FLOW OF TIME IN BST/BCONT MODELS AND RELATED SEMANTICAL OBSERVATIONS

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ABSTRACT. First the Branching Space-time and Branching Continuations models are briefly presented. We compare their properties with the traditional definition of a Flow of Time from physics and we point out the difficulties of it in relativistic time. A solution of a Flow of Time in the given models is then proposed.

1. INTRODUCTION

This paper explores the semantical issues and possibilities of two temporal branching structures, Branching Space-time (BST) and Branching Continuations (BCont). In order for the models to be of any use, they should include our intuitive notions connected with time. One of these notions is the Flow of Time (FoT). These attempts to incorporate FoT into the BST/BCont models also yield interesting results about the semantical nature of these structures. We mention the basic idea of BST/BCont in the first section. The second section is devoted to the presentation of the traditional FoT definition and the problems it faces in relativistic physics, and in the end we give a possible solution how to incorporate it into BST/BCont.

2. BST AND BCONT

The two formal approaches to space-time studied in this article are Branching space-time, first introduced by N. Belnap in [1], and Branching Continuations, formed by T. Placek in [5]. The notion B-models is used in this article in cases where we refer to both models. We present only basic rudiments of both systems here.

2.1. **BST.** The basis of a BST model is $\langle W, \leq \rangle$. In other words, we have a set of point events, called *Our World* and denoted W, that is partially ordered by $\leq$. This set represents all the possibilities and options of the world. We can find in W a particular kind of sets of events, called histories. These sets are the maximal directed subsets of W. The property directed describes sets that have for every two of their members a common upper bound. These histories represent possible scenarios of events. It was T. Müller, who later prepared a version of BST called Minkowski branching structure (MBS) where histories are isomorphic to Minkowski space-time. We leave any further motivations, details, or explanations to the reader. They can all be found in [2] or [6].

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2.2. **BCont.** BCont was introduced as an attempt to create a version of BST capable of coping with spatiotemporal holes (such as singularities). The basic change lies in the construction of histories. Instead of a BST history, we content ourselves with any nonempty consistent subsets of W. The question of consistency is founded on the connectivity of two points in the model via a snake-link. A snake-link is a path of point-events where every two point-events are comparable by $\leq$. These replacements of histories are then called large events or just l-events. We recommend the reader to pay attention to the proposed semantics of BCont - the so called Branching Time+*Instants*-like models[1] in the original paper [5]. These semantics take BCont closer to the original idea of Branching Time, i.e. a usual tree structure, and thus force l-events to be chains.

3. Flow of Time

A common perception of time is seeing it as a succession of events, in other words a flow of time (FoT). We aim to incorporate this notion in some way into the B-models. We now address the question of how the formal notion of a flow of time in logic differs from the flow of time as it is understood in physics and present the basic requirements for a FoT.

We follow two introductory articles to temporal logics [8] and [4]. In these articles, we can find the term 'flow of time' described as a pair $(T, \prec)$, where T is a non-empty set of time points and $\prec$ is a irreflexive and transitive binary relation on T, i.e. a strict partial order. Actually, FoT means any kind of temporal structure in the usual temporal logics. It can mean also branching or circular models. However, we would like to take our motivation from physics. The founding article of BST [2] can serve as a good transition as Belnap argues in it that there is a 'classical view' on time - as a succession of infinite Euclidean spaces.

A modern view on FoT, as judged by physics, can be found in [3]. Dieks describes how we can assign a different now-point to every world line and generate a partial order based on the linear order of the now-points on the world lines. This process has one important restriction: no now-point should lie in the interior of the conjunction of past lightcones of other now-points. This construction of FoT can be used, if one stays careful, even in special relativity. Let us sum up what this notion entails according to Dieks:

Definition 1 Generalized FoT

A **generalized flow of time** fulfils the following:

(1) world lines with a linear order of now-points
(2) ontological definiteness of past and present
(3) a relation (or set) of now-points on world lines respecting ontological definiteness

At this point, we can turn to the B-models to see how generalized flow of time can be formalized in those models.

4. Does time flow in B-models?

To sum up our point of departure we can see that if we intend to study FoT in B-models we must ask ourselves what kind of FoT we want to study. As we

[1]Abbreviated as BT+I.

already know, studying the pure logical meaning of the term is useless as it covers also branching structures. However, we can study a different type of FoT. In our case it is the generalized FoT. We follow the idea of D. Dieks and we leave behind the notion from temporal logic. A basic formalization of a generalized FoT could be $(wl, N, W, \leq)$, where wl are world lines, N is the set of now-points, W are point-events with $\leq$ being the partial ordering of these points. In addition, the set N follows the rule of ontological definiteness. As we might already see from this draft formalization, our primary view is closer to McTaggart's B-series and relies on an ordering of events. We investigate how a generalized FoT can relate to B-models and how could we incorporate the basic formalization into B-models. We address the models in the following order: BT+I-like models, BCont, and BST.

