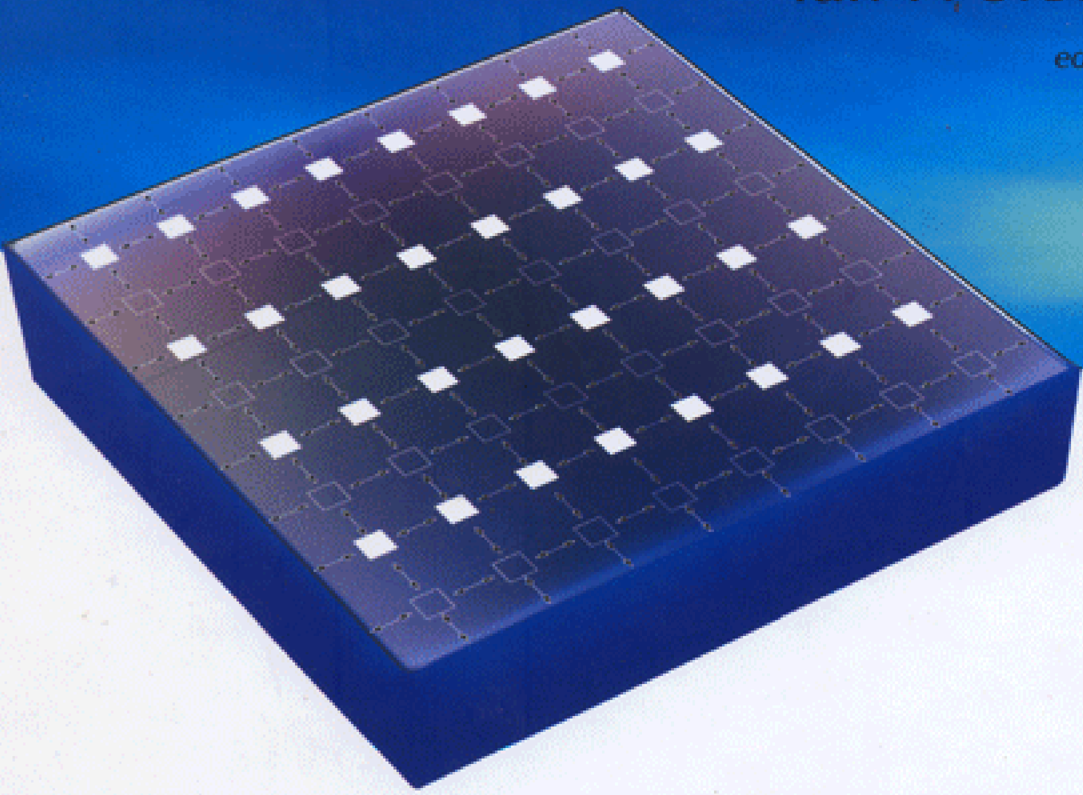
Series in Contemporary Applied Mathematics
CAM 8

Some Topics in Industrial and Applied Mathematics

Rolf Jeltsch
Ta-Tsien Li
Ian H Sloan
editors



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Rolf Jeltsch
ETH Zurich, Switzerland

Ta-Tsien Li
Fudan University, China

Ian H Sloan
University of New South Wales, Australia

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Rolf Jeltsch
Seminar of Applied Mathematics
ETH Zürich
CH-8092 Zürich
Switzerland

Tatsien Li
School of Mathematical Sciences
Fudan University
220, Handan Road
Shanghai, 200433
China

Ian Hugh Sloan
School of Mathematics
University of New South Wales
Sydney NSW 2052
Australia

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Preface

On the occasion that the Officers' Meeting and the Board Meeting of ICIAM (International Council for Industrial and Applied Mathematics) was held in Shanghai from May 26 to May 27, 2006, many famous industrial and applied mathematicians gathered in Shanghai from different countries. The Shanghai Forum on Industrial and Applied Mathematics was organized from May 25 to May 26, 2006 at Shanghai Science Hall for the purpose of inviting some of them to present their recent results and discuss recent trends in industrial and applied mathematics. Sixteen invited lectures have been given for this activity. This volume collects the material covered by most of these lectures. It will be very useful for graduate students and researchers in industrial and applied mathematics.

The editors would like take this opportunity to express their sincere thanks to all the authors in this volume for their kind contribution. We are very grateful to the Shanghai Association for Science and Technology (SAST), Fudan University, the National Natural Science Foundation of China (NSFC), The China Society for Industrial and Applied Mathematics (CSIAM), the Shanghai Society for Industrial and Applied Mathmatics (SSIAM), the Institut Sino-Français de Mathématiques Appliquées (ISFMA) and the International Council for Industrial and Applied Mathematics (ICIAM) for their help and support. Our special thanks are also due to Mrs. Zhou Chunlian for her efficient assistance in editing this book.

Rolf Jeltsch, Ta-Tsien Li, Ian H. Sloan

April 2007

On the Martingale Representation Theorem and on Approximate Hedging a Contingent Claim in the Minimum Deviation Square Criterion

Nguyen Van Huu

Vietnam National University, Hanoi

Vuong Quan Hoang

ULB Belgium.

Abstract

In this work we consider the problem of the approximate hedging of a contingent claim in the minimum mean square deviation criterion. A theorem on martingale representation in case of discrete time and an application of the result for semi-continuous market model are also given.

