# Supervaluationism and Paraconsistency

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*Abstract.* Supervaluational semantics have been applied rather successfully to a variety of phenomena involving truth-value gaps, such as vagueness, lack of reference, sortal incorrectedness. On the other hand, they have not registered a comparable fortune (if any) in connection with truth-value gluts, i.e., more generally, with semantic phenomena involving overdeterminacy or inconsistency as opposed to indeterminacy and incompleteness. In this paper I review some basic routes that are available for this purpose. The outcome is a family of semantic systems in which (i) logical truths and falsehoods retain their classical status even in the presence gaps and gluts, although (ii) the general notions of satifiability and refutability are radically non-classical.

## 1. Introduction

Since its first appearance in van Fraassen's semantics for free logic [1966a, 1966b], the notion of a supervaluation has been regarded by many as a powerful tool for dealing with truth-value gaps and, more generally, with phenomena involving semantic indeterminacy, partiality, deficiency of meaning. Unlike three-valued semantics, whose truth-functional character induced a proliferation of competing variants, supervaluational semantics appeared to offer a uniform way of keeping these phenomena under control; and work in the following decades has supported this expectation with a fair deal of significant developments. These include, among others, applications to such diverse domains as quantum logic, temporal logic, vagueness, presuppositions, sortal incorrectness, or the semantic paradoxes.[1]

In light of this prosperity, it is remarkable that supervaluation-like methods have not registered a comparable fortune (if any) in connection with truth-value gluts, i.e., more generally, with phenomena involving semantic overdeterminacy or inconsistency as opposed to indeterminacy and incompleteness. Part of the explanation is, of course,

[1] See Van Fraassen [1968, 1969], Skyrms [1968], Lambert [1969], Thomason [1970, 1972], Fine [1975], Kamp [1975, 1981], Bencivenga [1978, 1980b, 1981], Klein [1980], Pinkal [1983], van Bendegem [1993], McGee and McLaughlin [1994] among others. The discussion of vagueness in Lewis [1970], Grant [1974], Dummett [1975], and Przełęcki [1976] also contain supervaluational ideas, although the connection is not made explicit. In this sense, the approach seems to trace back to Mehlberg [1958], §29.

to be found in the greater hostility that such forms of semantic anomaly have registered on the whole. However, truth-functional methods *have* been applied to semantics with gluts, or with gaps and gluts alike. For instance, four-valued generalizations of Kleene's [1938] three-valued matrices have become rather popular among those who view gaps and gluts as two sides of the same coin, two complementary ways in which a semantics may fail to be uniquely determined.[2] One wonders, then, whether and to what extent similar generalizations are available—at least in principle—to the friends of supervaluationism.[3]

The purpose of this paper is to show that one can actually go quite far in that direction. There is no intrinsic difficulty in applying the supervaluational insight to interpret a language in the presence of semantic inconsistency. More generally, there is no intrinsic difficulty in generalizing the concept of a supervaluation so as to deal with both kinds of semantic anomaly—gaps *and* gluts. In fact there are many options available; and in all cases the distinguishing features of van Fraassen's original technique can be preserved: (i) the general notions of satifiability and refutability are non-classical, but (ii) logical truths and logical falsehoods (and more generally entailment relations between sentences) retain their classical status.

## 2. Supervaluations, Subvaluations, Ultravaluations

There is no question that the notion of a supervaluation *as such* is unsuitable for dealing with semantic inconsistency. Typically, the supervaluation induced by an incomplete interpretation, or model, $M$ is construed as a function of the valuations induced by a certain class of complete *extensions* of $M$.[4] If $M$ is at worst incomplete, i.e., incomplete but consistent, such extensions are bound to be classical models, since they result from "filling in" some gaps in an otherwise classical model. (This is not trivially true and may have to be qualified depending on the syntax of the language, but the basic idea is unproblematic.) Accordingly, if a sentence $A$ involves expressions with respect to which $M$ is undefined, one can just look at the values $A$ takes on such complete extensions (values that can be determined by a classical evaluation procedure) and then register

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2 See e.g. Dunn [1976], Belnap [1977],Woodruff [1984], Muskens [1989], Fitting [1992], and Gupta and Belnap [1993]. Most semantics for paraconsistent logics can also be viewed in this light (see Priest & Routley [1989] for a survey).

3 Some hints can be found in Grant [1975], Belnap [1977], Rescher & Brandom [1980], Lewis [1982], and Visser [1984]. My own preliminary forays are in Varzi [1991, 1994, 1997]. See also Hyde [1997] for an application to vagueness.

4 In van Fraassen's original formulation, this construction is actually left on the background, as the basic valuations are introduced directly by assigning arbitrary truth-values to those atomic formulas that are left indetermined by the given interpretation. With few exceptions, however, that account has been abandoned in favor of the construction outlined here. See Skyrms [1968] and Bencivenga [1980a] for explicit motivations, and Herzberger [1982] for further discussion and comparisons.

their pattern of agreement: (i) if *A* is true in every extension, then certify it true in *M*; (ii) if *A* is false in every extension, then certify it false in *M*; (iii) otherwise leave the value of *A* in *M* undefined, for there appears to be no definite way of making up for the indeterminacy of *M*. This is roughly how a supervaluation works: it reduces the problem of evaluating a sentence on a non-classical model to that of evaluating it on a family of classical models. However, it is crucial for this method that we start with a model that is "only" incomplete. If we begin instead with a model that is inconsistent, then its extensions will be of no help: extending an inconsistent structure can never yield classical valuations, as inconsistency is preserved under extension. And without classical valuations, the main rationale for construing a supervaluation breaks down[5].

Note that the argument does not depend on the specific notion of a model at issue (hence, ultimately, on the relevant notion of a language). Simply, it emphasizes what Brian Skyrms [1968] once called the "Aristotelian notion of Redemption" offered by supervaluations in response to the "Fregean notion of Sin" represented by a gapful modeling: if we get the same outcome no matter how we fill in the gaps, then the gaps don't matter. But, of course, if there is no gap to begin with there is no redemption either. If we therefore want to restore some balance between gaps and gluts we cannot just look for a straightforward generalization of supervaluations, at least not in the sense in which, say, a four-valued Kleene semantics is a straightforward generalization of the three-valued version.[6] Rather, we must consider extending the initial notion (e.g., by analogy) in such a way as to preserve the "spirit" if not the "letter" of supervaluational semantics as usually understood.

Now, if we keep at this intuitive level, one obvious solution suggests itself. If an incomplete model *M* induces a *super*valuation defined on the basis of *M*'s *extensions* (specifically its *complete* extensions, each of which corresponds to some way of filling in the gaps in *M*), likewise an inconsistent model *M* may be taken to induce a sort of *sub*valuation defined on the basis of *M*'s *restrictions* (specifically its *consistent* restrictions, each of which reflects some way of "weeding out" *M*'s semantic gluts). This is intuitive, especially if we emphasize the duality between incompleteness and inconsistency. In particular, if the guiding idea behind the notion of a supervaluation is that an incomplete model is essentially the *meet* of a series of complete extensions thereof, so that the supervaluation is construed essentially as the meet of the classical valuations induced by those extensions, likewise an inconsistent model may be viewed as the *join* of

[5] I take 'classical' valuations to be gapless and glutless assignments of truth values, leaving room for the possibility that these deviate locally from classical logic (as for instance in Bencivenga's semantics for free logic [1980a, 1981]).