4.1. **BT+I-like models of BCont.** The reason why we start with BT+I-like models of BCont should not be a surprising fact as their basis, the original Prior/Thomason models, were close to the intuitive notion of a flow of time. Let us have a model $\langle W, \leq, S\rangle$. A member of S, an instant, is in this case a spatio-temporal location as defined in definition 2. These are ordered by the relation $\precsim$. Thanks to the facts 15 and 16 from [5], we know that this ordering is dense, partial and even linear. Our first observation is thus straightforward and simple. A BT+I-like model comes very close to fulfil our formal demands for a flow of time. We just need to find a way to put that into a formal statement. We list some definitions for reference[2]:

Definition 2 S-t locations [5]

We say that a model $\langle W, \leq\rangle$ of BCont has *spatio-temporal locations* iff there is a partition S of W such that

(1) For each l-event A and each $s \in S$, the intersection $A \cap s$ contains at most one element;
(2) S respects the ordering $\leq$, that is, for all l-events A, B, and all $s_1, s_2 \in S$, if all the intersections $A \cap s_1, A \cap s_2, B \cap s_1$ and $B \cap s_2$ are nonempty, and $A \cap s_1 = A \cap s_2$, then $B \cap s_1 = B \cap s_2$;
(3) similarly for the strict ordering $<$;
(4) if $e_1 \leq e_2 \leq e_3$, then for every l-event A such that $s(e_1) \cap A \neq \emptyset$ and $s(e_3) \cap A \neq \emptyset$, there is an l-event A' such that $A \subseteq A'$ and $s(e_2) \cap A' \neq \emptyset$, where $s(e_i)$ stands for a (unique) $s \in S$ such that $e_i \in s$;
(5) if L is a chain of choice events in $\langle W, \leq\rangle$ upper bounded by e_0 and such that $\exists s \in S \forall x \in L \exists e \in W : (x < e \wedge s(e) = s)$, then $\exists e^* \left(e^* \in \bigcap_{x \in L} \Pi_x \langle e_0\rangle \wedge s(e^*) = s\right)$.

Definition 3 Ordering of s-t locations [5]

For $s_1, s_2 \in S$, S being the set of s-t locations, let $s_1 \precsim s_2$ iff $\exists e_1, e_2 (e_1 \in s_1 \wedge e_2 \in s_2 \wedge e_1 \leq e_2)$.

BT+I-like models were used by Placek as a neat way to introduce semantics and valuation on the BCont framework (or at least on a special kind of BCont). It seems suitable to use the motivation of BT+I-like models and treat FoT as a question of valuation. It suffices to use definition 23 from [5]:

[2] All referenced material has a bibliography reference number in its title. If the referenced definition is altered in some way, this reference number is accompanied by an apostrophe.

Definition 4 Point fulfils formula [5]

For given $e_C, e/A$ and the model $\mathfrak{M} = \langle \mathfrak{G}, \mathcal{I} \rangle$.Then:

(1) if $\psi \in$ Atoms:$\mathfrak{M}, e_C, e/A \Vdash \psi$ iff $e \in \mathcal{I}(\phi)$[3];
(2) if ψ is $\neg\varphi : \mathfrak{M}, e_C, e/A \Vdash \psi$ iff it is not the case that $\mathfrak{M}, e_C, e/A \Vdash \varphi$;
(3) for $\wedge, \vee, \rightarrow$ also in the usual manner;
(4) if ψ is $F_x\varphi$ for $x > 0 : \mathfrak{M}, e_C, e/A \Vdash \psi$ iff there are $e' \in W$ and $e^* \in A$ such that $e' \leq e*$ and $int(e', e, x)$, and $\mathfrak{M}, e_C, e'/A \Vdash \varphi$;
(5) if ψ is $P_x\varphi, x > 0 : \mathfrak{M}, e_C, e/A \Vdash \psi$ iff there is $e' \in W$ such that $e' \cup A \in$ l-events and $int(e', e, x)$ and $\mathfrak{M}, e_C, e'/A \Vdash \varphi$;
(6) if ψ is $Sett : \varphi : \mathfrak{M}, e_C, e/A \Vdash \psi$ iff for every evaluation point e/A' from fan $\mathcal{F}_{e/A}$ and $\mathfrak{M}, e_C, e/A' \Vdash \varphi$;
(7) $Poss : \psi := \neg Sett : \neg\psi$;
(8) if ψ is Now $: \varphi : \mathfrak{M}, e_C, e/A \Vdash \psi$ iff there is $e' \in s(e_C)$ such that $e' \cup A \in$ l-events and $\mathfrak{M}, e_C, e'/A \Vdash \varphi$.

It is the last clause that should catch our attention. Observe that either $e' = e_c$ or the two events are inconsistent, in other words belong to different continuations of a choice point earlier than e_c.

Lemma 5

For given $e_C, e/A$ and the model $\mathfrak{M} = \langle \mathfrak{G}, \mathcal{I} \rangle$. If $\mathfrak{M}, e_C, e/A \Vdash$ Now $: \varphi$ and thus there is a e' such that $\mathfrak{M}, e_C, e'/A \Vdash \varphi$ then the point-event e' mentioned in definition 4 is either equal to e_C or is inconsistent with it.

