The model constructed in Corollary 20 is not isomorphic to the principal model of A_2. If we let a be the smallest model number, we shall obtain a β-model for A_2 of power $2^{\aleph_0}$, which is not elementarily equivalent to the principal model of A_2. Namely, our model will not contain any other model of A_2 as an element.

COROLLARY 21. *Without Martin's Axiom we can prove the existence of uncountable β-models of A_2 of height a for every model number a.*

An analogous result concerning the Kelley-Morse set theory was proved in the same way in the author's doctoral thesis.

Added in proof. A proof of a much stronger theorem was recently given by H. Friedman in his unpublished paper *Uncountable models of set theory*.

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Reçu par la Rédaction le 3. 7. 1973

Two notes on abstract model theory
I. Properties invariant on the range of definable relations between structures

by

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Abstract. Suppose L is any model-theoretic language satisfying the many-sorted interpolation property and that $\mathfrak{R}$ is an L-definable or even L-projective relation between L-structures. It is shown that if (1) an L-sentence φ holds in $\mathfrak{N}_1$ just in case it holds in $\mathfrak{N}_2$ whenever $\mathfrak{R}(\mathfrak{M}, \mathfrak{N}_1)$ and $\mathfrak{R}(\mathfrak{M}, \mathfrak{N}_2)$ then (2) there is an L-sentence ψ such that φ holds in $\mathfrak{N}$ if and only if ψ holds in $\mathfrak{M}$ whenever $\mathfrak{R}(\mathfrak{M}, \mathfrak{N})$. This has various results of Beth, Robinson, Gaifman, Barwise and Rosenthal for familiar languages as immediate corollaries.

Introduction. *Abstract* (or *general*) *model-theory* deals with notions that are applicable to all *model-theoretic languages* L. Each such L is determined by a relation $\mathfrak{M} \models_L \varphi$, called its *satisfaction relation*, in which $\mathfrak{M}$ ranges over a collection Str_L of *structures* for L and φ ranges over a collection of objects Stc_L called the *sentences* of L. The notions of general model theory are just those which can be expressed in terms of these basic ones (using ordinary set-theoretical concepts). Examples of such are: *elementary class*, *projective class*, *Löwenheim-Skolem properties*, *Hanf number*, *interpolation property*, *compactness properties*, *categoricity*. Typically, the results apply to all L satisfying some simple conditions or characterize some given L_0 by means of such conditions.

Lindström [L2] provided the first work clearly of this character. Its point of departure (via [L1]) was Mostowski's characterization of $L_{\omega,\omega}$ among certain languages with generalized quantifiers [Mo]. Since then, the subject of general model theory has been especially developed by Barwise [B2]–[B4]. The present two notes are a sequel to my own contribution in § 3 of [F2], making essential use, as there, of *many-sorted* structures.

(1) Guggenheim Fellow 1972–73. The author is indebted to the Guggenheim Foundation and to the U.E.R. de Mathématiques, Université Paris VII, for their generous assistance during the period in which these notes were prepared for publication.

The first of these notes deals (for reasonably arbitrary L) with *L-(projectively) definable relations $\mathfrak{R}$ between structures*. That is, the class of $[\mathfrak{M}, \mathfrak{N}]$ for which $\mathfrak{R}(\mathfrak{M}, \mathfrak{N})$ holds is assumed to be a projective class for L. A sentence φ of L is said to be *invariant on the range* of $\mathfrak{R}$ if for every $\mathfrak{M}$, $\mathfrak{N}_1$, $\mathfrak{N}_2$:

(1) $\mathfrak{R}(\mathfrak{M}, \mathfrak{N}_1)$ and $\mathfrak{R}(\mathfrak{M}, \mathfrak{N}_2)$ implies $[\mathfrak{N}_1 \models \varphi$ if and only if $\mathfrak{N}_2 \models \varphi]$.

It is shown that if L has the many-sorted interpolation property, then φ is invariant on the range of $\mathfrak{R}$ just in case it can be *uniformly reduced to a property ψ of the domain of $\mathfrak{R}$*, in the sense that for every $\mathfrak{M}$, $\mathfrak{N}$:

(2) $\mathfrak{R}(\mathfrak{M}, \mathfrak{N})$ implies $[\mathfrak{N} \models \varphi$ if and only if $\mathfrak{M} \models \psi]$.

