On the Semantics of Graded Modalities

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Abstract

We enrich propositional modal logic with operators $\Diamond^{>n}$ $(n \in \mathbb{N})$ which are interpreted on Kripke structures as "there are more than n accessible worlds for which ...", thus obtaining a basic graded modal logic **GrK**. We show how some familiar concepts (such as subframes, p-morphisms, disjoint unions and filtrations) and techniques from modal model theory can be used to obtain results about expressiveness (like graded modal equivalence, correspondence and definability) for this language. On the basis of the class of linear frames we demonstrate that the expressive power of the language is considerably stronger than that of classical modal logic. We give a class of formulas for which a first-order equivalent can be systematically obtained, but also show that the set of formulas for which such an equivalence exists is in some sense a proper subset of the set of so called Sahlqvist formulas, a syntactically defined set of modal formulas for which a corresponding formula is guaranteed to exist. Finally we show how, combining the technique of 'filtration' with a notion of 'copying worlds' — in view of the "more than n" interpretation, one cannot simply collapse worlds — for some graded modal logics (**GrK**, **GrT**, . . .), the finite model property (and also decidability) is obtained.

1 Introduction

We undertake some investigations in (specifically) the semantics of modal logic **GrK**, (**GrT**, **GrKD4**, . . .) which is obtained by augmenting the classical modal logic **K** (**T**, **KD4**, . . .) with graded modalities $\Diamond^{>n}$ $(n \in \mathbb{N})$. $\Diamond^{>n}\varphi$ is interpreted on Kripke structures as "more than n accessible worlds verify φ". This language was already mentioned in [Gob70] and studied in [Kap70] — as an extension of **S5** — and [Fin72] in the seventies, and rediscovered in [FatCar85] in the eighties, where the main concern of those contributions was to obtain a sound and complete logic for this operator. In [Ben87a] a fixed point theorem for this formalism was proven, which then was used to turn recursive implicit definition of *finite (tree) automata* into explicit ones. Other applications of this enriched language are to be found in the areas of *epistemic logic* (chapter five) and that of generalized quantifiers (chapter six).

Here, we are concerned with the expressibility of the graded modal language and we whink that the greater expressive power of the language (over that of modal logic) is appreciated when 'standard' modal techniques are applied to it in order to lay bare and discuss (a-)similarities with modal logic. We present some expressibility results of this logic which are already interesting on their own, but we consider the main contribution of our chapter lies in the use of adapted, or, in some cases new developed techniques that are provided to obtain those results.

In section 2 we define our base logic **GrK** and derive some properties. We also introduce its proper semantics and give some basic definitions. In section 3 we test our graded modal logic (GML) against some so called *reflection* and *preservation* operations known from modal logic (ML). It appears that, with some adjustments, most of the results remain valid.

We then use those results to establish some expressibility results for GML in sections 4 and 5. Also, we apply classical preservation results to argue that GML can distinguish more (properties of) frames than ML. In section 4, we give some examples of such properties, but we also show that there exist non-isomorphic frames which, graded modally, still cannot be told apart. We illustrate matters using binary frames and linear orders. We show that, even in this simple class of frames, GML is quite stronger than ML: although ML cannot distinguish between strict, weak and ordinary linear orders, the graded modal theories of all those classes are different. We also show that *within* the class of strict linear orders, however, the two languages are equally expressible.

Correspondence, yet another theme from classical modal logic, is studied in section 5. We show that more first-order properties become definable in our enriched system. However, section 5.1 shows that there are first-order properties that remain undefinable in GML. In section 5.2 we present a class of GML-formulas, of which the first-order corresponding formula can be obtained systematically. This class is a subclass of a (syntactically defined) set of ML-formulas for which such a result is known: the so called Sahlqvist formulas. A negative example shows that it is indeed a strict subclass, leaving the open question where the exact borders have to be drawn.

In section 6 we adapt the famous filtration technique of ML for our purposes. We provide a technique to filtrate a model for a given formula φ into a *finite* model for that formula, obtaining the finite model property (and hence, decidability) for several classes of models. Throughout the chapter, we mention some open problems and directions for further research.

2 Language and semantics

In this section we define the language L for our graded modal logic **GrK** and provide it with a semantics. L resembles very much that of ordinary modal logic. Its semantics is given by means of Kripke structures, which is also standard (cf. [Che80, HugCre68]). The main difference is that we have a possibility operator $\Diamond^{>n}$ for each $n \in \mathbb{N}$ with intended meaning of $\Diamond^{>n}\varphi$: "φ is true in more than n possible worlds".

Definition 2.1. The language L for our graded modalities is built according to the following rules:

(i) $\mathsf{P} = \{p, q, r, \ldots\} \subseteq L$
(ii) $\varphi \in L \;\Rightarrow\; \neg\varphi \in L$
(iii) $\varphi, \psi \in L \;\Rightarrow\; \varphi \vee \psi \in L$
(iv) $\varphi \in L \;\Rightarrow\; \Diamond^{>n}\varphi \in L$, for each $n \in \mathbb{N}$.

We also use brackets and standard abbreviations such as $(\varphi \to \psi) \equiv (\neg\varphi \vee \psi)$, $\bot \equiv (p \wedge \neg p)$, $\bot \equiv \neg\top$. Moreover, we add:

- $\Diamond^{\geqslant 0}\varphi \equiv \top$ and $\Diamond^{\geqslant n}\varphi \equiv \Diamond^{>(n-1)}\varphi$;
- $\Diamond^{\leqslant n}\varphi \equiv \neg\Diamond^{>n}\varphi$ and $\Diamond^{<n}\varphi \equiv \Diamond^{\leqslant(n-1)}\varphi$;
- $\Diamond^{=0}\varphi \equiv \Diamond^{\leqslant 0}\varphi$ and $\Diamond^{=n}\varphi \equiv \Diamond^{\leqslant n}\varphi \wedge \Diamond^{\geqslant n}\varphi$, for $n \geqslant 1$;
- $\Box^{\leqslant n}\varphi \equiv \Diamond^{\leqslant n}\neg\varphi$ and $\Box^{<n}\varphi \equiv \Diamond^{<n}\neg\varphi$.

With the intended meaning of $\Diamond^{>n}\varphi$ given, it is easily verified that the reading of $\Diamond^{=n}\varphi$ is "at exactly n accessible worlds, φ is the case", and of $\Box^{\leqslant n}\varphi$ is "in at most n accessible worlds, $\neg\varphi$ is the case" (or "φ is the case in all accessible worlds except for at most n of them"). We will refer to formulas of L as GML-formulas, as opposed to 'pure' modal formulas (ML-formulas). This terminology is self-explanatory and extends to notions such as GML (ML) theorems, GML (ML) properties, etc.

The semantics of L is based on Kripke structures $\langle W, R, \pi\rangle$ (see [Che80, HugCre68]), where $W \neq \varnothing$ is a set of worlds, $R \subseteq W \times W$ is an accessibility relation, and $\pi : W \to (\mathsf{P} \to \{\mathtt{tt}, \mathtt{ff}\})$ a valuation.

Definition 2.2. For a Kripke structure M we define the *truth of* φ at $w \in W$ inductively:

(i) $(M,w) \vDash p$ iff $\pi(w)(p) = \mathtt{tt}$, for all $p \in \mathsf{P}$;
(ii) $(M,w) \vDash \neg\varphi$ iff not $(M,w) \vDash \varphi$;
(iii) $(M,w) \vDash \varphi \vee \psi$ iff $(M,w) \vDash \varphi$ or $(M,w) \vDash \psi$;
(iv) $(M,w) \vDash \Diamond^{>n}\varphi$ iff $\left|\{x \in W\colon\ Rwx \text{ and } (M,x) \vDash \varphi\}\right| > n, \quad n \in \mathbb{N}$.

Definition 2.3. We say that φ is *true* in M at w if $(M,w) \vDash \varphi$. A formula φ is true in M ($M \vDash \varphi$) if $(M,w) \vDash \varphi$ for all $w \in W$, and φ is called *valid* ($\vDash \varphi$) if $M \vDash \varphi$ for all M. A tuple $F = \langle W, R\rangle$ is called a *frame*. $(F,w) \Vdash \varphi$ ($F \Vdash \varphi$) means that for all π, $(\langle F,\pi\rangle, w) \vDash \varphi$ ($\langle F,\pi\rangle \vDash \varphi$, respectively). When a model M (or frame F) is discussed and we argue about a valuation π or world w, it is assumed that $M = \langle W, R, \pi\rangle$ ($F = \langle W, R\rangle$) for some W and R, and that $w \in W$.

Remark 2.4. Note that $(M,w) \vDash \Box^{\leqslant n}\varphi$ iff $\left|\{x \in W\colon\ Rwx \text{ and } (M,x) \vDash \neg\varphi\}\right| \leqslant n$. The modal operators $\Diamond$ and $\Box$ are special cases of our indexed operators: $\Diamond\varphi \equiv \Diamond^{>0}\varphi$ and $\Box\varphi \equiv \Box^{\leqslant 0}\varphi$.

Definition 2.5. For a world w we define $R(w) = \{v \mid Rwv\}$. $R^n wv$ is inductively defined to be $(w = v)$ if $n = 0$ and $\exists z\,(R^{n-1}wz \,\&\, Rzv)$ if $n > 0$. We say that v is (R-)*reachable* from w if $R^n wv$ for some $n \geqslant 0$. The model *generated* by w, $\langle \overrightarrow{w} \rangle$, is the model containing all worlds that are reachable from w, and in which the valuation and accessibility relation are the restriction of the original valuation and relation, respectively, to those worlds.

The system **K** is known to be the weakest of all common modal systems. We call its graded analogue **GrK**, and give its axioms and rules (from now on, $n, m \in \mathbb{N}$).

Definition 2.6. The system **GrK** has the following axioms and inference rules.

A1 the axioms of propositional logic
A2 $\Diamond^{>n+1}\varphi \to \Diamond^{>n}\varphi$
A3 $\Box(\varphi \to \psi) \to (\Diamond^{>n}\varphi \to \Diamond^{>n}\psi)$
A4 $\neg\Diamond(\varphi \wedge \psi) \to \big((\Diamond^{=n}\varphi \wedge \Diamond^{=m}\psi) \to \Diamond^{=n+m}(\varphi \vee \psi)\big)$
R1 $\vdash \varphi, \vdash \varphi \to \psi \;\Rightarrow\; \vdash \psi$
R2 $\vdash \varphi \;\Rightarrow\; \vdash \Box\varphi$

To see the system in action, we derive some of its theorems.

Proposition 2.7. *The following are derivable in* **GrK**.

(1). $\Box(\varphi \to \psi) \to (\Box^{\leqslant n}\varphi \to \Box^{\leqslant n}\psi)$
(2). $\Diamond^{=n}\varphi \to \Diamond^{\leqslant m}\varphi \qquad (n \leqslant m)$
(3). $\Diamond^{=n}\varphi \to \neg\Diamond^{=m}\varphi \qquad (n \neq m)$
(4). $\Diamond^{\leqslant n}\varphi \leftrightarrow (\Diamond^{=0}\varphi \vee \ldots \vee \Diamond^{=n}\varphi)$
(5). $\Diamond^{>n}(\varphi \wedge \psi) \to (\Diamond^{>n}\varphi \wedge \Diamond^{>n}\psi)$
(6). $(\Diamond^{>n}\varphi \vee \Diamond^{>n}\psi) \to \Diamond^{>n}(\varphi \vee \psi)$
(7). $\Diamond^{>n+m}(\varphi \vee \psi) \to (\Diamond^{>n}\varphi \vee \Diamond^{>m}\psi)$
(8). $\Diamond^{\geqslant n}(\varphi \wedge \psi) \wedge \Diamond^{\geqslant m}(\varphi \wedge \neg\psi) \to \Diamond^{\geqslant n+m}\varphi$

Proof. 'sub' denotes the (derivable) rule of substitution: $\vdash \alpha \leftrightarrow \beta \Rightarrow \vdash \varphi \leftrightarrow \varphi[\alpha/\beta]$.

(1) Recall that $\Box^{\leqslant n}\varphi \equiv \neg\Diamond^{>n}\neg\varphi$. Then:
$\Box(\varphi \to \psi) \overset{\text{sub}}{\Leftrightarrow} \Box(\neg\psi \to \neg\varphi) \overset{\text{A3}}{\Rightarrow} (\Diamond^{>n}\neg\psi \to \Diamond^{>n}\neg\varphi) \overset{\text{A1}}{\Leftrightarrow} (\neg\Diamond^{>n}\neg\varphi \to \neg\Diamond^{>n}\neg\psi) \overset{\text{def}}{\Leftrightarrow} (\Box^{\leqslant n}\varphi \to \Box^{\leqslant n}\psi)$.

(2) $\Diamond^{=n}\varphi \overset{\text{def}}{\Leftrightarrow} (\Diamond^{\leqslant n}\varphi \wedge \Diamond^{\geqslant n}\varphi) \overset{\text{A1}}{\Rightarrow} \Diamond^{\leqslant n}\varphi \overset{\text{A2}}{\Rightarrow} \Diamond^{\leqslant m}\varphi$.

(3) Without loss of generality, $n < m$. Then:
$(\Diamond^{=n}\varphi \wedge \Diamond^{=m}\varphi) \overset{\text{A1}}{\Rightarrow} (\Diamond^{\leqslant n}\varphi \wedge \Diamond^{\geqslant m}\varphi) \overset{\text{def}}{\Leftrightarrow} (\neg\Diamond^{>n}\varphi \wedge \Diamond^{>(m-1)}\varphi) \overset{\text{A2}}{\Rightarrow} (\neg\Diamond^{>n}\varphi \wedge \Diamond^{>n}\varphi) \Rightarrow \bot$.