Proof. If $e' = e_C$ then by definition $e' \in s(e_C)$. However, $e' \cup A \in$ l-events holds only if A is consistent with e_C. If they are not consistent then e_C cannot be e' as the evaluation point needs compatibility of e' and A. Therefore let us choose an e' different from e_C. If it were consistent with e_C, A would also need to be consistent with e_C. However, $s(e_C) \cap A'$ for any A' has according to definition 2 at most one member. If they are consistent, we can choose $A' = \{e'\} \cup \{e_C\} \cup A$ and thus only one possible outcome of the intersection, namely e_C. Hence supposing e' and e_C are consistent but different also leads to a contradiction. □

We can now introduce the basic idea of generalized FoT (gFoT). We understand gFoT as an ordering of sets of events deemed as contemporary to a given point-event e. The sets are ordered according to a world line. We also need to follow the points mentioned in the definition 1. The promise of the introduction of gFoT is that it could alleviate the work with the original BCont model. As Placek writes in [5]:

> We take the Kripke/Prior/Thomason semantics for our reference theory, since it is relatively simple and we have some intuitions concerning tenses. We do not have comparable intuitions concerning relativistic notions, and for this reason it will not be revealing to take BST for our reference theory.

We could place weaker constrains on BCont models by using gFoT instead of BT and thus exploit much more from their potential. We first introduce the notions in the context of the familiar and easier BT+I-like models, later to be re-evaluated in pure BCont and also in BST models.

[3] We use $\Vdash$ instead of Placek's $\not\approx$ purely for technical reasons, the meaning is the same.

Some important choices have to be done at this point. Namely, how do we want to identify a given set of 'now'-points; do we want it to also reach different continuations or be only valid in one continuation? The point e_C already serves in the original approach as a point of reference, thus there seems to be nothing wrong in using this point again. Addressing the second question, we saw in def. 1 that there should be a difference between indefinite future and definite past. If we were to hold only onto one continuation, we could not make this distinction. On the contrary, making use of the available branching structure seems as a natural way to combine gFoT and branching, therefore now-points should also include possible continuations.

Definition 6 Setting of now-points

X_e is a setting of now-points for the point-event e iff $e \in W$ and for $X_e \subseteq W$ it holds that (1) $e \in X_e$, (2) $\forall x, y \in X_e : x \nless y \wedge y \nless x$, and (3)$\forall x \in X_e : x$ is consistent with e.[4]

This is just a general linking of sets of now-points connected to a point of reference in a given continuation. We could also make the definition shorter by using the term space-like related (SLR) mentioned in [5]. Two points are SLR if they are consistent but incomparable. Thus a setting of now-points is a set containing a reference point and points that are SLR with each other. We can also have a maximal setting of now-points, bearing in mind that being maximal does not yield uniqueness. This partially covers the need to avoid point-events from the past or future of a given point-event. However, as the construction of gFoT asks us for a world line as well, we need to present a notion of world line-like nature. We use a more general approach here than the BT+I-like models would need, where l-events are already chains. A more general definition, however, does not cause any harm. In fact, it can be used later for all BCont models.

Notation 7 World line

A set $Wl \subseteq W$ is a world line iff Wl is a chain. We denote $Wl_{(e)}$ a world line containing the point-event e.

We give the usual BCont chains a new name, although the two terms do not differ in any significant way. This definition might seem superfluous but we add it in order to comply with the original definition 1 vocabulary.

Definition 8 Setting of now-points for a world line

The X_{Wl} is a setting of now-points for a world line iff Wl is a world line and $X_{Wl} = \{X_e | e \in Wl \wedge \forall e, e' \in Wl (X_e \cap X_{e'} = \emptyset)\}$.

In other words, the setting of a world line is constituted of a disjunctive set of settings for the points of the world line. One should keep in mind that this is a set of settings and thus a quite different notion from X_e. We also note some observations concerning the simplicity of these notions in the currently studied models.

Lemma 9

In BT+I-like models, a setting of now-points for e is a singleton $\{e\}$.

Proof. The proof follows trivially from the given definitions. □

[4] Although the letter may seem as the letter 'X', it is the Greek Chi.

This observation is good to be kept in mind if one were to judge on the sense of introducing a new concept to BT+I-like models. The gFoT concept does not bring much new in these models. However, BT+I-like models also work with possible continuations. For this reason (and for future use in BCont) we take into account other continuations.

Definition 10 Setting of now-points in continuation
The set $X_{e,A}$, the setting of now-points in continuation A with respect to point-event e, is equal to the set $X_{e'}$ for some $e' \in s(e)$ such that e' is consistent with A.

Definition 11 Setting of now-points for a world line in continuation
The set $X_{Wl,A}$ is equal to the set $X_{Wl'}$, where Wl' is a chain consistent with A, constituted of events $e' \in W$ such that $\forall e' \in Wl' \exists ! e \in Wl : e' \in s(e)$ and $\forall e \in Wl \exists ! e' \in Wl' : e \in s(e')$.

Lemma 12
In BT+I-like models, if e is inconsistent with A then $X_{e,A}$ has a single point-event, namely $e' \in s(e)$, where e' is consistent with A.

Proof. The proof follows trivially from the given definitions and lemma 5. □

There is a common idea to both definitions. One transforms the original points using spatiotemporal locations to the points consistent with the given l-event and constructs settings for these. With the exception of our approach there seems, in general, no reason to relate $X_{e,A}$ and $X_{e,A'}$, where A is consistent with e and A' is not. One could, based on the physical motivations behind the whole project, for example imagine that continuation A' leads the observer to some gravitational field and thus his setting of now-points should, quite naturally, be different from the observer in the continuation A.