[6] A four-valued Kleene semantics can be obtained from the three-valued semantics simply by dropping the assumption that truth and falsehood are mutually exclusive, just as the three-valued semantics is itself obtained from classical semantics by dropping the assumption that truth and falsehood are exhaustive. (The usual clauses work as long as we state those for falsity along with those for truth.) This was the gist of Dunn's remarks in [1976].

a series of consistent restrictions thereof, and the subvaluation can be construed as the join of the classical valuations induced by those restrictions. In the general case, where a model may be both incomplete and inconsistent (i.e., underdetermined with respect to some basic expressions and overdetermined with respect to others), one can then rely on a suitable combination of these two complementary ideas. If *M* is such a model, neither its complete extensions nor its consistent restrictions are classical. However, the complete extensions of its consistent restrictions are complete and consistent, hence classical, and one can use this fact to obtain the desired generalization (the "ultravaluation", to give it a name): just take the second-order subvaluation based on the admissible supervaluations; or, alternatively, take the second-order supervaluation based on the admissible subvaluations.[7]

To illustrate, suppose we follow Belnap [1977] and think of a model as a data base, or as an epistemic state of a computer. (An epistemic conception of gaps and gluts is of course best suited to supervaluationism.) Sam and Elisabeth enter the relevant information. If neither says anything about *P*, then there is a gap. If the two of them enter contradictory information about *P*, then there is a glut—not a desirable situation, but certainly a conceivable one. Now consider the following three cases:

*A* = true
*B* = false
*C* = gap.

If you ask the computer about the disjunction '*A* or *C*', it should treat it as true in spite of the gap, for the gap is ultimately irrelevant: no matter how the computer fills in its lack of information, the disjunction comes out true by virtue of the first disjunct's being true. However, if you ask the computer about the disjunction '*B* or *C*', then the computer would treat it as neither true nor false: opposite ways of filling in the gap yield opposite outcomes, and there is no way for the computer to choose one outcome over the other. This is how supervaluations work. On the other hand, consider the following set up:

*A* = true
*B* = false
*C* = glut.

[7] This is the suggestion I examine in Varzi [1991, 1994]. The idea of a subvaluation can be found in various formats in the works cited in note 3 and bears some obvious connections with the non-adjunctive logics pioneered by Jaśkowski [1948] (compare e.g. Da Costa & Dubikajtis [1977], Kotas & Da Costa [1979], Jennings & Schotch [1984], Schotch & Jennings [1989].) Epistemic semantics exploiting similar intuitions may be found in Levesque [1984, 1990], Konolige [1985] and Fagin & Halpern [1987] (see Vardi [1986] for connections). By contrast, the alternative approaches considered in the sequel have not to my knowledge been considered in the literature, with the only exception mentioned in note 8 below.

In this case the disjunction '*A* or *C*' will still be true, but '*B* or *C*' will be treated as both true and false. For the computer has been told that *B* and *C* are both false, so the disjunction must be false; and it has been told also that *C* is true, so the disjunction must be true. There is no way for the computer to disregard one value or the other: both Sam and Elisabeth are equally reliable. Thus both values must be accepted. And this is the subvaluation.

Note that other strategies may be considered. For example, one could program the computer so as to suspend judgment on '*B* or *C*' also in the second case. The computer cannot choose between Sam and Elisabeth—one could argue—hence it cannot answer your query.[8] Formally, this alternative way of dealing with gluts amounts to treating all complete and consistent structures into which a given model *M* can be sharpened as being *on a par* with one another. If there is no precise information, and if this lack of precise information turns out to be relevant, then the sentence cannot be evaluated, regardless of whether the imprecision is due to a gap or to a glut. (Think of a certain customary way of analysing failure of existence and failure of uniqueness in the case of definite descriptions: surely these are opposite semantic accidents, but they both lead eventually to the same situation of indeterminacy). Accordingly, one could construe the general valuation induced by a model *M* as the supervaluation registering the pattern of agreement among *all admissible sharpenings* of *M*, be they complete extensions or consistent restrictions (or complete extensions of consistent restrictions).

Now both strategies—I submit—correspond to some way of implementing a basic idea that is intrinsically supervaluational, viz. the idea of construing the valuation on a model as a function of the valuations induced by a family of models strictly related to the former in terms of semantic content. The two strategies differ as to the specific format of the function (meet versus join), and one can imagine other formats as well. But precisely this seems to be the core of the supervaluationary approach: from an abstract perspective, its distinguishing feature is the purely functional characterization of the process whereby a language is evaluated—not a function of the compositional structure of the language's syntax (as in truth-functional semantics), but a function of the relative complementarity of the language's models. Depending on the variety of functions one considers, a corresponding variety of semantics ensues.

## 3. Example: Propositional Logic

Let us illustrate the above in connection with a concrete example. To avoid lengthy definitions, it will be convenient to focus on a simple case study—say a propositional language *L* built up as usual from a set $\mathcal{A}$ of atomic sentences by means of finitary connectives such as negation (~), conjunction ( ), disjunction ( ), etc. It would be instruc-

[8] See Lewis [1978: 46] for a suggestion in this vein, though Lewis [1983] favors instead a subvaluational account of the sort discussed above.

tive to work with arbitrary or more complex languages, but spelling out the details at the proper abstract level would lead us too far afield. I shall confine myself to some occasional remarks where this choice of syntax may have the effect of hiding something important.[9]

With respect to such a language, a model as classically understood is of course nothing but a bivalent map associating each atomic sentence with a definite truth-value, i.e., a total function from $\mathcal{A}$ to, say, 2={T, F}. By contrast, here we are interested in the possibility of relaxing this notion by allowing a model to be undefined, or even overdefined, with respect to some atomic sentences. This means that, in general, a model need not be a *total* function on the domain $\mathcal{A}$, and it need not be a total *function* either:

(1) A model for $L$ is any relation $M \quad \mathcal{A}\times 2$.

(Alternatively, we could also define a model to be a set-valued function, e.g., a mapping from $\mathcal{A}$ to the power set of 2: the two policies are isomorphic.)

Clearly (1) subsumes the classical notion of a model as a limit case. More generally, it is apparent that the class of all models, $Mod(L)$, is partially ordered in terms of "definiteness" by the inclusion relation , and we may single out complete, consistent, and sharp (i.e., classical) models as distinguished special cases:

(2a) $M$ is complete [$M \quad Comp(L)$] iff $M[p]\neq \quad$ for all $p \quad \mathcal{A}$;
(2b) $M$ is consistent [$M \quad Cons(L)$] iff $M[p]\neq 2$ for all $p \quad \mathcal{A}$;
(2c) $M$ is sharp [$M \quad Shrp(L)$] iff it is both complete and consistent

(where $M[p]$ is the image of $p$ under $M$). Even more generally, one may define various related notions such as, for instance, completeness and consistency relative to single sentences, or sets of sentences. These generalizations are obvious and would be particularly useful in the context of languages with greater semantic complexity, where there may be no guarantee that all gaps in an incomplete model can be filled in—or all gluts in an inconsistent model weeded out—simultaneously. This would be the case, for instance, if $L$ were a first-order language with definite descriptions treated as *bona fide* singular terms, or a language with an abstraction operator. For $L$ as assumed, however, we may ignore these complications.