(4) Easy direction: $\Diamond^{=i}\varphi \to \Diamond^{\leqslant n}\varphi$, for any $i \leqslant n$, by (2). Hence $(\Diamond^{=0}\varphi \vee \ldots \vee \Diamond^{=n}\varphi) \to \Diamond^{\leqslant n}\varphi$.
Hard direction (we omit φ everywhere): we will use that $\neg\Diamond^{=0} \equiv \Diamond^{>0}$ and for $i > 0$:
$\neg\Diamond^{=i} \Leftrightarrow \neg(\Diamond^{\geqslant i} \wedge \Diamond^{\leqslant i}) \Leftrightarrow (\neg\Diamond^{\geqslant i} \vee \neg\Diamond^{\leqslant i}) \Leftrightarrow (\neg\Diamond^{>(i-1)} \vee \Diamond^{>i}) \Leftrightarrow (\Diamond^{>(i-1)} \to \Diamond^{>i})$. Now we have:
$\neg(\Diamond^{=0} \vee \ldots \vee \Diamond^{=n}) \overset{\text{A1}}{\Leftrightarrow} (\neg\Diamond^{=0} \wedge \ldots \wedge \neg\Diamond^{=n}) \Leftrightarrow \Diamond^{>0} \wedge (\Diamond^{>0} \to \Diamond^{>1}) \wedge \ldots \wedge (\Diamond^{>(n-1)} \to \Diamond^{>n}) \Rightarrow \Diamond^{>n} \Leftrightarrow \neg\Diamond^{\leqslant n}$.

(5) Apply (A3) to $\Box((\varphi \wedge \psi) \to \varphi)$ and $\Box((\varphi \wedge \psi) \to \psi)$.

(6) Apply (A3) to $\Box(\varphi \to (\varphi \vee \psi))$ and $\Box(\psi \to (\varphi \vee \psi))$.

(7) By contraposition, we shall prove: $(\Diamond^{\leqslant n}\varphi \wedge \Diamond^{\leqslant m}\psi) \to \Diamond^{\leqslant n+m}(\varphi \vee \psi)$. We have:
$\Diamond^{\leqslant n}\varphi \Rightarrow_{(4)} \Diamond^{=k}\varphi$, for some $k \leqslant n$ (a).
$\Diamond^{\leqslant m}\psi \Rightarrow_{(A3)} \Diamond^{\leqslant m}(\neg\varphi \wedge \psi) \Rightarrow_{(4)} \Diamond^{=\ell}(\neg\varphi \wedge \psi)$, for some $\ell \leqslant m$ (b).
Since φ and $\neg\varphi \wedge \psi$ are incompatible: $\neg\Diamond(\varphi \wedge (\neg\varphi \wedge \psi))$, from ($a$) and ($b$) by (A4) we infer: $\Diamond^{=k+\ell}(\varphi \vee (\neg\varphi \wedge \psi))$, or equivalently, $\Diamond^{=k+\ell}(\varphi \vee \psi)$. Finally, by (2), we obtain $\Diamond^{\leqslant n+m}(\varphi \vee \psi)$.

(8) Suppose (a) $\Diamond^{\geqslant n}(\varphi \wedge \psi)$ and (b) $\Diamond^{\geqslant m}(\varphi \wedge \neg\psi)$. For the sake of contradiction, assume $\Diamond^{<n+m}\varphi$.
$\Diamond^{<n+m}\varphi \Rightarrow_{(5)} \Diamond^{<n+m}(\varphi \wedge \psi) \Rightarrow_{(4)} \Diamond^{=k}(\varphi \wedge \psi)$, for some $k < n + m$; by (a), $k \geqslant n$.
Similarly, $\Diamond^{=\ell}(\varphi \wedge \neg\psi)$, for some $\ell \geqslant m$. Since $(\varphi \wedge \psi)$ and $(\varphi \wedge \neg\psi)$ are incompatible, by (A4) we infer: $\Diamond^{=k+\ell}((\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi))$, or equivalently, $\Diamond^{=k+\ell}\varphi$. Since $k + \ell \geqslant n + m$, this implies $\Diamond^{\geqslant n+m}\varphi$ by (A2). This contradicts to our assumption and hence proves the claim. □

Note that we can indeed consider **GrK** to be an extension of **K**, since we have all propositional tautologies (A1), Necessitation (R2) and the **K**-axiom (Proposition 2.7(1) with $n = 0$). We end this introduction into graded modalities by stating that the logic and its semantics perfectly fit:

Theorem 2.8 (Completeness, [Fin72, FatCar85])**.** *For any* GML*-formula* φ*, we have:*
GrK $\vdash \varphi$ *iff* $\vDash \varphi$.

3 Elementary model theory: preservation

An important tool when studying elementary equivalences (section 4) and deriving characterization results (section 5) is the property of *preservation*. It helps us to derive definability results.

Definition 3.1. Let $\mathrm{Th}_{\mathsf{GML}}(M) = \{\varphi \in L \mid M \vDash \varphi\}$ be the *graded modal theory* of a model M and $\mathrm{Th}_{\mathsf{ML}}(M)$ the *modal theory* of M.

Definition 3.2. We say that a relation Q between models *preserves* GML*-validity* if $(QM_1M_2 \Rightarrow \mathrm{Th}_{\mathsf{GML}}(M_1) \subseteq \mathrm{Th}_{\mathsf{GML}}(M2))$. If, instead of '$\subseteq$', '$\supseteq$' holds, we have *anti-preservation* (or *reflection*).

These definitions easily extend to frames and to ML-formulas. We start out with an easy example of a preserving and reflecting relation (we omit the proof by induction of the complexity of formulas).

Fact 3.3. $\cong$ *(being isomorphic with) preserves and anti-preserves* GML*-validity.*

An example of an ML-preserving operation is provided by the notion of p-morphism, as defined in section 2.13 of chapter 3. Theorem 2.14 of chapter 3 guarantees that, for models M and M' of figure 1, and all modal formulas φ, we have $(M, w) \vDash \varphi$ iff $(M', w') \vDash \varphi$. That p-morphisms do not preserve *graded* modal formulas is seen by observing that in w, $\Diamond^{>1}\top$ is true, whereas it is not so in w'.

In [Hoe91d], we generalized this notion of p-morphism to a notion of p^+-morphism, which does preserve truth of graded modal formulas. Here, we will generalize the notion of *bisimulation* (cf. [Ben83]) to the graded language, which is slightly more general than that of p-morphism.

Definition 3.4. For $n \in \mathbb{N}$, we use the following abbreviations:

$$\begin{array}{lcl} \exists^{\neq} x_1 \ldots x_n\, \varphi & \equiv & \exists x_1 \ldots x_n \left(\bigwedge_{1\leqslant i\neq j\leqslant n}(x_i \neq x_j) \;\;\&\;\; \varphi\right) \\ \forall^{\neq} x_1 \ldots x_n\, \varphi & \equiv & \forall x_1 \ldots x_n \left(\bigwedge_{1\leqslant i\neq j\leqslant n}(x_i \neq x_j) \;\;\to\;\; \varphi\right) \end{array}$$

Definition 3.5. Suppose that $Z \subseteq W \times W'$ is a relation between worlds of the models $M = \langle W, R, \pi\rangle$ and $M' = \langle W', R', \pi'\rangle$.

(i). Z satisfies *n-forth choice* if

$$\forall x \in W\, \forall x' \in W'\;\; \forall^{\neq} y_0 \ldots y_n \in W \Big((Zxx' \wedge \bigwedge_{0\leqslant i\leqslant n} Rxy_i) \;\Rightarrow\; \exists^{\neq} y'_0 \ldots y'_n \in W' \bigwedge_{0\leqslant i\leqslant n} (Zy_iy'_i \wedge R'x'y'_i) \Big)$$

(ii). Z satisfies *n-back choice* if

$$\forall x \in W\, \forall x' \in W'\;\; \forall^{\neq} y'_0 \ldots y'_n \in W' \Big((Zxx' \wedge \bigwedge_{0\leqslant i\leqslant n} R'x'y'_i) \;\Rightarrow\; \exists^{\neq} y_0 \ldots y_n \in W \bigwedge_{0\leqslant i\leqslant n} (Zy_iy'_i \wedge Rxy_i) \Big)$$

(iii). Z is called GML*-bisimulation* between M and M' if it satisfies n-forth choice and n-back choice for all $n \in \mathbb{N}$, and moreover that $\forall x \in W\, \forall x' \in W' (Zxx' \Rightarrow \pi(x) = \pi'(x'))$. We say that M and M' GML*-bisimulate* each other, $M \approx M'$, if there exists a GML-bisimulation between them (as is easily see, if Z is a GML-bisimulation, then so is Z^{-1}). We then also say that M and M' are GML*-bisimulated by* Z. The relation Z is a GML*-bisimulation* between frames $F = (W, R)$ and $F' = (W', R')$ if it satisfies both n-forth choice and n-back choice for all $n \in \mathbb{N}$.

(iv). A GML-bisimulation Z that moreover satisfies that $\mathsf{domain}(Z) = W$ and $\mathsf{range}(Z) = W'$ is called a GML *zigzag connection* between M and M', and we write $M \leftrightarrows M'$ if such a relation exists between M and M'. In this dissertation, if Z is a function, we will call it a p^+*-morphism* (in fact, the p^+-morphisms of [Hoe91d] are a special case of them). These notions also apply to frames.

The notion of zigzag connection (which is the relevant notion for standard modal logic) is also called a *p-relation* (cf. [Ben83]). It was Segerberg who introduced the notion of p-morphism for standard modal logic ([Seg70a]). If M and N are GML-bisimulated by B, then the condition on the valuations, together with the fact that from any two worlds w and v for which Bwv holds, we can go to the same number of accessible worlds (that are identified by B), yields the following result:

Theorem 3.6. *If models M and M' are GML-bisimulated by Z, then, for all graded modal formulas φ and all $w \in W$ and $w' \in W'$ with Zww', $M, w \vDash \varphi$ iff $M', w' \vDash \varphi$.*

Proof. Suppose Bwv. If $\varphi = p \in \mathsf{P}$, the theorem follows immediately from 3.5(iii). For $\varphi = \neg\psi$ or $\varphi = \psi \vee \chi$, the induction hypothesis is applied straightforwardly. Assume $\varphi = \Diamond^{>n}\psi$, and the theorem proven for ψ. ($\Rightarrow$) If $M, w \vDash \Diamond^{>n}\psi$, then w has distinct R-successors $w_0, \ldots, w_n$ such that $M, w_i \vDash \psi$, $i = 0 \ldots n$. Since B satisfies *n-forth choice*, there are distinct S-successors $v_0, \ldots, v_n$ of v, in which, using induction, ψ is true. So, $N, v \vDash \Diamond^{>n}\psi$. The $\Leftarrow$ part is proven in the same way, using *n-back choice.* □

Corollary 3.7. *Suppose $f{:}\, M \to M'$ is a p^{+}-morphism of models.*
Then, for all graded modal formulas φ, $M \vDash \varphi \;\Rightarrow\; N \vDash \varphi$.

Proof. If $M' \nvDash \varphi$, there is some $w' \in W'$ such that $M', w' \vDash \neg\varphi$. Since f is by definition surjective, we can find a w for which $f(w) = w'$ and , using 3.6, $M, w \vDash \neg\varphi$, i.e., $M \nvDash \varphi$. □

Example 3.8. Let $F = \langle \mathbb{N}, \{\langle n, n{+}1\rangle \mid n \in \mathbb{N}\}\rangle$ and $F' = \langle\{0,1\}, \{\langle 0,1\rangle, \langle 1,0\rangle\}\rangle$. The function f defined by $f(n) = n \bmod 2$ is a p^{+}-morphism from F onto F', not an isomorphism. Note that the mapping f of figure 1 (see discussion after 3.3) is (indeed) not a p^{+}-morphism.

Corollary 3.9. *Suppose $f{:}\, F \to F'$ is a p^{+}-morphism of frames. Then, for all GML formulas φ:*
(i) *For all $w \in W$, $F, w \Vdash \varphi \;\Rightarrow\; F', f(w) \Vdash \varphi$*
(ii) $F \Vdash \varphi \;\Rightarrow\; F' \Vdash \varphi$

Remark 3.10. Although not all p^{+}-morphisms are isomorphisms, on the class of finite transitive generated frames, the two concepts are equivalent. To see this, note that if F is generated by w, and $f{:}\, F = (W, R) \to F' = (W', R')$ is a p^{+}-morphism, then f must be a bijection between w's successors and $f(w)$'s successors. However, since F is transitive and generated by w, the set of w's successors is just W. We leave it to reader to verify that F' is transitive and generated by $f(w)$. Hence f is a bijection between $W = \langle\overrightarrow{w}\rangle$ and $\langle\overrightarrow{f(w)}\rangle = W'$.

We can use 3.6 to establish related preservation results for the graded language.

Definition 3.11. $M = \langle W, R, \pi\rangle$ is a *generated submodel* of $M' = \langle W', R', \pi'\rangle$, $M \hookrightarrow M'$, if $W \subseteq W'$ and $\forall x \in W\ \forall y \in W'$: $R'xy \Rightarrow y \in W$. Moreover, $R = R'|_W$ (the restriction of R' to W) and $\pi = \pi'|_W$. This definition naturally extends to frames.