Following our attempt to maintain a general approach, we also address the question of the interval function used in the definitions. The interval function was defined in the original paper [5] for some point events e, e' and a coordinalization $\mathcal{X}$ as follows:

$$int(e, e', t) \text{iff} \mathcal{X}(s(e')) - \mathcal{X}(s(e)) = t \quad (1)$$

As we can see, this definition explicitly builds on two notions that have a specific character in BT-like models. First, the ordering of S in these models and the second the coordinalization. The ordering is quite simple, coordinalization is therefore easily made and allows us to determine t with mere subtraction. If we want to have a more general approach applicable also in the BCont models we can use world lines.

Definition 13
For $e, e' \in W$, A and l-event such that e' is consistent with it, $Wl_{(e)}$ a world line, and $\mathcal{X}$ being a coordinalization on the world line: $\text{int}(e, e', Wl_{(e)}, t)$ is equivalent to:

(1) $\exists e'' : X_{e'',A} \in X_{Wl_{(e)},A} \wedge e' \in X_{e'',A}$
(2) $\mathcal{X}(s(e'')) - \mathcal{X}(s(e)) = t$

We refer here to the same coordinalization as in the original article, i.e. an order preserving bijection $\mathcal{X}$ between $\langle S, \precsim \rangle$ mapped to the dense subset of $\mathfrak{R}$.

The general notion of interval does not, like the other altered definitions, change anything in the context of the simple BT+I models. We can show this in a simple lemma.

Lemma 14

In BT+I models: $int(e, e', t) \equiv (\exists Wl_{(e)})(int(e, e', Wl_{(e)}, t))$.

Proof. The core idea that t is equal to the difference of two points is the same in both definitions. The question is the identity of the two points. Let $int(e, e', t)$ be true. Also let e, e' be consistent. Then there exists an l-event they belong to, let it be A. L-events are chains in BT+I, thus there exists a chain they both belong to. Let this chain be $Wl_{(e)}$. Because $e' \in A$ according to our assumptions, there exists e'' such that $X_{e'',A} \in X_{Wl_{(e)}} \wedge e' \in X_{e''}$. If we take into account lemma 9 then there is only the option that $e' = e''$. Now if e and e' are inconsistent we have an l-event A consistent with e' (in the worst case it is the singleton of e'). Construct a chain $Wl_{(e)}$ such that $Wl_{(e)} \cap s(e') \neq \emptyset$. It follows from definition 2 and from fact 16 in [5][5] that there is one element in BT+I models in this intersection, let it be e''. Therefore $e'' \in s(e')$ and hence $\mathcal{X}(s(e'')) - \mathcal{X}(s(e)) = t$ is the same as $\mathcal{X}(s(e')) - \mathcal{X}(s(e)) = t$. Point-event e'' does verify all that we expect from it, by the definitions and fact 16 it holds that $X_{e'',A} = \{e'\}$ and $X_{e'',A} \in X_{Wl_{(e)},A}$.

The other direction needs to be verified also. Let $\exists Wl(e) : int(e, e', Wl_{(e)}, t)$ be true and may the events be consistent. Then we ask: does it hold that $e' = e''$? And it does, as $X_{e''} \cap e' = e'$ in BT+I models. If the two events are inconsistent then obviously $e' \neq e''$ but for our proof it is enough to show that $s(e') = s(e'')$. This follows from $X_{e'',A} \cap e' = e'$ by definition and lemma 12.

□

At this point we can prepare a new evaluation. We return to the definition of how a point fulfils a formula but alter it to use the newly introduced notions.

Definition 15 Point fulfils formula - BT+*I* and FoT

For given $e_C, e/A$ and the model $\mathfrak{M} = \langle \mathfrak{G}, \mathcal{I} \rangle$, the definition that a point fulfils a formula is the same as in def. 4 with the exception of the following:

- if $\psi \in$ Atoms:$\mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \Vdash \psi$ iff $e \in \mathcal{I}(\phi)$;
- if ψ is $\neg\varphi : \mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \Vdash \psi$ iff it is not the case that $\mathfrak{M}, e_C, e/A \Vdash \varphi$;
- for $\wedge, \vee, \rightarrow$ also in the usual manner;
- if ψ is $F_x\varphi$ for $x > 0 : \mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \Vdash \psi$ iff there are $e' \in \bigcup X_{Wl_{(e_C)},A}$ and $e^* \in A$ such that $e' \leq e^*$ and $int(e, e', Wl_{(e)}, x)$, and $\mathfrak{M}, e_C, e'/A \Vdash \varphi$;
- if ψ is $P_x\varphi$ for $x > 0 : \mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \Vdash \psi$ iff there is $e' \in \bigcup X_{Wl_{(e_C)},A}$ such that $e' \cup A \in$ l-events and $int(e', e, Wl_{(e)}, x)$ and $\mathfrak{M}, e_C, e'/A \Vdash \varphi$;
- if ψ is $Sett : \varphi : \mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \Vdash \psi$ iff for every evaluation point e/A' from fan $\mathcal{F}_{e/A} : \mathfrak{M}, e_C, e/A', X_{Wl_{(e_C)},A} \Vdash \varphi$;
- if ψ is Now : $\varphi : \mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \Vdash \psi$ iff there is $e' \in X_{e_C,A}$ such that $e' \cup A \in$ l-events and $\mathfrak{M}, e_C, e'/A, X_{Wl_{(e_C)},A} \Vdash \varphi$.

