# WALLY AXIOMATICS OF BRANCHING CONTINUATIONS

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Abstract. We give a brief introduction to the axiomatization of temporal logics. Branching continuations are shortly presented thereafter and the possibility of their clear syntactical axiomatization in a Hilbert-style system is investigated as last. Some basic preliminary observations and suggestions, how such axiomatization could start, are presented.

## 1. Introduction

Branching continuations (BCont) is a temporal logic first introduced by T. Placek in his article from 2010[6]. It represents a descendant of the Branching time logics of A. Prior and its direct predecessor is the so called Branching spacetime logic from N. Belnap[1]. Although BST was extensively studied since its introduction, there weren't presented any clear axioms of the theory neither in Hilbert-style, nor in Gentzen-style axioms. Every account of this logic and its relatives is at most accompanied by a definition starting with the words "A model of the theory of branching spacetime is a pair $\mathcal{W} = (W, \leq)$ that satisfies the following axioms" followed by a list of conditions concerning the structure of the model. Thus some axioms are present but their syntactical form is hidden in plain sight the same way as Wally[1] usually is. This article presents a generaly informal attempt to find these hidden axioms of BCont and thereby externalize the debate of this topic from the 'Prague dynamic group'[2].

## 2. Time and axioms

As a short repetition of the core concepts of temporal logics, let us say that the language consists usually of four temporal operators added to classical propositional logic formulae. These operators are based on the modal $\Box$ and $\Diamond$ operators. The main difference being that they are not limited to one direction of the accessibility relation. Thus we get the operators **H** and **G** being the future and past equivalent of $\Box$ and **F**, **P** as temporal equivalents of the diamond. They are read in the following way:

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[1]For those who are not familiar with Wally, it is a cartoon figure used in childrens' books. He is hidden in a crowd and one must find him, which is not always easy. Although he has a quite unique appearance, the reader is deceived by similarly dressed false bait. Please look for "Where's Wally?" for additional information and pictures.

[2]Many thanks go to O. Majer, M. Peliš and others that participated in the debates concerning the nature of BCont and BST.

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|---|---|
| $\mathbf{F}\varphi$ | *it will sometimes be that* $\varphi$ |
| $\mathbf{P}\varphi$ | *it was sometimes that* $\varphi$ |
| $\mathbf{G}\varphi$ | *it always will be that* $\varphi$ |
| $\mathbf{H}\varphi$ | *it always was that* $\varphi$ |

The relation between operators is similar as with their modal counterparts, e.g.: $\neg\mathbf{H}\neg\varphi \equiv \mathbf{P}\varphi$. Therefore we can add just two operators $(\mathbf{H}, \mathbf{G})$ to the language of propositional logic and treat $\mathbf{P}, \mathbf{F}$ as abbreviations. There is no simple relation between the operators going into opposite directions of time. It is also possible to read the two necessity operators as incorporating the present moment, e.g.: $\mathbf{H}\varphi$ would mean 'it always was and is now that' but we do not use this interpretation.

There are multiple accounts of axioms of temporal logic. Most of them, however, are concerned with linear temporal logic. This can be useful to some limited extent also for our purpose. We mainly use three sources for this part of the debate. Garson in his book [4] gives a clear and vivid account of modal logics also with regards to temporal notions. The second helping hand comes from a source whose name already seems promising - Axioms for branching time by Reynolds[9]. Although Reynolds focuses, as many others, on the use of temporal logics in computer science his work remains a useful source of inspiration. Third but still of golden value is the account of Burgess[2] describing the properties of temporal logics from a more philosophical perspective.