Keywords Hedging, contingent claim, risk neutral martingale measure, martingale representation

1. Introduction

The activity of a stock market takes place usually in discrete time. Unfortunately such markets with discrete time are in general incomplete and so super-hedging a contingent claim is requires usually an initial price two great, which is not acceptable in practice.

The purpose of this work is to propose a simple method for approximate hedging a contingent claim or an option in minimum mean square deviation criterion.

Financial market model with discrete time:

Without loss of generality let us consider a market model described by a sequence of random vectors $\{S_n,\ n=0,1,\cdots,N\}$, $S_n \in \mathbb{R}^d$, which are discounted stock prices defined on the same probability space $\{\Omega,\ \Im,\ P\}$ with $\{F_n,\ n=0,1,\cdots,N\}$ being a sequence of sigma-algebras of information available up to the time n, whereas "risk free" asset chosen as a numeraire $S_n^0 = 1$.

A F_N-measurable random variable H is called a contingent claim (in the case of a standard call option $H = \max(S_n - K, 0)$, K *is strike price*).

Trading strategy:

A sequence of random vectors of d-dimension $\gamma = (\gamma_n,\ n = 1,2,\cdots,N)$ with $\gamma_n = (\gamma_n^1, \gamma_n^2, \cdots, \gamma_n^d)^T$ (A^T denotes the transpose of matrix A), where γ_n^j is the number of securities of type j kept by the investor in the interval $[n-1,\ n)$ and γ_n is F_{n-1} -measurable (based on the information available up to the time $n-1$), then $\{\gamma_n\}$ is said to be predictable and is called *Portfolio or trading strategy* .

Assumptions:

Suppose that the following conditions are satisfied:

(i) $\Delta S_n = S_n - S_{n-1},\ H \in L_2(P),\ n = 0,1,\cdots,N.$

(ii) Trading strategy γ is self-financing, i.e. $S_{n-1}^T\gamma_{n-1} = S_{n-1}^T\gamma_n$ or equivalently $S_{n-1}^T\Delta\gamma_n = 0$ for all $n = 1,2,\cdots,N$.
Intuitively, this means that the portfolio is always rearranged in such a way its present value is preserved.

(iii) The market is of *free arbitrage*, that means there is no trading strategy γ such that $\gamma_1^T S_0 := \gamma_1 S_0 \le 0,\ \gamma_N S_N \ge 0,\ P\{\gamma_N S_N > 0\} > 0$.

This means that with such trading strategy one need not an initial capital, but can get some profit and this occurs usually as the asset $\{S_n\}$ is not rationally priced.

Let us consider

$$G_N(\gamma) = \sum_{k=1}^{N} \gamma_k \Delta S_k \text{ with } \gamma_k \Delta S_k = \sum_{j=1}^{d} \gamma_k^j \Delta S_k^j.$$

This quantity is called the *gain* of the strategy γ.

The problem is to find a constant c and $\gamma = (\gamma_n,\ n = 1, 2, \cdots, N)$ so that

$$E_P(H - c - G_N(\gamma))^2 \to \min . \tag{1.1}$$

Problem (1.1) have been investigated by several authors such as H. Folmer, M. Schweiser, M. Schal, M. L. Nechaev with $d = 1$. However, the solution of problem (1.1) is very complicated and difficult for application if $\{S_n\}$ is not a $\{\mathbf{F}_n\}$-martingale under P, even for $d = 1$.

By the fundamental theorem of financial mathematics, since the market is of free arbitrage, there exists a probability measure $Q \sim P$ such that under Q, $\{S_n\}$ is an $\{\mathbf{F}_n\}$-martingale, i.e. $E_Q(S_n|F_n) = S_{n-1}$ and the measure Q is called *risk neutral martingale probability measure*.

We try to find c and γ so that

$$E_Q(H - c - G_N(\gamma))^2 \to \min \text{ over } \gamma. \tag{1.2}$$

Definition 1.1 $(\gamma_n^*) = (\gamma_n^*(c))$ minimizing the expectation in (1.2) is called Q- optimal strategy in the minimum mean square deviation (MMSD) criterion corresponding to the initial capital c.

The solution of this problem is very simple and the construction of the Q-optimal strategy is easy to implement in practice.

Notice that if $L_N = dQ/dP$ then

$$E_Q(H - c - G_N(\gamma))^2 = E_P[(H - c - G_N)^2 L_N]$$

can be considered as an weighted expectation under P of $(H - c - G_N)^2$ with the weight L_N. This is similar to the pricing asset based on a risk neutral martingale measure Q.

In this work we give a solution of the problem (1.2) and a theorem on martingale representation in the case of discrete time.

It is worth to notice that the authors M.Schweiser, M.Schal, M.L. Nechaev considered only the problem (1.1) with S_n of one-dimension and M.Schweiser need the additional assumptions that $\{S_n\}$ satisfies non-degeneracy condition in the sense that there exists a constant δ in

(0, 1) such that

$$(E[\Delta S_n|F_{n-1}])^2 \le \sigma E[(\Delta S_n)^2|F_{n-1}] \qquad \text{P-a.s. for all } n = 1, 2, \cdots, N.$$

and the trading strategies γ_n's satisfy:

$$E[\gamma_n \Delta S_n]^2 < \infty,$$

while in this article $\{S_n\}$ is of d-dimension and we need not the preceding assumptions.