Theorem 16

Definition 4 is equivalent to definition 15 in BT+I-like models.

[5]Linearity in BT+I models.

Proof. Some of the points from the definition 4 were not changed in any significant way. Those points obviously hold as we only added a new notion to the right side of $\Vdash$ but it does not influence in any way the valuation in those cases.

The F operator's equivalence: we need to prove that $e' = e'_{FoT}$, where $e' \in W$ comes from definition 4 and $e'_{FoT} \in \bigcup X_{Wl_{(e_C)},A}$ is from definition 15. We are using lemma 14. Let us have $e'_{FoT} \in \bigcup X_{Wl_{(e_C)},A}$. It is a subset of W and it also meets all the requirements of definition 4 and hence it is an e' point as required by it. According to def. 13 and def. 2 it even has to be equal to that point. For the other direction let us have $e' \in W$ that suits the definition 4. Using downward directedness of BT+I models for e_{FoT}, e' there exists e_0 comparable to both events as both events are consistent with A, there can be an l-event A' that both events belong to. The intersection of a s-t location and an l-event is only one point-event. Hence only one point-event fulfils $int(e, e', Wl_{(e)}, x)$ for a given x and so $e' \in W = e'_{FoT}$. And a similar combination of BT+I models' linearity and downward directedness leads to the equivalence of the P operator.

For the Now operator, if we have e' from def. 15 then this exact e' fulfils all that is needed for def. 4 to work. For the other direction let us have e' based on def. 4. However, based on lemma 12, $X_{e_c,A}$ is always a singleton. If A is consistent with e_C then it has e_C as its single member and from the definition of setting of now-points (def. 10) this single member must be e', thus $e' = e_C$ and it fulfils all requirements of def. 15. If e_C is inconsistent with A then the only change is that $e' \neq e_C$ but it is again a member of $X_{e_C,A}$. □

With this theorem, let us shift our attention to the other B-models as the results there promise to be less trivial. Our work in BT+I served merely as a didactical or pragmatic training field before we enter the main battlefield of gFoT for which all the general approach was meant.

4.2. **BCont.** Our first attempt of use of the tested definitions is in the BCont structures without the limitations imposed on them by the BT-likeness. L-events do not have to be chains anymore. This is the main and most important change from the BT+I framework. We use the pure BCont models with the addition of s-t locations as presented in [5]. All the definitions stay unaltered in any way by the introduction of gFoT. However, we need to rethink the following definitions and theorems from [5]:

- $int(e, e', t)$
- extensions of evaluation points
- fan of evaluation points
- point fulfils formula
- definite truth
- three values of definiteness

Some changes were already made in the previous section (the interval definition for example), other changes are inherited (e.g. extension of an evaluation point simply uses the new interval definition) and a few need to be completely redone with regards to BCont models. This section is devoted to the changes that weren't presented already in the BT+I part or differ from those mentioned there. We also use the same structure and language as before. As a short reminder, the language $\mathcal{L}$ is made out of present tensed atomic formulas, classical logical connectives ($\wedge$ etc.), two metric temporal operators (F_x,P_x), two modal

operators ($Sett$:,$Poss$:), and the operator Now. The semantical model is the same. The model for $\mathcal{L}$ is $\mathfrak{M} = \langle \mathfrak{G}, \mathfrak{I} \rangle$, where $\mathfrak{G} = \langle \mathcal{W}, \mathcal{X} \rangle$ is the structure and $\mathfrak{I} : Atoms \to \mathcal{P}(W)$ is an interpretation function. The structure is composed from a BCont model
$\mathcal{W} = \langle W, \leq, S \rangle$ and $\mathcal{X}$, a real coordinalization of S.

Definition 17 Extensions of evaluation points [5]'
e/A goes at least x-units-above e $(0 \leq x)$ iff $\exists e_1 \in W \exists e_2 \in A \exists Wl \subseteq W$: $(e_1 \leq e_2 \wedge e_1 \in Wl \wedge e_2 \in Wl \wedge int(e_1, e_2, Wl, x)$
e/A' is an x-units-above e extension of $e/A, (0 \leq x)$ iff $A \subseteq A' \subseteq W$ and e/A' goes at least x-units-above e.

For the extension of evaluation points we only added the new definition of an interval. The fan of evaluation points and the so called instant-wise isomorphism can be left as they were in the original paper.

Definition 18 Fan of evaluation points [5]
Two l-events A_1, A_2 of W are isomorphic instant-wise iff $\forall e_1 \in A_1 \exists e_2 \in A_2$: $s(e_1) = s(e_2)$ and $\forall e_2 \in A_2 \exists e_1 \in A_1$: $s(e_1) = s(e_2)$;
A fan of evaluation points for e/A is a set of evaluation points where $e/A' \in \mathcal{F}_{e/A}$ iff e/A' is an evaluation point in $\mathfrak{G}$ and A, A' are isomorphic instant-wise.