Using a suitable definition of compactness property, this theorem is also extended to certain infinite sets of sentences. It then has various results of Beth, Robinson, Gaifman, Barwise, and Rosenthal as immediate corollaries.

More briefly, the succeeding note deals with a model-theoretic generalization of recursion theory employing a notion: *S is semi-invariantly implicitly definable* in L, in place of: S is recursively enumerable. Simple sufficient conditions (which apply to a variety of familiar languages) are found for the class of *L-valid sentences* to be s.i.i.d. in L.

§ 1. Some basic notions for abstract model theory. The following set-up in Secs. 1.1–1.4 was presented in [F2] (²). It is repeated here for convenience and to make some minor but useful modifications; cf. § 3 (loc. cit.) for further details. The material on compactness properties in Sec. 1.5 is new.

1.1. Types and structures. Each *similarity type* τ may be considered to be a *disjoint union*

$$\tau = \mathrm{Sort}(\tau) \cup \mathrm{Symb}(\tau)$$

where $J = \mathrm{Sort}(\tau)$ is a non-empty set of sorts and $\mathrm{Symb}(\tau)$ is a set of relation, operation and constant symbols. A structure $\mathfrak{M}$ of type τ may then be considered to be a function with domain τ which assigns to each $j \in J$ a non-empty domain $M_j = \mathfrak{M}(j)$ and to each symbol $\underset{\sim}{R}$, $\underset{\sim}{F}$ or $\underset{\sim}{c}$ in τ, a relation R, operation F, or individual c of the appropriate kind among the M_j. We write

$$\mathfrak{M} = (\langle M_j \rangle_{j \in J}; \ldots).$$

$\mathrm{Str}(\tau)$ denotes the collection of all structures of type τ.

$\tau, \tau', \ldots$ range throughout over *finite* similarity types. $\sigma, \sigma', \ldots$ are used to range over *subsets* of such types, and need not themselves be

(²) A similar setup has been elaborated by Barwise in his notes [B4]. There are minor (but possibly confusing) terminological differences from [B2], [B4]: where Barwise uses "language", "logic", I use "similarity type" and "language", respectively.

types. If $\mathfrak{M} \in \mathrm{Str}(\tau)$ and $\sigma \subseteq \tau$, then $\mathfrak{M} \restriction \sigma$ is the function $\mathfrak{M}$ restricted to σ, which need not be a structure. $\tau = [\sigma_0, \sigma_1, \ldots, \sigma_n]$ is written when the σ_i are disjoint and $\tau = \sigma_0 \cup \sigma_1 \cup \ldots \cup \sigma_n$. Then for a structure $\mathfrak{M}$ of type τ, we write

$$\mathfrak{M} = [\mathfrak{M}_0, \mathfrak{M}_1, \ldots, \mathfrak{M}_n]$$

where $\mathfrak{M}_i = \mathfrak{M} \restriction \sigma_i$. These notations are modified slightly when the σ_i are singletons, e.g., $\mathfrak{M} = [\mathfrak{M}_0, c_1, \ldots, c_n]$ when $\sigma_i = \{\underset{\sim}{c_i}\}$ $(i \geqslant 1)$ and $\mathfrak{M} = [\mathfrak{M}_0, R_1, \ldots, R_n]$ when $\sigma_i = \{\underset{\sim}{R_i}\}$, etc.

Let $\tau \subseteq \tau'$, $\mathfrak{M} \in \mathrm{Str}(\tau)$, $\mathfrak{M}' \in \mathrm{Str}(\tau')$. Then $\mathfrak{M}'$ is called a *τ'-expansion* of $\mathfrak{M}$ (and $\mathfrak{M}$ a *τ-retract* of $\mathfrak{M}'$) if $\mathfrak{M} = \mathfrak{M}' \restriction \tau$; this is also written $\mathfrak{M}' \in \tau' - \mathrm{Exp}(\mathfrak{M})$. Given $\mathfrak{K} \subseteq \mathrm{Str}(\tau')$, the *projection* of $\mathfrak{K}$ *along* $(\tau' - \tau)$ is defined to be the class

(1) $$\exists_{(\tau'-\tau)} \mathfrak{K} = \{\mathfrak{M}\colon \exists \mathfrak{M}' \in \tau' - \mathrm{Exp}(\mathfrak{M})[\mathfrak{M}' \in \mathfrak{K}]\}.$$