The organization of this article is as follows:

The solution of the problem (1.2) is fulfilled in paragraph 2. (Theorem 2.1) and a theorem on the representation of a martingale in terms of the differences ΔS_n (Theorem 3.1) will be also given (the representation is similar to the one of a martingale adapted to a Wiener filter in the case of continuous time).

Some examples are given in paragraph 3.

The semi-continuous model, a type of discretization of diffusion model, is investigated in paragraph 4.

2. Finding the optimal portfolio

Notation. Let Q be a probability measure such that Q is equivalent to P and under Q $\{S_n,\ n = 1, 2, \cdots, N\}$ is an integrable square martingale and let us denote $E_n(X) = E_Q(X|F_n)$, $H_N = H$, $H_n = E_Q(H|F_n) = E_n(H)$; $\text{Var}_{n-1}(X) = [\text{Cov}_{n-1}(X_i, X_j)]$ denotes the conditional variance matrix of random vector X when F_{n-1} is given, Γ is the family of all predictable strategies γ.

Theorem 2.1 *If $\{S_n\}$ is an $\{F_n\}$-martingale under Q then*

$$E_Q(H - H_0 - G_N(\gamma^*))^2 = \min\{E_Q(H - c - G_N(\gamma))^2 : \gamma \in \Gamma\}, \quad (2.1)$$

where γ_n^* is a solution of the following equation system:

$$[\text{Var}_{n-1}(\Delta S_n)]\gamma_n^* = E_{n-1}((\Delta H_n \Delta S_n) \qquad \text{P- a.s.}, \qquad (2.2)$$

Proof: At first let us notice that the right side of (2.1) is finite. In fact, with $\gamma_n = 1$ for all n, we have

$$E_Q(H - c - G_N(\gamma))^2 = E_Q\left(H - c - \sum_{n=1}^{N}\sum_{j=1}^{d}\delta S_n^j\right)^2 < \infty.$$

Furthermore, we shall prove that $\gamma^*\Delta S_n$ is integrable square under Q.

Recall that (see [Appendix A]) if $Y, X_1, X_2, \cdots, X_d$ are $d+1$ integrable square random variables with $E(Y) = E(X_1) = \cdots = E(X_d) = 0$ and if $\widehat{Y} = b_1X_1 + b_2X_2 + \cdots + b_dX_d$ is the optimal linear predictor of Y on the basis of $X_1, X_2, \cdots, X_d$ then the vector $b = (b_1, b_2, \cdots, b_d)^T$ is the solution of the following equations system:

$$\mathrm{Var}(X)b = E(YX), \tag{2.3}$$

and as $\mathrm{Var}(X)$ is non-degenerated b is defined by

$$b = [\mathrm{Var}(X)]^{-1}E(YX), \tag{2.4}$$

and in all cases

$$bE(YX) \le E(Y^2), \tag{2.5}$$

where $X = (X_1, X_2, \cdots, X_k)^T$.

Furthermore,

$$Y - \widehat{Y}\perp, \text{ i.e. } E[X_i(Y - \widehat{Y})] = 0,\ i = 1, \cdots, k. \tag{2.6}$$

Applying the above results to the problem of conditional linear prediction of ΔH_n on the basis of $\Delta S_n^1, \Delta S_n^2, \cdots, \Delta S_n^d$ as F_n is given we obtain from (2.3) the formula (2.2) defining the regression coefficient vector γ^*. On the other hand we have from (2.3) and (2.5):

$$\begin{aligned}E(\gamma_n^*\Delta S_n)^2 &= EE_{n-1}(\gamma_n^*\Delta S_n^T\Delta S_n\gamma_n^{*T}) = E(\gamma_n^*\mathrm{Var}_{n-1}(\Delta S_n)\gamma_n^{*T})\\ &= E(\gamma_n^*E_{n-1}(\Delta H_n\Delta S_n^T)) \le E(\Delta H_n)^2 < \infty.\end{aligned}$$

With the above remarks we can consider only, with no loss of generality, trading strategies γ_n such that

$$E_{n-1}(\gamma_n\Delta S_n)^2 < \infty.$$

We have:

$$H_N = H_0 + \Delta H_1 + \cdots + \Delta H_N$$

and

$$E_{n-1}(\Delta H_n - \gamma_n^T \Delta S_n)^2 = E_{n-1}(\Delta H_n)^2 - 2\gamma_n^T E_{n-1}((\Delta H_n \Delta S_n) + \gamma_n^T E_{n-1}(\Delta S_n \Delta S_n^T)\gamma_n.$$

This expression takes the minimum value when $\gamma_n = \gamma_n^*$.