We are ready to present how the fulfilment of a formula works in BCont with FoT. We can actually use the same definition as for BT+I. However, we present the whole definition to have it in one place:

Definition 19 Point fulfils formula - BCont and FoT [5]'
For given $e_C, e/A$ and the model $\mathfrak{M} = \langle \mathfrak{G}, \mathcal{I} \rangle$, then:

- if $\psi \in$ Atoms : $\mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \Vdash \psi$ iff $e \in \mathcal{I}(\phi)$;
- if ψ is $\varphi \wedge \phi$: $\mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \Vdash \psi$ iff $\mathfrak{M}, e_C, e/A \Vdash \varphi$ and $\mathfrak{M}, e_C, e/A \Vdash \phi$;
- for $\vee, \to$ in the usual manner;
- if ψ is $F_x\varphi$ for $x > 0$: $\mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \Vdash \psi$ iff there are $e' \in \bigcup X_{Wl_{(e_C)},A}$ and $e^* \in A$ such that $e' \leq e^*$ and $int(e, e', Wl_{(e)}, x)$ and $\mathfrak{M}, e_C, e'/A \Vdash \varphi$;
- if ψ is $P_x\varphi$ for $x > 0$: $\mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \Vdash \psi$ iff there is $e' \in \bigcup X_{Wl_{(e_C)},A}$ such that $e' \cup A \in$ l-events and $int(e', e, Wl_{(e)}, x)$ and $\mathfrak{M}, e_C, e'/A \Vdash \varphi$;
- if ψ is $Sett$: φ : $\mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \Vdash \psi$ iff for every evaluation point e/A' from fan $\mathcal{F}_{e/A}$: $\mathfrak{M}, e_C, e/A', X_{Wl_{(e_C)},A} \Vdash \varphi$;
- if ψ is Now : φ : $\mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \Vdash \psi$ iff there is $e' \in X_{e_C,A}$ such that $e' \cup A \in$ l-events and $\mathfrak{M}, e_C, e'/A, X_{Wl_{(e_C)},A} \Vdash \varphi$.

Definition 20 Definite truth [5]'
ψ is definite at $\mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A}$, written $\mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \models \psi$, iff there is an $0 \leq x$ such that for every x-units-above e extension of e/A' of e/A: $\mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \Vdash \psi$.
ψ is indefinitely true at $\mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A}$, written $\mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \overset{?}{\models} \psi$, iff there is no $0 \leq x$ such that for every x-units-above e extension of e/A' of e/A: $\mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \Vdash \psi$ or for every x-units-above e extension of e/A' of e/A: $\mathfrak{M}, e_C, e/A, X_{Wl_{(e_C)},A} \Vdash \neg\psi$.

Theorem 21 **Three options** [5]
For any formula ψ and any evaluation point e/A, exactly one of the following three options must hold:

$$e/A \models \psi \text{ or } e/A \models \neg\psi \text{ or } e/A ?\!\!\models \psi \quad (2)$$

Proof. The proof is the same as used by Placek in [5]. □

This presents our main topic with only one more addition. Placek has shown in his paper [5] a few examples (he calls them puzzles) of how BT+I valuation works. We use two of those examples, Peircean future and 'Was Einstein born a Nobel Prize winner' to demonstrate how the FoT valuation allows general BCont models to address similar topics.

4.2.1. *Peircean future.* Peircean approach means that a sentence in the future tense being true at e means that it is true in every possible history to which e belongs and thus fails to distinguish between what will happen and what will necessarily happen. As Placek's BT+I models are able to distinguish between the two cases, so are we. We need to capture the difference between $F_x\psi$ and $Sett : F_x\psi$.

Similarly to the original paper, we can demonstrate this using a model $\mathfrak{M}$, where $\mathfrak{M}, e_C, e_C/A, X_{Wl_{(e_C)},A} \models F_1\psi$ and $\mathfrak{M}, e_C, e_C/A, X_{Wl_{(e_C)},A} \models \neg Sett : F_1\psi$.

An exemplar model is visualized on fig. 1 and it asks for some explanation. Although it does not incorporate all the notions it is already quite crowded.

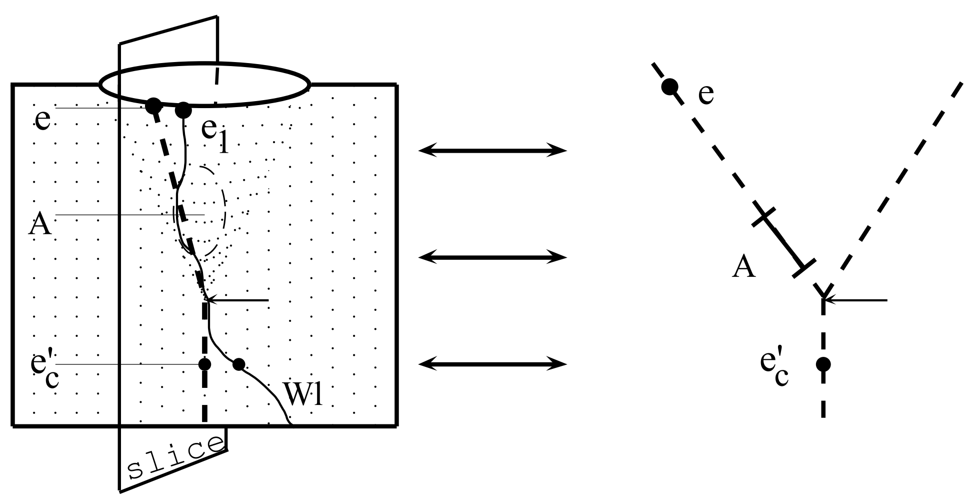

Figure 1. A 2D BCont model for Peircean future.