A *renaming* of types is given by any function $\gamma\colon \tau \xrightarrow[\text{onto}]{1-1} \tau'$ for which $\gamma\colon \mathrm{Sort}(\tau) \xrightarrow[\text{onto}]{1-1} \mathrm{Sort}(\tau')$ and γ sends each symbol $\underset{\sim}{s}$ of τ into a symbol of corresponding kind in τ'. We then write $\tau \equiv_\gamma \tau'$. This induces a relation $\mathfrak{M} \equiv_\gamma \mathfrak{M}'$ between $\mathfrak{M} \in \mathrm{Str}(\tau)$ and $\mathfrak{M}' \in \mathrm{Str}(\tau')$, which holds whenever $\mathfrak{M}(s) = \mathfrak{M}'(\gamma(s))$ for each $s \in \mathrm{Symb}(\tau)$.

1.2. Languages. A *model-theoretic language L* is defined to be a system

(1) $$L = (\mathrm{Typ}_L, \mathrm{Str}_L, \mathrm{Stc}_L, \models_L)$$

where Typ_L is a non-empty set of similarity types, called the *admitted types* of L, and Str_L, Stc_L, $\models_L$ are functions with domain Typ_L such that for each admitted τ:

(i) $\mathrm{Str}_L(\tau)$ is a sub-collection of $\mathrm{Str}(\tau)$, called the *admitted structures for $L(\tau)$*,

(ii) $\mathrm{Stc}_L(\tau)$ is a collection, called the *sentences* of $L(\tau)$, and

(iii) $\models_{L,\tau}$ is a sub-relation of $\mathrm{Str}_L(\tau) \times \mathrm{Stc}_L(\tau)$, called the *satisfaction* (or *truth*) *relation of $L(\tau)$*.

Throughout the following, $\varphi, \psi, \ldots$ range over $\mathrm{Stc}_L = \bigcup_{\tau \in \mathrm{Typ}_L} \mathrm{Stc}_L(\tau)$. Where possible without ambiguity, τ and/or L are omitted.

L is said to be *regular* if it satisfies the following conditions for all admitted types τ, τ':

(2) (i) Expansion. $\tau \subseteq \tau' \Rightarrow \mathrm{Stc}(\tau) \subseteq \mathrm{Stc}(\tau')$; $\mathfrak{M}' \in \mathrm{Str}_L(\tau') \Rightarrow \mathfrak{M} = (\mathfrak{M}' \restriction \tau) \in \mathrm{Str}_L(\tau)$ and $\varphi \in \mathrm{Stc}(\tau) \Rightarrow [\mathfrak{M} \models \varphi \Leftrightarrow \mathfrak{M}' \models \varphi]$.

(ii) Renaming. Each $\tau \equiv_\gamma \tau'$ induces a 1-1 correspondence $\bar{\gamma}\colon \mathrm{Stc}(\tau) \to \mathrm{Stc}(\tau')$ such that if $\mathfrak{M} \in \mathrm{Str}_L(\tau)$ and $\mathfrak{M}' \in \mathrm{Str}(\tau')$ and $\mathfrak{M} \equiv_\gamma \mathfrak{M}'$, then $\mathfrak{M}' \in \mathrm{Str}_L(\tau')$ and $\mathfrak{M} \models \varphi \Leftrightarrow \mathfrak{M}' \models \bar{\gamma}(\varphi)$.

(iii) Isomorphism. If $\mathfrak{M} \in \mathrm{Str}_L(\tau)$ and $\mathfrak{M}' \in \mathrm{Str}(\tau)$ and $\mathfrak{M} \cong \mathfrak{M}'$, then $\mathfrak{M}' \in \mathrm{Str}_L(\tau)$ and $\mathfrak{M} \models \varphi \Leftrightarrow \mathfrak{M}' \models \varphi$.

In practice, formulas are introduced before sentences in setting up a language. But it is possible to treat formulas in terms of sentences in this general framework. Namely, by a *formula* φ of $L(\tau)$ we mean a sentence of any $L([\tau, \underset{\sim}{c}_1, \ldots, \underset{\sim}{c}_n])$. In this case we indicate φ by $\varphi(x_1, \ldots, x_n)$ (with "variables" x_i of the same sort as $\underset{\sim}{c}_i$) and take

(3) $$\mathfrak{M} \models \varphi(a_1, \ldots, a_n) \underset{\mathrm{Def}}{\Leftrightarrow} [\mathfrak{M}, a_1, \ldots, a_n] \models \varphi .$$

1.3. Examples. The following indicates how familiar languages are to be construed as model-theoretic languages in the sense of 1.2 (1).