Theorem 3.12. *Suppose that $M \hookrightarrow M'$, $w \in W$ and $\varphi \in L$. Then:*
(i) $M, w \vDash \varphi \;\Leftrightarrow\; M', w \vDash \varphi$
(ii) $M' \vDash \varphi \;\Rightarrow\; M \vDash \varphi$

Proof. For item *(i)*, observe that $\{\langle w, w\rangle \mid w \in W\}$ is a GML-bisimulation between M and M', and use 3.6. Item *(ii)* then follows easily. □

Corollary 3.13. *Let $F \hookrightarrow F'$, $w \in W$ and $\varphi \in L$. Then:*
(i) $F, w \Vdash \varphi \;\Leftrightarrow\; F', w \Vdash \varphi$
(ii) $F' \Vdash \varphi \;\Rightarrow\; F \Vdash \varphi$

Definition 3.14. Let $\{M_i = \langle W_i, R_i, \pi_i\rangle \mid i \in I\}$ be a set of models and $M'_i = \langle W'_i, R'_i, \pi'_i\rangle$, with $W'_i = \{\langle i, w\rangle \mid w \in W_i\}$, $R'_i = \{(\langle i, w\rangle, \langle i, v\rangle) \mid R_i wv\}$ and $\pi'_i(\langle i, w\rangle) = \pi_i(w)$. The *disjoint union* $\biguplus_{i\in I} M_i$ of models $\{M_i \mid i \in I\}$ is the model $\langle \bigcup_{i\in I} W'_i, \bigcup_{i\in I} R'_i, \bigcup_{i\in I} \pi'_i\rangle$. The *disjoint union* $\biguplus_{i\in I} F_i$ of frames $\{F_i \mid i \in I\}$ is the frame $\langle \bigcup_{i\in I} W'_i, \bigcup_{i\in I} R'_i\rangle$.

Theorem 3.15. *Let $\{M_i \mid i \in I\}$ be as above, $w \in W_i$ for some $i \in I$, and $\varphi \in L$. Then:*
(i) $\biguplus_{i\in I} M_i, \langle i, w\rangle \vDash \varphi \;\Leftrightarrow\; M_i, w \vDash \varphi$
(ii) $\biguplus_{i\in I} M_i \vDash \varphi \;\Leftrightarrow\;$ *for all* $i \in I$*:* $M_i \vDash \varphi$

Proof. Observe that $M_i \cong M'_i$ and $M'_i \hookrightarrow \biguplus_{i\in I} M_i$. Now apply 3.3 and 3.12. □

Corollary 3.16. $\biguplus_{i\in I} F_i \Vdash \varphi \;\Leftrightarrow\;$ *for all* $i \in I$*:* $F_i \Vdash \varphi$.

Isomorphisms preserve GML-validity, but for modal logic we need not impose such a stringent condition on mappings between frames to achieve this. In the literature, ([Seg71a]), a weakening of isomorphisms to pseudo isomorphisms (or simply p-morphisms) is known which already is sufficient.

In the literature, one more anti-preservation result for modal formulas is known: that of ultrafilter extensions (we refer to [ChaKei73] for the definition of ultrafilter). Although (in [Hoe91d]) we were able, by using some ad hoc arguments, to prove that taking the ultrafilter extension of the frame $\mathbb{N}$ anti-preserves graded modal formulas, as far as we know, the question about anti-preservation in general (for the graded language) has not been settled yet.[1]

4 Expressive power 1: graded modal equivalence

We use the results of section 3 in obtaining expressibility results for **GrK** in section 4 and 5, using the following general pattern: suppose we have a frame $G = \langle W, R\rangle$ satisfying some property A of the accessibility relation R, and a frame $G' = \langle W', R'\rangle$ that is obtained from G using some GML-validity preserving operation, and of which R' does not satisfy A. Then there is no way we can define A in our graded language. For, the assumption

$$\text{for all frames } F\text{:}\quad F \Vdash \varphi \text{ iff } F \text{ satisfies } A(R) \qquad (*)$$

justifies the following argument: if G satisfies $A(R)$ then, by $(*)$, $G \Vdash \varphi$. Since G' is obtained from G using some validity preserving operation, we have $G' \Vdash \varphi$. Using $(*)$ once more, we conclude that G' satisfies $A(R')$ — a contradiction.

From the preservation theorem for p-morphisms (cf. theorem 3.14, chapter 3) we know that the existence of a p-morphism $f\colon F \to F'$ implies $\mathrm{Th}_{\mathsf{ML}}(F) \subseteq \mathrm{Th}_{\mathsf{ML}}(F')$. The frames F_1 and F_2 based on the models of figure 1 then are examples of frames for which $\mathrm{Th}_{\mathsf{ML}}(F_1) \subseteq \mathrm{Th}_{\mathsf{ML}}(F_2)$, but at the same time $\mathrm{Th}_{\mathsf{GML}}(F_1) \nsubseteq \mathrm{Th}_{\mathsf{GML}}(F_2)$. This implies that using graded modalities, more frames can be distinguished than in traditional modal logic.

Definition 4.1. Two frames F and F' are *graded modally equivalent*, $F \equiv_{\mathsf{GML}} F'$, if $\mathrm{Th}_{\mathsf{GML}}(F) = \mathrm{Th}_{\mathsf{GML}}(F')$. If $\mathrm{Th}_{\mathsf{GML}}(F) \subseteq \mathrm{Th}_{\mathsf{GML}}(F')$, we say that F and F' are GML-*comparable*. Analogously, we define *modal equivalence* ($F \equiv_{\mathsf{ML}} F'$ if $\mathrm{Th}_{\mathsf{ML}}(F) = \mathrm{Th}_{\mathsf{ML}}(F')$) and ML-*comparability* ($\mathrm{Th}_{\mathsf{ML}}(F) \subseteq \mathrm{Th}_{\mathsf{ML}}(F')$). The definitions can also be applied to models.

So we can rephrase the above observation by saying that the two frames F_1 and F_2 are ML-comparable. Since, moreover, $\mathrm{Th}_{\mathsf{GML}}(F_1) \nsubseteq \mathrm{Th}_{\mathsf{GML}}(F_2)$, (this will become obvious in the following subsection) they are not GML-comparable.

Some interesting questions now immediately break surface. For instance, what does it mean for two frames to have the same theory? Are there modally equivalent frames that are not graded

[1] Typist's comment: it has been settled in a paper in AiML 2010.

modally equivalent? If so, are there finite examples with that property? We will decide on this in this section, showing how the results of section 3 can be fruitfully used here. We start out by comparing ordinary equivalence with graded modal equivalence. Besides some simple finite frames, the class of binary frames is the scenery in which we settle some questions. The restriction to binary frames is not essential, but is rather made because of its conceptual simplicity and mathematical neatness. In fact, in chapter 3 we already gave an example of the expressive power of the graded language. At the end of this section, we will focus on a (mathematically) interesting subclass of all frames: that of linear orders, which, on it turn, is important when studying logics of time (cf. [Ben82]).

On the class of *finite frames*, $\equiv_{\mathsf{GML}}$ and even $\equiv_{\mathsf{ML}}$ appear to be quite strong properties:

Theorem 4.2 ([Ben84a]). *Let F and F' be finite, and generated by one element. Then:*
$F \cong F'$ iff $F \equiv_{\mathsf{ML}} F'$.

Corollary 4.3. *If F and F' are finite and generated by one element, then:*
$F \cong F'$ iff $F \equiv_{\mathsf{GML}} F'$.

Proof. '⇒' is obvious; for '⇐', it is sufficient to notice that $F \equiv_{\mathsf{GML}} F'$ implies $F \equiv_{\mathsf{ML}} F'$. □

We will see that there are non-isomorphic (even point-generated) frames that still cannot be told apart, graded modally. Corollary 4.3 implies that they should be infinite. Another consequence of 4.2 and 4.3 is that if we are searching for possible frames that are ML-equivalent, but not GML-equivalent, they have to be either not point-generated or infinite. It will turn out that in both cases (4.4 and 4.6, respectively) such frames exist. We start by giving an example of two finite frames.

Theorem 4.4. *There exist finite frames G and H for which the following hold:*
(a) $G \equiv_{\mathsf{ML}} H$, but also (b) $G \not\equiv_{\mathsf{GML}} H$.

Proof. Consider the following frames:

$$\begin{array}{rcl} F & = & \langle\, \{a,b\},\ \{\langle a,b\rangle\}\,\rangle \\ G & = & \langle\, \{x,y,z\},\ \{\langle x,y\rangle,\langle x,z\rangle\}\,\rangle \\ H & = & F \uplus G \end{array}$$

There is a p-morphism $G \to F$, hence $\mathrm{Th}_{\mathsf{ML}}(G) \subseteq \mathrm{Th}_{\mathsf{ML}}(F)$. Then:

$$\mathrm{Th}_{\mathsf{ML}}(H) \ = \ \mathrm{Th}_{\mathsf{ML}}(F) \cap \mathrm{Th}_{\mathsf{ML}}(G) \ = \ \mathrm{Th}_{\mathsf{ML}}(G),$$

and thus $G \equiv_{\mathsf{ML}} H$. At the same time, $G \not\equiv_{\mathsf{GML}} H$, since $\Diamond^{=0}\top \vee \Diamond^{=2}\top$ is valid in G, not in H. □

In the above proof $G \hookrightarrow H$, hence $\mathrm{Th}_{\mathsf{GML}}(H) \subsetneq \mathrm{Th}_{\mathsf{GML}}(G)$, and there is no p^{+}-morphism $G \to F$.

Definition 4.5. For each $n \in \mathbb{N}$, we define $\mathbb{F}^n = \{F \mid F \Vdash \Diamond^{=n}\top\}$ the class of frames in which every element has exactly n successors. We call $\mathbb{F}^2$ the class of *binary frames*. A special binary frame in $\mathbb{F}^2$ is $F_{\mathrm{bin}} = \langle W_{\mathrm{bin}}, R_{\mathrm{bin}}\rangle$ which can be viewed as a binary tree: it has a generating element *root* ($= \varepsilon$) and each element has one unique predecessor. (See figure 3). Formally, $W_{\mathrm{bin}} = \{0,1\}^*$, i.e., all sequences containing 0's and 1's, including the empty sequence ε. Finally, $R_{\mathrm{bin}} = \{\langle x,y\rangle \mid y = x0 \text{ or } y = x1\}$.

First, we use F_{bin} to discriminate between $\equiv_{\mathsf{ML}}$ and $\equiv_{\mathsf{GML}}$:

Theorem 4.6. *There exist point-generated frames F_1 and F_2 for which:*
$F_1 \equiv_{\mathsf{ML}} F_2$, but $F_1 \not\equiv_{\mathsf{GML}} F_2$.

Proof. Let $F_1 = F_{\mathrm{bin}}$ and let F_2 be F_{bin} preceded by an element w. That is, we extend F_{bin} with a world w and stipulate that $R_2 = R_{\mathrm{bin}} \cup \{(w, \mathit{root})\}$. Then $\mathrm{Th}_{\mathsf{ML}}(F_2) \subseteq \mathrm{Th}_{\mathsf{ML}}(F_1)$ since $F_1 \hookrightarrow F_2$, and $\mathrm{Th}_{\mathsf{ML}}(F_1) \subseteq \mathrm{Th}_{\mathsf{ML}}(F_2)$ by the existence of a p-morphism $f\colon F_1 \to F_2$: $f(\varepsilon) = w$, $f(0x) = f(1x) = x$ for all $x \in \{0,1\}^*$. However, $\mathrm{Th}_{\mathsf{GML}}(F_1) \nsubseteq \mathrm{Th}_{\mathsf{GML}}(F_2)$, for which $\Diamond^{=2}\top$ is a witness. $(*)$ □

The latter argument, $(*)$, is easily generalized: if $F_1 \in \mathbb{F}^n$ and $F_2 \notin \mathbb{F}^n$ for some $n \in \mathbb{N}$, then $\mathrm{Th}_{\mathsf{GML}}(F_1) \nsubseteq \mathrm{Th}_{\mathsf{GML}}(F_2)$. We cannot reverse this observation. In terms of $\mathbb{F}^2$ again, not all binary frames possess the same graded modal theory. There are $F_1, F_2 \in \mathbb{F}^2$ for which $\mathrm{Th}_{\mathsf{GML}}(F_1) \nsubseteq \mathrm{Th}_{\mathsf{GML}}(F_2)$. This is shown in 4.7(b). Theorem 4.7(a) shows that F_{bin} cannot be graded modally defined. According to 4.7(c), F_{bin} is the 'weakest' frame in $\mathbb{F}^2$.

Theorem 4.7.

(a) *There exists a frame* $F_1 \ncong F_{\mathrm{bin}}$ *with* $F_1 \equiv_{\mathsf{GML}} F_{\mathrm{bin}}$.

(b) *There are frames* $F_1, F_2 \in \mathbb{F}^2$ *for which* $F_1 \not\equiv_{\mathsf{GML}} F_2$.

(c) $F \in \mathbb{F}^2 \;\Leftrightarrow\; \mathrm{Th}_{\mathsf{GML}}(F_{\mathrm{bin}}) \subseteq \mathrm{Th}_{\mathsf{GML}}(F)$.

Proof. Let F_1 and F_2 be as in figure 4.

(a). The reader is encouraged to verify that there exists a p^+-morphism $f: F_{\mathrm{bin}} \to F_1$ and that also $F_{\mathrm{bin}} \hookrightarrow F_1$. Although obviously $F_1 \ncong F_{\mathrm{bin}}$, we can use 3.9 and 3.13(*ii*) to conclude $F_1 \equiv_{\mathsf{GML}} F_{\mathrm{bin}}$.

(b). Note that $\Diamond\Diamond^{>1}p \to \Diamond^{>1}\Diamond p \in \mathrm{Th}_{\mathsf{GML}}(F_2) \setminus \mathrm{Th}_{\mathsf{GML}}(F_{\mathrm{bin}})$.