The usual focus of temporal logics treats some form of linear temporal logic for the purposes of computer science. We can borrow some ideas from these approaches but we need to focus on branching structures. For this reason we introduce also a distinction that is central to the topic of branching temporal logics. Past is always regarded as a settled case with given truth values. Valuation of formulae that speak about the future, however, presents two options. The so called Ockhamist perspective claims that we need to specify what possible course of events $h$ we are speaking about and only then we can assign truth values. For a given event $e$ and a course of events $h$ $\mathbf{F}\varphi$ is true if sometime in the future of $e$ in $h$ the proposition $\varphi$ is true. The other option is the Peircean view that claims that we cannot assign truth values to future sentences, the exception being necessarily true statements. The formula $\mathbf{F}\varphi$ would be read as 'for all possible $h$, $\varphi$ is at some point true in the future'. This is a modal notion and one can make the distinction clearly visible by introducing the modal operator $\Box$ to symbolize this quantification over the set of possible $h$. Ockhamist logic can distinquish between the following three: for a given $h, e$ $\mathbf{F}\varphi$, $\Box\mathbf{F}\varphi$, $\Diamond\mathbf{F}\varphi$. On the other hand it holds that $\Box\mathbf{F}_{Ockhamist}\varphi \equiv \mathbf{F}_{Peircean}\varphi$ and hence Peircean logic cannot make the same distinction. This shows how modal notions can be naturally introduced into temporal logic. We can quote the axioms of Prior's Ockhamist branching time logic (OBTL)[3]:

**Definition 1** **(Axioms of OBTL**[9]**)**

Let $p, q$ be propositional atoms, $\perp$ being a constant for false, $\varphi, \psi$ formulae of the language of temporal logics with modal operators, then the axioms are the following.

[3]It is our convention to cite the origin of a given definition or theorem next to its name.

Inference rules of substitution, modus ponens, temporal and path generalization, and an IRR-rule:

$$\frac{\varphi}{\varphi[\psi/q]} \quad \frac{\varphi, \varphi \to \psi}{\psi} \quad \frac{\varphi}{\mathbf{G}\varphi} \quad \frac{\varphi}{\mathbf{H}\varphi} \quad \frac{\varphi}{\Box\varphi} \quad \frac{(p \wedge \mathbf{H}\neg p) \to \varphi}{\varphi}$$

if $p$ does not appear in $\varphi$.

The following axioms:

**L0:** any propositional tautology
**L1:** $\mathbf{G}(\varphi \to \psi) \to (\mathbf{G}\varphi \to \mathbf{G}\psi)$ - distributivity
**L2:** $\mathbf{G}\varphi \to \mathbf{GG}\varphi$ - transitivity
**L3:** $\varphi \to \mathbf{GP}\varphi$ - converse
**L4:** $\mathbf{F}\varphi \to \mathbf{G}(\mathbf{F}\varphi \vee \varphi \vee \mathbf{P}\varphi)$ - branch linearity

and the 'mirror images' of L1 - L4 that means switching temporal operators for their duals (e.g.: **H** with **G**). Followed by modal axioms of S5:

**S1:** $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ - distributivity
**S2:** $\Box\varphi \to \Box\Box\varphi$ - transitivity
**S3:** $\Box\varphi \to \varphi$ - reflexivity
**S4:** $\varphi \to \Box\Diamond\varphi$ - "B" axiom

and some axioms for the relation between operators:

**HN1:** $p \to \Box p$ - persistence on propositional atoms
**HN2:** $\Box\mathbf{H}\varphi \leftrightarrow \mathbf{H}\Box\varphi$ - non-branching past
**HN3:** $\mathbf{P}\Box\varphi \to \Box\mathbf{P}\varphi$ -
**HN4:** $\Box\mathbf{G}\varphi \to \mathbf{G}\Box\varphi$ -
**HN5:** $\mathbf{G}\bot \to \Box\mathbf{G}\bot$ - maximality of branches

This example is also important because of Reynolds' observation that the axioms are sound with regards to branching trees but they are not complete. He proved completeness with regards to Kamp frames and bundled trees. It is not our aim to create new semantics for BCont based on our attempt to find the appropriate axioms, hence we take this only as an indication of the property of these axioms.

However, the main importance of this example lays in the hybrid approach to branching time. We have temporal operators mainly capturing the relations in branches (or histories from the BST vocabulary), i.e. capturing temporal relations. On the other hand we have modal operators that capture the possibilities or options, i.e. focusing not on temporal succession but on the consistency of two events. This is well expressed by Garson[4]:

> *The demand for an open future is really a demand for openness in what is determined by the present, and should not be treated as a condition on the structure of time. Those who argue for an open "future" are really interested in the structure of determination, not the structure of time.* (pg. 103-4)

This quote is worth remembering even out of our current context as it reflects also Belnap's so called "indeterminism without choice" [1], i.e. that future events are determined also by events that are space-like related to our point of evaluation. Keeping in mind that branching structures represent such studies of determinism, we can follow up these general observations by some concrete notions from BCont.

