

## THE PLAN OF A SQUARE

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Amidst many discussions on supervaluational algebras and their philosophical applications—on which I was writing my dissertation—Hans and I once paused to ponder the mystical experience of the square. I mean A Square, the hero of *Flatland*. I mean that perfectly two-dimensional being, with no depth whatsoever, citizen of an equally two-dimensional depthless world, who one day had the good fortune of receiving a visit from a Sphere. What's more, he had the fortune of being able to visit, albeit briefly, the undreamt-of three-dimensional world his guest came from—which is to say, our three-dimensional world. He visited and experienced our world before falling back for all eternity into the total flatness of his: Flatland, the plane world, the world with no aboves and no belows, the world in which cars and airplanes alike, so to speak, belong to the same category, and everything, literally everything, is reduced to fragile shadows on an enormous and eternally illuminated floor. (Which does not mean that Flatland was a perfectly democratic world. Power was in the hands of the caste of Circles, most certainly not in the hands of the infamous Irregular Polygons.)

I said that the Square's experience was a mystical one, and this was the point of my discussion with Hans. It is almost as if we were given the opportunity to visit a four-dimensional world, a world which we have no knowledge of, and whose shapes, beauties, and dangers we could not quite imagine. For us the three dimensions are everything: they constitute a habitat so natural that to us it seems impossible to envision different spaces, new and unknown

dimensions, shapes that don't even fit into our imagination. And I am not thinking of the temporal dimension: that dimension we know far too well. I am thinking specifically of a fourth dimension of space, which I wouldn't even know what to call, just as the Square did not know what the Sphere's greatly extolled "depth" was. Now, some will say this is just science fiction, a play of wit, late-Victorian satire at best. Perhaps. But I'm sure Hans would rather speak of mental horizons. And the capacity to extend our mental horizon is not a matter of science fiction, for it is there that we can measure our provincialism. It is there that we can see if we are truly able to think freely, to push ourselves beyond the obvious. And it is there that we can test our sense of possibility—that sense of possibility upon which Reverend Abbott invited us to reflect in such an unusual way but which is the driving force, I think, in many a writing by Hans himself, if not in his way of being a philosopher.

But I don't intend to dwell on these topics here. I simply wish to honor Hans by presenting him with a note that I happened to find while rummaging through some old papers in the attic and that speaks for itself. For I have reasons to believe that the author is no one else than the Square. Indeed I am certain about that. The date on the note is not readable, but it will be apparent that it is posterior to the Sphere's visit, even if Abbott doesn't mention it explicitly in his novel. So here it is, or rather here is the text, which I've strived to transcribe with the utmost accuracy.

\* \* \*

*The Sphere said that our world is flat, but can have different shapes. She said this more than once and for a long while I tried to understand what this could possibly mean. If the world is flat, is this not enough? What else could we add, if not to bemoan our limits and to recognize the scarcity of our horizons? Isn't it enough to say that the land in which we are forced to live is akin to a vast sheet of paper, with no width or depth? I will admit that some time ago I would have said "the universe in which we are forced to live", but now my mind has been opened to a larger view of things. And yet I had always asked myself what else*

*one could say of our country if not this: that we cannot lift ourselves out from it, nor can we immerse ourselves in it—that all in all we are shadows, as the Sphere was able to show me.*

*But now I understand. Now my intellect has finally seen the light. And even though my compatriots still cannot understand the nature of the Third Dimension and continue to profess their utter disbelief in the existence of the Sphere, I continue to hope, though I do not know how, that these memories of mine will someday reach the minds of the people of Some Other Dimension and inspire the actions of those who refuse to accept the confines of a limited Dimensionality.*

*Our world is flat, this is certain. But this is not the same thing as asserting its shape. I should have realized this long ago, when the Scalene Triangle returned from his daring escape. He left (my contemporaries will remember this well) on a search for faraway lands. But he returned. Or rather, he continued on his way until, *mirabile dictu*, he found himself back at the place where he had begun his journey long before. After years and years of travel, he reappeared on our horizon. To be sure, this was not the horizon on which he had set sail; the Triangle appeared on the opposite horizon, the horizon on the opposite side. Those who were there to greet him had great fun mocking him, showering him with the cruelest public humiliation. “You went in circles!” they said. “You made a loop!” The Triangle’s words were worthless; he continued to insist that the route he had taken was perfectly rectilinear. “I marched in a straight line,” he said, “without ever taking my gaze off the horizon.” Alas, how staunchly did he insist on this claim, from that moment until the end of his days, despite our obstinate disbelief. The sad thought of that injustice still weighs on me. I still suffer, in my nocturnal visions, from the memory of the stupidity we were all victim to, and remorse harasses me like a Sphinx devouring my soul.*