(c). If $\varphi \notin \mathrm{Th}_{\mathsf{GML}}(F)$, there is some π and w such that $\langle F, \pi\rangle, w \vDash \neg\varphi$. Since $F \notin \mathbb{F}^2$, we can 'unravel' the frame F along F_{bin}, in a way that *root* validates the same graded modal formulas as w. Let us denote this unravelling with f. Then $f(w) = \mathit{root}$, and if $f(v) = x \in F_{\mathrm{bin}}$, we map the R-successors of v onto the R_{bin}-successors of x. The valuation π' on F_{bin} is defined by $\pi'(f(v)) = \pi(v)$. A simple induction shows $\langle F_{\mathrm{bin}}, \pi'\rangle, \mathit{root} \vDash \neg\varphi$. □

Corollary 4.8. *From 4.7(a) we see that 4.3 is generally not true for infinite frames: even for point-generated frames,* $\equiv_{\mathsf{GML}}$ *does not imply* $\cong$.

A natural question now is, whether the frames F, for which $F \equiv_{\mathsf{GML}} F_{\mathrm{bin}}$, can be characterized. The positive answer is to be found in [Hoe91d].

Now we make a move to the realm of linear orders which have an incessant appeal on mathematics, both for their simplicity and wide applicability. Also in modal logic it makes sense to study them, especially in logic for time. A systematic modal approach to linear orders is to be found in [Ben82]. This section is, although much briefer and sketchier, very much inspired by it. Suggestively, we will write '$\leqslant$' for the linear order, and '$<$' if it is strict.

Definition 4.9. Let $\mathbb{F}^{\mathrm{lo}}$ be the class of frames of which the accessibility relation is a linear order and $\mathbb{F}^{\mathrm{slo}}$ the class of strictly linear frames. Finally, $\mathbb{F}^{\mathrm{wlo}}$ is the class of weak linear orders: here, the relation need not be anti-symmetric.

Perhaps the most appealing frames in $\mathbb{F}^{\mathrm{lo}}$ are $\langle \mathbb{N}, <\rangle$ and $\langle \mathbb{N}, \leqslant\rangle$. However, they appear to be rather rigid structures, witnessing the following theorems.

Theorem 4.10 ([Hoe91d])**.** *If* $f: \langle \mathbb{N}, <\rangle \to \langle \mathbb{N}, <\rangle$ *is a p-morphism, then* $f \equiv I$ *(identity).*

Of course, there are frames $\langle W,R\rangle$ that are non-trivial p-morphic image of $\langle \mathbb{N},<\rangle$, like $\langle\{0\},\{\langle 0,0\rangle\}\rangle$. However, p^+-morphisms even exclude this:

Theorem 4.11. *If* $f: \langle \mathbb{N}, <\rangle \to \langle W, R\rangle$ *is a* p^+*-morphism, then* $\langle W, R\rangle \cong \langle \mathbb{N}, <\rangle$ *and* $f \equiv I$.

Proof. We recall that transitivity, seriality and linearity are modally definable. Then, by 3.9(*ii*), R must also have these properties. So, $\langle W, R\rangle$ is a linear structure, which possibly contains some clusters (in which the relation is symmetric, and whence universal), cf. figure 5.

We consider the following a priori possibilities.

- R contains clusters C somewhere 'halfway'. This cluster must be finite, and so there is a greatest $n \in \mathbb{N}$ with $f(n) = c \in C$. By definition of cluster, Rcc, so by definition of p^+-morphism, there must be an $m \in \mathbb{N}$ with $f(m) = c$ and $m > n$, contradicting our assumption about n. Conclusion: if $\langle W, R\rangle$ contains a cluster, it must be at the end.

- Assuming finiteness of this final cluster, it would yield finiteness of W. However, $(\mathbb{N},<) \Vdash \Diamond^{>n}\top$, and since f is a p^+-morphism, we would have $(W,R) \Vdash \Diamond^{>n}\top$ for all $n \in \mathbb{N}$, which obviously contradicts the finiteness of W.

- $\langle W,R\rangle$ has an infinite cluster, possibly R-preceded by a finite initial fragment. Without loss of generality, we assume that this fragment is empty. f is surjective, so W is countable. This situation does allow p-, but not p^+-morphisms: let us write $0', 1', 2', \ldots$ for the members of W. The function $f\colon \mathbb{N} \to W$ that assigns $0', 1', 0', 1', 2', 0', \ldots$ to the natural numbers $0, 1, 2, 3, 4, 5, \ldots$ is a p-morphism. However, using 3.9, there cannot be any p^+-morphism: $(\Diamond^{=1}p \to \Diamond\Box\neg p) \in \mathrm{Th}_{\mathsf{GML}}(\mathbb{N},<) \smallsetminus \mathrm{Th}_{\mathsf{GML}}(W,R)$. □

If we drop the strictness condition on $\langle\mathbb{N},<\rangle$, theorem 4.10 does not hold for p-morphisms. But still then, p^+-morphisms are more rigid, according to theorem 4.12.

Theorem 4.12.
(a) *Not all p-morphisms* $f\colon\langle\mathbb{N},\leqslant\rangle \to \langle\mathbb{N},\leqslant\rangle$ *equal* I.
(b) *If* $f\colon\langle\mathbb{N},\leqslant\rangle \to \langle\mathbb{N},\leqslant\rangle$ *is a* p^+*-morphism, then* $f \equiv I$.

Proof. An example for (a) is $f(n) = n/2$ if n is even, $(n-1)/2$ else. (b) follows from 4.11. □

Let us leave $\mathbb{N}$ as it is and turn to the classes $\mathbb{F}^{\mathrm{wlo}}$ and $\mathbb{F}^{\mathrm{slo}}$. Using an argument of Segerberg ([Seg71a]), one can 'bulldoze' a symmetric cluster into asymmetric orders (with the same modal theory), so that:

Theorem 4.13 ([HugCre84])**.** $\mathrm{Th}_{\mathsf{ML}}(\mathbb{F}^{\mathrm{lo}}) \subseteq \mathrm{Th}_{\mathsf{ML}}(\mathbb{F}^{\mathrm{wlo}})$. *And hence,* $\mathrm{Th}_{\mathsf{ML}}(\mathbb{F}^{\mathrm{lo}}) = \mathrm{Th}_{\mathsf{ML}}(\mathbb{F}^{\mathrm{wlo}})$.

For GML, this bulldozing technique does not work, because it does not leave the number of successors the same. Let us illustrate this by a simple example. Consider the frame F and its 'bulldozed' result f' of figure 6 (R is the transitive closure of the relation denoted with arrows). If the valuations of F' respect those of F, it is obvious how a modal formula, true in a model on F, can be transferred to F'. For instance, if ML-formula $\varphi = \Box((p \to \Diamond\neg p) \wedge (\neg p \to \Diamond p))$ is true at w, we get an alternating sequence of p and $\neg p$ worlds in F'. However, it is impossible to do this with the GML-formula $\varphi \wedge \Diamond^{=2}\top$, showing that the bulldozing technique is inadequate for GML. Indeed, in section 5.1, we shall show that $\mathrm{Th}_{\mathsf{GML}}(\mathbb{F}^{\mathrm{lo}}) \nsubseteq \mathrm{Th}_{\mathsf{GML}}(\mathbb{F}^{\mathrm{wlo}})$.

Now we will show that in $\mathbb{F}^{\mathrm{slo}}$ graded modalities do not supply a greater expressive power than modalities (in section 5 we will see, that GML *is* important in *defining* $\mathbb{F}^{\mathrm{slo}}$, though).

Example 4.14. Consider $F = \langle W,R\rangle \in \mathbb{F}^{\mathrm{slo}}$ and suppose that $\langle F,\pi\rangle, w \vDash \Diamond^{>1}p$. Then w is accessible to at least two p-worlds. Since R is antisymmetric, linear and transitive, it follows that one of those worlds is reached *before* the other, whence $\langle F,\pi\rangle, w \vDash \Diamond(p \wedge \Diamond p)$. Conversely, if $\Diamond(p \wedge \Diamond p)$ is true at w, we use transitivity and irreflexivity of R to conclude that $\langle F,\pi\rangle, w \vDash \Diamond^{>1}p$. We thus have a way to translate GML-formulas to equivalent ML-formulas.

Definition 4.15. We define the translation $T\colon \mathsf{GML} \to \mathsf{ML}$ as follows:
- $T(p) = p$, for propositional atoms p;
- $T(\neg\varphi) = \neg T(\varphi)$, $T(\varphi \vee \psi) = T(\varphi) \vee T(\psi)$;
- $T(\Diamond^{>0}\varphi) = \Diamond T(\varphi)$ and $T(\Diamond^{>n+1}\varphi) = \Diamond(T(\varphi) \wedge T(\Diamond^{>n}\varphi))$.

Theorem 4.16. *Let* $F = \langle W,R\rangle \in \mathbb{F}^{\mathrm{slo}}$ *and* T *as above. Then for all* GML*-formulas* φ*:*
(*i*) *for all* π *and* w*:* $\langle F,\pi\rangle, w \vDash \varphi$ *iff* $\langle F,\pi\rangle, w \vDash T(\varphi)$*;*
(*ii*) *for all* π*:* $\langle F,\pi\rangle \vDash \varphi$ *iff* $\langle F,\pi\rangle \vDash T(\varphi)$*;*
(*iii*) $F \Vdash \varphi$ *iff* $F \Vdash T(\varphi)$.

Proof. Immediate (as in example 4.14, note that $T(\Diamond^{>1}p) = \Diamond(p \wedge \Diamond p)$), using the definition of translation T and of strict linear order. □

We conclude this section by giving two examples of orders on the cartesian product $F_1 \times F_2$, where F_1 and F_2 are linear orders. In the first (4.18), GML appears to be richer, in the other (4.19), we use 4.16 to show that ML is as adequate as GML here.

Definition 4.17. Let $F_1 = \langle W_1, R_1 \rangle$ and $F_2 = \langle W_2, R_2 \rangle$ be two linear orders. The *direct product* of F_1 and F_2 is defined as $F_1 \,\&\, F_2 = \langle W_1 \times W_2, R \rangle$, where $R(x_1,x_2)(y_1,y_2)$ iff both $R_1 x_1 y_1$ and $R_2 x_2 y_2$. The *lexicographical product* of F_1 and F_2 is defined as $F_1 \odot F_2 = \langle W_1 \times W_2, R \rangle$, where $R(x_1,x_2)(y_1,y_2)$ iff $R_1 x_1 y_1$ or ($x_1 = y_1$ and $R_2 x_2 y_2$).

In what follows, $\mathbb{Z}$ stands for the frame $\langle \mathbb{Z}, < \rangle$ with the strict order.

Theorem 4.18.
(*i*) $\mathrm{Th}_{\mathsf{ML}}(F_1 \,\&\, F_2) \subseteq \mathrm{Th}_{\mathsf{ML}}(F_1)$ *and* $\mathrm{Th}_{\mathsf{ML}}(F_1 \,\&\, F_2) \subseteq \mathrm{Th}_{\mathsf{ML}}(F_2)$.
(*ii*) *There are* F_1 *and* F_2 *for which* $\mathrm{Th}_{\mathsf{GML}}(F_1 \,\&\, F_2) \nsubseteq \mathrm{Th}_{\mathsf{GML}}(F_1)$.

Proof. (*i*) It easily verified that the projections $\pi_\ell\colon F_1 \& F_2 \to F_1$ and $\pi_r\colon F_1 \& F_2 \to F_2$ are p-morphisms.

(*ii*) We claim that $\mathrm{Th}_{\mathsf{GML}}(\mathbb{Z} \,\&\, \mathbb{Z}) \nsubseteq \mathrm{Th}_{\mathsf{GML}}(\mathbb{Z})$. To see this, observe that $\Diamond\Diamond\Diamond p \to \Diamond^{>3}\Diamond p$ is valid on $\mathbb{Z} \,\&\, \mathbb{Z}$, though not on $\mathbb{Z}$. For, if $\Diamond\Diamond\Diamond p$ is true at some world $(x,y) \in \mathbb{Z} \times \mathbb{Z}$, then p is true for some (x', y') with $x' \geqslant x + 3$ and $y' \geqslant y + 3$. But then there are at least four successors of (x,y) for which $\Diamond p$ is true: $(x{+}1, y{+}1)$, $(x{+}1, y{+}2)$, $(x{+}2, y{+}1)$, $(x{+}2, y{+}2)$. Finally, define π on $\mathbb{Z}$ such that $\pi(x)(p) = \mathtt{tt}$ iff $x = 4$. Then, at 0, $\Diamond\Diamond\Diamond p$ is true, but not $\Diamond^{>3}\Diamond p$, hence, $\langle \mathbb{Z}, < \rangle \nVdash \Diamond\Diamond\Diamond p \to \Diamond^{>3}\Diamond p$. □

Theorem 4.19.
(*i*) $\mathrm{Th}_{\mathsf{ML}}(\mathbb{Z} \odot \mathbb{Z}) = \mathrm{Th}_{\mathsf{ML}}(\mathbb{Q} \odot \mathbb{Z})$.
(*ii*) $\mathrm{Th}_{\mathsf{GML}}(\mathbb{Z} \odot \mathbb{Z}) = \mathrm{Th}_{\mathsf{GML}}(\mathbb{Q} \odot \mathbb{Z})$.

Proof. (*i*) This is proven in [Ben82].