Furthermore, since $\{H_n - c - G_n(\gamma)\}$ is an $\{F_n\}$- integrable square martingale under Q,

$$\begin{aligned}
E_Q(H_N - c - G_N(\gamma))^2 &= E_Q\left[H_0 - c - \sum_{n=1}^{N}(\Delta H_n - \gamma_n \Delta S_n)\right]^2 \\
&= (H_0 - c)^2 + E_Q\left[\sum_{n=1}^{N}(\Delta H_n - \gamma_n \Delta S_n)\right]^2 \\
&= (H_0 - c)^2 + \sum_{n=1}^{N} E_Q(\Delta H_n - \gamma_n \Delta S_n)^2 \\
&\quad \text{(for } \Delta H_n - \gamma_n \Delta S_n \text{ is martingale difference)} \\
&= (H_0 - c)^2 + E_Q \sum_{n=1}^{N} E_{n-1}(\Delta H_n - \gamma_n \Delta S_n)^2 \\
&\geq (H_0 - c)^2 + E_Q \sum_{n=1}^{N} E_{n-1}(\Delta H_n - \gamma_n^* \Delta S_n)^2 \\
&= (H_0 - c)^2 + E_Q \sum_{n=1}^{N} (\Delta H_n - \gamma_n^* \Delta S_n)^2 \\
&= (H_0 - c)^2 + E_Q\left[\sum_{n=1}^{N}(\Delta H_n - \gamma_n^* \Delta S_n)\right]^2 \\
&\geq E_Q(H_N - H_0 - G_n(\gamma^*))^2.
\end{aligned}$$

So $E_Q(H_N - c - G_N(\gamma))^2 \geq E_Q(H_N - H_0 - G_n(\gamma^*))^2$ and the inequality become the equality if $c = H_0$ and $\gamma = \gamma^*$. □

3. Martingale representation theorem

Theorem 3.1. *Let $\{H_n,\ n=0,1,2,\cdots\}$, $\{S_n,\ n=0,1,2,\cdots\}$ be arbitrary integrable square random variable defined on the same probability space $\{\Omega,\Im,P\}$, $F_n^S=\sigma(S_0,\cdots,S_n)$. Denoting by $\Pi(S,P)$ the set of probability measures Q such that $Q\sim P$ and that $\{S_n\}$ is $\{F_n^S\}$ integrable square martingale under Q, then if $F=\bigvee_{n=0}^{\infty}F_n^S$, $H_n,\ S_n\in L_2(Q)$ and if $\{H_n\}$ is also a martingale under Q we have:*

$$1.\ H_n=H_0+\sum_{k=1}^{n}\gamma_k\Delta S_k+C_n \qquad \text{a.s.,} \tag{3.1}$$

where $\{C_n\}$ is a $\{F_n^S\}$–Q-martingale orthogonal to the martingale $\{S_n\}$, i.e. $E_{n-1}((\Delta C_n\Delta S_n)=0$, *for all $n=0,1,2,\cdots$, whereas $\{\gamma_n\}$* is *$\{F_{n-1}^S\}$- predictable.*

$$2.\ H_n=H_0+\sum_{k=1}^{n}\gamma_k\Delta S_k:=H_0+G_n(\gamma) \qquad \text{P-a.s..} \tag{3.2}$$

for all n finite iff the set $\Pi(S,P)$ consists of only one element.

Proof: According to the proof of Theorem 2.1, putting

$$\Delta C_k=\Delta H_k-\gamma_k^*\Delta S_k,\ C_n=\sum_{k=1}^{n}\Delta C_k,\ C_0=0, \tag{3.3}$$

then $\Delta C_k\perp\Delta S_k$, by (2.6).

Taking summation of (3.3) we obtain (3.1).

The conclusion 2 follows from the fundamental theorem of financial mathematics. □

Notice 1. By the fundamental theorem of financial mathematics a security market has no arbitrage opportunity and is complete iff $\Pi(S,P)$ consists of the only element and in this case we have (3.2) with γ defined by (2.2). Furthermore, in this case the conditional probability distribution of S_n given F_{n-1}^S concentrates at most $d+1$ points of R^d (see [2], [3]), in particular for $d=1$, with exception of binomial or generalized binomial market models (see [2], [7]), other models are incomplete.

Notice 2. We can choose the risk neutral martingale probability measure Q so that Q has minimum entropy in $\Pi(S,P)$ as in [2] or Q is near P as much as possible.

Example 3.1. Let us consider a stock with the discounted price S_0 at $t=0$, S_1 at $t=1$, where

$$S_1 = \begin{cases} 4S_0/3 & \text{with prob. } p_1 \\ S_0 & \text{with prob. } p_2, \quad p_1, p_2, p_3 > 0, \quad p_1+p_2+p_3 = 1 \\ 5S_0/6 & \text{with prob. } p_3 \end{cases}$$

Suppose that there is an option on the above stock with the maturity at $t=1$ and with strike price $K = S_0$. We shall show that there are several probability measures $Q \sim P$ such that $\{S_0, S_1\}$ is, under Q, a martingale or equivalently $E_Q(\Delta S_1) = 0$.

In fact, suppose that Q is a probability measure such that under Q S_1 takes the values $4S_0/3$, S_0, $2S_0/3$ with positive probability q_1, q_2, q_3, respectively. Then $E_Q(\Delta S_1) = 0 \Leftrightarrow S_0(q_1/3 - q_3/6) = 0 \Leftrightarrow 2q_1 = q_3$, so Q is defined by $(q_1,\ 1-3q_1,\ 2q_1)$, $0 < q_1 < 1/3$.