*Our world is flat insofar as we are only permitted on its surface. We are perfectly and completely superficial beings. But the surface on which we live can have millions of shapes, and I—who have had the privilege of three-dimensional illumination—I know I am speaking the truth. The Triangle did not return to his point of departure. He arrived there. And the explanation for his arrival was not to be found in the direction of his route, but in the very nature of Flatland. Our world is flat, but our universe is curved and, with its surface, it envel-*

*ops a Three Dimensional space that is inaccessible to us. This also explains the colossal mystery of our Great Books, where it is said that Flatland is necessarily infinite, seeing as no one has ever reached or come near to its outer limits. Flatland is flat for us, but is curved for those who observe us from Above, the Above that my compatriots are unable to imagine. The courage of the Triangle revealed the error, obtuse as our blindness might have been. [ACV: Evidently, the Square's reasoning is still not entirely correct. There still exists the possibility that Flatland is not in fact the surface of a solid. It could very well be, for instance, that it has the shape of a tube, like a sheet of paper rolled up and taped together at the edges. The Triangle would have circumnavigated his world, so to speak, but if he had gone in another direction he could have come upon its outer bounds, or he might have gone on forever if, as the Great Books would imply, the tube were infinitely long. Despite this oversight, we must recognize the Galilean perspicacity of the Square, who was able to understand the source of the error of those who mocked the Triangle. Indeed, one wonders what the Square would have thought had the Triangle returned mirror-reversed. We know what that would have meant: Flatland could be shaped like a Möbius Strip, or better (to avoid edges, since the mystery of the Great Books tells us it is unbounded) like the surface of a Klein Bottle. And we obviously know how to tell whether a scalene Triangle has been mirror-reversed. But so could the folks of Flatland: the Triangle might have started out so configured that they would pass his shortest, intermediate, and longest sides in order when going around him clockwise, and he might have returned so configured that to pass his sides in the same order would require going around him counter-clockwise. They would have still made fun of him, and felt no need for an explanation. But what about the Square? Would he have seen the light? Would he have come to grasp what even Kant failed to recognize, namely, that a spatial manifold may be embedded in higher-order space in non-orientable fashion?]*

*Now I understand. And I understand what the Sphere's words meant. She said that our world could have many different shapes. Flatland could be the*

*surface of a perfectly spherical globe, she said, but it could also be the uneven surface of a potato, or of a long tuber. She explained those concepts very clearly, and I think I now understand. She also said, however, that Flatland could have holes. And this is what I have had difficulty understanding until now. There is no room for holes in Flatland. But over there, in their three-dimensional world, holes abound. I saw them with my very own eyes. They are horrid and frightful things, with no substance at all. Their essence is absence, their medium is nothingness. According to the Sphere, they can also have a function, but on this we did not dwell for long and my limitations prevent me from understanding the meaning of her claims. Now, I have long reflected upon the possibility that Flatland has holes, and have concluded that in the unfortunate event that this were true, the holes would still not be part of our world. Rather, they would have to be holes in the Space whose surface is constituted by our world, not holes in the surface itself. (Otherwise the edges of the holes would be the edges of Flatland, and this goes against the Great Books.) Flatland could very well be the surface of a globe or of a large potato, but it could also be the surface of a large “donut,” as they call it, or of a big “colander” (another concept whose meaning I cannot fully grasp, though I believe it refers to an object with many holes, a sort of multiply perforated donut). The adventures of the Triangle revealed to us the closed nature of our universe. But it revealed no details as to its actual shape. And although I fear it is impossible to accurately determine whether it is an uneven potato or a perfectly rotund globe, I wonder if it is at least possible to determine whether it is an intact shape or a perforated one, which is to say donut-like. The Triangle could have passed by a hole without realizing it. Indeed, while traveling adventurously around the world (an entirely appropriate description) he would never have been able to imagine this possibility in any way whatsoever. But the reader who has read up to this point will have the pleasure of finding out how I was finally able to figure out a way of addressing this dilemma. If I am successful in this endeavor, I intend to dedicate my discovery to the memory of the Scalene Triangle as a way to make amends for the injustices he suffered, which I, too, experienced following my encounter with the Sphere.*

*So this is how I intend to proceed. I will leave tomorrow at dawn, taking the very same route chosen by the Triangle. Like him, I will make sure to march in rectilinear fashion, so as to avoid any lapse of concentration. But un-*