E1. L_A (for A a transitive class) has formulas generated by the operations $\sim\varphi$, $\bigwedge x^{(j)}\varphi$ and $\bigwedge\bigwedge_{\varphi \in a} \varphi$ for each set of formulas $a \in A$. (For $\varkappa$ regular, $L_{\varkappa,\omega}$ is $L_{H(\varkappa)}$ where $H(\varkappa)$ consists of all sets of hereditary cardinality $< \varkappa$.) $\mathrm{Typ}_L = \mathrm{Typ}$ and $\mathrm{Stc}_L(\tau) = \mathrm{Str}(\tau)$ for these languages. Sentences and satisfaction are defined as usual.

E2. $L_A(Q_\varkappa)$ makes use in addition of the operation $(Q_\varkappa x^{(j)})\varphi$ interpreted in the definition of satisfaction as, "there are at least $\varkappa$ elements $x^{(j)}$ satisfying φ." There is no change in Typ_L, Str_L.

E3. $L_{\omega,\omega}(\omega)$ or *ω-logic in $L_{\omega,\omega}$* takes Typ_L to consist of all $\tau \supseteq \tau_0$ where τ_0 is the type of $\mathfrak{N} = (\omega, 0, ')$ and $\mathrm{Str}_L(\tau)$ consists of all $\mathfrak{M}$ with $\mathfrak{M} \restriction \tau_0 \cong \mathfrak{N}$. Then Stc_L, $\models_L$ are the restrictions to these particular types and structures of $\mathrm{Stc}_{L_{\omega,\omega}}$ and $\models_{L_{\omega,\omega}}$, resp. (In the same way we can associate an ω-logic $L_A(\omega)$ with any L_A.)

E4. $L^{(2)}_{\omega,\omega}$ or *2nd-order logic in $L_{\omega,\omega}$* takes Typ_L so that in each admitted τ the set of sorts is divided into two equivalent parts

$$\mathrm{Sort}(\tau) = \mathrm{Ind}(\tau) \cup \mathrm{Sub}(\tau);$$

each $j \in \mathrm{Ind}(\tau)$ is considered a sort for "individuals" and each corresponding $j^* \in \mathrm{Sub}(\tau)$ the sort for "subsets" of these. Among the relation symbols of τ are to be a binary $\underset{\sim}{\epsilon}_j$ of type $j \times j^*$ for each $j \in \mathrm{Ind}(\tau)$. The admitted structures (up to $\cong$) are those for which $M_{j^*} = \mathfrak{S}(M_j)$ and ϵ_j is the membership relation in $M_j \times \mathfrak{S}(M_j)$. The sentences and satisfaction are then the restriction from $L_{\omega,\omega}$ to these admitted types and structures. (Again we can associate a 2nd-order logic $L^{(2)}_A$ with any L_A.)

E5. $L_{\varkappa,\lambda}$ $(\lambda > \omega)$ may be treated similarly.

Remark. Even though the present notions *prima facie* provide only for *two-valued logics*, we can also construe *many-valued logics*, *Boolean-valued logic*, etc. as model-theoretic languages in the above sense. For

example, a $\mathfrak{B} = (B; \ldots)$-*valued structure* $\mathfrak{M} = (M; \ldots)$ is simply taken to be a certain kind of two-sorted structure

$$\mathfrak{M}[\mathfrak{B}] = (M, B; \ldots),$$

where for each n-ary R of $\mathfrak{M}$, $R\colon M^n \to B$ [3]. Then with each sentence φ is associated a formula $\varphi^*(x)$ for which

$$[[\varphi]] = b \Leftrightarrow \mathfrak{M}[\mathfrak{B}] \models \varphi^*(b).$$

In this way, further notions of a (prospective) abstract multi-valued model theory could be reduced to those for two-valued models. But a simple, smooth-running development of such a theory may require giving the value structure $\mathfrak{B}$ an explicit role. A theory of this kind has been started by Waszkiewicz and Węglorz [W, W] [4].