In the above market, the payoff of the option is

$$H = (S_1 - K)_+ = (\Delta S_1)_+ = \max(\Delta S_1, 0).$$

It is easy to get a Q-optimal portfolio

$$\gamma^* = E_Q[H\Delta S_1]/E_Q(\Delta S_1)^2 = 2/3,\ E_Q(H) = q_1 S_0/3,$$

$$E_Q[H - E_Q(H) - \gamma^* \Delta S_1]^2 = q_1 S_0^2 (1-3q_1)/9 \to 0 \text{ as } q_1 \to 1/3.$$

However we can not choose $q_1 = 1/3$, because $q = (1/3,\ 0,\ 2/3)$ is not equivalent to P. It is better to choose $q_1 \cong 1/3$ and $0 < q_1 < 1/3$.

Example 3.2 Let us consider a market with one risky asset defined by:

$$S_n = S_0 \Pi Z_i, \text{ or } S_n = S_{n-1} Z_n, n = 1, 2, \cdots, N,$$

where $Z_1, Z_2, \cdots, Z_N$ are the sequence of i.i.d. random variables taking the values in the set $\Omega = \{d_1, d_2, \cdots, d_M)$ and $P(Z_i = d_k) = p_k > 0$, $k = 1, 2, \cdots, M$. It is obvious that a probability measure Q is equivalent to P and under Q $\{S_n\}$ is a martingale if and only if $Q\{Z_i = d_k) = q_k > 0$, $k = 1, 2, \cdots, M$ and $E_Q(Z_i) = 1$, i.e.

$$q_1 d_1 + q_2 d_2 + \cdots + q_M d_M = 1.$$

Let us recall the integral Hellinger of two measure Q and P defined on some measurable space $\{\Omega^*, F\}$:

$$H(P,Q) = \int_{\Omega^*} (dP \cdot dQ)^{1/2}.$$

In our case we have

$$\begin{aligned} H(P,Q) &= \sum \{P(Z_1 = d_{i1},\ Z_2 = d_{i2}, \cdots, Z_N = d_{iN}) \cdot \\ &\qquad Q(Z_1 = d_{i1},\ Z_2 = d_{i2}, \cdots, Z_N = d_{iN})\}^{1/2} \\ &= \sum \{p_{i1} q_{i1}\ p_{i2} q_{i2} \cdots p_{iN} q_{iN}\}^{1/2}, \end{aligned}$$

where the summation is extended over all $d_{i1}, d_{i2}, \cdots, d_{iN}$ in Ω or over all $i_1, i_2, \cdots, i_N$ in $\{1, 2, \cdots, M\}$. Therefore

$$H(P,Q) = \{\sum_{i=1}^{M} (p_i q_i)^{1/2}\}^N.$$

We can define a distance between P and Q by

$$||Q - P||^2 = 2(1 - H(P,Q)).$$

Then we want to choose Q^* in $\Pi(S,P)$ so that $||Q^* - P|| = \inf\{||Q - P|| : QP(S,P)\}$ by solving the following programming problem:

$$\sum_{i=1}^{M} p_i^{1/2} q_i^{1/2} \to \max$$

with the constraints :

a. $q_1 d_1 + q_2 d_2 + \cdots + q_M d_M = 1.$

b. $q_1 + q_2 + \cdots + q_M = 1.$

c. $q_1,\ q_2, \cdots,\ q_M > 0.$

Giving $p_1,\ p_2, \cdots,\ p_M$ we can obtain a numerical solution of the above programming problem. It is possible that the above problem has not a solution. However, we can replace the condition c) by the condition

c'. $q_1,\ q_2, \cdots,\ q_d \geq 0,$

then the problem has always the solution $q^* = (q_1^*,\ q_2^*, \cdots,\ q_M^*)$ and we can choose the probabilities $q_1, q_2, \cdots, q_M > 0$ are sufficiently near to $q_1^*, q_2^*, \cdots, q_M^*$.

4. Semi-continuous market model (discrete in time continuous in state)

Let us consider a financial market with two assets:

+ Free risk asset $\{B_n,\ n=0,1,\cdots,N\}$ with dynamics

$$B_n = \exp\left(\sum_{k=1}^{n} r_k\right),\ 0 < r_n < 1. \tag{4.1}$$

+ Risky asset $\{S_n,\ n=0,1,\cdots,N\}$ with dynamics

$$S_n = S_0 \exp\left(\sum_{k=1}^{n}[\mu(S_{k-1}) + \sigma(S_{k-1})g_k]\right), \tag{4.2}$$

where $\{g_n,\ n=0,1,\cdots,N\}$ is a sequence of i.i.d. normal random variable $N(0,1)$. It follows from (4.2) that

$$S_n = S_{n-1}\exp(\mu(S_{n-1}) + \sigma(S_{n-1})g_n), \tag{4.3}$$

where S_0 is given and $\mu(S_{n-1}) := a(S_{n-1}) - 2(S_{n-1})/2$, with $a(x)$, $\sigma(x)$ being some functions defined on $[0,\infty)$.

The discounted price of risky asset $\tilde{S}_n = S_n/B_n$ is equal to

$$\tilde{S}_n = S_0 \exp\left(\sum_{k=1}^{n}[\mu(S_{k-1}) - r_k + \sigma(S_{k-1})g_k]\right). \tag{4.4}$$

We try to find a martingale measure Q for this model.

It is easy to see that $E_P(\exp(\lambda g_k)) = \exp(\lambda^2/2)$, for $g_k \sim N(0,1)$, hence

$$E\exp\left(\sum_{k=1}^{n}[\beta_k(S_{k-1})g_k - \beta_k(S_{k-1})^2/2]\right) = 1, \tag{4.5}$$

for all random variable $\beta_k(S_{k-1})$.