## 3. Branching continuations

As already mentioned, every account of this logic has its model formulation and so do continuations. We introduce here only the basic definitions. The fundament of branching theories is $(W, \leq)$ where $W$ is a set of so called point-events and $\leq$ is their partial ordering. There are some important definitions that we need to list in order to make the reader familiar with the basic idea of BCont.

**Definition 2** **(Snake-link**[6]**)**
The properties and basic definitions of *snake-links*:

(1) $\langle e_1, e_2, \ldots, e_n, \rangle \subseteq W$ $(1 \leq n)$ is a snake-link iff
$$\forall i : 0 < i < n \rightarrow (e_i \leq e_{i+1} \vee e_{i+1} \leq e_i)$$
(2) A snake-link is above (below) $e \in W$ if every element of it is strictly above (below) $e$.
(3) Let $W' \subseteq W$ and $x, y \in W'$. $x$ and $y$ are snake-linked in $W'$ iff there is a snake-link $\langle e_1, e_2, \ldots, e_n, \rangle$ such that such that $x = e_1$ and $y = e_n$ and $e_i \in W'$ for every $i : 0 < i \leq n$.
(4) For $x, y \in W$, x and y are snake-linked above $e$ $(x \approx_e y)$, iff there is a snake-link $\langle e_1, e_2, \ldots, e_n, \rangle$ above $e$ such that $x = e_1$ and $y = e_n$.

Obviously the fourth definition is a special case of the third and can be altered for other relations. The relation $\approx_e$ is reflexive, symmetrical, and transitive, hence an equivalence relation on the set $W_e = \{e' \in W | e < e'\}$. We have some more definitions at this point.

**Definition 3** **(Set of possible continuations**[6]**)**
*Set of possible continuations* of $e$ $(\Pi_e)$ is the partition of $W_e$ induced by the relation $\approx_e$.
$\forall e < x : \Pi_e \langle x \rangle$ is the unique continuation of $e$ to which the given $x$ belongs.

**Definition 4** **(Set $CE$ of choice events**[6]**)**
For $e \in W$, $e \in CE$ iff $card(\Pi_e) > 1$.

**Definition 5** **(Consistency**[6]**)**
For $e, e' \in W$, let there be $W_e := \{x \in W | \forall c(c \in CE \wedge c < e \rightarrow c < x)\}$ and a similar for $e'$. Then $e, e'$ are *consistent* iff they are snake-linked within $W_e \cup W_{e'}$. A set $A \subseteq W$ is then consistent if every two elements of $A$ are consistent. A set $A$ is inconsistent iff it is not consistent.

**Definition 6** **(Large events, l-events**[6]**)**
$A \subseteq W$ is an *l-event* iff $A \neq \emptyset$ and A is consistent.

**Definition 7** **(Model of BCont**[6]**)**
A model of the theory of BCont is a pair $\mathcal{W} = (W, \leq)$ that satisfies the following axioms:

(1) $W$ is a non-empty set partially ordered by $\leq$;
(2) the ordering $\leq$ is dense on $W$;
(3) $W$ has no maximal elements;
(4) every lower bounded chain $C \subseteq W$ has an infimum;
(5) if a chain $C \subseteq W$ is upper bounded and $C \leq b$, then there is a unique minimum in $\{e \in W | C \leq e \wedge e \leq b\}$;

(6) for every $x, y, e \in W$, if $e \not< x$ and $e \not< y$, then $x$ and $y$ are snake-linked in the subset $W_{e\not\leq} := \{e' \in W | e \not\leq e'\}$ of $W$;
(7) if $x, y \in W$ and $W_{\leq xy} := \{e \in W | e \leq x \wedge e \leq y\} \neq \emptyset$, then $W_{\leq xy}$ has a maximal element;
(8) for every $x_1, x_2 \in W$, if $\forall c : c \in CE \to c \not< x_i$, then $x_1, x_2$ are snake-linked in the subset $W_{\not> CE} := \{e \in W | \forall c \in CE e \not> c\}$ of W.

