)$. With part (i) of the lemma it follows that $\Sigma(D) = \mathcal{S}(v_D)$.

Second, suppose that there are two nodes $h, j \in N$, $h \neq j$, with $P_D(h) = P_D(j)$, $\#P_D(j) \geq 2$. Given (i), to prove (ii) it is sufficient to show that $Conv(\Sigma(D)) \subset Conv(\mathcal{S}(v_D))$. We prove this by showing that every extreme point of $Conv(\Sigma(D))$ belongs to $Conv(\mathcal{S}(v_D))$. Obviously, every extreme point of $Conv(\Sigma(D))$ belongs to $\Sigma(D)$. Take $x \in \Sigma(D)$. It is sufficient to prove that $x \in \mathcal{S}(v_D)$ or x is not an extreme point of $Conv(\Sigma(D))$. Equivalently, it is sufficient to show that $x \notin \mathcal{S}(v_D)$ implies that x is not an extreme point of $Conv(\Sigma(D))$. Therefore, assume that $x \in \Sigma(D) \setminus \mathcal{S}(v_D)$. Since, by assumption, there are no two nodes with the same set of predecessors, there is an $A \in Sim(D)$ with $x = \sigma(A)$ and $P_A(h) = i$, $P_A(j) = g$ for $i, g \in P_D(j)$, $i \neq g$. Define A', A'' by $A' = (A \setminus \{(g, j)\} \cup \{(i, j)\}$ and $A'' = (A \setminus \{(i, h)\}) \cup \{(g, h)\}$. Then $A', A'' \in Sim(D)$ and $x = \sigma(A) = \frac{1}{2}(\sigma(A') + \sigma(A''))$ with $\sigma(A') \neq \sigma(A'')$. So, x is not an extreme point of $Conv(\Sigma(D))$. □

A well-known result that has been shown be various authors (see Vasil'ev (1981) and Derks, Haller and Peters (2000)) is that $Core(v) = Sel(v)$ if and only if v is almost positive. Since a digraph game v_D is almost positive we have $Core(v_D) = Sel(v_D)$. With Lemma 4.1 this gives as a corollary that the core of a digraph game v_D is the convex hull of all simple score vectors in D.

THEOREM 4.2. *For every $D \in \mathcal{D}^N$ it holds that $Core(v_D) = Conv(\Sigma(D))$.*

Since, in general, $\mathcal{M}(v) \subset \mathcal{S}(v)$ for any $v \in \mathcal{G}^N$, Lemma 4.1 in particular shows that each marginal vector of a digraph game v_D is a simple score vector in D. The converse however need not hold as is seen in the following example.

EXAMPLE 4.3. Consider the digraph game $v_D = u_{\{1\}} + u_{\{1,2\}} + u_{\{1,2,3\}}$ corresponding to the digraph D of Example 3.2. Then $\mathcal{M}(v_D) = \{(1, 1, 1), (1, 2, 0), (2, 0, 1), (3, 0, 0)\}$, while $\Sigma(D) = \mathcal{M}(v_D) \cup \{(2, 1, 0)\}$. The simple score vector $(2, 1, 0)$ corresponds to both A_2 and A_4 of Example 3.2. □

It turns out that equality between the set $\Sigma(D)$ of simple score vectors in D and the set $\mathcal{M}(v_D)$ of marginal vectors depends on whether or not the digraph competition D contains an *anti-directed semi-circuit*, i.e., whether or not there is a sequence $(i_1, r_1, \dots, i_t, r_t, i_1)$, $t \geq 2$, such that

(i) $i_k \in N$ and $r_k \in D$ for all $k \in \{1, \ldots, t\}$,

(ii) $r_t = (i_1, i_t)$, and for all $k \in \{1, \ldots, t-1\}$,

$$r_k = \begin{cases} (i_k, i_{k+1}) & \text{if } k \text{ is odd} \\ (i_{k+1}, i_k) & \text{if } k \text{ is even,} \end{cases}$$

(iii) $i_k \neq i_l$ if $k \neq l$ and $k + l$ is even (i.e., if $k, l, k \neq l$, are both odd or are both even).