Relative to this simple setting, the intuitive ideas outlined above can be made precise as follows. (I shall begin with the first strategy, and then move on to consider the second and other strategies in the next sections.) First of all, every unsharp model can be sharpened in various ways: if it is incomplete (but consistent), its sharpenings are its complete extensions (*completions*, for short); if it is inconsistent (but complete), its sharpenings are its consistent restrictions (or *constrictions*); and if it is both incomplete

[9] I deal with these matters at some length in my [1994].

and inconsistent, one may take as sharpenings the completions of its constrictions (or the constrictions of its completions). More precisely:

- (3a) $M'$ is a completion of $M$ [$M' \quad Comp(M)$] iff $M'$ is -minimal in the class {$M_i \quad Comp(L)$: $M \quad M_i$};
- (3b) $M'$ is a constriction of $M$ [$M' \quad Cons(M)$] iff $M'$ is -maximal in the class {$M_i \quad Cons(L)$: $M_i \quad M$};
- (3c) $M'$ is a sharpening of $M$ [$M' \quad Shrp(M)$] iff $M'$ is a completion of some constriction of $M$.

Note that every complete model qualifies as a completion of itself, and every consistent model as a constriction of itself. In fact, the cardinality of *Comp*(*M*), *Cons*(*M*), and *Shrp*(*M*) will always be a power of 2, as an undefined or overdefined atomic sentence can only be sharpened in one of two ways. Note also that -minimality and -maximality are required for the simple purpose of avoiding undue loss of information in the process of sharpening a given model. In the present case this only becomes relevant when the model is both inconsistent and incomplete, but the issue could be more substantial in the context of different model classes. Finally, and more importantly, note that the sharpenings of any model are always sharp.

Now, every sharp model $M': \mathcal{A} \rightarrow 2$ extends to a straightforward classical valuation, i.e., a Boolean map $V': L \rightarrow 2$. Using the above notions, we then extend every model $M \quad \mathcal{A} \times 2$ to a corresponding valuation $V \quad L \times 2$ as follows:

- (4a) if $M \quad Cons(L)$, set $V = \bigcap\{V': M' \quad Comp(M)\}$ (*supervaluation*);
- (4b) if $M \quad Comp(L)$, set $V = \bigcup\{V': M' \quad Cons(M)\}$ (*subvaluation*);
- (4c) in general, set $V = \bigcup\{\bigcap\{V': M' \quad Comp(M'')\}: M'' \quad Cons(M)\}$ (*ultravaluation*).

Thus, in general a sentence is certified true (at least) on a model *M* iff it comes out classically true in every completion of some constriction of *M*, and it is certified false (at least) on *M* iff it comes out classically false in every completion of some constriction of *M*. (For simplicity, I am following the radical course of treating *all* completions and constrictions as admissible; a more general account would require a relation of admissibility to be defined among models, so as to constrain valuations accordingly. I shall come back to such generalizations below.)

It is immediate that the notion defined in (4c) subsumes the ones in (4a)–(4b) as limit cases. That is, relative to models that are consistent or complete, the ultravaluation reduces to the supervaluation and to the subvaluation, respectively, each of which reduces to the classical valuation when the model is both complete and consistent, i.e., sharp. This justifies treating ultravaluations as embodying and generalizing the spirit of supervaluationism, and guarantees that the resulting semantics is normal, i.e., reduces to classical semantics when all the classical requirements are jointly satisfied.

To get a flavor of the sort of generalization afforded by (4a)–(4c), it is now sufficient to look at some instances of the dual behavior of super- and subvaluations. For example, it is clear that truth and falsity are neither exhaustive nor exclusive: just as a supervaluation on an incomplete model will leave certain sentences undefined as a result of conflicting outputs in the model's completions, likewise a subvaluation on an inconsistent model will leave certain sentences overdefined as a result of conflicting outputs in the relevant constrictions. On the other hand, just as a supervaluation may have the effect of dismissing (or "redeeming") a gap in case it turns out to be irrelevant, likewise a subvaluation may have the effect of dismissing an irrelevant glut (think again of the computer's behavior in response to Sam's and Elisabeth's inputs). Here are some illustrative examples:

| (5a) If $M[p]=\{\mathrm{T}\}$ and $M[q]=\ $ : | (5b) If $M[p]=\{\mathrm{F}\}$ and $M[q]=2$: |
|---|---|
| $V[p]=\{\mathrm{T}\}$ | $V[p]=\{\mathrm{F}\}$ |
| $V[q]=$ | $V[q]=2$ |
| $V[\sim p]=\{\mathrm{F}\}$ | $V[\sim p]=\{\mathrm{T}\}$ |
| $V[\sim q]=$ | $V[\sim q]=2$ |
| $V[p\ \ q]=\{\mathrm{T}\}$ | $V[p\ \ q]=2$ |
| $V[p\ \ q]=$ | $V[p\ \ q]=\{\mathrm{F}\}$ |
| $V[p\ \ \sim p]=V[q\ \ \sim q]=\{\mathrm{T}\}$ | $V[p\ \ \sim p]=V[q\ \ \sim q]=\{\mathrm{T}\}$ |
| $V[p\ \ \sim p]=V[q\ \ \sim q]=\{\mathrm{F}\}$ | $V[p\ \ \sim p]=V[q\ \ \sim q]=\{\mathrm{F}\}$ |

In particular, note that just as a supervaluation on an incomplete model has the effect of making a tautology such as '$p\ \ \sim p$' true even if '$p$' is neither true nor false (i.e., even if neither disjunct is true), a subvaluation on an inconsistent model has the effect of making a contradiction such as '$p\ \ \sim p$' false even if '$p$' is both true and false (i.e., even if both conjuncts are true). For just as in the first case the process of filling in the gap guarantees that either '$p$' or '$\sim p$' is true in every possible completion of the model, in the latter case the process of weeding out the glut guarantees that either '$p$' or '$\sim p$' is false in every possible constriction. This restores a perfect symmetry between one of the most characteristic features of the supervaluationist semantics for incomplete models—the distinction between the logical law of *excluded middle* and the semantic principle of *bivalence*—and what seems to be its natural counterpart for inconsistent models.[10]

More generally, here is how the classical truth conditions for compound sentences are weakened on the present account:

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[10] The distinction was first emphasized by Mehlberg [1956], §29, and (independently) by McCall [1966] and especially van Fraassen [1966a], §8. The case for its counterpart can be traced back to Jaśkowski [1948] and Przełęcki [1964], but see also Rozeboom [1962]; more recent sources are Belnap [1977], Rescher and Brandom [1980], Lewis [1982], and Urchs [1994]. I discuss it at some length in Varzi [1997].

(6a) $\mathrm{T} \quad V[\sim A]$ iff $\mathrm{F} \quad V[A]$
$\mathrm{F} \quad V[\sim A]$ iff $\mathrm{T} \quad V[A]$

(6b) $\mathrm{T} \quad V[A \quad B]$ only if $\mathrm{T} \quad V[A] \cap V[B]$
$\mathrm{F} \quad V[A \quad B]$ if $\mathrm{F} \quad V[A] \quad V[B]$

(6c) $\mathrm{T} \quad V[A \quad B]$ if $\mathrm{T} \quad V[A] \quad V[B]$
$\mathrm{F} \quad V[A \quad B]$ only if $\mathrm{F} \quad V[A] \cap V[B]$.