Large events are meant to capture something close to the familiar notion of possible courses of events. For example if we think about a possible outcome of a sea battle, we can take an l-event $A$ to be the set of statements like { the weather is nice, the general of the red army slept well, the blue army had a bad breakfast }. These statements point out one of the possible futures. They represent a set of consistent events, if two events belong to different l-events they are necessarily inconsistent.

As we see, these aren't the axioms we are looking for. In search of lost axioms, we need to address the following points: how do we understand the 'axioms' of BCont, and how do we represent them in the language of BCont. First we compare the demanded structures with those given by priorean or modal formulae with hopes of finding suitable representations of our demands. This attempt explores the relation to the usual propositional temporal logic or at most variations as Reynolds' hybrid-like branching axioms, we do not take into account axioms made for example with first order temporal logic, nor any higher order logic.

## 4. BCont axioms

Explaining some of the axioms is straightforward. BCont asks for $\leq$ to be a partial order. We know formulae that capture reflexivity ($[*]\varphi \to \varphi$) and transitivity ($[*]\varphi \to [*][*]\varphi$), where $*$ stands for some temporal or modal operator and the brackets indicate if it's a necessity or possibility operator.[4] However, there is no modal formula that would be able to capture antisymmetry. A glimpse of hope comes from hybrid logic as it is able to describe antisymmetric structures thanks to its use of nominals[3] with $c \to \Box(\Diamond c \to c)$. Thus our attention turns to hybrid formulae. Obviously combining only the set of future ($\mathbf{F}, \mathbf{G}$) or past ($\mathbf{P}$, $\mathbf{H}$) temporal operators is not sufficient as it would be equivalent to the use of modal operators and thus futile. This extends, however, even to the group as a whole. A troubling bisimilarity persists even with all six operators from Reynolds. A structure composed of distinct states with either $\varphi$ or $\neg\varphi$ being true, and a linear accessibility relation is bisimilar to a structure with two states, a symmetric accessibility relation between them, and where $\varphi$ holds in one state and $\neg\varphi$ holds in the other. We would need something similar to nominals from hybrid logic and we see later that BCont's language can give us such tools and we attempt to find an axiom capturing antisymmetry later.

Thankfully formulae can quite easily capture density, the second property mentioned in the definition 7. The formula for density is $[*][*]\varphi \to [*]\varphi$.

In order to capture that there is no maximal point we could refer to the negation of the formula $G\bot \to FG\bot$ mentioned by Venema[11]. This formula claims that there is a maximal point and although it was meant for linear temporal models,

[4] We need to substitute the symbol $*$ for operators from the language of BCont in order to investigate the final form of the axioms. At this moment this notation is sufficient.

we could use it to describe our "temporal" part. A similar purpose is fulfilled also by the formula $\mathbf{G}\bot \to \Box\mathbf{G}\bot$ from definition 1. Our preference, however, lays with $\mathbf{G}\varphi \to \mathbf{F}\varphi$ from [2]. This formula might be also laden with the burden of being meant for linear flows of time but it does not introduce a new symbol into our language.

The following two axioms, the fourth and fifth, are of a particular nature. They speak about the structure with respect to chains of events. So far we did not manage to find a way how to work with propositional formulae in order to speak about subsets of point events with specific properties nor how these chain properties could be described in the context of the whole structure.

The sixth axiom tells us that every event has at most one antinuation (or in other words there is no branching towards the past). This is captured by one of the Reynolds' formulae, namely HN2: $\Box\mathbf{H}\varphi \leftrightarrow \mathbf{H}\Box\varphi$.

Axiom number seven uses the notion of a choice point. That should be a point allowing to distinquish at least two possible and distinct futures. Although we can capture the existence of two distinct futures by for example $\Diamond\mathbf{G}\varphi\wedge\Diamond\mathbf{G}\neg\varphi$. However, in this case we are speaking only about a specific type of distinct possibilities and for example eliminates the possibility of alternating valuations. We postpone further attempts until we find a formal tool capable of handling the idea of choice points.

The last axiom, claiming that there is an original point, could be given by $\mathbf{H}p \vee \mathbf{PH}p$. Although this axiom was originally meant for linear flows of time, we can add as an antecedent the sixth axiom that there is only one past and hence accomodate the idea for our branching structures. Without this antecedent the formula alone would allow for multiple points of origin.