Therefore, putting

$$L_n = \exp\left(\sum_{k=1}^{n}[\beta_k(S_{k-1})g_k - \beta_k(S_{k-1})^2/2]\right),\ n=1,\cdots,N \tag{4.6}$$

and if Q is a measure such that $dQ = L_N dP$ then Q is also a probability measure. Furthermore,

$$\frac{\tilde{S}_n}{\tilde{S}_{n-1}} = \exp(\mu(S_{n-1}) - r_n + \sigma(S_{n-1})g_n). \tag{4.7}$$

Denoting by E^0, E expectation operations corresponding to P, Q, $E_n(.) = E[(.)|F_n^S]$ and choosing

$$\beta_n = -(a(S_{n-1}) - r_n)/\sigma(S_{n-1}) \tag{4.8}$$

then it is easy to see that

$$E_{n-1}[\bar{S_n}/\bar{S_{n-1}}] = E^0[L_n\bar{S_n}/\bar{S_{n-1}}|F_n^S]/L_{n-1} = 1$$

which implies that $\{\bar{S_n}\}$ is a martingale under Q.

Furthermore, under Q, S_n can be represented in the form

$$S_n = S_{n-1}\exp((\mu^*(S_{n-1}) + \sigma(S_{n-1})g_n^*). \tag{4.9}$$

Where $\mu^*(S_{n-1}) = r_n - \sigma^2(S_{n-1})/2,\ g_n^* = -\beta_n + g_n$ is Gaussian $N(0,1)$. It is not easy to show the structure of $\Pi(S,P)$ for this model.

We can choose a such probability measure Q or the weight function L_N to find a Q- optimal portfolio.

Notice 3. The model (4.1), (4.2) is a type of discretization of the following diffusion model:

Let us consider a financial market with continuous time consisting of two assets:

+Free risk asset:

$$B_t = \exp\left(\int_0^t r(u)du\right). \tag{4.10}$$

+Risky asset : $dS_t = S_t[a(S_t)dt + \sigma(S_t)dWt]$, S_0 is given, where $a(.), \sigma(.) : (0,\infty) \to R$ such that $xa(x)$, $x\sigma(x)$ are Lipschitz. It is obvious that

$$S_t = \exp\left\{\int_0^t [a(S_u) - \sigma^2(S_u)/2]du + \int_0^t \sigma(S_u)dW_u\right\},\ 0 \le t \le T. \tag{4.11}$$

Putting

$$\mu(S) = a(S) - \sigma^2(S)/2, \tag{4.12}$$

and dividing $[0,T]$ into N intervals by the equidistant dividing points $0, \Delta,\ 2\Delta, \cdots,\ N\Delta$ with $N = T/\Delta$ sufficiently great, it follows from (4.10), (4.11) that

$$S_{n\Delta} = S_{(n-1)\Delta} \exp\left\{ \int_{(n-1)\Delta}^{n\Delta} \mu(S_u)du + \int_{(n-1)\Delta}^{n\Delta} \sigma(S_u)dW_u \right\}$$
$$\cong S_{(n-1)\Delta} \exp\{\mu(S_{(n-1)\Delta})\Delta + (S_{(n-1)\Delta})[W_{n\Delta} - W_{(n-1)\Delta}]\}$$
$$\cong S_{(n-1)\Delta} \exp\{(S_{(n-1)\Delta})\Delta + \sigma(S_{(n-1)\Delta})\Delta^{1/2} g_n\}$$

with $g_n = [W_{n\Delta} - W_{(n-1)\Delta}]/\Delta^{1/2}$, $n = 1, \cdots,\ N$, being a sequence of the i.i.d. normal random variables of the law $N(0,1)$, so we obtain the model:

$$S^*_{n\Delta} = S^*_{(n-1)\Delta} \exp\{(S_{(n-1)\Delta})\Delta + (S_{(n-1)\Delta})\Delta^{1/2} g_n\}. \tag{4.13}$$

Similarly we have

$$B^*_{n\Delta} \cong B^*_{(n-1)\Delta} \exp(r_{(n-1)\Delta}\Delta). \tag{4.14}$$

According to (4.10), the discounted price of the stock S_t is

$$\tilde{S}_t = \frac{S_t}{B_t} = S_0 \exp\left\{ \int_0^t [\mu(S_u) - r_u]d_u + \int_0^t \sigma(S_u)dW_u \right\}. \tag{4.15}$$

By Girsanov Theorem, the unique probability measure Q under which $\{\tilde{S}_t,\ F^S_t,\ Q\}$ is a martingale is defined by

$$(dQ/dP)|F^S_T = \exp\left(\int_0^T \beta_u dW_u - \frac{1}{2} \int_0^T \beta_u^2 d_u \right) := L_T(\omega), \tag{4.16}$$

where

$$\beta_s = -\frac{((a(S_s) - r_s)}{\sigma(S_s)},$$

and $(dQ/dP)|F^S_T$ denotes the Radon-Nikodym derivative of Q w.r.t. P limited on F^S_T. Furthermore, under Q

$$W^*_t = W_t + \int_0^t \beta_u du$$

is a Wiener process. It is obvious that L_T can be approximated by

$$L_N := \exp\left(\sum_{k=1}^{N} \beta_k \Delta^{1/2} g_k - \Delta\beta_k^2/2\right), \tag{4.17}$$

where

$$\beta_n = -\frac{[a(S_{(n-1)\Delta}) - r_{n\Delta}]}{\sigma(S_{(n-1)\Delta})}. \tag{4.18}$$

Therefore the weight function (4.14) is approximate to Radon-Nikodym derivative of the risk unique neutral martingale measure Q w.r.t. P and Q is used to price derivatives of the market.