The supervaluational treatment of gaps is responsible for the weakening in the second half of (6b) and the first half of (6c); the subvaluational treatment of gluts is responsible for the weakenings in the other two halves.[11]

It is then easy to see that these symmetries extend throughout the framework. In particular, a simple dualization of the usual argument establishing the adequacy of supervaluational semantics to classical propositional logic will show a corresponding adequacy of subvaluational and, more generally, ultravaluational semantics. To be more precise, where *K* is any set of models and *A* any sentence, define:

(7a) $\mathrm{T}_K(A) = \{M \quad K\text{: T} \quad V[A]\}$ (the *K-truth set* of *A*)
(7b) $\mathrm{F}_K(A) = \{M \quad K\text{: F} \quad V[A]\}$ (the *K-falsity set* of *A*).

Thus, the *K*-truth set of a sentence *A* is the set of all *K*-models in which *A* is at least true, and similarly for its *K*-falsity set. This allows us to generalize the ordinary notions of logical truth, falsity, etc. in the obvious way:

(8a) *A* is *K-true* iff $\mathrm{T}_K(A) = K$
(8b) *A* is *K-false* iff $\mathrm{F}_K(A) = K$
(8c) *A* is *K-satisfiable* iff $\mathrm{T}_K(A) \neq$
(8d) *A* is *K-refutable* iff $\mathrm{F}_K(A) \neq$ .

Then the following is easily seen to hold whenever $Shrp(L) \quad K \quad Mod(L)$:

(9) A sentence *A* is *K*-true/false/satisfiable/refutable iff *A* is *Shrp*(*L*)-true/false/satisfiable/refutable, respectively.

For if $Shrp(L) \quad K$ and *A* is *K*-true, then obviously *A* is also *Shrp*(*L*)-true. On the other hand, if A is not *K*-true, then there must exist some model $M \quad K$ with the property that every constriction $M'' \quad Cons(M)$ has some completion $M' \quad Comp(M'')$ in which *A* is not true, i.e., such that $\mathrm{T} \quad V'[A]$. But completions of constrictions are complete and consistent. Hence the above amounts to saying that there must exist a model

[11] The strengthening obtained by restoring biconditionals throughout (6b) and (6c) would correspond to the above-mentioned four-valued generalization of Kleene's three-valued scheme.

$M'$   $Shrp(L)$ in which $A$ is not true, which means that $A$ is not $Shrp(L)$-true. The proofs for the other equivalences is similar.

Thus, the account preserves and generalizes another important feature of supervaluational semantics. It retains *holus bolus* the classical notions of logical truth and falsity as well as the classical notions of satisfiability and refutability—not only in the presence of gaps, but also in the presence of gluts, or both.

Of course, these results do not necessarily carry over to languages of greater semantic complexity. More importantly, they cannot be extended immediately to arbitrary *sets* of $L$-sentences so as to cover other important logical or model-theoretic notions. It is here that the non-standardness of the semantics shows up. For instance, it follows from (9) that the classical notion of logical consequence continues to hold if we restrict it to pairs of sentences; however, this notion becomes rather wild as soon as we go beyond this simplest case. To be more precise, consider the obvious candidates for defining entailment between sets of sentences in the presence of gaps and gluts:

(10a) $\Sigma$ $K$-entails $\Gamma$ positively [$\Sigma \vDash_{K/\mathrm{T}} \Gamma$] iff $\bigcap\{\mathrm{T}_K(A): A \quad \Sigma\} \quad \bigcup\{\mathrm{T}_K(A): A \quad \Gamma\}$
(10b) $\Sigma$ $K$-entails $\Gamma$ negatively [$\Sigma \vDash_{K/\mathrm{F}} \Gamma$] iff $\bigcap\{\mathrm{F}_K(A): A \quad \Gamma\} \quad \bigcup\{\mathrm{F}_K(A): A \quad \Sigma\}$;
(10c) $\Sigma$ $K$-entails $\Gamma$ [$\Sigma \vDash_K \Gamma$] iff $\Sigma$ $K$-entails $\Gamma$ both positively and negatively.

(In classical logic these notions of entailment coincide, as one verifies by taking $K = Shrp(L)$; but in the general case they need not agree. This assimilates our semantics to most general semantics with gaps and/or gluts.[12]) Then the following clearly holds whenever $Shrp(L) \quad K \quad Mod(L)$:

(11a) $A \vDash_K B$ iff $A \vDash_{Shrp(L)} B$.

Moreover, the following hold for arbitrary sets $\Sigma$ and $\Gamma$:

(11b) $\Sigma \vDash_{Cons(L)} B$ iff $\Sigma \vDash_{Shrp(L)} B$
(11c) $A \vDash_{Comp(L)} \Gamma$ iff $A \vDash_{Shrp(L)} \Gamma$.

However, these equivalences may not hold when the sentences $A$ and $B$ are replaced by two or more sentences. For instance, the entailments corresponding to the rules of adjunction and disjunction may fail:

(12a) $A, {\sim}A \nvDash_{K/\mathrm{T}} A \quad {\sim}A$   whenever $\mathrm{T}_K(A) \cap \mathrm{F}_K(A) \neq$
(12b) $A, {\sim}A \nvDash_{K/\mathrm{F}} A \quad {\sim}A$   whenever $\mathrm{T}_K(A) \quad \mathrm{F}_K(A) \neq K$
(12c) $A \quad {\sim}A \nvDash_{K/\mathrm{T}} A, {\sim}A$   whenever $\mathrm{T}_K(A) \quad \mathrm{F}_K(A) \neq K$
(12d) $A \quad {\sim}A \nvDash_{K/\mathrm{F}} A, {\sim}A$   whenever $\mathrm{T}_K(A) \cap \mathrm{F}_K(A) \neq$  .

[12] See e.g. Blamey [1986] and Muskens [1995]. The significance of this situation has been emphasized in Dunn [1997].

It is precisely this that makes the overall framework paraconsistent (and paracomplete). It follows from (11a) that contradictions “explode” consequentially, just like in classical logic, and tautologies implode. However, (12a)–(12d) show that, in general, inconsistencies and incompletenesses do not metastasize throughout the entire language. For accepting unsatisfiable premiss sets or irrefutable conclusion sets is not tantamount to endorsing unsatisfiable (contradictory) premisses or irrefutable (tautologous) conclusions; and this protects the relations of entailment defined in (10a)–(10c) from logical trivialization. (At present, the exact characterization of the patterns that are valid in the class of all models is still an open problem.)

## 4. Up or Down?

Before considering our second strategy for generalizing supervaluations, it is worth pointing out an important connection between the above account (and the intuitive rationale behind it) and certain structural properties of the notion of a model.

*Mod*($L$) is partially ordered in terms of “definiteness” by the inclusion relation . In fact, is a complete lattice ordering. This means that if we take any models for $L$ and put them together, either by union or by intersection, we still get a model for $L$. More specifically, the whole class of models can be regarded as being “generated” from the class of sharp models by closing it under these two operations:

(13a) *Cons*($L$) is the closure of *Shrp*($L$) under $\bigcap$;
(13b) *Comp*($L$) is the closure of *Shrp*($L$) under $\bigcup$;
(13c) *Mod*($L$) is the closure of *Shrp*($L$) under both $\bigcap$ and $\bigcup$.

Each single model, in turn, can be regarded as being generated from a certain class of sharp models, viz. the class of its sharpenings:

(14a) if $M$ *Cons*($L$), then $M = \bigcap\{M' : M'$ *Comp*($M$)$\}$;
(14b) if $M$ *Comp*($L$), then $M = \bigcup\{M' : M'$ *Cons*($M$)$\}$;
(14c) in general, $M = \bigcup\{\bigcap\{M' : M'$ *Comp*($M''$)$\} : M''$ *Cons*($M$)$\}$.