1.4. Elementary and projective classes; interpolation. Given any L and $\tau \in \mathrm{Typ}_L$, define

(1) (i) $\mathrm{Mod}_\tau(\varphi) = \{\mathfrak{M}\colon \mathfrak{M} \in \mathrm{Str}_L(\tau) \text{ and } \mathfrak{M} \models \varphi\}$ for $\varphi \in \mathrm{Stc}(\tau)$,
(ii) $\mathrm{EC}(\tau) = \{\mathrm{Mod}_\tau(\varphi)\colon \varphi \in \mathrm{Stc}(\tau)\}$,
(iii) $\mathrm{PC}(\tau) = \{\mathfrak{H}_{(\tau'-\tau)}\mathcal{K}\colon \tau \subseteq \tau' \text{ and } \mathcal{K} \in \mathrm{EC}(\tau')\}$,
(iv) $\mathrm{PC}^0(\tau) = \{\mathfrak{H}_{(\tau'-\tau)}\mathcal{K}\colon \tau \subseteq \tau',\ \mathrm{Sort}(\tau) = \mathrm{Sort}(\tau') \text{ and } \mathcal{K} \in \mathrm{EC}(\tau')\}$.

Then the members of

$$\mathrm{EC} = \bigcup \mathrm{EC}(\tau)[\tau \in \mathrm{Typ}_L] \quad \text{and} \quad \mathrm{PC} = \bigcup \mathrm{PC}(\tau)[\tau \in \mathrm{Typ}_L]$$

are called, respectively, the *L-elementary classes* and the *L-projective classes*, while those of $\mathrm{PC}^0 = \bigcup \mathrm{PC}^0(\tau)\ [\tau \in \mathrm{Typ}_L]$ are called the *strictly L-projective classes*. These last are the projective classes in the usual sense of word, where no domains of new sort are added.

We may express that *L is $L_{\omega,\omega}$-closed* with these notions as follows: (i) each atomic sentence determines an L-elementary class; (ii) each $\mathrm{EC}(\tau)$ is closed under intersection and complementation relative to $\mathrm{Str}_L(\tau)$; and (iii) whenever $\mathcal{K} \in \mathrm{EC}(\tau')$ where $\tau' = [\tau, c]$, then $\mathfrak{H}_{(\tau'-\tau)}\mathcal{K} \in \mathrm{EC}(\tau)$.

The following is the main property to be considered here.

L-Interpolation. *For each $\tau \in \mathrm{Typ}_L$ and $\mathcal{K}_1, \mathcal{K}_2 \in \mathrm{PC}_L(\tau)$, if $\mathcal{K}_1 \cap \mathcal{K}_2 = 0$, then there exists $\mathcal{E} \in \mathrm{EC}_L(\tau)$ with $\mathcal{K}_1 \subseteq \mathcal{E}$ and $\mathcal{E} \cap \mathcal{K}_2 = 0$.*

In words: any two disjoint L-projective classes can be separated by an L-elementary class. For familiar languages, this statement is equiva-

[3] Indeed, it seems to me preferable to think of $\mathfrak{B}$-valued structures in this way.
[4] Cf. also the review of [W, W] in Mathematical Reviews 41 (1971), #8220.

lent to the *many-sorted interpolation theorem* for L (5). The statement restricted to strictly L-projective $\mathfrak{K}_1, \mathfrak{K}_2$ is equivalent to Craig's original form of interpolation theorem extended to L (6). With reference to the examples E1-E5: L-Interpolation holds for all L_A with $A \subseteq H(\aleph_1)$ and A admissible or a union of admissibles; it is false for E2-E5 (cf. [F2] for detailed references).

1.5. Compactness properties. For any set S of sentences of $L(\tau)$, we write

(1) (i) $\mathrm{Mod}_\tau(S) = \bigcap \mathrm{Mod}_\tau(\varphi)[\varphi \in S]$,
(ii) $\mathfrak{M} \models S$ if $\mathfrak{M} \in \mathrm{Mod}(S)$, and
(iii) $S \vdash \varphi \Leftrightarrow \varphi \in \mathrm{Stc}(\tau)$ and $\forall \mathfrak{M}[\mathfrak{M} \models S \Rightarrow \mathfrak{M} \models \varphi]$.