(*ii*) $\mathbb{Z} \odot \mathbb{Z}$ and $\mathbb{Q} \odot \mathbb{Z}$ are both strict linear orders, hence we can apply theorem 4.16 yielding:
$\varphi \in \mathrm{Th}_{\mathsf{GML}}(\mathbb{Z} \odot \mathbb{Z}) \quad \Leftrightarrow_{4.16} \quad T(\varphi) \in \mathrm{Th}_{\mathsf{ML}}(\mathbb{Z} \odot \mathbb{Z}) \quad \Leftrightarrow_{4.19(i)}$
$T(\varphi) \in \mathrm{Th}_{\mathsf{ML}}(\mathbb{Q} \odot \mathbb{Z}) \quad \Leftrightarrow_{4.16} \quad \varphi \in \mathrm{Th}_{\mathsf{GML}}(\mathbb{Q} \odot \mathbb{Z})$. □

5 Expressive power 2: correspondence

In this section we study first-order definability of graded modal formulas, i.e., the correspondence between GML-formulas on the one side and (first-order) properties of the accessibility relation on the other. In section 5.1 we give examples of first-order (f.o.) properties that can be graded modally defined (the most interesting are of course those that were not modally definable). Then, negative examples show some limitations on the expressive power of GML. For doing so, the tools developed in section 3 and 4 appear to be helpful.

Section 5.2 is devoted to deriving f.o. properties from GML formulas. Then we know already examples from section 5.1, but we will show how for some classes of graded modal formulas the corresponding f.o. property can be derived systematically. We show that this is only one side of the picture: there are modal formulas (the so called 'Sahlqvist formulas') that correspond to some f.o. property, but as soon as graded modal operators are plugged in for ordinary modal operators, the correspondence result is smashed up.

5.1 Definability of first-order properties

Definition 5.1. Let $\varphi \in L$ and $A(R)$ be a first-order property of relation R (possibly using '='). We say that φ corresponds to A iff

$$\text{for all frames } F\colon \quad F \Vdash \varphi \text{ iff } F \text{ satisfies } A(R) \qquad (*)$$

We then also say that φ *defines* $A(R)$. This definition easily extends to systems with multiple modalities, allowing φ (a GML-formula in operators $\Diamond_1^{>n}, \dots, \Diamond_m^{>n}$ to correspond with $A(R_1, \dots, R_m)$. We have *relative correspondence* if $(*)$ applies for all frames in some class $\mathbb{F}$ of frames. Then φ defines $A(R)$ *in* $\mathbb{F}$.

Example 5.2. Graded modalities seem to pre-eminently suited to define 'having at least (at most, exactly) n R-successors' (called "seriality" for $n = 1$). In chapter six of this thesis we explore this feature to show that on models, any first-order quantifier is definable. From chapter 3 we also know that the property $R_3 = R_1 \cap R_2$ does not correspond to any (ordinary) modal formula, although it does correspond to a GML-formula — provided that we first add an appropriate rule.

In this section, we will see that there exists a GML-formula that corresponds to transitivity and irreflexivity, viz. $\Diamond(p \wedge \Diamond p) \rightarrow \Diamond^{>1} p$ (cf. 5.21). In section 4 we used the bi-implication $\Diamond(p \wedge \Diamond p) \leftrightarrow \Diamond^{>1} p$ to 'unfold' GML-formulas into equivalent ML-formulas on strict linear orders. It follows that the expressive power of GML is equal to that of ML on $\mathbb{F}^{\text{slo}}$. It turns out that this unfolding yields equivalent formulas only on $\mathbb{F}^{\text{slo}}$.

Theorem 5.3. $F \in \mathbb{F}^{\text{slo}}$ *iff* $F \Vdash \Diamond^{>1} p \leftrightarrow \Diamond(p \wedge \Diamond p)$.

Proof. This follows immediately from our remark that $\Diamond(p \wedge \Diamond p) \rightarrow \Diamond^{>1} p$ defines irreflexivity and transitivity and the fact that right-linearity is defined by $\Diamond^{>1} p \rightarrow \Diamond(p \wedge \Diamond p)$. The latter property is also ML-definable by $\Box(\Box p \rightarrow q) \vee \Box(\Box q \rightarrow p)$, but GML turns out to be more economical here: in ML one cannot do this by using one propositional atom ([Ben83]). □

Now that $\mathbb{F}^{\text{slo}}$ is GML-definable, we can distinguish the strict linear order from the weak linear orders.

Corollary 5.4. $\text{Th}_{\mathsf{GML}}(\mathbb{F}^{\text{slo}}) \not\subseteq \text{Th}_{\mathsf{GML}}(\mathbb{F}^{\text{wlo}})$, *although* $\text{Th}_{\mathsf{ML}}(\mathbb{F}^{\text{slo}}) \subseteq \text{Th}_{\mathsf{ML}}(\mathbb{F}^{\text{wlo}})$.

Proof. $\Diamond^{>1} p \leftrightarrow \Diamond(p \wedge \Diamond p) \in \text{Th}_{\mathsf{GML}}(\mathbb{F}^{\text{slo}})$, but falsified on the $\mathbb{F}^{\text{wlo}}$-frame F of figure 6: make p true only in the world w_1, then w satisfies $\Diamond(p \wedge \Diamond p)$, but not $\Diamond^{>1} p$. For the second part, cf. theorem 4.13. □

Theorem 5.5. $\text{Th}_{\mathsf{GML}}(\mathbb{F}^{\text{lo}}) \not\subseteq \text{Th}_{\mathsf{GML}}(\mathbb{F}^{\text{wlo}})$, *although* $\text{Th}_{\mathsf{ML}}(\mathbb{F}^{\text{lo}}) \subseteq \text{Th}_{\mathsf{ML}}(\mathbb{F}^{\text{wlo}})$.

Proof. $(\Diamond^{=2} p \rightarrow \Diamond \Diamond^{=1} p) \in \text{Th}_{\mathsf{ML}}(\mathbb{F}^{\text{lo}})$, but it is denied on the $\mathbb{F}^{\text{wlo}}$-frame F of figure 6: make p true in both w_1 and w_2. □

We summarize the definability results of $\mathbb{F}^{\text{lo}}$, $\mathbb{F}^{\text{slo}}$ and $\mathbb{F}^{\text{wlo}}$ in the following theorem.

Theorem 5.6. *For all classes* $\mathbb{F}$ *and* $\mathbb{F}'$ *from* $\{\mathbb{F}^{\text{lo}}, \mathbb{F}^{\text{slo}}, \mathbb{F}^{\text{wlo}}\}$, *we have:*

(a) $\text{Th}_{\mathsf{ML}}(\mathbb{F}) = \text{Th}_{\mathsf{ML}}(\mathbb{F}')$
(b) $\mathbb{F} \neq \mathbb{F}' \;\Rightarrow\; \text{Th}_{\mathsf{GML}}(\mathbb{F}) \neq \text{Th}_{\mathsf{GML}}(\mathbb{F}')$
(c) $\text{Th}_{\mathsf{GML}}(\mathbb{F}) \subseteq \text{Th}_{\mathsf{GML}}(\mathbb{F}') \;\Rightarrow\; \mathbb{F}' \subseteq \mathbb{F}$

Apparently, on transitive frames, GML is quite stronger than ML. We give an application of this. It is useful in logics of time, in which, in addition to an operator for the future, there is one for the past. So we assume to have graded modalities $\Diamond^{>n}$ and $\overleftarrow{\Diamond}^{>n}$, $n \in \mathbb{N}$. We call the system time-GML.

Definition 5.7. A frame $F = \langle W, R \rangle$ is *connected* if for all $w, w' \in W$ there is a finite sequence $w = w_1, \dots, w_n = w'$ such that, for all $i < n$, either $Rw_i w_{i+1}$ or $Rw_{i+1} w_i$.

Theorem 5.8. *On the class of connected frames the structure* $\langle \mathbb{Z}, < \rangle$ *is time-*GML *definable (up to isomorphism).*

Proof. Consider the following axioms for future modalities, and similar ones for past modalities:

(1). $\Diamond\top$
(2). $\Diamond\Diamond p \to \Diamond p$
(3). $\Box(\Box p \to q) \lor \Box(\Box q \to p)$
(4). $\Diamond(p \land \Diamond p) \to \Diamond^{>1} p$
(5). $\big(\Diamond(p \land \Box\neg p) \to \Box\Box\neg p\big) \ \lor \ \Diamond(\Diamond p \land \neg\Diamond\Diamond p)$
(6). $\Box(\Box p \to p) \to (\Diamond\Box p \to \Box p)$

Then, each connected frame F that satisfies (1)–(6) is isomorphic to $\langle \mathbb{Z}, < \rangle$. For, if (1)–(6) are valid at a world w, then, by (1), $\langle \overrightarrow{w} \rangle$ is serial (we initially consider the right-successors of w and use the axioms with forward modalities, whereas the case of the past is argued symmetrically), by (2) transitive, by (3) linear. Formula (4), with theorem 5.3 guarantees irreflexivity. On strict linear orders, discreteness is defined by (5) (cf. [Ben82]). So, $\langle \overrightarrow{w} \rangle$ satisfies the axioms of $\langle \mathbb{N}, < \rangle$. Its nonstandard models are excluded by (6). Since $\langle \overleftarrow{w} \rangle$ satisfies the same properties, and F is connected, we have $F \cong \langle \mathbb{Z}, < \rangle$. □

In ML, $\langle \mathbb{Z}, < \rangle$ is not definable (cf. [Hoe91d]). After this positive result about GML, we have to say something about the limitations of definability (even on transitive frames) in GML now.

Theorem 5.9. *The following f.o. properties of R do not correspond to any* GML*-formula:*

(a)	linearity:	$\forall x\, y\, (x = y)$
(b)	discreteness:	$\forall x\, (\exists y\, Rxy \to \exists y\, (Rxy \land \neg\exists z\, (Rxz \land Rzy)))$
(c)	left-seriality:	$\forall x\, \exists y\, Ryx$
(d)	'selected' reflexivity:	$\forall x\, \exists y\, (Rxy \land Ryy)$

Proof. (a) Suppose linearity corresponds to φ. Then $\langle \mathbb{Z}, < \rangle \Vdash \varphi$ and, by 3.16, $\langle \mathbb{Z}, < \rangle \uplus \langle \mathbb{Z}, < \rangle \Vdash \varphi$, yielding, with our assumption, that the latter is linear, which is obviously not.

(b) The structure $\langle \mathbb{N}, \{(n, n{+}1) \mid n \in \mathbb{N}\} \rangle$ is discrete, its p^{+}-morphic image $\langle \{0\}, \{(0,0)\} \rangle$ is not. Now use 3.9 and argue as in (a).

(c) $\forall x\, \exists y\, Rxy$ is true in $\langle \mathbb{Z}, < \rangle$ but not in $\langle \mathbb{N}, < \rangle$. Observe that $\langle \mathbb{N}, < \rangle \hookrightarrow \langle \mathbb{Z}, < \rangle$ and use 3.13.

(d) Here we use our reflection result about ultrafilter extensions as stated in section 3: the ultrafilter extension $\mathfrak{ue}\langle \mathbb{N}, < \rangle$ of $\langle \mathbb{N}, < \rangle$ satisfies $\forall x \exists y\, (Rxy \land Ryy)$, whereas $\langle \mathbb{N}, < \rangle$ does not. □

Corollary 5.10. *There are f.o. properties that do not have a corresponding* GML*-formula, even if we restrict ourselves to transitive frames.*

Proof. A witness is the part (d) of the previous theorem. □

We end this section by stating another negative correspondence result, now adapting a technique that was described for multi-modal logic in chapter 3, called unraveling. It appears that this technique is useful also for GML. The definition of how to obtain unraveled model $M^*_w = \langle W^*, R^*, \pi^* \rangle$, given $M = \langle W, R, \pi \rangle$, is given in 4.4 of chapter 3 of this thesis. Recall that each world of M is copied in M^*_w as many times as it is reachable from w: if in M we have Rwv, Rvu and Rwu, we distinguish in M^*_w between an u-world that is a successor of v and one that is a successor of w, i.e., we get $R^*\langle w \rangle \langle wv \rangle$, $R^*\langle wv \rangle \langle wvu \rangle$, $R^*\langle w \rangle \langle wu \rangle$ (cf. figure 7). We can also consider M to be the result of an identification of different worlds in M^*_w, according to the following theorem.

Theorem 5.11. *Let M be a model with a world w and M^*_w the unraveled result.*
Then the function $f\colon M^*_w \to \langle \overrightarrow{w} \rangle$ defined by $f(\langle w, w_1, \ldots, w_n \rangle) = w_n$ is a p^{+}-morphism.

Proof. From chapter 3 we know that f is a p-morphism. To show that it satisfies 'n-forth choice' and 'n-back choice', we show that, for each $v \in M^*_w$, f is a bijection between $R^*(v)$ and $R(f(v))$. Let '$\circ$' denote the concatenation of sequences. To see that f is injective, take $x \neq x' \in R^*(v)$. By definition of R^*, we get $x = v \circ \langle y \rangle$ and $x' = v \circ \langle y' \rangle$ for some $y, y' \in R(f(v))$. Since $v \circ \langle y \rangle = x \neq x' = v \circ \langle y' \rangle$, we have $f(x) = y \neq y' = f(x')$. □

Corollary 5.12. *For all models M and worlds w:*
$\mathrm{Th}_{\mathsf{GML}}(M) \subseteq \mathrm{Th}_{\mathsf{GML}}(\langle\overrightarrow{w}\rangle)$ *and* $\mathrm{Th}_{\mathsf{GML}}(M^*_w) \subseteq \mathrm{Th}_{\mathsf{GML}}(\langle\overrightarrow{w}\rangle)$.