Note that these conditions imply that $t \geq 4$, t is even, and $r_k \neq r_l$ for $k, l \in \{1, \ldots, t\}$, $k \neq l$. The digraph D of Example 4.3 contains the anti-directed semi-circuit $(1, (1, 2), 2, (2, 2), 2, (2, 3), 3, (1, 3), 1)$.

THEOREM 4.4. *Let $D \in \mathcal{D}^N$. Then $\mathcal{M}(v_D) = \Sigma(D)$ if and only if D has no anti-directed semi-circuit.*

Theorem 4.4 follows from the following more general result on the relation between marginal vectors and selectope vectors for almost positive TU-games. We know that if $v \in \mathcal{G}^N$ is almost positive (and hence, convex) then $Sel(v) = Core(v)$, i.e., $Conv(\mathcal{S}(v)) = Conv(\mathcal{M}(v))$. However, as is illustrated by the almost positive digraph game of Example 4.3, almost positivity of v does not guarantee that $\mathcal{S}(v) = \mathcal{M}(v)$. It turns out that the class of games for which these two sets coincide can be determined using the concept of a game semi-circuit.

Let $v \in \mathcal{G}_+^N$. A sequence $(E_1, i_1, \ldots, E_t, i_t, E_1)$, with $t \geq 2$, is a *game semi-circuit* in v if

(i) $E_k \subset N$, $\#E_k \geq 2$ and $\Delta_v(E_k) \neq 0$ for all $k \in \{1, \ldots, t\}$,

(ii) $i_t \in E_t \cap E_1$, and $i_k \in E_k \cap E_{k+1}$ for all $k \in \{1, \ldots, t-1\}$,

(iii) $\#\{i_1, \ldots, i_t\} = \#\{E_1, \ldots, E_t\} = t$.

THEOREM 4.5. *Let $v \in \mathcal{G}_+^N$. Then $\mathcal{M}(v) = \mathcal{S}(v)$ if and only if v has no game semi-circuit.*

The proof of this theorem is postponed to the appendix. Here we only show how Theorem 4.4 follows from Theorem 4.5 and Lemma 4.1.

Proof of Theorem 4.4. We distinguish the following two cases:

1. Suppose that there is a $T \subset N$ with $\#T \geq 2$ and $\Delta_{v_D}(T) \notin \{0, 1\}$. Then there are $h, j \in N$, $h \neq j$, with $P_D(h) = P_D(j)$

and $\#P_D(j) \geq 2$. For $g_1, g_2 \in P_D(h)$, $g_1 \neq g_2$, it then follows that $(g_1, (g_1, h), h, (g_2, h), g_2, (g_2, j), j, (g_1, j), g_1)$ is an anti-directed semi-circuit in D.

To show that $\mathcal{M}(v_D) \neq \Sigma(D)$ we construct an $A \in Sim(D)$ satisfying:

(i) $P_A(h) = g_1$,

(ii) $P_A(j) = g_2$,

(iii) $P_A(i) \subset \{g_1, g_2\}$ if $P_D(i) \cap \{g_1, g_2\} \neq \emptyset$.

Then $\sigma_{g_1}(A) \geq \#S_D(g_1)\backslash S_D(g_2) + 1 > \#S_D(g_1)\backslash S_D(g_2)$, and similarly $\sigma_{g_2}(A) > \#S_D(g_2)\backslash S_D(g_1)$. Take $\pi \in \Pi(N)$. Suppose without loss of generality that $\pi(g_1) < \pi(g_2)$. Then $m^{\pi}_{g_1}(v_D) \leq \#S_D(g_1)\backslash S_D(g_2) < \sigma_{g_1}(A)$. Thus, $\mathcal{M}(v_D) \neq \Sigma(D)$.