By a *compactness property* δ for L we mean a pair

$$\delta = (\boldsymbol{F}, \boldsymbol{I})$$

of collections having the following properties (generalizing the properties of the collections of *finite* and possibly *infinite* sets of sentences in $L_{\omega,\omega}$, resp.):

(2) (i) $\boldsymbol{F} \subseteq \boldsymbol{I} \subseteq \mathfrak{S}(\mathrm{Stc}_L)$,
(ii) $\varphi \in \mathrm{Stc}_L \Rightarrow \{\varphi\} \in \boldsymbol{F}$,
(iii) if $S \in \boldsymbol{F} \cap \mathfrak{S}(\mathrm{Stc}(\tau))$, then $\mathrm{Mod}(S) \in \mathrm{EC}$,
(iv) if $X \in \boldsymbol{F}$, then $X \cap \mathrm{Stc}(\tau) \in \boldsymbol{F}$,
(v) $\boldsymbol{I}$ is closed under union and under renaming,
(vi) if $S \in \boldsymbol{I}$, $S \subseteq \mathrm{Stc}(\tau)$ and $\forall X[X \subseteq S$ and $X \in \boldsymbol{F} \Rightarrow \mathrm{Mod}_\tau(X) \neq 0]$, then $\mathrm{Mod}_\tau(S) \neq 0$.

We then define

(3) (i) $\mathrm{EC}^\delta(\tau) = \{\mathrm{Mod}_\tau(S)\colon S \in \boldsymbol{I} \cap \mathfrak{S}(\mathrm{Stc}_\tau)\}$,
(ii) $\mathrm{PC}^\delta(\tau) = \{\mathfrak{R}_{(\tau'-\tau)}\mathfrak{K}\colon \tau \subseteq \tau'$ and $\mathfrak{K} \in \mathrm{EC}^\delta(\tau')\}$,

(5) This is called the *simple form* of many-sorted interpolation in [F2], § 1. A refinement needed for applications to preservation theorems used and stated there also gave information about the location of universal and existential quantifiers in the interpolant. This is not needed here. It is worth mentioning though that Stern [S2] (cf. also [S1]) has found a more general theorem in which opposite conditions on the location of quantifiers may be arranged for certain sorts. This permits applications to the same preservation theorems but without requiring as in [F2] the use of an equality relation between arbitrary sorts. Stern's proof is by means of a model-theoretic forcing argument, but this is an inessential difference; the result can be established by the same methods as indicated in [F2], § 1.

(6) There is no obvious simple way in general model theory to state forms of interpolation with syntactic refinements, such as Lyndon's for positive and negative occurrences, or the form involving location of quantifiers mentioned in ftn. (5).

and $\mathrm{EC}^\delta = \bigcup \mathrm{EC}^\delta(\tau)[\tau \in \mathrm{Typ}_L]$, $\mathrm{PC}^\delta = \bigcup \mathrm{PC}^\delta(\tau)[\tau \in \mathrm{Typ}_L]$. Note by (2) (iii) that $\mathrm{EC} \subseteq \mathrm{EC}^\delta$.

The following are examples of compactness properties:

C1. $L_{\omega,\omega}$. Take $\boldsymbol{F} = \mathfrak{S}_{<\omega}(\mathrm{Stc}) =$ the collection of all finite sets of sentences and $\boldsymbol{I} = \mathfrak{S}(\mathrm{Stc})$. (6) (v) is then equivalent to the usual compactness theorem.

C2. $L_{\omega,\omega}(Q_{\aleph_1})$. By Keisler's compactness theorem [K] we can use the same $\boldsymbol{F}, \boldsymbol{I}$ as in (*a*).

C3. L_A for A countable admissible. Barwise's compactness theorem [B1] shows that $(\boldsymbol{F}, \boldsymbol{I})$ is a compactness property where $\boldsymbol{F} = \mathfrak{S}(\mathrm{Stc}) \cap A$ and $\boldsymbol{I}$ consists of all $\Sigma_1^{(A)}$ definable sets of sentences. Again by [K] this extends to $L_A(Q_{\aleph_1})$.

C4. Any language for which EC is closed under $\cap$ has the *trivial compactness property*: $\boldsymbol{F} = \boldsymbol{I} = \mathfrak{S}_{<\omega}(\mathrm{Stc})$. $\mathrm{EC}^\delta = \mathrm{EC}$ for this.

Question. Are there any non-trivial compactness properties for L_A with A uncountable or for any of the examples E3-E5? A less precise but more interesting question is, what simple conditions on L ensure the existence of a natural compactness property, which includes C1-C3 as special cases?

When δ is a compactness property we take *L^δ-Interpolation* to be the statement of L-Interpolation with the hypothesis weakened to $\mathfrak{K}_1, \mathfrak{K}_2 \in \mathrm{PC}_L^\delta(\tau)$.