These facts are illustrated in Figure 1.

Now, (14a)–(14c) correspond very closely to the definitions in (4a)–(4c). And since a model is essentially an evaluation of atomic sentences, one could take this correspondence very seriously: those definitions may be viewed as a natural extension of these very facts to the task of evaluating compound sentences. That is to say, (4a)–(4c) may be viewed as being *patterned after* (14a)–(14c). Just as every model is the meet/join of a certain class of sharp models, likewise the valuation on any model is the meet/join of the valuations induced by those sharp models.

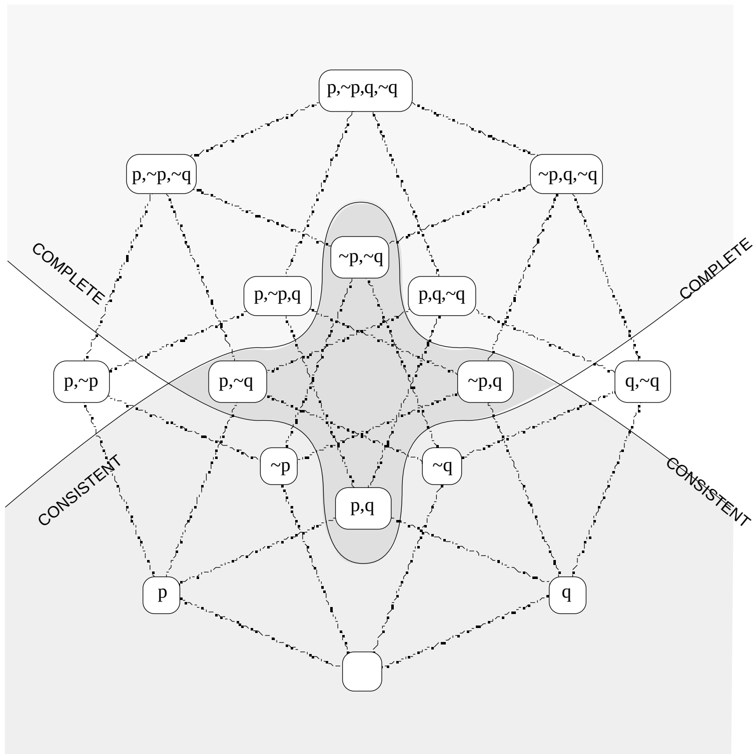


*Figure 1*. Lattice of models of a propositional language with only two atomic sentences, *p* and *q*. Each node corresponds to a possible model, with ‘p’ and ‘~p’ indicating the presence of T and F respectively in the values for ‘*p*’ (likewise for ‘q’ and ‘~q’). The ordering goes uphill along the dotted lines: models below the curved line in the bottom half (grey region) are consistent; models above the other curved line, in the top half (light grey region), are complete. Classical (sharp) models lie in the intersection of these two regions (dark grey region).

I am not sure exactly how much weight to put on this line of argument, though it does appear to bring out an interesting way of explaining the intuitive rationale behind (4a)–(4c). At the same time, it falls short of providing an exclusive account. For as I mentioned earlier, there is an obvious variant of (4c). It is obtained by redefining the ultravaluation induced by a model *M* to be the relation

(4c') $V = \bigcap\{\bigcup\{V' : M' \quad Cons(M'')\} : M'' \quad Comp(M)\}$.

Intuitively, this corresponds to a valuational policy where the things to be done are reversed: here one must *first* fill in the gaps (i.e., look at the completions of the given model) and *then* see whether there is any way of weeding out the gluts (looking at its constrictions) which agree on some common value. In short: a sentence is true (false) on an arbitrary model $M$ iff it is true (false) on some constriction of every completion of $M$. This policy seems just as natural as the one defined in (4c). And indeed, along with (14c) we can also prove the following general equality, matching (4c'):

$$\text{(14c')}\ \text{if}\ M \quad Mod(L), \text{then}\ M = \bigcap\{\bigcup\{M' : M' \quad Cons(M'')\} : M'' \quad Comp(M)\}.$$

Thus, if (14c) provides grounds for (4c), (14c') should provide grounds for (4c'); and it would be natural to expect that the two definitions yield the same semantic relation (the same ultravaluation).

Rather surprisingly, however, this is not the case. To see this, let $V_{\cup\cap}$ be the ultravaluation defined in (4c) (for arbitrary $M$), and let $V_{\cap\cup}$ be its dual defined in (4c') Both valuations agree on the partial truth conditions for compound sentences, (6a)–(6c). And it is easy to verify that both valuations coincide as far as the main logical properties are concerned—for instance, (9) to (12d) would continue to hold even if we redefined (8a)–(8d) in terms of $V_{\cap\cup}$. However, with respect to the truth values that a sentence may take on a model that is both incomplete *and* inconsistent, the two valuations may disagree. For example, if $M$ is overdefined with respect to an atom, $p$, and undefined with respect to another atom, $q$ (as in the leftmost node of the lattice in Figure 1), then the biconditional '$p \leftrightarrow q$' is evaluated in two different ways.[13] It is evaluated as neither true nor false if we use the first schema, $V_{\cup\cap}$. For there are only two constrictions of $M$, one where '$p$' is true and one where '$p$' is false; and each constriction has two completions, one where '$q$' is true and one where '$q$' is false. Thus, for each constriction there is a completion where '$p \leftrightarrow q$' is true and a completion where '$p \leftrightarrow q$' is false, leaving the biconditional truth-valueless. On the one hand, if we use the alternative schema, $V_{\cap\cup}$, then '$p \leftrightarrow q$' is evaluated as both true and false. For there are two completions of $M$, one where '$q$' is true and one where '$q$' is false; and in each completion '$p \leftrightarrow q$' is both true and false, since each completion has a constriction where '$p$' is true and one where '$p$' is false. Thus $V_{\cup\cap}[p \leftrightarrow q] = \quad$ but $V_{\cap\cup}[p \leftrightarrow q] = 2$. More generally, it turns out that $V_{\cup\cap}$ is always included in $V_{\cap\cup}$ (since every completion of any constriction of a model $M$ is extended by some constriction of some completion of $M$), whereas the converse only holds if $M$ is either consistent or complete (for in such cases both $V_{\cup\cap}$ and $V_{\cap\cup}$ reduce to the same supervaluation and the same subvaluation, respectively).

This result of course weakens the above argument. For if the point of the argument is that the way a valuation is construed from sharp valuations may be grounded directly upon the way an arbitrary model can be represented in terms of sharp models, then the question arises of *what criteria* could eventually justify a choice between two equally grounded valuational policies. And that seems to have no principled answer. Nonetheless, there is a good explanation for this *impasse*. For in the final analysis this situation

13 The example is based on a remark from Visser [1984: 194].

ties in directly with the assumption that inconsistency and incompleteness are two sides of the same coin. If we take that standpoint, it should come as no surprise that when gluts and gaps occur simultaneously, the strategy we are examining forces us to choose among two possible policies: we can either weed out the gluts first, or we can start by filling in the gaps. And although in general these two policies yield equal results, in those cases when the very expression to be evaluated amounts to a direct *comparison* between gluts and gaps (as with '$p \leftrightarrow q$' in the indicated model) the difference must become apparent: on one policy gluts go first, hence gaps prevail; on the other gaps go first and gluts prevail. (This point becomes clearer if one considers that, in a standard propositional language like $L$, a biconditional '$p \leftrightarrow q$' is provably the only type of sentence in two variables—up to logical equivalence—on which the two policies may disagree.)