Proof. From 5.11 we know the existence of a p^+-morphism from M^*_w to $\langle\overrightarrow{w}\rangle$, which on its turn is a generated submodel of M. The corollary then follows from 3.7 and 3.12. □

Theorem 5.13. *There are no* GML*-formulas that correspond to antisymmetry, irreflexivity, asymmetry or intransitivity.*

Proof. Let $A(x)$ be any of the four properties, and let F be a frame that does not satisfy $A(x)$ at w. Then $\langle\overrightarrow{w}\rangle$ does not satisfy $A(x)$. If A would correspond to φ, then $\langle\overrightarrow{w}\rangle \nvDash \varphi$. By 5.12, $M^*_w \nvDash \varphi$. However, M^*_w *does* satisfy A. For, any world in the unraveled model M^*_w, which is a sequence (of length, say, n) of worlds of $\langle\overrightarrow{w}\rangle$, has only access to sequence of length n+1. □

Example 5.14. Figure 7 shows how a frame F is unraveled (from w) into a frame F'. On easily verifies that, although $\langle w\rangle$ is an irreflexive, intransitive, asymmetric and antisymmetric world, its p^+-morphic image w is not.

5.2 First-order definability of modal principles

Each positive result of 5.1 attaches a f.o. property to a given GML-formula. Two questions now emerge: does there always exist such a f.o. property, and, if yes, is there any systematics in deriving them? In order o systematically attach f.o. properties to GML, the following Standard Translation (adapted from [Ben83], where it was introduced for ML) is useful.

Definition 5.15. ST translates GML-formulas to f.o. properties as follows:

- $\mathrm{ST}(p) = Px$
- $\mathrm{ST}(\neg\varphi) = \neg\,\mathrm{ST}(\varphi)$
- $\mathrm{ST}(\varphi \vee \psi) = \mathrm{ST}(\varphi) \vee \mathrm{ST}(\psi)$
- $\mathrm{ST}(\Diamond^{>n}\varphi) = \exists^{\neq} y_0 \ldots y_n \bigwedge_{0\leqslant i\leqslant n} (Rxy_i \wedge [y_i/x]\,\mathrm{ST}(\varphi))$,

where in the last item no y_i ($i = 0 \ldots n$) occurs in $\mathrm{ST}(\varphi)$, and $[y/x]\alpha$ is α with each free occurrence of x replaced by y. For the definition of $\exists^{\neq}$ and $\forall^{\neq}$, see 3.4.

Example 5.16.

(a) $\mathrm{ST}(\Box^{\leqslant n}\varphi) \;=\; \forall^{\neq} y_0 \ldots y_n \big(\bigwedge_{0\leqslant i\leqslant n} Rxy_i \;\to\; \bigwedge_{0\leqslant i\leqslant n} [y_i/x]\,\mathrm{ST}(\varphi)\big)$
(b) $\mathrm{ST}(\Box p \to p) \;=\; \forall y\,(Rxy \to Py) \to Px$
(c) $\mathrm{ST}(\Diamond^{>1}\Diamond p \to \Diamond^{>1} p) \;=\; \exists^{\neq} y, y' \big(Rxy \wedge Rxy' \wedge \exists u\,(Ryu \wedge Pu) \wedge \exists u'\,(Ry'u' \wedge Pu')\big) \to$
$\exists^{\neq} z, z'\,(Rxz \wedge Rxz' \wedge Pz \wedge Pz')$

Theorem 5.17. *For all models M, frames F, and formulas φ in the propositions $p_1, \ldots, p_n$:*

(*i*). $M, w \vDash \varphi$ *iff* $M \vDash \mathrm{ST}(\varphi)[w]$
(*ii*). $M \vDash \varphi$ *iff* $M \vDash \forall x\,\mathrm{ST}(\varphi)$
(*iii*). $F, w \Vdash \varphi$ *iff* $F \vDash \forall P_1 \ldots \forall P_n\,\mathrm{ST}(\varphi)[w]$
(*iv*). $F \Vdash \varphi$ *iff* $F \vDash \forall x\,\forall P_1 \ldots \forall P_n\,\mathrm{ST}(\varphi)$

Proof. Is an easy generalization of [Ben83]. □

Combining 5.16(b) and 5.17(*iii*), we get: $F, w \Vdash \Box p \to p$ iff $F \vDash \forall P\big(\forall y\,(Rwy \to Py) \to Pw\big)$, giving correspondence between a GML-formula and a second-order property. We give a sufficient condition for turning this into a f.o. property (theorem 5.20) and illustrate the proof by an example.

Definition 5.18. $\pi \leqslant_{p,F} \pi'$ if for all $w \in W$: $\pi(w)(p) = \mathtt{tt} \Rightarrow \pi'(w)(p) = \mathtt{tt}$. For each $k \in \mathbb{N}$, $\pi \leqslant_{p_1,\dots,p_k,F} \pi'$ if $\pi \leqslant_{p,F} \pi'$ for all $i \leqslant k$. A formula $\varphi \in L$ is called *positive* if it is equivalent to a formula built with $\top, \bot, \wedge, \vee, \Diamond^{>n}, \Box^{\leqslant n}$ and propositional atoms. φ is *monotonic in* $p_1, \dots, p_k$ if φ's truth is enduring under extended valuations; formally: $\langle W, R, \pi\rangle, w \vDash \varphi$ and $\pi \leqslant_{p_1,\dots,p_k,F} \pi'$ imply $\langle W, R, \pi'\rangle, w \vDash \varphi$.

Lemma 5.19 ([Ben83]). *Any positive formula is monotonic in all its propositional atoms.*

Theorem 5.20. *Let φ be a* GML*-formula built using $p_1, \dots, p_k, \wedge, \vee, \Diamond^{>r}$, and ψ positive in $p_1, \dots, p_k$. Then the* GML*-formula $(\varphi \to \psi)$ corresponds to a f.o. formula, which can be systematically obtained.*

Proof. We proceed in several steps, and we will exemplify each step (n) with the formula $\Diamond(p \wedge \Diamond p) \to \Diamond^{>1} p$ in step (n'); 5.22 provides other examples. The variable s ranges over $\{1, \dots, k\}$.

1. Obtain $\mathrm{ST}(\varphi \to \psi) = \mathrm{ST}(\varphi) \to \mathrm{ST}(\psi)$ using different variables for each quantifier. Since φ only consists of $p_1, \dots, p_k, \wedge, \vee, \Diamond^{>r}$, the formula $\mathrm{ST}(\varphi)$ only contains existential quantifiers, which all can be moved to front, giving $\exists y_1 \dots y_n \, \varphi'$.

1′. We get $\mathrm{ST}(\Diamond(p \wedge \Diamond p) \to \Diamond^{>1} p) =$
$\exists y_1 \, (Rxy_1 \wedge Py_1 \wedge \exists y_2 \, (Ry_1y_2 \wedge Py_2)) \to \exists^{\neq} u, v \, (Rxu \wedge Rxv \wedge Pu \wedge Pv)$.
Here $\mathrm{ST}(\varphi)$ can be rewritten as $\exists y_1 \exists y_2 \, (Rxy_1 \wedge Py_1 \wedge Ry_1y_2 \wedge Py_2)$.

2. Using f.o. logic, we can rewrite $\mathrm{ST}(\varphi \to \psi)$ into $\forall y_1 \dots y_n (\varphi' \to \mathrm{ST}(\psi)) =: \alpha$.

2′. $\alpha = \forall y_1 \, y_2 \big((Rxy_1 \wedge Py_1 \wedge Ry_1y_2 \wedge Py_2) \to \exists^{\neq} u, v \, (Rxu \wedge Rxv \wedge Pu \wedge Pv)\big)$.

3. Let $P_sY = \{y_i \mid P_sy_i \text{ occurs in } \varphi'\}$. This set is finite, and we abbreviate $\bigvee_{u \in P_sY}(y = u)$ as $y \in P_sY$. Let the f.o. formula $\Phi \to \Psi$ be the result of replacing each occurrence of P_sz in α by $(z \in P_sY)$. For the antecedent, this has only the effect that each occurrence of P_sy_i (in φ') is replaced by $\top$. $(\Phi \to \Psi)[w]$ is the result of replacing the free variable x in $(\Phi \to \Psi)$ by w.

3′. $PY = \{y_1, y_2\}$, so $\Phi \to \Psi =$
$\forall y_1, y_2 \big((Rxy_1 \wedge Ry_1y_2) \to \exists^{\neq} u, v \, (Rxu \wedge Rxv \wedge u \in \{y_1, y_2\} \wedge v \in \{y_1, y_2\})\big)$

4. We claim that $\Phi \to \Psi$ corresponds to $\varphi \to \psi$, i.e., for all frames F and worlds w, we have: $F, w \Vdash \varphi \to \psi$ iff $F \vDash (\Phi \to \Psi)[w]$.

4a. $(\Rightarrow)$ If $F, w \Vdash \varphi \to \psi$, then, by 5.17$(iii)$, $F \vDash \forall P_1 \dots P_k \, \forall y_1 \dots y_n \, (\varphi' \to \mathrm{ST}(\psi))[w]$, and $F \vDash \forall y_1 \dots y_n \, \forall P_1 \dots P_k \, (\varphi' \to \mathrm{ST}(\psi))[w]$. But $(\Phi \to \Psi)$ is only an instantiation of this formula: take $P_sz = (z \in P_sY)$. Thus, $F \vDash (\Phi \to \Psi)[w]$.

4a′. $(\Rightarrow)$ If $F, w \Vdash \Diamond(p \wedge \Diamond p) \to \Diamond^{>1} p$, then, by 5.17$(iii)$, $F \vDash \alpha(w)$, see step 2′ for α. Taking an instance of P as 'being equal to either y_1 or y_2' yields:
$F \vDash \forall y_1, y_2 \big((Rwy_1 \wedge Ry_1y_2) \to \exists^{\neq} u, v \, (Rwu \wedge Rwv \wedge u \in \{y_1, y_2\} \wedge v \in \{y_1, y_2\})\big)$.

4b. $(\Leftarrow)$ Suppose $F \vDash (\Phi \to \Psi)[w]$ and $\langle F, \pi\rangle, w \vDash \varphi$. Since $\langle F, \pi\rangle \vDash \mathrm{ST}(\varphi)[w]$ and $\mathrm{ST}(\varphi)$ has the prefix $\exists y_1 \dots y_n$, there are worlds $w_1, \dots, w_n$ (we denote the set of them by W_φ) for which φ' is true in $\langle F, \pi\rangle$. Recall that $\varphi'(w, w_1, \dots, w_n)$ is built using only $\wedge$, $R_jw'w''$ and Pw' for some $w', w'' \in W_\varphi$. Let $P_sW = \{w' \in W_\varphi \mid P_sw' \text{ occurs in } \varphi'\}$. This is the minimal set (in p_s) to make φ' true for the choices of W_φ. We define π' according to this minimal fulfillment: $\pi'(x)(p_s) = \mathtt{tt}$ iff $x \in P_sW$. Then also $\langle F, \pi'\rangle \vDash \varphi'(w, w_1, \dots, w_n)$ (we did not change the assignment for p at those whitewashes for φ'). Let Φ° be the result of replacing each occurrence of P_sw' in $\varphi'(w, w_1, \dots, w_n)$ with $(w' \in P_sW)$. By definition of π', $\langle F, \pi'\rangle \vDash \Phi^\circ(w, w_1, \dots, w_n)$. Since Φ° does not refer to any P_s, we even have $F \vDash \Phi^\circ(w, w_1, \dots, w_n)$. (This amounts to saying that at w, there are witnesses that allow a valuation to make φ' true). Φ° is an instantiation of Φ, so, by our assumption, we have $F \vDash \Psi^\circ$, where Ψ° is obtained from Ψ by performing the same substitution. Ψ° contains subformulas of the form $(z \in P_sW)$, which was exactly the extension of p_s under π'. If we replace P_sz for each $(z \in P_sW)$ in Ψ° (giving back $\mathrm{ST}(\psi)$), we have

$\langle F, \pi' \rangle \vDash \mathrm{ST}(\psi)[w]$. With 5.17(*iii*), we conclude that $\langle F, \pi' \rangle, w \vDash \psi$. Since ψ is positive (in p_s) and $\pi' \leqslant_{p_s,F} \pi$, we have $\langle F, \pi \rangle, w \vDash \psi$.

4b′. ($\Leftarrow$) Suppose $F \vDash \forall y_1, y_2 \big((Rwy_1 \wedge Ry_1y_2) \to \exists^{\neq} u, v\,(Rwu \wedge Rwv \wedge u \in \{y_1, y_2\} \wedge v \in \{y_1, y_2\})\big)$ and $\langle F, \pi \rangle, w \vDash \Diamond(p \wedge \Diamond p)$. Since w satisfies $\mathrm{ST}(\Diamond(p \wedge \Diamond p))$, there are worlds w_1, w_2 such that $\varphi'(w, w_1, w_2) = (Rww_1 \wedge Rw_1w_2 \wedge Pw_1 \wedge Pw_2)$ is true in $\langle F, \pi \rangle$. Obviously, $PW = \{w_1, w_2\}$. So, π' makes p true only at w_1 and w_2, and $\langle F, \pi' \rangle \vDash (Rww_1 \wedge Rw_1w_2 \wedge Pw_1 \wedge Pw_2)$. After substitution, we have $\langle F, \pi' \rangle \vDash Rww_1 \wedge Rw_1w_2$. Clearly, $F \vDash Rww_1 \wedge Rw_1w_2$. The assumption yields $F \vDash \Psi^\circ[w]$, i.e., $F \vDash \exists^{\neq} u, v\,(Rwu \wedge Rwv \wedge u \in \{w_1, w_2\} \wedge v \in \{w_1, w_2\})$. Since under the minimal assignment π', p is true exactly at PW, we get $\langle F, \pi' \rangle \vDash \exists^{\neq} u, v\,(Rwu \wedge Rwv \wedge Pu \wedge Pv)$, i.e., $\langle F, \pi' \rangle \vDash \mathrm{ST}(\psi)[w]$. So, we have $\langle F, \pi' \rangle, w \vDash \psi$ ($= \Diamond^{>1} p$). Since π only extends π', we conclude $\langle F, \pi \rangle, w \vDash \Diamond^{>1} p$. □

Corollary 5.21. $\Diamond(p \wedge \Diamond p) \to \Diamond^{>1} p$ *defines the conjunction of transitivity and irreflexivity.*

Proof. From 5.20 item $4a'$, this formula corresponds to

$$\forall y_1, y_2 \big((Rwy_1 \wedge Ry_1y_2) \to \exists^{\neq} u, v\,(Rwu \wedge Rwv \wedge u \in \{y_1, y_2\} \wedge v \in \{y_1, y_2\})\big).$$

The latter is equivalent to the property of R being both transitive and irreflexive (at x). The proof of the equivalence of these f.o. properties in predicate logic is left to the reader. □

Example 5.22.