We see that finding equivalents to the original BCont axioms encounters some significant difficulties. However, these are not the only ones. We need to address also the second point mentioned in the begining of this section, namely the use of language.

## 5. BCont language

There is one particular reason why BCont is used for the first attempt to make an axiomatization of branching spatiotemporal models derived from Branching space-time. This reason is that there are semantics for BCont that work with operators same (or very close) to those encountered in temporal logics[10]. Other works, for example [1],[8], do not introduce any priorean operators or formulae and concentrate only on studying the structure. Those accounts usually lack also any reference to a language for the theory and work merely with the structure itself.

The original BCont language consists of the classical logical language, the operator $Sett:$, the operator $Now:$, and the temporal operators $\mathbf{P}_x, \mathbf{F}_x$. In short, $\mathbf{P}_x\varphi$ means that $x$ units of time in the past $\varphi$ is true. Similarly for $\mathbf{F}_x$. The operator $Sett:\varphi$ means that for all possible continuations $\varphi$ is true[5]. The formula $Now:\varphi$ means that $\varphi$ holds at some event contemporary to the event of evaluation. We see that these operators slightly differ from the classical temporal operators. They can be related, however, with the classical ones.

The classical operator $\mathbf{F}\varphi$ would be equivalent to the formula $\exists x\mathbf{F}_x\varphi$, close to an actual BCont formula. We can take $Sett:$ as similar to $\Box$. There is the option

[5]In the original article the definition of $Sett:$ must be a little bit more complicated as it takes into account the metric indicators of $\mathbf{F}$ and $\mathbf{P}$. For our purpose we can omit this point.

of defining the operator $Poss : \varphi \equiv \neg Sett : \neg\varphi$, in other words an equivalent to $\Diamond$. There remains $Now : \varphi$, which does not have a simple equivalent in classical temporal logic. We can observe that this exception is not a troubling one as $Now :$ can serve, to some extent, a little bit as a nominal from hybrid logic. These operators were introduced mainly for practical purposes and with a motivation from physics. We want to axiomatize BCont and hence we stay with the operators from BCont, using the original axioms as inspiration and theorems from modal logic as support for our work with frames and structures.

The translation between the classical language of temporal logics and BCont, however, still remains a challenge. The language of BCont does not know any operator that would be equivalent to the classical operators $\mathbf{H}, \mathbf{G}$, although we saw that they were used extensively in temporal axioms. Even without introducing the exact definition of $\mathbf{F}_x, \mathbf{P}_x$ we could show that opposed to classical temporal operators $\mathbf{F}_x \equiv \neg\mathbf{G}_x\neg$ does not hold without some additional comments. However, using the exact formalisms is necessary later on. We present the definition of point fulfilling a formula from the original BCont paper in form used for so called BT+*Instants*-like semantics of BCont. This is simple but sufficient way of capturing the general idea of our procedure. We recommend [6] or [10] for more details about BCont semantics.

**Definition 8 Point fulfils formula** [6]

For given $e_C, e/A$ and the model $\mathfrak{M} = \langle \mathfrak{G}, \mathcal{I} \rangle$ we define:

(1) if $\psi \in$ Atoms:$\mathfrak{M}, e_C, e/A \Vdash \psi$ iff $e \in \mathcal{I}\,(\phi)^6$;
(2) if $\psi$ is $\neg\varphi : \mathfrak{M}, e_C, e/A \Vdash \psi$ iff it is not the case that $\mathfrak{M}, e_C, e/A \Vdash \varphi$;
(3) for $\wedge, \vee, \rightarrow$ also in the usual manner;
(4) if $\psi$ is $\mathbf{F}_x\varphi$ for $x > 0 : \mathfrak{M}, e_C, e/A \Vdash \psi$ iff there are $e' \in W$ and $e^* \in A$ such that $e' \leq e*$ and $int(e', e, x)$, and $\mathfrak{M}, e_C, e'/A \Vdash \varphi$;
(5) if $\psi$ is $\mathbf{P}_x\varphi, x > 0 : \mathfrak{M}, e_C, e/A \Vdash \psi$ iff there is $e' \in W$ such that $e' \cup A \in$ l-events and $int(e', e, x)$ and $\mathfrak{M}, e_C, e'/A \Vdash \varphi$;
(6) if $\psi$ is $Sett : \varphi : \mathfrak{M}, e_C, e/A \Vdash \psi$ iff for every evaluation point $e/A'$ from fan $\mathcal{F}_{e/A}$ and $\mathfrak{M}, e_C, e/A' \Vdash \varphi$;
(7) if $\psi$ is Now $: \varphi : \mathfrak{M}, e_C, e/A \Vdash \psi$ iff there is $e' \in s(e_C)$ such that $e' \cup A \in$ l-events and $\mathfrak{M}, e_C, e'/A \Vdash \varphi$.