To be sure, one can now see that for any incomplete and inconsistent model $M$ there is a whole family of equally legitimate (but increasingly stronger) potential valuations: a family of ultravaluations forming a complete lattice under the inclusion relation , with $V_{\cup\cap}$ at the bottom and $V_{\cap\cup}$ at the top. The intermediate elements of this lattice need not be very "natural", as they would somewhat arbitrarily shift between $V_{\cup\cap}$ and $V_{\cap\cup}$ for different (compound) sentences. Nevertheless, the semantic complexity of such a proliferation has some intrinsic interest and could be a rewarding subject for further exploration.

## 5. Gluts as Gaps

Both variants of the strategy described above exploit the intrinsic duality between gaps and gluts. Let us now consider the second strategy for generalizing supervaluations mentioned in Section 2. This is based on an idea that is somewhat opposite to duality, viz. that semantic gaps and gluts are phenomena *of the same sort*. As I already mentioned, the intuition operating here is reminiscent of a certain way of dealing with these phenomena in connection with the semantics of singular terms such as definite descriptions: if there are *no* objects or *more than one* object satisfying the identifying property of the description, then the description is left without a denotation, i.e., its interpretation is left undefined. Likewise, one could advocate a radical policy with respect to the interpretation of other expressions, including valuation of sentences: if there is either a gap or a glut in a sentence's pattern of reference then—one could argue—the preconditions for a truth-value assignment are violated.

Formally, the shift of attitude corresponding to this view is only a matter of changing the final definitions: subvaluations on inconsistent models become standard supervaluations, i.e., meets of classical valuations, albeit induced by constrictions rather than completions. More generally, one can obtain a semantics in which gluts are treated as gaps of a kind by redefining the ultravaluation induced by an arbitrary model $M$ to be the relation

(4c") $V=\bigcap\{\bigcap\{V': M' \quad Comp(M'')\}: M'' \quad Cons(M)\}$.

Thus, on this view a sentence is true (false) on a model $M$ iff it is true (false) on every completion of every constriction of $M$; if it is true on every completion of some constriction, and false on every completion of some other constriction, then the sentence is left truth-valueless.

Let us use $V_{\cap\cap}$ to indicate the ultravaluation defined by (4c"). The main effects of such an amendment are not difficult to imagine. Gluts are cured with the same medicine that cures gaps. And since the classical valuations from which $V_{\cap\cap}$ is construed are functions, in the end $V_{\cap\cap}$ will simply be a (possibly partial) function, i.e., it will involve no gluts. In other words, unlike $V_{\cup\cap}$ and $V_{\cap\cup}$, $V_{\cap\cap}$-valuations on inconsistent models will never treat any sentence as both true and false, as the presence of conflicting outputs in the relevant constrictions will be treated as a sign of indefiniteness. In fact, it is easy to prove that $V_{\cap\cap}$ is always bound to be included in the restriction of $V_{\cup\cap}$ to the set of sentences $\{A$: $V_{\cup\cap}[A]\neq 2\}$. (This set is included in $\{A$: $V_{\cap\cup}[A]\neq 2\}$, so the same could be said of $V_{\cap\cup}$.) As an example, here is a comparison between the two strategies in the most critical case, where $M$ is both inconsistent and incomplete:

(15a) If $M[p]=$ and $M[q]=2$:

$V_{\cup\cap}[p]=$
$V_{\cup\cap}[q]=2$
$V_{\cup\cap}[\sim p]=$
$V_{\cup\cap}[\sim q]=2$
$V_{\cup\cap}[p \quad q]=\{T\}$
$V_{\cup\cap}[p \quad q]=\{F\}$
$V_{\cup\cap}[p \quad \sim p]=V_{\cup\cap}[q \quad \sim q]=\{T\}$
$V_{\cup\cap}[p \quad \sim p]=V_{\cup\cap}[q \quad \sim q]=\{F\}$

(15b) If $M[p]=$ and $M[q]=2$:

$V_{\cap\cap}[p]=$
$V_{\cap\cap}[q]=$
$V_{\cap\cap}[\sim p]=$
$V_{\cap\cap}[\sim q]=$
$V_{\cap\cap}[p \quad q]=$
$V_{\cap\cap}[p \quad q]=$
$V_{\cap\cap}[p \quad \sim p]=V_{\cap\cap}[q \quad \sim q]=\{T\}$
$V_{\cap\cap}[p \quad \sim p]=V_{\cap\cap}[q \quad \sim q]=\{F\}$

Another important consequence is that in this case the variant definition corresponding to the reversion of the valuational process does not yield anything new, i.e., the following is perfectly equivalent to (4c"):

(4c‴) $V=\bigcap\{\bigcap\{V': M' \quad Cons(M'')\}: M'' \quad Comp(M)\}$.

In fact, both definitions reduce to

(4c⁗) $V=\bigcap\{V': M' \quad Shrp(M)\}$,

where the notion of a model's sharpening may be interpreted equivalently as in (3c) or as its obvious dual, meaning completion of a constriction rather than constriction of a completion. In other words, if we treat all sharpenings in the same way, it does not matter whether we get to them *via* completions or *via* constrictions: the final class of models is always the same.

This equivalence among possible equally legitimate variants may be regarded as an advantage of the present strategy over the strategy examined in Sections 3 and 4. On the other hand, the connection with the lattice-theoretic properties of *Mod*(*L*) is now much weaker—in fact, inexistent. For not only are (4c″)–(4c‴) plainly in contrast with (14c)–(14c'). On this account, *every* model inconsistency results in valuational incompleteness, i.e., $M[p]=2$ always implies $V_{\cap\cap}[p]=$ , which means that valuations on inconsistent models are not direct extensions of the models themselves. More generally, on this account valuations are not monotonic: if $M$ $M'$, then $V_{\cap\cap}$ $V'_{\cap\cap}$ if $M'$ is consistent, while $V'_{\cap\cap}$ $V_{\cap\cap}$ if $M$ is complete. By contrast, both variants of the first strategy above are perfectly monotonic, i.e., $M$ $M'$ always implies $V_{\cup\cap}$ $V'_{\cup\cap}$ and $V_{\cap\cup}$ $V'_{\cap\cup}$. If we stick to the suggested reading of as an ordering going up hill in terms of degrees of definiteness, this is of course an important advantage of that strategy over the one presently under consideration.

Be it as it may, it will be clear that these differences do not drastically effect the logical flavor of the semantics. In particular, the general facts expressed by (9) will continue to hold under the present account as well: the basic notions of logical truth and falsehood, we may say, as well as the notions of sentence satisfiability and refutability, do not depend on our specific attitude towards gaps and gluts. Still, concerning entailment the situation is slightly different, as the following is now provable (for *Shrp*(*L*) $K$ *Mod*(*L*)):

(16) A set of sentences Σ is *K*-satisfiable/refutable iff Σ is *Shrp*(*L*)-satisfiable/ refutable, respectively

where a set is said to be *K*-satisfiable/refutable if its elements are all simultaneously *K*-satisfiable/refutable. This means that the rule of adjunction is now valid. More precisely, one can prove that an entailment is (positively or negatively) *K*-valid in the semantics based on $V_{\cap\cap}$ iff it is (positively or negatively) $K \cap Cons(L)$-valid in the semantics based on $V_{\cup\cap}$ or $V_{\cap\cup}$. Thus, unlike the previous stategy, the strategy under examination is fully equivalent—as far as logic goes—to ordinary supervaluational semantics.