(a) $\Diamond^{>1} p \to \Diamond(p \wedge \Diamond p)$ corresponds to right-linearity: $\forall^{\neq} y, y'\,((Rxy \wedge Rxy') \to (Ryy' \vee Ry'y))$;
(b) $\Diamond^{\geqslant n} p \to \Box^{\leqslant k} p$ corresponds to having at most $(n{+}k)$ successors ($n \geqslant 1$), i.e., to $\Diamond^{\leqslant (n+k)} \top$.

Proof. We proceed along the lines of 5.20.

a1. $\mathrm{ST}(\varphi \to \psi) = \exists^{\neq} y, y'\,(Rxy \wedge Rxy' \wedge Py \wedge Py') \to \exists u(Rxu \wedge Pu \wedge \exists v\,(Ruv \wedge Pv))$

a2. $\forall y, y'\,(\varphi' \to \mathrm{ST}(\psi)) = \forall^{\neq} y, y' \big((Rxy \wedge Rxy' \wedge Py \wedge Py') \to \exists u(Rxu \wedge Pu \wedge \exists v\,(Ruv \wedge Pv))\big)$

a3. $PY = \{y, y'\}$, so $(\Phi \to \Psi)$ becomes
$\forall^{\neq} y, y' \big((Rxy \wedge Rxy') \to \exists u(Rxu \wedge u \in \{y, y'\} \wedge \exists v\,(Ruv \wedge v \in \{y, y'\}))\big)$
The latter formula is equivalent to right-linearity (it says that every two successors of x must be accessible to each other (in at least one direction)).

b1. $\mathrm{ST}(\varphi \to \psi) = \exists^{\neq} y_1 \dots y_n \bigwedge_{1 \leqslant i \leqslant n} (Rxy_i \wedge Py_i) \to \forall^{\neq} z_0 \dots z_k (\bigwedge_{0 \leqslant j \leqslant k} Rxz_j \to \bigvee_{0 \leqslant j \leqslant k} Pz_j)$

b2. $\forall^{\neq} y_1 \dots y_n \big(\bigwedge_{1 \leqslant i \leqslant n} (Rxy_i \wedge Py_i) \to \forall^{\neq} z_0 \dots z_k (\bigwedge_{0 \leqslant j \leqslant k} Rxz_j \to \bigvee_{0 \leqslant j \leqslant k} Pz_j)\big)$

b3. $\forall^{\neq} y_1 \dots y_n \big(\bigwedge_{1 \leqslant i \leqslant n} Rxy_i \to \forall^{\neq} z_0 \dots z_k (\bigwedge_{0 \leqslant j \leqslant k} Rxz_j \to \bigvee_{0 \leqslant j \leqslant k} z_j \in \{y_1, \dots, y_n\})\big)$
The latter formula expresses that every set of n successors of x must have a world common with any set of $(k{+}1)$ successors of x, which is equivalent to saying that there are $\leqslant(n{+}k)$ successors of x. □

The following theorem shows that we may (at least carefully) allow for negative parts in the antecedent. We omit a formal proof, since, with the substitution we provide, it is a technical exercise in the spirit of 2.5.6 of [Hoe91d]. An example clarifies some matters.

Theorem 5.23. *Let ψ be positive in p. Then $\Diamond^{=n} \neg p \to \psi$ is f.o. definable.*

Proof. The key observation is that, if we have exactly n successors of w that verify $\neg p$, we know that any successor of w satisfying p must be different from them. More formally, denoting $\varphi = \Diamond^{=n} \neg p$:

$$\mathrm{ST}(\varphi) \;=\; \exists^{\neq} y_1 \dots y_n \big(\bigwedge_{i \leqslant n} (Rxy_i \wedge \neg Py_i) \wedge \forall z\,(Rxz \wedge z \notin \{y_1, \dots, y_n\} \to Pz)\big)$$

Denoting $Y = \{y_1, \dots, y_n\}$, we replace each occurrence of Pu in $\mathrm{ST}(\varphi \to \psi)$ with $(u \notin Y)$, which yields a formula $\Phi \to \Psi$ with $\Phi \equiv \exists^{\neq} y_1 \dots y_n \bigwedge_{i \leqslant n} Rxy_i$. Now the proof is continued similarly to that of 5.20, and hence we will not spell it out here. (Note that the valuation π' must be taken the same as π: if the antecedent φ is true under π at w, this π is immediately minimal w.r.t. p, cf. example 5.26). □

There is a straightforward way to combine correspondence results to new ones:

Theorem 5.24 ([Hoe91d])**.** *Suppose $(\varphi \to \psi)$ corresponds to α and $(\varphi' \to \psi)$ to α'. Then $(\varphi \vee \varphi') \to \psi$ corresponds to $(\alpha \wedge \alpha')$.*

Corollary 5.25. *Let ψ be positive in p. Then $\Box^{\leqslant n} p \to \psi$ is f.o. definable.*

Proof. $\Box^{\leqslant n} p$ is equivalent to $\bigvee_{0 \leqslant i \leqslant n} \Diamond^{=i} \neg p$, and $\Diamond^{=i} \neg p \to \psi$ is f.o. definable by 5.23, so $\Box^{\leqslant n} p \to \psi$ is f.o. definable by 5.24. □

Example 5.26. $\Box^{\leqslant 2} p \to \Diamond p$ corresponds to $\exists^{\neq} y_1, y_2, y_3 \, (Rxy_1 \wedge Rxy_2 \wedge Rxy_3)$.

The results of 5.23 and 5.25 suggest that theorem 5.20 might be strengthened in that we might freely allow operators $\Diamond^{=n}$ and $\Box^{\leqslant n}$ in the antecedent. It appears that we may not, as we shall show now. In modal logic, a substantial class of formulas is determined from which a corresponding f.o. property can be obtained constructively. The class has been given the name of 'Sahlqvist' formulas, after its 'discoverers', i.e., the following theorem was independently proven in [Sah75] and [Ben76]. In 5.28, we show that the restriction to 'non-graded' modalities is essential.

Theorem 5.27 ([Sah75, Ben76])**.** *Let ψ be a positive ordinary modal formula. Then $\Diamond^k \Box^n p \to \psi$ is f.o. definable.*

Theorem 5.28. *The formula* $\Box\Box^{\leqslant 2}p \to \Diamond\Diamond\Box^{\leqslant 1}p$ *is not f.o. definable.*

The same holds for the formula $\Box\Box\Diamond^{\geqslant 2}p \to \Diamond\Diamond^{\geqslant 3}p$*, by contraposition.*

Proof. The central tool here is the theorem of Löwenheim (cf. [ChaKei73]):

(LöSk) *For each frame* $F = \langle W, R\rangle$ *and a countable* $U \subseteq W$*, there is a frame* $F' = \langle W', R'\rangle$ *such that:*

- W' *is countable and* $U \subseteq W'$*,*
- F *and* F' *have the same f.o. theory.*

Let $\varphi = \Box\Box^{\leqslant 2}p \to \Diamond\Diamond\Box^{\leqslant 1}p$. To use (LöSk), we construct a non-countable frame $F = \langle W, R\rangle$ with

(1) $F \Vdash \varphi$, and
(2) indicate a countable $U \subseteq W$ such that for no countable W' with $U \subseteq W' \subseteq W$, we have $\langle W', R'\rangle, w \Vdash \varphi$. (WHICH R' and w????)

We define $F = \langle W, R\rangle$ as follows (cf. figure 8). Let the index set $I = \{1, 2, 3\}$, $NI = \mathbb{N} \times I$, and $FI = \{f \mid f{:}\mathbb{N} \to I\}$.

$W = \{w\} \cup \{u, a, b, c\} \cup \{y_n \mid n \in \mathbb{N}\} \cup \{y_{ni} \mid n \in \mathbb{N}, i \in I\} \cup \{z_f \mid f \in FI\}$.
$R = \{(w, y_n) \mid n \in \mathbb{N}\} \cup \{(w, u)\} \cup \{(y_n, y_{ni}) \mid n \in \mathbb{N}, i \in I\} \cup \{(y_{ni}, a), (y_{ni}, b) \mid n \in \mathbb{N}, i \in I\} \cup \{(a, a), (b, b), (c, c)\} \cup \{(u, z_f), (z_f, c), (z_f, y_{f(n)}) \mid n \in \mathbb{N}, f \in FI\}$.

For this frame we prove (1) and (2).

(1) We show that $F \Vdash \Box\Box^{\leqslant 2}p \to \Diamond\Diamond\Box^{\leqslant 1}p$. Fix any valuation π on F, and denote $M = \langle F, \pi\rangle$. $\Box^{\leqslant 1}p$ is true at a, b, c, since these worlds have only one successor. Hence $\Diamond\Diamond\Box^{\leqslant 1}p$ (and, thus, φ) is true at $a, b, c, z_f, u, y_n, y_{ni}$ for all $f \in FI$, $i \in I$, $n \in \mathbb{N}$. It remains to check that φ is true at w. If $M, w \vDash \Box\Box^{\leqslant 2}p$, then for each y_n there must be at least one successor validating p, say, $\{y_{n,f(n)} \mid n \in \mathbb{N}\}$ for some function $f \in FI$. Then all z_f's successors that differ from c satisfy p, so $M, z_f \vDash \Box^{\leqslant 1}p$. Then, since Rwu and Ruz_f, also $M, w \vDash \Diamond\Diamond\Box^{\leqslant 1}p$, thus $M, w \vDash \varphi$.

(2) Let $U = \{w, a, c, b, y_n, y_{ni} \mid i \in I, n \in \mathbb{N}\}$. Any countable $W' \supseteq U$ lacks at least one element of FI, say $z_f \notin W'$. Define $\pi(v)(\pi) = \mathtt{tt}$ iff $v \in \{y_{n,f(n)}, z_g \mid n \in \mathbb{N}, z_g \in W'\}$. Denote $M' = \langle F', \pi'\rangle$. Then (WHERE IS R' DEFINED???):

   i. $M', w \vDash \Box\Box^{\leqslant 2}p$. ($\Box p$ being true at u, $\Box^{\leqslant 2}$ at y_n, $n \in \mathbb{N}$)

   ii. $M', w \nvDash \Diamond\Diamond\Box^{\leqslant 1}p$. To see this, we will show that there is no world v that is both an R'-successor of an R'-successor of w and satisfying $\Box^{\leqslant 1}p$: v cannot be y_{ni}, since each $y_{ni} \vDash \Diamond^{>1}\neg p$ (p being false at a and b). v also differs from each $z_g \in W'$: we have $R'z_gc$ (and p false at c), but z_g has yet another R'-successor satisfying $\neg p$. Since $z_g \in W'$ and $z_f \notin W'$, we have $z_g \neq z_f$ implying that, for some $k \in \mathbb{N}$, $g(k) \neq f(k)$. By definition of π, p is false at $y_{k,g(k)}$. Conclusion: $\Diamond^{>1}\neg p$ is true at z_g. □

6 Filtration

We will now discuss a way to obtain a *finite* model for **GrK**-consistent formulas. Not all extensions of **GrK** satisfy this property. For instance, adding the two schemes $\Diamond\top$ and $\Diamond^{>n}\top \to \Diamond^{>(n+1)}\top$ is sufficient to exclude all finite models. For ML, one way to distill a finite model M^* that satisfies φ from an arbitrary model M for φ if to *filter* M *through (the subformulas of)* φ (cf. [HugCre84]). The gist of this technique is that worlds w of M are compressed to equivalence classes $[w]$ containing worlds that verify the same subformulas of φ.

6.1 General filtration

Example 6.1. Consider $M = \langle \mathbb{N}, <, \pi \rangle$, where π makes p true in all even numbers, and q in all multiples of 3. Suppose we filtrate through $p \wedge \Diamond q$ which is true at, for instance, 2. In essence, there are only four kind of worlds: each verifying one Boolean combination of p and q (the worlds all verify the same *modal formulas*). Let us denote these worlds by $\underline{1}$, $\underline{2}$, $\underline{3}$ and $\underline{4}$, the worlds of M^*. Since for each of these classes $\underline{x}$ and $\underline{y}$, in M each x-world has access to a y-world, we take the accessibility relation on M^* to be universal. If π^* treats $\underline{x}$ as π does x with respect to p and q, it is easily seen that $(p \wedge \Diamond q)$ is true at $\underline{1}$.