Hence we see that the negation of any of the two operators does not yield the same result as the negations of the classical $\mathbf{F}, \mathbf{P}$. To some extent this is a desired property because the original operators $\mathbf{H}, \mathbf{G}$ make reference to the whole structure while BCont semantics always work with only localized notions. A simple negation of $\mathbf{F}, \mathbf{P}$ is not satisfactory either, because then $\neg\mathbf{F}_x\neg\varphi \equiv \mathbf{G}_x\varphi$ would mean that at the point $x$ units in the future $\varphi$ holds. This result goes to the other extreme and changes the operator to something similar to $\mathbf{F}_x$. If we would attempt to translate the original temporal axioms into the language of BCont

We suggest therefore a different interpretation of the original definition. If $\mathbf{F}_x\varphi$ would be read as "there is a state at most $x$ units far in the future where $\varphi$ holds". This reading does not change the definition of $\mathbf{F}_x$ too much but it allows us to define $\mathbf{G}_x$ via negation and read as "there are states up to $x$ units far in the future where $\varphi$ holds". This was the only operator in need of introduction or altering, so we present the definition of it in BT+*Instants*-like semantics of BCont.

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[6] We use $\Vdash$ instead of Placek's $\not\approx$ purely for technical reasons, the meaning is the same.

**Definition 9** **F, G, P, H operators in BT+*Instants*-like semantics**
For given $e_C, e/A$ and the model $\mathfrak{M} = \langle \mathfrak{G}, \mathcal{I} \rangle$[7] we define:

- if $\psi$ is $\mathbf{F}_x\varphi$ for $x > 0 : \mathfrak{M}, e_C, e/A \Vdash \psi$ iff there are $e' \in W$, $e^* \in A$, and $x' \leq x$ such that $e' \leq e*$ and $int(e', e, x')$, and $\mathfrak{M}, e_C, e'/A \Vdash \varphi$;
- if $\psi$ is $\mathbf{P}_x\varphi, x > 0 : \mathfrak{M}, e_C, e/A \Vdash \psi$ iff there is $e' \in W$ and $x' \leq x$ such that $e' \cup A \in$ l-events and $int(e', e, x')$ and $\mathfrak{M}, e_C, e'/A \Vdash \varphi$;
- if $\psi$ is $\mathbf{G}_x\varphi$ for $x > 0 : \mathfrak{M}, e_C, e/A \Vdash \psi$ iff for all $e' \in W$, $e^* \in A$, and $x' \leq x$ such that $e' \leq e*$ and $int(e', e, x')$, $\mathfrak{M}, e_C, e'/A \Vdash \varphi$ holds;
- if $\psi$ is $\mathbf{H}_x\varphi, x > 0 : \mathfrak{M}, e_C, e/A \Vdash \psi$ iff for all $e' \in W$ and $x' \leq x$ such that $e' \cup A \in$ l-events and $int(e', e, x')$, $\mathfrak{M}, e_C, e'/A \Vdash \varphi$ holds.

At this point we can attempt to formulate the axioms from section 4 in the language of BCont. The first axiom of BCont was composed of multiple demands. In the language of BCont reflexivity would be represented with $Sett : \varphi \to \varphi$. In this way we do capture the demanded reflexivity of the relation without risking reflexive temporal relations. Here we see once again the earlier mentioned distinction between determinancy relations (as reflexivity is in this case) and purely temporal relations by the means of operator use.