## 6. Completing the Map

Nothing at this point prevents us from considering further possible ways of generalizing the basic notion of a supervaluation. For instance, generally speaking there is no definite reason to dismiss the (pairwise equivalent) variants obtained by replacing $\bigcap$ with $\bigcup$ in (4c′)–(4c″) above. For if one can provide arguments to the effect that gluts should be treated as gaps, then one may be willing to consider the dual attitude and regard gaps as a type of gluts.

There are other possible variants. The overall picture is given by the family of ultra-valuations generated by the following scheme:

(17) $V_{f_1 C_1 f_2 C_2} = f_1\{f_2\{V': M' \quad C_2(M'')\}: M'' \quad C_1(M)\}$,

where each $f_i$ is either $\bigcup$ or $\bigcap$, and each $C_i$ is either *Comp* or *Cons* ($C_1 \neq C_2$). The diagram in (18) shows the ordering of the resulting valuations in terms of relative strength. (Here I write '↑' and '↓' to abbreviate '*Comp*' and '*Cons*', respectively. Thus, $V_{\cup\downarrow\cap\uparrow}$ is just the old $V_{\cup\cap}$, and so on.)

$$V_{\cup\uparrow\cap\downarrow} \quad V_{\cap\downarrow\cup\uparrow}$$

(18) $V_{\cap\uparrow\cap\downarrow} = V_{\cap\downarrow\cap\uparrow} \qquad\qquad V_{\cup\downarrow\cup\uparrow} = V_{\cup\uparrow\cup\downarrow}$

$$V_{\cup\downarrow\cap\uparrow} \quad V_{\cap\uparrow\cup\downarrow}$$

As before, each relation of strict inclusion leaves room for an entire lattice of intermediate (but somewhat arbitrary) valuations, which we need not examine here. And again, all options turn out to share the same basic properties expressed by (9). Thus, the above remark to the effect that such properties as $K$-truth, $K$-falsity, etc. do not seem to depend on one's specific attitude towards gaps and gluts (i.e., on the specific cure that gaps and gluts require) extends to the full spectrum of strategies compatible with the supervaluational "spirit". On the other hand, as regards logical entailment, all schemes induce a departure from classical logic, but in different degrees. The adjunctive pattern $A, {\sim}A \vDash A \quad {\sim}A$ holds only for the two leftmost (purely supervaluational) schemes, and its disjunctive dual $A \quad {\sim}A \vDash A, {\sim}A$ holds only for the rightmost (purely subvaluational) pair.

There is more room for generalization. For instance, all the strategies discussed so far and captured by (17) require that in order to evaluate an $L$-sentence $A$ one considers the value assigned to $A$ in *every* sharpening of the given model. However, this is a simplification that one may want to relax. In general, one might want to assume a suitable relation $R$ to be defined on $Mod(L)$ so that only models in the image $R[M]$ would qualify as *admissible* refinements of a given model $M$. For instance, we may suppose that certain ways of filling in certain gaps in a model are incompatible with certain ways of filling in other gaps, or with certain ways of weeding out gluts, on account of certain "penumbral connections" (in the terminology of Fine [1975]).[14] We may in these cases speak of $R$-completions and $R$-constrictions, respectively. From this point of view, the policy followed above corresponds to the limit case $R = Mod(L) \times Mod(L)$, but it may be interesting to see what can be gained by considering other cases as well. Moreover, other means for construing valuations may be considered beside meet ($\bigcap$) and join ($\bigcup$). In principle one may take any pair of functions $f_1$ and $f_2$, or even any ordered sequence $F = f_0, \ldots, f_n$ , where each $f_i$ may be thought of as having a relevant admissibility rela-

[14] This point is detailed in Varzi [1997], §3.3.

tion $R_i$ associated with it. Thus, taking all this into account, (17) can be further generalized by considering the family of ultravaluations generated by the following scheme:

$$(19)\quad V_{FR} = f_0\{f_1\{\ldots\{f_n\{V_n\!: M_n \quad R_n[M_{n-1}]\}\!: \ldots\}\!: M_1 \quad R_1[M_0]\}\!: M_0 \quad R_0[M]\}$$

(where $V_n$ is again the relevant classical Boolean assignment).

This yields a sufficiently abstract setting for further investigations. In fact, it emphasizes what is arguably a distinguishing feature of the general philosophy behind the supervaluational method, viz. the purely functional characterization of the process whereby a language is evaluated. All of these valuations are not truth-functional, i.e., they do not assign values as a function of the compositional structure of the language's syntax. They do, rather, establish the value of any given sentence as a function of the relative completeness/consistency of the language's models. Depending on the variety of functions one considers, a corresponding variety of semantics ensues. And as (19) suggests, the variety may be very wide indeed.[15]

## References


Belnap, N. D. Jr. [1977], 'A Useful Four-Valued Logic', in J. M. Dunn and G. Epstein (eds.), *Modern Uses of Multiple-Valued Logics*, Dordrecht, Reidel, pp. 8-37.

Bencivenga, E. [1978], 'Free Semantics for Indefinite Descriptions', *Journal of Philosophical Logic* 7, 389-405.

— [1980a], 'Truth, Correspondence, and Non-Denoting Singular Terms', *Philosophia* 9, 219-29.

— [1980b], 'Free Semantics for Definite Descriptions', *Logique et Analyse* 23, 393-405.

— [1981], 'Free Semantics', in M. L. Dalla Chiara (ed.), *Italian Studies in the Philosophy of Science*, Dordrecht, Reidel, pp. 31-48.

Blamey, S. [1986], 'Partial Logic', in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic, III: Non-Classical Logics*, Dordrecht, Reidel, pp. 1–70.

Da Costa, N. C. A., and Dubikajtis, L. [1977], 'On Jaśkowski's Discussive Logic', in A. I. Arruda, N. C. A. Da Costa, and R. Chuaqui (eds.), *Non-Classical Logic, Model Theory and Computability*, Amsterdam, North Holland, pp. 37-56.

Dummett, M. [1975], 'Wang's Paradox', *Synthese* 30, 301-24.

Dunn, M. J. [1976], 'Intuitive Semantics for First-degree Entailments and "Coupled Trees"', *Philosophical Studies* 29, 149–68.

— [1997], 'Partiality and Its Dual', talk at the *First World Congress on Paraconsistency*, Gent (Belgium), July 1997.

Fagin, R., and Halpern, J. Y. [1987], 'Belief, Awareness, and Limited Reasoning', *Artificial Intelligence* 34, 39-76.

Fine, K. [1975], 'Vagueness, Truth and Logic', *Synthese* 30, 265-300.

Fitting, M. [1992], 'Kleene's Logic, Generalized', *Journal of Logic and Computation* 18, 225-56.

Gupta, A., and Belnap, N. D. Jr. [1993], *The Revision Theory of Truth*, Cambridge, Mass., MIT Press/ Bradford Books.


[15] A preliminary version of this paper appeared as Varzi [1995]. I am grateful to the editors for allowing me to make use of that material.