However, if we want to filtrate through a *graded* modal formula, it is obvious that we cannot simply identify worlds with their equivalence classes, because in GML, we may want typically *a number of* some classes. Of course, we may use copies at will, but how many? We give some preliminary definitions before constructing the canonical model $M^* = \langle W^*, R^*, \pi^* \rangle$. These definitions will be used throughout this section, without explicit reference.

Definition 6.2. Let Φ be the set of all subformulas of φ. The function $S{:}W \to 2^\Phi$ defines the equivalence classes on W (through Φ):

(1) $S(w) = \{\alpha \in \Phi \mid M, w \vDash \alpha\}$

(2) Let $\mathsf{range}(S) = \{S_1, \dots, S_m\}$. Then $m \leqslant 2^{|\Phi|}$. From now, i and j will range over $\{1, \dots, m\}$. If $w \in S_i$, we will call w an S_i-world, or of type S_i.

(3) $H = 1 + \max\{n \in \mathbb{N} \mid \text{for some formula } \beta,\ \Diamond^{>n}\beta \in \Phi\}$.

(4) $n_j(w) = \min(H, |\{v\colon\ Rwv \text{ and } S(v) = S_j\}|)$. Obviously, $n_j(w) \leqslant H$.

How many copies of each class S_j must be R^*-accessible from $S(w)$ in our filtrate? We have to take into account the possibility that $S(w) = S(w') = S_i$, although in the original modal w and w' are R-accessible to a different number of S_j-worlds. It turns out that we may freely choose a representative for $S(w)$ to determine the number of S_j-successors, but, in order to be able to reason about the original modal M again (cf. 6.4), we have to do some bookkeeping about which choice we make.

(5) For each class S_i, we choose a unique representative w_i, for which $S(w_i) = S_i$.

Finally, to build W^*, the proper number of copies of each class S_j should be available, so we define N_i, the maximum number of S_i-worlds accessible from any relevant world:

(6) $N_j = \max\{1, n_j(w_1), \dots, n_j(w_m)\}$.

Definition 6.3. Suppose for a model $M = \langle W, R, \pi \rangle$, we have $M, w \vDash \varphi$. We define the *filtrate* $M^* = \langle W^*, R^*, \pi^* \rangle$ of M through Φ, the set of subformulas of φ, as follows.

- W^* of each $S_i \in \mathsf{range}(S)$, W^* contains N_i copies of S_i: $S_i^1, \dots, S_i^{N_i}$
- R^* for any $i, j \leqslant m$, the copy S_i^k of S_i $(1 \leqslant k \leqslant N_i)$ is R^*-related to the first $n_j(w_i)$ copies of S_j
- π^* for each $S_i^k \in W^*$, let $\pi^*(S_i^k)(p) = \mathtt{tt}$ iff $p \in S_i^k$.

Note that all copies of any S_i are R^*-related to the same worlds (but not accessible *from* the same worlds). The number of S_j-copies that are R^*-accessible from any S_i-copy is completely determined by the number of S_j-type worlds that are R-accessible from w_i (which may be zero!).

Lemma 6.4. *Let M^* be a filtrate of M through Φ. Then, for all $\alpha \in \Phi$:*
$M^*, w^* \vDash \alpha$ *iff* $\alpha \in w^*$.

Proof. The sets $w^* \in W^*$ act like maximal consistent sets with respect to the formulas of Φ: we have that for all $(\varphi_1 \vee \varphi_2) \in w^*$ and $\neg\psi \in \Phi$:
(*i*) $(\varphi_1 \vee \varphi_2) \in w^*$ iff $\varphi_1 \in w^*$ or $\varphi_2 \in w^*$, and
(*ii*) $\neg\psi \in w^*$ iff $\psi \notin w^*$.

Now, the lemma for $\alpha = p$ immediately follows from the definition of π^*. The cases $\alpha = (\varphi_1 \vee \varphi_2)$ and $\alpha = \neg\psi$ are easy with (*i*) and (*ii*). We check $\alpha = \Diamond^{>n}\psi$. Note that, since $\alpha \in \Phi$, we have $n < H$.

($\Leftarrow$) Assume $\alpha \in w^*$, so $\Diamond^{>n}\psi$ must be true at the unique w_i with $S(w_i) = S_i$. So w_i in M had at least n+1 R-successors at which ψ is true. We distinguish two cases:

1) There is $j \leqslant m$ such that $\psi \in S_j$ and $n_j(w_i) = H$. Using the definition of R^* and the induction hypothesis for ψ, we get $M^*, w^* \vDash \Diamond^{>H}\psi$. Since $H > n$, then $M^*, w^* \vDash \Diamond^{>n}\psi$.

2) No such j exists. Let $j_1, \dots, j_r \leqslant m$ be such that all ψ-successors of w_i are of type $S_{j_1}, \dots, S_{j_r}$. Since w_i is of type S_i and $\Diamond^{>n}\psi \in S_i$, $n_{j_1}(w_i) + \dots + n_{j_r}(w_i) > n$, such that, by definition of R^*, each S_i^k ($k \leqslant N_i$) is accessible to more than n worlds that contain ψ and thus, using the induction hypothesis, at which which ψ is true. Thus, $M^*, w^* \vDash \Diamond^{>n}\psi$.

($\Rightarrow$) If $M^*, w^* \vDash \Diamond^{>n}\psi$, then (again let $w^* = S_i^k$) more than n R^*-successors of S_i^k verify ψ and, by induction, contain ψ. Let, for $j \leqslant m$, $r_j = n_j(w_i)$ if $\psi \in S_j$, 0 else. Then $r_1 + \dots + r_m$ is the number of R^*-successors of w^* that contain ψ, so $r_1 + \dots + r_m > n$. Using the definition of $n_j(w_i)$ and that of r_j, we see $M, w_i \vDash \Diamond^{>n}\psi$. Hence, $\Diamond^{>n}\psi \in S_i$, as required. □

Corollary 6.5. *Let M^* be a filtrate of M through Φ. Then $M, w \vDash \varphi$ iff $M^*, S(w) \vDash \varphi$.*

Proof. $M^*, S(w) \vDash \varphi$ iff $\varphi \in S(w)$ iff $M, w \vDash \varphi$. □

It is important to observe that the filtrate M^* is constructed through a *finite* set Φ of formulas. For instance, there is no such a filtrate through $\{\Diamond^{>n}p \mid n \in \mathbb{N}\}$.

Corollary 6.6.
$\mathrm{Th}_{\mathsf{GML}}(\{M \mid M \text{ is a model}\}) = \mathrm{Th}_{\mathsf{GML}}(\{M \mid M \text{ is a finite model}\})$;
$\mathrm{Th}_{\mathsf{GML}}(\{F \mid F \text{ is a frame}\}) = \mathrm{Th}_{\mathsf{GML}}(\{F \mid F \text{ is a finite frame}\})$.

Definition 6.7. A (graded) modal system S has the *finite model property* if each non-theorem of S is falsified in a *finite* model.

Corollary 6.8. GrK *has the finite model property.*

Proof. Immediate from 2.8 and 6.6. □

Theorem 6.9. GrK *is decidable.*

Since **GrK** is not finitely axiomatized (cf. (A2)–(A4) of definition 2.6, we cannot immediately apply 6.8. However, from our construction of a filtrate, we can easily compute an upper bound on the number of models to be considered when seeking a finite model for a consistent φ: it is $2^{|\Phi|} \cdot H$, where Φ and H are defined in 6.2.

6.2 Adding special conditions

Definition 6.10. For each class $\mathbb{F}$ of frames we define $\mathbb{F}_{\mathsf{fin}} \subseteq \mathbb{F}$ as all of $\mathbb{F}$'s finite elemets. We say that $\mathbb{F}$ is *characterized by its finite elements* (c.f.e.) if $\mathrm{Th}_{\mathsf{GML}}(\mathbb{F}) = \mathrm{Th}_{\mathsf{GML}}(\mathbb{F}_{\mathsf{fin}})$.

We can now restate 6.6 by saying that the class $\mathbb{F}$ of all frames is c.f.e. We even have $\mathrm{Th}_{\mathsf{GML}}(\mathbb{F}) = \mathrm{Th}_{\mathsf{GML}}(\mathbb{F}_{\mathsf{fin}}) = \mathrm{Th}_{\mathsf{GML}}((\mathbb{F}_{\mathsf{fin}})^c)$ (where $()^c$ is the complement). It is worthwhile to note that this c.f.e.-property is not immediately inherited for subclasses of frames. The filtrate F^* of $F \in \mathbb{F}$ need not itself be a member of $\mathbb{F}$ (cf. 6.15). We now give some classes for which c.f.e. holds. As we proceed, the application of the filtration lemma becomes less straightforward.

Theorem 6.11. *The class $\mathbb{F}^n$ (cf. definition 4.5) is c.f.e.*

Proof. We filtrate M for which $M, w \models \varphi$ through $(\varphi \wedge \Diamond^{=n}\top)$. □

Theorem 6.12. *Let $\mathbb{F}_U$ be the class of frames in which R is universal: $\mathbb{F}_U = \{F = \langle W, R\rangle \mid \forall x, y\,(Rxy \wedge Ryx)\}$. Then $\mathbb{F}_U$ is c.f.e.*

Proof. We show that the filtrate M^* of a universal model M is itself universal. Take two worlds S_i^k and S_j^ℓ in W^*. Since W^* has at least k copies of S_i, there was a world $w \in W$ for which $n_i(w) \geqslant k$. Since R was universal, we have for all $w' \in W$ that $n_i(w') \geqslant k$, in particular, $n_i(w_j!) \geqslant k$. By definition of R^*, we see that each copy S_j^ℓ is accessible to at least the first k copies of S_i, implying $R^* S_j^\ell S_i^k$. □

Remark 6.13. We thus have $\mathrm{Th}_{\mathsf{GML}}(\mathbb{F}_U) = \mathrm{Th}_{\mathsf{GML}}((\mathbb{F}_U)_{\mathsf{fin}})$. Although we also know that for each universal model M with $M, w \models \varphi$, there is a finite universal model M^* and its world w^* with $M^*, w^* \models \varphi$, the converse is not true. For, if M is finite, we have $M, w \models \Diamond^{\leqslant n}\top$ for some n, which obviously fails in any infinite universal model.

Corollary 6.14. *The system* **GrS5** $=$ **GrK** $+ \{\Box p \to p\} + \{\Diamond^{>n} p \to \Box\Diamond^{>n} p \mid n \in \mathbb{N}\}$ *has the finite model property and is decidable.*

Proof. Each **GrS5**-consistent formula is satisfiable in some universal model ([Kap70, Fin72]). □

In chapter six, the system **GrS5** is studied in more detail. It appears to provide a natural context to study generalized quantifiers. It is shown that it has some very neat normal forms (for instance, embedded modalities are superfluous) and also that the complexity of deciding whether a given GML-formula is satisfiable, is PSpace. A special kind of semantic normal forms provide a technique with which some questions, familiar from the field of generalized quantifiers, like obtaining a Lyndon theorem (finding a syntactic characterization of upward-monotonicity), are very easily settled. We think that applying those techniques to systems like **GrK** and **GrT**, although more complicated by the lack of the mentioned normal forms, is worthwhile studying.

We now consider a class of frames which is not closed under filtration, but for which we can still prove the c.f.e.-property.

Theorem 6.15. *The class $\mathbb{F}_R$ of reflexive frames is c.f.e.. Hence,* **GrT** *is decidable.*

Proof. We now have $n_i(w_i!) \geqslant 1$. Suppose S_i^k is R^*-connected to the first t copies of S_i: $S_i^1, \ldots S_i^t$, with $t \geqslant 1$. The problem is that, if S_i^{t+1} exists, it is not accessible to itself. We simply change R^* as follows: for each $s > t$, we withdraw $R^* S_i^s S_i^1$ and replace it with $R^* S_i^s S_i^s$. Clearly this modified R^* is reflexive and, since we did not change any number of successors of any world, lemma 6.4 and hence corollary 6.5 still hold. □

The c.f.e. property is established for one more class of frames:

Theorem 6.16 ([Hoe91d]). *$\mathbb{F}^{\mathrm{lo}}$, the class of linear frames, is c.f.e.*

In order to warn the reader for some possible pitfalls, we round off with a negative example.

Theorem 6.17. *Both the class of transitive irreflexive frames, and that of transitive antisymmetric frames, are not c.f.e.*

Proof. There is no finite transitive irreflexive model in which $\Diamond\top \wedge \Box\Diamond\top$ is true in any world. For, suppose W is finite, and φ true at w. Then w and all its successors must have at least one successor. Let $W_1, w_2, \ldots$ be a sequence such that $w = w_1$ and Rw_iw_{i+1}. By transitivity, Rw_iw_{i+n} for each $i, n \in \mathbb{N}$. Since W is finite, there must be n and i such that $Rw_{i+n}w_i$. Transitivity now yields Rw_iw_i, contradicting our assumption about irreflexivity.

For transitive antisymmetric frames, we apply a similar argument to $\Diamond^{>1}\top \wedge \Box\Diamond^{>1}\top$. We need a higher 'grade' in order to deal with 'reflexive endpoints' now: we find a pair for which Rw_iw_{i+n} and $Rw_{i+n}w_i$ *and* $w_i \neq w_{i+n}$! □

So, it is still an open question which transitive frames are c.f.e. (like those with a universal relation, cf. 6.12) and which are not (like those with an irreflexive, or antisymmetric relation, cf. 6.17).

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Figure 1: Figure

Figure 2: Figure

Figure 3: Figure

Figure 4: Figure

Figure 5: Figure

Figure 6: Figure

Figure 7: Figure

Figure 8: Figure