Transitivity is already a property common to both views, hence we could use the form $[*]\varphi \to [*][*]\varphi$, where $[*] \in \{\mathbf{H}_x, \mathbf{G}_x, Sett :\}$.[8] This axiom, however, won't have the same meaning as in 1 as we use the limited reach temporal operators of BCont. While the classical operators are not limited in their reach per se, their BCont versions speak only about a limited part of the structure. In these limits they still remain the same valid. We can reformulate the axiom schema for the temporal operators as $[*]_x\varphi \to [*]_y[*]_z\varphi$, where $y + z \leq x$. In other words the antecedent tells us on what scale it "quarantees" that transitivity works.

In the last section antisymmetry presented a great problem and we promised an attempt to solve it by using the language of BCont. However, our attempts face similar problems as were present in the previous section. We could benefit either from the nature of the BCont temporal operators or from the new operator $Now$ :.

Density is primarily a property of determination but also of time, therefore $[*][*]\varphi \to [*]\varphi$ could stay as a schema for this axiom with the addition of BCont operators $[*]_y[*]_x\varphi \to [*]_x\varphi$.

**Example 10**
Let there be a linear model where $\mathbf{G}_y\mathbf{G}_x\varphi \to \mathbf{G}_x\varphi$ holds. We see that if $\mathbf{G}_y\mathbf{G}_x\varphi$ holds in a state, this claims that in the state itself and in $y$ succeeding states $\mathbf{G}_x\varphi$ holds also. Similarly as before, $\mathbf{G}_y$ represents a kind of "quaranteed" size for the given statement.

The lack of maximal points was captured with the formula $\mathbf{G}\varphi \to \mathbf{F}\varphi$. Suddenly the limited scope of our operators presents a possible obstacle. We cannot use $\mathbf{G}_x\varphi \to \mathbf{F}_{x+1}\varphi$ because then we would rule out alternating values of $\varphi$. Thankfully we do not need to look beyond the guarantees of $\mathbf{G}_x$. We can use $\mathbf{G}_x\varphi \to \mathbf{F}_x\varphi$. It is fulfilled also in structures with maximal points when we choose $x$ correctly but it fails for arbitrary $x$. If $\mathbf{G}_x\varphi$ encounters the end of a branch, however, this formula

[7] As usual $\mathfrak{G}$ is the structure and $\mathcal{I}$ the interpretation of the model.

[8] In the following parts $*$ fulfills the same role as earlier with the addition of those with a subscript stands for a BCont temporal operator (for example $< * >_x$ could be replaced with $\mathbf{F}_x$ or $\mathbf{P}_x$).

still holds but the succedent of the schema won't hold because there is no point $e'$ demanded by definition 9. Hence the structure cannot have maximal points.

The operators $\mathbf{G}_x$ and $\mathbf{H}_x$ represent in BT+*Instants*-like semantics chains, hence we could come up with the idea to use them to interpret the fourth and fifth axiom. This property is limited to these semantics, thus axioms based on it would not be complete with regards to other semantics (for example BCont+generalized flow of time semantics). However, unless we start using some first-order or probably even second-order language, supremum or infimum still remain out of our reach.

The original sixth axiom translation into logic was Reynolds' axiom HN2: $\Box\mathbf{H}\varphi \leftrightarrow \mathbf{H}\Box\varphi$. Similarly as with the no-maximal points axiom, our BCont operators can guarantee us only a limited view of the structure. Nonetheless, they can still maintain the demanded property on this limited field of view. Thus we can reformulate the axiom as $Sett : \mathbf{H}_x\varphi \leftrightarrow \mathbf{H}_x Sett : \varphi$.

We left the investigation of the seventh axiom hoping to find some way how to express that there is a choice point. BCont does give us some new tools that can describe some properties important for the characterization of a choice point. For example $Poss : \mathbf{F}_1\varphi \wedge Poss : \mathbf{F}_1\neg\varphi$ brings us one step closer to a choice point as it shows two possible but distinct futures, but a choice point has to be maximal in the set of common past points of the two possibilities (which from the perspective of this formula means that the subscript should be limitely close to zero). Hence we can capture the existence of two distinct futures but not the fact that there is a choice point between them for similar reasons as we could not formalize the suprema and infima of two previous axioms.