This two dimensional model (one spatial dimension along the x axis, one time dimension along the y axis) is a subset of W. We see a world line Wl, the curve going from the bottom to the top of the plane, and a point e_C that belongs to the world line. The point e_1 is a point such that $int(e_C, e_1, Wl_{(e_C)}, 1)$ holds. The cone shows two possible continuations from a given choice event (an arrow points to it). The event e represents then a member of $X_{e_1,A}$. If we wanted to see the two possible continuations, we could slice the model with a plane and get the image on the right side. There we can see the l-event A (in the first picture it is the oval shape). The point e' shows a member of $X_{e_1,A'}$. If at e we fulfilled ψ but at e' $\neg\psi$ would hold, then we have a model that suits our purpose. $F_1\psi$ is fulfiled by every 1-units-above-e_C extension of e_C/A, while every fan determined by each x-units-above-e_C extension of e_C/A has an element on the far side of the cone, where $\neg\psi$ is true at the given distance and setting of now-points.

This example also shows the usefulness of the gFoT notions. The world line allows us to have some measure for intervals and the settings of now-points allow us to relate SLR points to each other and thus evaluate sentences using the operators of $\mathcal{L}$.

4.2.2. *Natural born Nobel Prize winner.* In the second example the sentence "Einstein was born a Nobel Prize winner" is analysed. We assert it in the year 2012 and grant that Einstein might have failed to receive the Nobel Prize in 1921. The critical question to examine is the relation between $Sett : P_{100}F_9\psi$, called (S), and $P_{100}Sett : F_9\psi$, called (P), where ψ stands for the given sentence. Placek has shown in [5] that (S) does not imply (P). The figure 2 helps us imagine the situation. The final figure is actually very similar to the figure shown in [5]. Once again, we start out from the figure on the left with a similar description. The difference is now in the position of e_C and the event we are looking for, namely the Nobel Prize ceremony of 1921 (marked N).

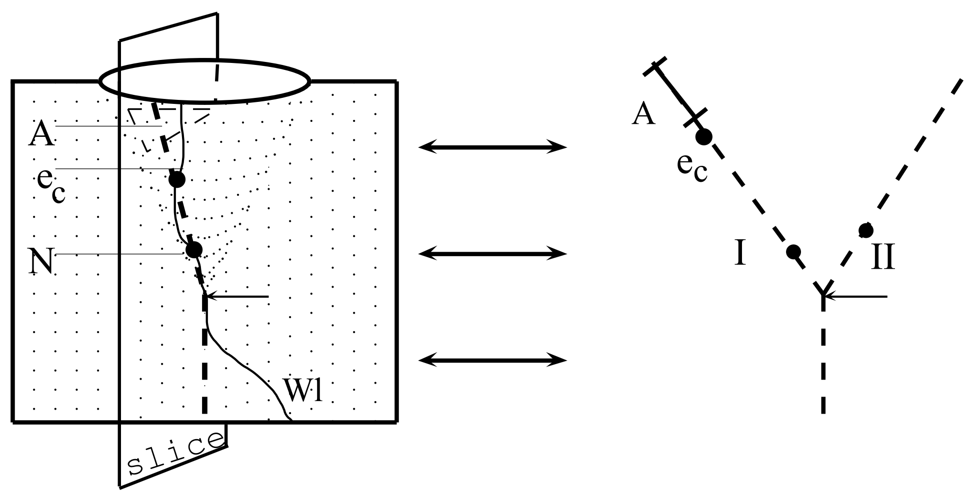

FIGURE 2. A 2D BCont model for $Sett : P_{100}F_9\psi \not\to P_{100}Sett : F_9\psi$.

For every x-units-above e_C extension ψ holds, it means that fans determined by any extension are made out of events that have N in their past. This means that (S) holds in the given model, on the contrary to (P) that does not hold. Our model is made in such a way that for every e for which $int(e_C, e, Wl_{(e_C)}, 100)$ holds, this event does not have the choice event in its 'past' while it has either N or $\neg N$ in its 'future'[6]. We need to say for every event e as the settings of now-points might be

[6]Neither of these was formally introduced, but are simple placeholders for the defined terms of 'strictly above' and 'strictly below'

different. For reasonable settings of now-points and a reasonable Wl we would just take the event 100 units to the past on the world line. Hence we are evaluating $Sett : F_9\psi$ in the event e. However, fans determined by any extension e/A' from this point-event also have the elements from the possible continuation (II) where N did not occur and hence (P) does not hold because neither holds $\mathfrak{M}, e_C, e/A', X_{Wl_{(e_C)},A'} \models Sett : F_9\psi$ nor does $\mathfrak{M}, e_C, e_C/A', X_{Wl_{(e_C)},A'} \models P_{100}Sett : F_9\psi$.