2. Suppose that $\Delta_{v_D}(T) \in \{0, 1\}$ for all $T \subset N$ with $\#T \geq 2$. Then there are no two nodes $h, j \in N$, $h \neq j$, with $P_D(h) = P_D(j)$ and $\#P_D(j) \geq 2$. In a similar way as done in the proof of Lemma 4.1.(ii) it follows that for every $\sigma(A) \in \Sigma(D)$, taking the selector $\alpha \in \mathcal{A}(N)$ given by $\alpha(E) = P_A(i)$ if $E = P_D(i)$ for some $i \in S_D(N)$, and $\alpha(E) \in E$ for all other coalitions E, it follows that $\sigma_i(A) = \#S_A(i) = \#\{j \in N \mid i \in P_A(j)\} = s^{\alpha}_i(v_D)$ for all $i \in N$, and thus $\sigma(A) \in \mathcal{S}(v_D)$. So, $\Sigma(D) \subset \mathcal{S}(v_D)$, and with Lemma 4.1.(i) it then follows that $\mathcal{S}(v_D) = \Sigma(D)$. With Theorem 4.5 it then is sufficient to prove that D has an anti-directed semi-circuit if and only if v_D has a game semi-circuit.

(Only if) Suppose that $(i_1, r_1, \ldots, i_t, r_t, i_1)$, $t \geq 4$, t even, is an anti-directed semi-circuit in D. Then $(E_1, j_1, \ldots, E_{\frac{1}{2}t}, j_{\frac{1}{2}t}, E_1)$ with $E_k = P_D(i_{2k})$ for $k \in \{1, \ldots, \frac{1}{2}t\}$, $j_k = i_{2k+1}$ for $k \in \{1, \ldots, \frac{1}{2}t - 1\}$, and $j_{\frac{1}{2}t} = i_1$, is a game semi-circuit in v_D.

(If) On the other hand, let $(E_1, i_1, \ldots, E_t, i_t, E_1)$ be a game semi-circuit in v_D. Then we can take $h_k \in S_D(i_k) \cap S_D(i_{k+1})$ for all $k \in \{1, \ldots, t-1\}$, and $h_t \in S_D(i_t) \cap S_D(i_1)$. (So, every E_k corresponds to $P_D(h_k)$.)

But then $(j_1, r_1, \ldots, j_{2t}, r_{2t}, j_1)$ with $j_{2k-1} = i_k$, $j_{2k} = h_k$ for $k \in \{1, \ldots, t\}$, $r_t = (j_1, j_t)$, and for all $k \in \{1, \ldots, t-1\}$, $r_k = (j_k, j_{k+1})$ if k is odd, and $r_k = (j_{k+1}, j_k)$ if k is even, is an anti-directed semi-circuit in D. □

5. APPLICATIONS

As mentioned in the introduction, measures for evaluating the strength of positions in digraph competitions can be applied to various types of situations. Here we discuss two of them.

First, they can be applied in ranking teams in sports competitions. To allow a draw as a result of a match we summarize the results of matches in a sports competition between the teams in the set N by digraph competition $D \in \mathcal{D}^N$ where $(i, j) \in D$ if and only if i did not lose the match it played against j. Moreover, we assume the digraph competition to be reflexive, i.e., $(i, i) \in D$ for all $i \in N$.