The final form our axiom number eight took was $\Box\mathbf{H}\varphi \leftrightarrow \mathbf{H}\Box\varphi \rightarrow \mathbf{H}p \vee \mathbf{PH}p$. We saw already the localized version of axiom six and we can actually use a similar localization for this axiom. The final axiom being $Sett : \mathbf{H}_x\varphi \leftrightarrow \mathbf{H}_x Sett : \varphi \rightarrow \mathbf{H}_x p \vee \mathbf{P}_y\mathbf{H}_x p$.

Let us sum up the result with regards to the BCont structure from [6] mentioning all our final ideas.

**Summary 11** **Hilbert-style Axioms of BCont**

| Axiom | $\mathcal{L}$ of hybrid temporal logic | $\mathcal{L}$ BCont |
|---|---|---|
| 1 | | |
| Reflexivity | $[*]\varphi \rightarrow \varphi$ | $Sett : \varphi \rightarrow \varphi$ |
| Transitivity | $[*]\varphi \rightarrow [*][*]\varphi$ | $[*]_x\varphi \rightarrow [*]_y[*]_z\varphi$ with $y + z \leq x$ |
| | $\Box\varphi \rightarrow \Box\Box\varphi$ | |
| Antisymmetry | None found | None found |
| 2 | $[*][*]\varphi \rightarrow [*]\varphi$ | $\Box\Box\varphi \rightarrow \Box\varphi$ |
| | | $[*]_y[*]_x\varphi \rightarrow [*]_x\varphi$ |
| 3 | $\mathbf{G}\varphi \rightarrow \mathbf{F}\varphi$ | $\mathbf{G}_x\varphi \rightarrow \mathbf{F}_x\varphi$ |
| 4 | None found | None found |
| 5 | None found | None found |
| 6 | $\Box\mathbf{H}\varphi \leftrightarrow \mathbf{H}\Box\varphi$ | $Sett : \mathbf{H}_x\varphi \leftrightarrow \mathbf{H}_x Sett : \varphi$ |
| 7 | None found | None found |
| 8 | Ax6 $\rightarrow \mathbf{H}p \vee \mathbf{PH}p$ | Ax6 $\rightarrow \mathbf{H}_x p \vee \mathbf{P}_y\mathbf{H}_x p$ |

We see that there are properties we did not manage to capture in our Hilbert-style axioms. As this paper aims to make the first investigation into the topic, this should not be taken as a grave fault of our approach. We managed to find some hints on how to capture BCont in an axiomatic form and.

## 6. Summary

Branching continuations were always presented without any classical axiomatic system that would be based on axioms or inference rules. Although there never existed any explicit reasoning why it is done so, we have shown in this article a few reasons why it seems reasonable to use the original BCont approach. The structures demanded for Branching continuations have some properties that are difficult to transform into temporal formulae. Namely the structure demands antisymmetry of its accessibility relation, which is usually a difficult property to model using modal logic, some properties demand notions and concepts not present in the language of the logic, for example references to chains or choice events. We managed to translate some of the BCont “axioms” into temporal propositional formulae and we even suggested a translation of formulae (and operators) from classical temporal logic to BCont. We also suggested a new interpretation of the operators $\mathbf{F}_x, \mathbf{P}_x$ in order to accomodate in BCont a version of the temporal box operators $\mathbf{G}, \mathbf{H}$. However, we did not arrive to a axiomatic system for BCont as we did not manage to capture some of the properties demanded of the structure via any propositional temporal formulae. It remains an open question if those demands can be formalized in temporal formulae altogether and what will be the properties of the resulting axiomatic system if there even is one. This paper also focused only on axioms and has left the question of inference rules aside for a while. It seems that BCont's Wally axioms remain still hidden and probably in a group of higher order temporal formulae. However, the trick to find them could be to use different goggles from the original BCont ones. We could for example take l-events as primitives (keeping in mind that a trivial l-event is a single point event) and start working with the theory in a similar way as with set theory. This switch of perspective could be all that is needed and it might also be desired. L-events reflect to a great extent how people usually speak about time, where usually a multitude of events or circumstances make up the context of evaluation or use. Because BCont pays homage also to our natural use of temporal language and concepts, this shift seems as a natural thing to do.

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