Notice 4. In the market model Black- Scholes we have $L_N = L_T$. We want to show now that for the weight function (4.17)

$$E_Q(H - H_0 - G_N(\gamma^*))^2 \to 0 \text{ as } N \to \infty \text{ or } \Delta \to 0.$$

Proposition Suppose that $H = K(S_T)/B_N$ is a integrable square discounted contingent claim. Then

$$E_Q(H - H_0 - G_N(\gamma^*))^2 \to 0 \text{ as } N \to \infty \text{ or } \Delta \to 0, \tag{4.19}$$

provided a and σ are constant (in this case the model (4.10), (4.11) is the model Black-Scholes).

Proof: It is well known (see[4],[5]) that for the model of complete market (4.10), (4.11) there exists a trading strategy $\varphi = (\varphi_t = \varphi(t, S(t)),\ 0 = t = T)$, hedging H, where $\varphi : [0,T] \times (0,\infty) \to R$ is continuously derivable in t and S, such that

$$H(S_T) = H_0 + \int_0^T \varphi_t d\tilde{S}(t) \qquad \text{a.s.}$$

On the other hand we have

$$
\begin{aligned}
&E_{Q_N}\left(H - H_0 - \sum_{k=1}^{N} \gamma^*_{(k-1)\Delta}\Delta\tilde{S}_{n\Delta}\right)^2 \\
&\leq E_{Q_N}\left(H - H_0 - \sum_{k=1}^{N} \phi_{(k-1)\Delta}\Delta\tilde{S}_{n\Delta}\right)^2 \\
&= E_Q\left(\int_0^T \phi_t d\tilde{S}(t) - \sum_{k=1}^{N} \phi_{(n-1)\Delta}\Delta\tilde{S}_{(n-1)\Delta}\right)^2 L_N/L_T \\
&= E_Q\left(\int_0^T \phi_t d\tilde{S}(t) - \sum_{k=1}^{N} \phi_{(k-1)\Delta}\Delta\tilde{S}_{(n-1)\Delta}\right)^2 \to 0 \text{ as } \Delta \to 0.
\end{aligned}
$$

(Since $L_N = L_T$ and by the definition of the stochastic integral Ito as a and σ are constant.) □

A Appendix A

Let $Y, X_1, X_2, \cdots, X_d$ be integrable square random variables defined on the same probability space $\{\Omega, F, P\}$ such that $EX_1 = \cdots = EX_d = EY = 0$.

We try to find a coefficient vector $b = (b_1, \cdots, b_d)^T$ so that

$$E(Y - b_1X_1 - \cdots - b_dX_d)^2 = E(Y - b^TX)^2 = \min_{a \in R^d}(Y - a^TX)^2. \quad \text{(A.1)}$$

Let us denote $EX = (EX_1, \cdots, EX_d)$, $\mathrm{Var}(X) = [\mathrm{Cov}(X_i, X_j),\ i, j = 1, 2, \cdots, d] = EXX^T$.

Proposition The vector b minimizing $E(Y - a^TX)^2$ is a solution of the following equation system:

$$\mathrm{Var}(X)b = E(XY). \quad \text{(A.2)}$$

Putting $U = Y - b^T X = Y - \hat{Y}$, with $\hat{Y} = b^T X$, then

$$E(U^2) = EY^2 - b^T E(XY) \geq 0. \tag{A.3}$$

$$E(UX_i) = 0 \text{ for all } i = 1, \cdots, d. \tag{A.4}$$

$$EY^2 = EU^2 + E\hat{Y}^2. \tag{A.5}$$

$$\rho = \frac{E\hat{Y}^2}{(EY^2 E\hat{Y}^2)^{1/2}} = \left(\frac{E\hat{Y}^2}{EY^2}\right)^{1/2}. \tag{A.6}$$

(ρ is called multiple correlation coefficient of Y relative to X.)

Proof: Suppose at first that $\mathrm{Var}(X)$ is a positively definite matrix. For each $a \in R^d$ We have

$$F(a) = E(Y - a^T X)^2 = EY^2 - 2a^T E(XY) + a^T EXX^T a \tag{A.7}$$

$$F(a) = -2E(XY) + 2\mathrm{Var}(X)a.$$

$$\left[\frac{\partial F(a)}{\partial a_i \partial a_j},\ i, j = 1, 2, \cdots, d\right] = 2\mathrm{Var(X)}.$$

It is obvious that the vector b minimizing $F(a)$ is the unique solution of the following equation:

$$F(a) = 0 \text{ or (A.2)}$$

and in this case (A.2) has the unique solution:

$$b = [\mathrm{Var}(X)]^{-1} E(XY).$$

We assume now that $1 \leq \mathrm{Rank}(\mathrm{Var}(X)) = r < d$.