EXAMPLE 5.1. At the UEFA European Soccer Championship (EURO 2000) sixteen teams were initially divided into four groups of four teams each. Group C consisted of Spain (Sp), Yugoslavia (Y), Norway (N) and Slovenia (Sl). The results of the matches yield the digraph competition $D = \{(Sp, Sp), (Y, Y), (N, N), (Sl, Sl), (Sp, Y), (Sp, Sl), (Y, N), (Y, Sl), (N, Sp), (N, Sl), (Sl, Y), (Sl, N)\}$. The UEFA ranked the teams by rewarding them with 3 points for a win, 1 point for a draw and 0 points for a loss. This resulted in the ranking: 1. Spain (6 points), 2. Yugoslavia (4 points), 3. Norway (4 points), 4. Slovenia (2 points). The ranking between Yugoslavia and Norway (who have the same UEFA score) is based on the result of the match between themselves. For the purpose of this paper these two teams can be considered equally ranked. If we apply the Shapley value to the corresponding digraph game v_D we obtain $\Phi(v_D) = (\Phi_{Sp}(v_D), \Phi_Y(v_D), \Phi_N(v_D), \Phi_{Sl}(v_D)) = \frac{1}{12}(13, 11, 13, 11)$. So, according to this ranking Norway would be ranked higher than Yugoslavia, and would even be ranked equal to Spain. One of the reasons for this is that in the UEFA ranking the number of points assigned to a team only depends on the number of matches this team won and lost. In the ranking by Shapley value the number of points assigned to a team not only depends on how many matches were won or lost, but also who were the opponents from whom is won or lost. Clearly, a win against a team that won many matches gives more points than winning from a team that won few matches. In this example we see that Norway does better in the ranking by Shapley value since it won the match from the 'strong' team Spain. □

A second example is the application to *social choice theory*. Preferences of an individual a over a set of n alternatives $N = \{1, \dots, n\}$ can be represented by a preference relation $\succeq$ on N. We denote $i \succeq_a j$ if individual a weakly prefers alternative i to alternative j. For a society consisting of m individuals a *preference profile* p is an m-tuple of such preference relations. A *social choice correspondence* assigns to every preference profile a non-empty subset of alternatives which can be viewed as the 'most prefered' alternatives by the society. A social choice correspondence is *majoritarian* if, for every preference profile p on the set of alternatives N and set of individuals A, it is based on the corresponding *simple majority win digraph* $D_p \subset N \times N$ given by $(i, j) \in D_p$ if and only if $\#\{a \in A \mid i \succeq_a j\} \geq \#\{a \in A \mid j \succeq_a i\}$, i.e., in D_p alternative i dominates alternative j if i defeats j by simple majority vote. Borm, van den Brink, Levínský and Slikker (2000) show that the social choice correspondence that is based on the Shapley value of the digraph game v_{D_p} is a *Pareto optimal* refinement of the *Top cycle* choice correspondence.

EXAMPLE 5.2. Consider the set of four alternatives $N = \{1, 2, 3, 4\}$ and the set of three agents $A = \{a, b, c\}$. The preference profile is described by the following three preference relations: (a) $1 \succ 2 \succ 3 \succ 4$, (b) $4 \succ 1 \succ 2 \succ 3$, and (c) $3 \succ 4 \succ 1 \succ 2$ (where $i \succ j$ means that $i \succeq j$ and not $j \succeq i$). The simple majority win digraph corresponding to this situation is $D = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (2, 3), (3, 4), (4, 1), (4, 2)\}$. The Shapley value of the digraph game $v_{D_p} = u_{\{1,4\}} + u_{\{1,2,4\}} + u_{\{1,2,3\}} + u_{\{3,4\}}$ is equal to $\Phi(v_{D_p}) = \frac{1}{6}(7, 4, 5, 8)$. So, alternative 4 is considered to be the unique best alternative. The top cycle consists of all four alternatives, although alternative 1 Pareto dominates alternative 2(, i.e., each individual prefers alternative 1 to alternative 2.) □

A related application is *voting* for candidates, for example electing presidents or chair persons. Each voter then has a preference relation over the candidates. The Shapley value of the digraph game corresponding to the simple majority win digraph of the preference profile can be used in electing one candidate.

Finally, we mention the application of relational power measures in social networks. Measuring power in social networks has been an important topic in the social network literature of the last decades. In particular directed networks have been studied in, e.g., Gould (1987) and White and Borgatti (1994). Both these papers concentrate on *betweenness* measures of *centrality* in such networks. The Shapley value of digraph games is treated as a *degree* measure of *domination* in van den Brink and Gilles (2000).