Grant, J. [1974], 'Incomplete Models', *Notre Dame Journal of Formal Logic* 15, 601-7.
—— [1975], 'Inconsistent and Incomplete Logics', *Mathematics Magazine* 48, 154-9.
Herzberger, H. G. [1982], 'The Algebra of Supervaluations', *Topoi* 1, 74-81.
Hyde, D. [1997], 'From Heaps and Gaps to Heaps and Gluts', *Mind* 106, 641-60.
Jaśkowski, S. [1948], 'Rachunek zdańdla systemów dedukcyjnych sprzecznych', *Studia Societatis Scientiarum Torunensis, Sectio A* 1, 8, 55–77. [Eng. trans. as 'Propositional Calculus for Contradictory Deductive Systems, *Studia Logica* 24, 1969, 143–57.]
Jennings, R. E., and Schotch, P. K. [1984], 'The Preservation of Coherence', *Studia Logica* 43, 89-106.
Kamp, H. [1975], 'Two Theories about Adjectives', in E. L. Keenan (ed.), *Formal Semantics and Natural Language*, Cambridge, Cambridge University Press, pp. 45-89.
—— [1981], 'The Paradox of the Heap', in E. Mönnich (ed.), *Aspects of Philosophical Logic*, Dordrecht, Reidel, pp. 225-77.
Kleene, S. K. [1938], 'On a Notation for Ordinal Numbers', *The Journal of Symbolic Logic* 3, 150-55.
Klein, E. [1980], 'A Semantics for Positive and Comparative Adjectives', *Linguistics and Philosophy* 4, 1-45.
Konolige, K. [1985], 'Belief and Incompleteness', in J. R. Hobbs and R. C. Moore (eds.), *Formal Theories of the Commonsense World*, Norwood, Ablex, 359-404.
Kotas, J., and Da Costa, N. C. A. [1979], 'A New Formulation of Discussive Logic', *Studia Logica* 38, 429-44.
Lambert, K. [1969], 'Logical Truth and Microphysics', in K. Lambert (ed.), *The Logical Way of Doing Things*, New Haven, Yale University Press, pp. 93-117.
Levesque, H. J. [1984], 'A Logic of Implicit and Explicit Belief', *Proceedings of AAAI-84. Fourth National Conference on Artificial Intelligence*, Austin, AAAI, pp. 198-202.
—— [1990], 'All I Know: A Study in Autoepistemic Logic', *Artificial Intelligence* 42, 263-309.
Lewis, D. K. [1970], 'General Semantics', *Synthese* 22, 18-67.
—— [1978], 'Truth in Fiction', *American Philosophical Quarterly* 15, 37–46.
—— [1982], 'Logic for Equivocators', *Noûs* 16, 431-41.
—— [1983], 'Postscripts to "Truth in Fiction"', in his *Philosophical Papers, Volume I*, New York and Oxford: Oxford University Press, pp. 276–80.
McCall, S. [1966], 'Excluded Middle, Bivalence, and Fatalism', *Inquiry* 4, 384-86.
McGee, V., and McLaughlin, B. [1994], 'Distinctions Without a Difference', *The Southern Journal of Philosophy*, 33 (Suppl.), 203–51.
Mehlberg, H. [1958], *The Reach of Science*, Toronto, Toronto University Press.
Muskens, R. [1995], *Meaning and Partiality*, Stanford, CA: CSLI Publications.
Pinkal, M. [1983], 'Towards a Semantics of Precization', in T. Ballmer and M. Pinkal (eds.), *Approaching Vagueness*, Amsterdam, North-Holland, pp. 13-57.
Priest, G., and Routley, R. [1989], 'Systems of Paraconsistent Logic', in G. Priest, R. Routley, and J. Norman (eds.), *Paraconsistent Logic. Essays on the Inconsistent*, München, Philosophia, pp. 151-86.
Przełęcki, M. [1964], 'Z semantyki pojec otwartych', *Studia Logica* 15, 189–220. [Eng. trans. as 'The Semantics of Open Concepts', in J. Pelc (ed.), *Semiotics in Poland, 1896–1969*, Dordrecht: Reidel, 1979, pp. 284–317.]\
—— [1976], 'Fuzziness as Multiplicity', *Erkenntnis* 10, 317-80.
Rescher, N., and Brandom, R. [1980], *The Logic of Inconsistency. A Study in Non-Standard Possible-World Semantics and Ontology*, Oxford, Basic Blackwell.

Rozeboom, W. W. [1962], 'The Factual Content of Theoretical Concepts', in H. Feigl and G. Maxwell (eds.), *Minnesota Studies in the Philosophy of Science, Volume III: Scientific Explanation, Space, and Time*, Minneapolis: University of Minnesota Press, pp. 273–357.

Schotch, P. K., and Jennings, R. E. [1989], 'On Detonating', in G. Priest, R. Routley, and J. Norman (eds.), *Paraconsistent Logic. Essays on the Inconsistent*, München, Philosophia, pp. 306-27.

Skyrms, B. [1968], 'Supervaluations: Identity, Existence, and Individual Concepts', *The Journal of Philosophy* 69, 477-82.

Thomason, R. H. [1970], 'Indeterministic Time and Truth-Value Gaps', *Theoria* 36, 264-81.

— [1972], 'A Semantic Theory of Sortal Incorrectness', *Journal of Philosophical Logic* 1, 209-58.

Urchs, M. [1994], 'On the Law of Excluded Contradiction', in P. Kolář and V. Svoboda (eds.), *Logica '93. Proceedings of the 7th International Symposium*, Praha, Filosofia, 1994, pp. 202-09.

van Bendegem, J. P. [1993], 'The Logical Analysis of Time and the Problem of Indeterminism', *Communication and Cognition* 26, 209–30.

Van Fraassen, B. C. [1966a], 'Singular Terms, Truth-Value Gaps, and Free Logic', *The Journal of Philosophy* 63, 481-95.

— [1966b], 'The Completeness of Free Logic', *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* 12, 219-34.

— [1968], 'Presupposition, Implication, and Self-Reference', *The Journal of Philosophy* 69, 136-52.

— [1969], 'Presuppositions, Supervaluations, and Free Logic', in K. Lambert (ed.), *The Logical Way of Doing Things*, New Haven, Yale University Press, pp. 67-91.

Vardi, M. Y. [1986], 'On Epistemic Logic and Logical Omniscience', in J. Y. Halpern (ed.), *Theoretical Aspects of Reasoning about Knowledge. Proceedings of the 1986 Conference*, Los Altos, Morgan Kaufmann, pp. 293-305.

Varzi, A. C. [1991], 'Truth, Falsehood and Beyond', in L. Albertazzi and R. Poli (eds.), *Topics in Philosophy and Artificial Intelligence*, Bozen, Mitteleuropäisches Kulturinstitut, pp. 39-50.

— [1994], *Universal Semantics*, Ph.D. Thesis, University of Toronto; revised version to appear as *An Essay in Universal Semantics*, Dordrecht/Boston/London: Kluwer Academic Publishers.

— [1995], '*Super-Duper* Supervaluationism', in T. Childers and O. Majer (eds.), *Logica '94. Proceedings of the 8th International Symposium*, Prague: Filosofia, pp. 17-40.

— [1997], 'Inconsistency Without Contradiction', *Notre Dame Journal of Formal Logic* 38, 621–638.

Visser, A. [1984], 'Four-Valued Semantics and the Liar', *Journal of Philosophical Logic* 12, 181-212.

Woodruff, P. W. [1970], 'Logic and Truth-Value Gaps', in K. Lambert (ed.), *Philosophical Problems in Logic*, Dordrecht, Reidel, pp. 121-42.

— [1984], 'Paradox, Truth and Logic, Part I: Paradox and Truth', *Journal of Philosophical Logic* 12, 213-32.