4.3. **BST.** It was already shown by Placek[5] that not every BCont model is a BST model. Hence we cannot simply take our results from Bcont and apply them to a supposed subset of BCont models because BST does not represent such a subset. We need to verify all the steps made with BCont models to make sure. They could also be valid in BST. However, the main goal of this article was achieved and these results are merely a sketch of how BST could be treated. The reason for interest in BST is its closer relation to special relativity and also its MBS interpretation.

We tested gFoT on BCont models. In order to introduce gFoT and a valuation in BST it is necessary to have all notions we need for this task. We follow the basic definitions of BCont that are used in our gFoT approach and try to find BST equivalents for them(using [2] or [7]). To prepare such list, one simply takes the definition 19 and decomposes all its members to the crucial definitions for their existence. We do not list definitions that were introduced in this article because if we have the following notions, we can introduce for example settings of now-points in a similar way as we did here. In this way we get the following list:

- partially ordered model
- chains of events
- alternatives of events
- space-like related events
- evaluation points
- coordinalization
- interval
- fan of evaluation points
- spatiotemporal locations
- an interpretation and a language

Let us explain which conditions are met and why.

P-ord model: This is met by the BST model definition.
Chains: BST does have chains (or causal tracks)
Alternatives: Obviously, alternatives are represented by histories.
SLR: BST has a definition of SLR.
Evaluation points: BST does not have evaluation points but they can be defined as a pair e/h with h being some history to which e belongs.
Coordinalization: BST lacks a coordinalization but it could be introduced based on world lines. Those are dense and linearly ordered and we could create an order-preserving bijection between the world line and a dense subset of some linearly ordered set.
Interval: BST does not have intervals. Given a coordinalization, we could measure intervals based on world lines.
S-t locations: BST does not have s-t locations but we can introduce them a similar way as in BCont - a partition of W with some specific properties.

Fan of evaluation points: BST does not have this notion. However, there is no obstacle to introduce it to BST, especially if we have already introduced s-t locations.

Interpretation and language: We can easily construct an interpretation. The language should be the same as in BCont and can be added to BST.

This sketch suggests that BST models could also incorporate the gFoT ideas we used in BCont. It is interesting to see that some ideas are missing in BST itself but they are present in MBS. This is, for example, the case of coordinalization (using Minkowski spacetime distance) or s-t locations (points from the Minkowski spacetime).

5. Discussion

The FoT approach taken in this article favours, as we have seen, the view that flow of time is merely an ordering and does not need anything more. However, there is a dichotomy present in our current work as we started out by having a flow of time that is merely an ordering but while evaluating for a given e_C we do have an ontological difference between the future and the past. This can be seen on the settledness of the past but not of the future for a given point-event.

The other choice we made was to work with BCont rather than with BST or MBS. One could argue that MBS is more suitable to welcome a term from physics (as FoT). In spite of this we chose to use gFoT on BCont hoping it would be a challenging trial for gFoT and the use in BST/MBS would be simpler compared to the BCont.

6. Summary

We presented a way of how to interpret the idea of generalized flow of time by D. Dieks in Branching models. We have shown that it can be accommodated for the use in Branching Continuation models. First we verified their compliance with the original valuation from T. Placek and then we have shown how flow of time based valuation works in Branching Continuations. We demonstrated these properties on two examples, one analysing the question of Peircean future and the other examining the settledness of past events. Thereafter we sketched that this valuation would shift only a little in the Branching space-time model. Finally we closed with remarks on the nature of flow of time with regards to Branching Continuations.

References

[1] BELNAP, Nuel. Branching Space-Time. *Synthese*. 1992, 92, 3, s. 385-434.

[2] BELNAP, Nuel. Branching space-time, postprint January, 2003. *Phil-sci Archive* [online]. 2003, unpublished, [cit. 2011-07-25]. Available at: http://philsci-archive.pitt.edu/1003/.

[3] DIEKS, Dennis. *Special relativity and the flow of time.* In: Philosophy of Science. 1988. pp. 456–460.

[4] HODKINSON, I.; REYNOLDS, M. Temporal Logic. In BLACKBURN, Patrick; BENTHEM, Johan van; VOLTER, Frank. *Handbook of Modal Logic*, Volume 3. Amsterdam : Elsevier Science, 2006. s. - .

[5] PLACEK, Tomasz. Possibilities Without Possible Worlds/Histories. In: Journal of Philosophical Logic. 2009. pp. 1–29. Available at: http://philsci-archive.pitt.edu/8633/.

[6] PLACEK, Tomasz; BELNAP, Nuel. Indeterminism is a modal notion: branching spacetimes and Earman's pruning. In: Synthese. 2011. pp. 1–29. Available at: http://philsci-archive.pitt.edu/8631/.

[7] PLACEK, Tomasz; WRONSKI, Leszek. On Infinite EPR-like Correlations. In: Synthese. 2009. 167. pp. 1–32. Available at: http://philsci-archive.pitt.edu/3362/.
[8] VENEMA, Yde. Temporal Logic. In GOBLE, Lou. *The Blackwell Guide to Philosophical Logic*. Hoboken : Wiley-Blackwell, 2001. s. 520. ISBN 06-3120-693-0.

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