6. CONCLUDING REMARKS

In the approach of measuring the strength of nodes in digraph competitions using cooperative digraph games we started with assigning to every node in a digraph competition an 'initial' value equal to one. Thus, we assume that initially all nodes are equally important. A possible generalization of this approach is to assign possibly different positive 'initial' values to the nodes, allowing them to differ in importance ex-ante. For a *weight vector* $\delta > 0$ we thus obtain the *weighted digraph game* corresponding to $D \in \mathcal{D}^N$ and δ which is given by

$$v_{D,\delta}(E) = \sum_{\substack{j \in S_D(E) \\ P_D(j) \subset E}} \delta_j \text{ for all } E \subset N.$$

For the Shapley value of the weighted digraph game we find that

$$\Phi_i(v_{D,\delta}) = \sum_{j \in S_D(i)} \frac{\delta_j}{\# P_D(j)}$$

for all $i \in N$. Moreover, it is equal to the average of the *weighted simple score vectors* $\sigma(A, \delta)$ over all simple subdigraphs given by $\sigma_i(A, \delta) = \sum_{j \in S_A(i)} \delta_j$ for all $i \in N$. A special type of these weighted Shapley values is the one that is obtained by the iterative procedure that starts with initial input weights one, and in every step takes the output scores of the previous step as new input weights. Also Theorems 4.2 and 4.4 can be extended, i.e., $Core(v_{D,\delta}) = Conv(\Sigma(D, \delta))$, and $\Sigma(D, \delta) = \mathcal{M}(v_{D,\delta})$ if and only if there is no anti-directed semi-circuit in D, with $\Sigma(D, \delta) = \{\sigma(A, \delta) \mid A \in Sim(D)\}$.

APPENDIX: PROOF OF THEOREM 4.5

Let $v \in \mathcal{G}_+^N$. From Vasil'ev (1981) and Derks, Haller and Peters (2000) it follows that $\mathcal{M}(v) \subset \mathcal{S}(v)$. Thus we have to prove that $\mathcal{S}(v) \subset \mathcal{M}(v)$ if and only if there is no game semi-circuit in v.

Only if

Let $\mathcal{S}(v) \subset \mathcal{M}(v)$. Also, suppose there is a game semi-circuit $(E_1, i_1, \dots, E_t, i_t, E_1)$ in v. We show that this leads to a contradiction by constructing a selector α on N such that there is no bijection π on N with $s^\alpha(v) = m^\pi(v)$.

Let $I = \{i_1, \dots, i_t\}$. Since $E_k \neq E_l$, for $k \neq l$, we can consider a selector $\alpha: 2^N \backslash \{\emptyset\} \to N$ satisfying
(i) $\alpha(E_k) = i_k$ for all $k = 1, \dots, t$, and
(ii) $\alpha(E) \in I$ if $E \cap I \neq \emptyset$.

Since all $i \in I$ get assigned the (non-negative) dividends of all coalitions that contain i and no other player from I, and moreover, every player $i_k \in I$ gets the positive dividend of coalition E_k we have $s_i^\alpha(v) > \sum_{\substack{E \subset N \\ E \cap I = \{i\}}} \Delta_v(E)$ for all $i \in I$.

Let $\pi \in \Pi(N)$. Suppose without loss of generality that $\pi(h) = \min_{i \in I} \pi(i)$.

Then $m_h^\pi(v) \leq \sum_{\substack{E \subset N \\ E \cap I = \{i\}}} \Delta_v(E) < s_h^\alpha(v)$. Thus there is no $\pi \in \Pi(N)$ such that $s^\alpha(v) = m^\pi(v)$.

If

Suppose there is no game semi-circuit in v, and let $\alpha: 2^N \backslash \{\emptyset\} \to N$ be a selector. We show that there is a $\pi \in \Pi(N)$ such that $s^\alpha(v) = m^\pi(v)$.