We denote by $e_1, e_2, \cdots, e_d$ the ortho-normal eigenvectors w.r.t. the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_d$ of $\mathrm{Var}(X)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_d$ and P is a orthogonal matrix with the columns being the eigenvectors $e_1, e_2, \cdots, e_d$, then we obtain:

$$\mathrm{Var}(X) = P\Lambda P^T, \text{ with } \Lambda = \mathrm{Diag}(\lambda_1, \lambda_2, \cdots, \lambda_d).$$

Putting

$$Z = P^T X = [e_1^T X, e_2^T X, \cdots, e_d^T X]^T,$$

Z is the principle component vector of X, we have

$$\mathrm{Var}(Z) = P^T \mathrm{Var}(X) P = \Lambda = \mathrm{Diag}(\lambda_1, \lambda_2, \cdots, \lambda_r, 0, \cdots, 0).$$

Therefore

$$EZ_{r+1}^2 = \cdots = EZ_d^2 = 0, \text{ so } Z_{r+1} = \cdots = Z_d = 0 \text{ P- a.s.}$$

Then

$$\begin{aligned} F(a) = E(Y - a^T X)^2 &= E(Y - (a^T P)Z)^2 \\ &= E(Y - a_1^* Z_1 - \cdots - a_d^* Z_d)^2 \\ &= E(Y - a_1^* Z_1 - \cdots - a_r^* Z_r)^2, \end{aligned}$$

where

$$a^{*T} = (a_1^*, \cdots, a_d^*) = a^T P, \ \ \mathrm{Var}(Z_1, \cdots, Z_r) = \mathrm{Diag}(\lambda_1, \lambda_2, \cdots, \lambda_r) > 0.$$

According to the above result $(b_1^*, \cdots, b_r^*)^T$ minimizing $E(Y - a_1^* Z_1 - \cdots - a_r^* Z_r)^2$ is the solution of

$$\begin{pmatrix} \lambda_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \lambda_r \end{pmatrix} \begin{pmatrix} b_1^* \\ \ldots \\ b_r^* \end{pmatrix} = \begin{pmatrix} EZ_1 Y \\ \ldots \\ EX_r Y \end{pmatrix} \tag{A.8}$$

or

$$\begin{pmatrix} \lambda_1 & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \lambda_r & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} b_1^* \\ \ldots \\ b_r^* \\ b_{r+1}^* \\ \ldots \\ b_d^* \end{pmatrix} = \begin{pmatrix} EZ_1 Y \\ \ldots \\ EZ_r Y \\ 0 \\ \ldots \\ 0 \end{pmatrix} = \begin{pmatrix} EZ_1 Y \\ \ldots \\ EZ_r Y \\ EZ_{r+1} Y \\ \ldots \\ EZ_d Y \end{pmatrix} \tag{A.9}$$

with $b_{r+1}^*, \cdots, b_d^*$ arbitrary.

Let $b = (b_1, \cdots, b_d)^T$ be the solution of $b^T P = b^{*T}$, hence $b = Pb^*$ with b^* being a solution of (A.9). Then it is follows from (A.9) that

$$\mathrm{Var}(Z) P^T b = E(ZY) = P^T E(XY)$$

or

$$P^T \mathrm{Var}(X) P P^T b = P^T E(XY) \text{ (since } \mathrm{Var}(Z) = P^T \mathrm{Var}(X) P)$$

or

$$\mathrm{Var}(X) b = E(XY)$$

which is (A.2). Thus we have proved that (A.2) has always a solution, which solves the problem (A.1).

By (A.7), we have

$$\begin{aligned} F(b) &= \min_a E(Y - a_T X)^2 \\ &= EY^2 - 2b^T E(XY) + b^T \mathrm{Var}(X) b \\ &= EY^2 - 2b^T E(XY) + b^T E(XY) \\ &= EY^2 - b^T E(XY) \geq 0. \end{aligned}$$

On the other hand

$$EUX_i = E(X_i Y) - E(X_i b^T X) = 0, \tag{A.10}$$

since b is a solution of (A.2) and (A.10) is the ith equation of the system (A.2).

It follows from (A.10) that

$$E(U\hat{Y}) = 0 \text{ and } EY^2 = E(U+\hat{Y})^2 = EU^2 + E\hat{Y}^2 + 2E(U\hat{Y}) = EU^2 + E\hat{Y}^2.$$

□

Example We can use Hilbert space method to prove the above proposition. In fact, let H be the set of all random variables ξ's such that $E\xi = 0$, $E\xi^2 < \infty$, then H becomes a Hilbert space with the scalar product $(\xi, \zeta) = E\xi\zeta$, and with the norm $\|\xi\| = (E\xi^2)^{1/2}$. Suppose that $X_1, X_2, \cdots, X_d$, $Y \in H$, L is the linear manifold generated by $X_1, X_2, \cdots, X_d$. We want to find a $\hat{Y} \in H$ so that $\|Y - \hat{Y}\|$ minimizes, that means $\hat{Y} = b^T X$ solves the problem (A.1). It is obvious that $\hat{Y}$ is defined by

$$\hat{Y} = \mathrm{Proj}_L Y = b^T X \text{ and } U = \hat{Y} - Y \in L.$$

Therefore $(Y - b^T X, X_i) = 0$ or $E(b^T X X_i) = E(X_i Y)$ for all $i = 1, \cdots, d$ or $b^T E(X^T X) = E(XY)$ which is the equation (A.2). The rest of the above proposition is proved similarly.

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