*like the Triangle, I will also make sure to mark my route with rizca. [ACV: This is a word I do not know. It probably refers to a kind of substance used by Flatlanders to make marks on the land they live on.] According to my calculations, if I move quickly I should be able to arrive back at my starting point, or somewhere in its vicinity, in about ten years. The thought of embarking on such a long journey scares me, but I am prepared to make many sacrifices. Meanwhile the small Rhombus, my loyal friend who supported me throughout these hard times, has agreed to go on a similar journey. We will leave at the same time, but he will travel along a route that is orthogonal to my own, not towards the Light, which is the direction I will be traveling in, but towards the Cold. [ACV: Here I believe the Square is alluding to what we would call the cardinal directions; the Square will travel East, so to speak, and the Rhombus will travel North.] He, too, will be sure to proceed in a straight fashion, and will also mark his route with rizca. Unfortunately, we don't know how long his journey will take, nor do we have any way of calculating it. Up until now, the Triangle is the only one who has "circumnavigated" Flatland, and his testimony allows us to determine only the approximate duration of my journey. But the Rhombus is a loyal friend and he will not give up, I am certain of this. Even if a complication should occur, he will instruct someone to complete the mission successfully.*

*Now for my theory. If we live on the surface of a spherical globe, or on a large potato-like surface, then sooner or later our two paths will cross. Either I will come upon the rizca marks left by the Rhombus, or he will find my marks, depending on who has the misfortune of having embarked on the longer journey. This is a mathematical certainty. On the other hand, if we live neither on a sphere nor on a potato-like surface, then it is possible that our paths will never cross. It is possible that we will both return to our starting point without ever crossing the marks left by the other. In that case, we would know for certain that Flatland is shaped like a donut, or anyway that it is a surface that envelops a space that has at least one hole. In fact, we would be able to deduce that one of our routes (the shorter one) will have passed "through" a hole, while the other route will have gone around it. At that point, I assure you, we would have a hard time explaining the results of our discovery to the others. But such is the destiny of science: motivation for action comes not from the thirst for glory, but*

*from the hunt for truth. It will be enough to present our findings in writing, putting our trust in the wisdom of future generations.*

*I said that, in case our world is not a spherical or potato-like surface, it is “possible” that our paths will never cross and that we will both return to our starting point. Regrettably, given what I know, this is all I am able say. In the case at issue, there still is a possibility that Flatland is shaped like a donut, but that neither of our paths will go through its hole. It could very well be the case that in order to pass through the hole, the respective inclination of each route would have to be less than 90 degrees, perhaps much less. All we can confidently state is that if the world is shaped like a donut, then there exist at least two ways of making a complete round trip so that each path does not intersect the other at any point, though we can’t be sure we will be able to identify such paths on our first try. For this, we will have to hope for good luck, for if we do not obtain any results, it will be necessary to try again, and hope that sooner or later someone will succeed.*

*It follows, my dear readers, that my speculations establish the impossibility of determining with absolute certainty whether Flatland is in fact spherical (or potato-like); if it is shaped in this manner, then we know that our paths will cross, but this does not mean that our paths will cross only if Flatland is shaped in this manner. If our world is the surface of donut-like space, I’m afraid we might spend the rest of our days trying to prove it, without ever succeeding, each time passing by the hole without ever going through it. The illusion of living on the surface of a solid globe could take hold of us, in the very same way that for thousands of years we believed we were living on a vast open sheet, and I fear that it will be impossible to liberate ourselves of this idea without the help of friends who watch us from Above. For us, sphericalness is not demonstrable; this is the sad truth. Only imperfections, holes, absences—precisely those things we can never have direct experience of—can be identified with absolute certainty. Only they know how to impose themselves upon us at first sight, so to speak. May destiny have mercy on us.*

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Alas, the note ends here. Since it is not complete, we don’t know how things turned out in the end. Even so, it is difficult not to feel

surprise and admiration for the Square's testimony—his metaphorical vigor, his perspectival strength, the incredible fusion of imaginative ardor and intellectual modesty that is evident in his every word. To us this may sound like Topology 101. But the fact that even a flat polygon can form the concept of an immaterial hole and apply it to experiments of cosmological import should motivate in all of us the profound desire to look beyond the limits of *our* superficiality. I feel very fortunate to have found such a valuable document. I also hope it helps explain to Hans why I ended up co-writing an entire book on holes. And I cannot think of a better way to conclude this tribute to Hans than to cite Abbott's own inscription to the text he published in 1884. May this, too, motivate us to not surrender to the confines of our limited Dimensionality:

To  
*The Inhabitants of SPACE IN GENERAL*  
*And H. C. IN PARTICULAR*  
*This Work is Dedicated*  
*By a Humble Native of Flatland*  
*In the Hope that*  
*Even as he was Initiated into the Mysteries*  
*Of THREE Dimensions*  
*Having been previously conversant*  
*With ONLY TWO*  
*So the Citizens of that Celestial Region*  
*May aspire yet higher and higher*  
*To the Secrets of FOUR FIVE OR EVEN SIX Dimensions*  
*Thereby contributing*  
*To the Enlargement of THE IMAGINATION*  
*And the possible Development*  
*Of that most rare and excellent Gift of MODESTY*  
*Among the Superior Races*  
*Of SOLID HUMANITY*