We recursively define the sets $L_k \subset N$, $k \in \mathbb{N} \cup \{0\}$, by $L_0 = \emptyset$ and for every $k \in \mathbb{N}$

$$L_k = \left\{ i \in N \backslash \bigcup_{l=0}^{k-1} L_l \;\middle|\; \begin{array}{c} \text{For every } E \subset N \text{ with } \Delta_v(E) > 0 \text{ and } \alpha(E) = i: \\ E \backslash \{i\} \subset (\bigcup_{l=0}^{k-1} L_l) \end{array} \right\}.$$

Let $k \in \mathbb{N}$ be such that $N \backslash \bigcup_{l=0}^{k-1} L_l \neq \emptyset$. We prove that $L_k \neq \emptyset$.

On the contrary suppose that $L_k = \emptyset$. Let $\widehat{N} := N \backslash \bigcup_{l=0}^{k-1} L_l$. Then, for every $i \in \widehat{N}$ there is an $E \subset N$ with $\Delta_v(E) > 0, \alpha(E) = i$ and $E \backslash \{i\} \not\subset \bigcup_{l=0}^{k-1} L_l$. From this it follows that we can construct a sequence of players $(i_1, i_2, \dots)$ starting with $i_1 \in \widehat{N}$ and for every

$k \in \mathbb{N}$ take $i_{k+1} \in \widehat{N}$ such that there exists an $E \subset N$, with $\{i_k, i_{k+1}\} \subset E$, $\Delta_v(E) > 0$ and $\alpha(E) = i_k$. By finiteness of $\widehat{N}$ we will find that $i_s = i_t$ for some $s \neq t$, thus creating a game semi-circuit.

So, we conclude that there is an $m \in \mathbb{N}$ such that the sets $L_1, \dots, L_m$ form a partition of N consisting of non-empty sets only.

Now we construct a $\pi \in \Pi(N)$ such that $m^\pi(v) = s^\alpha(v)$. If $i \in L_k$ and $j \in L_l$ with $k < l$ then take $\pi(i) < \pi(j)$. Consider $i \in L_k$ for some $1 \leq k \leq m$. By definition of the set L_k it follows for $E \subset N$, $i \in E$, $\alpha(E) \neq i$ and $\Delta_v(E) > 0$, that $\alpha(E) \in L_t$ for some $t > k$, and thus $\pi(i) < \pi(\alpha(E))$. Define $P^\pi(i) = \{j \in N \mid \pi(j) \leq \pi(i)\}$ for $i \in N$. Then, $E \subset P^\pi(i)$, $i \in E$ and $\alpha(E) \neq i$ implies that $\Delta_v(E) = 0$. But then

$$\begin{aligned} m_i^\pi(v) &= v(P^\pi(i)) - v(P^\pi(i)\backslash\{i\}) \\ &= \sum_{E \subset P^\pi(i)} \Delta_v(E) - \sum_{E \subset P^\pi(i)\backslash\{i\}} \Delta_v(E) = \sum_{\substack{E \subset P^\pi(i) \\ i \in E}} \Delta_v(E) \\ &= \sum_{\substack{E \subset P^\pi(i) \\ i \in E, \alpha(E)=i}} \Delta_v(E) + \sum_{\substack{E \subset P^\pi(i) \\ i \in E, \alpha(E)\neq i}} \Delta_v(E) = \sum_{\substack{E \subset P^\pi(i) \\ i \in E, \alpha(E)=i}} \Delta_v(E) \\ &= \sum_{\substack{E \subset N \\ \alpha(E)=i}} \Delta_v(E) = s_i^\alpha(v). \end{aligned}$$

□

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NOTES

1. The unanimity game of coalition $T \in 2^N \backslash \{\emptyset\}$ is given by $u_T(E) = 1$ if $T \subset E$, and $u_T(E) = 0$ otherwise.
2. The β-measure is introduced as a *relational power measure* for (irreflexive) directed networks in van den Brink and Gilles (2000).

3. Additivity of the Shapley value means that for two games v, $w \in \mathcal{G}^N$ it holds that $\Phi(v+w) = \Phi(v)+\Phi(w)$, where $(v+w) \in \mathcal{G}^N$ is given by $(v+w)(E) = v(E) + w(E)$ for all $E \subset N$.
4. This convex hull is also known as the *Weber set* of v.

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