Lemma. *Suppose L is regular and that δ is a compactness property for L. Then L-Interpolation implies L^δ-Interpolation.*

Proof. Let $\mathfrak{K}_1 \cap \mathfrak{K}_2 = 0$ where $\mathfrak{K}_i = \mathfrak{R}_{(\tau'-\tau)} \mathrm{Mod}_{\tau_i}(S_i)$ and $S_i \in \boldsymbol{I}$. By renaming we may assume $\tau_1 \cap \tau_2 = \tau$. Then let $\tau' = \tau_1 \cup \tau_2$ and $S' = S_1 \cup S_2$ so $S' \in \boldsymbol{I}$. $\mathrm{Mod}_{\tau'}(S') = 0$; for otherwise there exists $\mathfrak{M}'$ such that $\mathfrak{M}' \restriction \tau_i \in \mathrm{Mod}_{\tau_i}(S_i)$ and then $\mathfrak{M} = (\mathfrak{M}' \restriction \tau_1) \restriction \tau = (\mathfrak{M}' \restriction \tau_2)\tau \in \mathfrak{K}_1 \cap \mathfrak{K}_2$. It follows that there exists $X \in \boldsymbol{F}$ with $X \subseteq S'$ and $\mathrm{Mod}_{\tau'}(X) = 0$. Let $X_i = X \cap \mathrm{Stc}(\tau_i)$ and $\mathfrak{K}_i^* = \mathfrak{R}_{(\tau_i-\tau)} \mathrm{Mod}_{\tau_i}(X_i)$. Then $X_i \in \boldsymbol{F}$ and $X_i \subseteq S_i$ so $\mathfrak{K}_i^* \in \mathrm{PC}(\tau)$, $\mathfrak{K}_i \subseteq \mathfrak{K}_i^*$ and $\mathfrak{K}_1^* \cap \mathfrak{K}_2^* = 0$.

Thus results depending on L-Interpolation can be strengthened in the presence of non-trivial compactness properties.

§ 2. Definable relations and invariant properties.

Throughout the remainder of the paper we assume that *L is any regular, $L_{\omega,\omega}$-closed language, and that δ is any compactness property for L.* (Actually we shall only need closure of EC_L under Boolean operations in this note.)

2.1. Elementary and projective relations between structures. We wish to deal with definable relations $\mathcal{R} \subseteq' \mathrm{Str}(\tau_0) \times \mathrm{Str}(\tau_1)$. By renaming we may assume $\tau_0 \cap \tau_1 = 0$. Then $\mathcal{R}$ can be identified with the class of all $[\mathfrak{M}, \mathfrak{N}]$ such that $\mathcal{R}(\mathfrak{M}, \mathfrak{N})$. This determines the meaning of

(1) $$\mathcal{R} \in \mathrm{EC}, \quad \mathcal{R} \in \mathrm{PC}, \quad \mathcal{R} \in \mathrm{PC}^0$$

which we read: $\mathcal{R}$ is an *elementary*, resp., *projective*, resp., *strictly projective relation between structures* (7). In addition this identification determines the meaning of

(2) $$\mathcal{R} \in \mathrm{EC}^\delta, \quad \mathcal{R} \in \mathrm{PC}^\delta.$$

The following are simple examples of these notions, where the relations to begin with may not be given on disjoint types.

R1. The *substructure relation* is elementary in $L_{\omega,\omega}$. That is $\mathcal{R}(\mathfrak{M}, \mathfrak{N}) \Leftrightarrow \mathfrak{M} \subseteq \mathfrak{N}$, for which we have a sentence Ext, written as in [F2], § 1, with $\mathcal{R}(\mathfrak{M}, \mathfrak{N}) \Leftrightarrow [\mathfrak{M}, \mathfrak{N}] \models \mathrm{Ext}$.

R2. The *embedding relation* is strictly projective in $L_{\omega,\omega}$.

$$\mathcal{R}(\mathfrak{M}, \mathfrak{N}) \Leftrightarrow \exists H (H\colon \mathfrak{M} \cong H(\mathfrak{M}) \text{ and } H(\mathfrak{M}) \subseteq \mathfrak{N})$$
$$\Leftrightarrow \exists H([\mathfrak{M}, \mathfrak{N}, H] \models \mathrm{Emb})$$

for a suitable sentence Emb.

R3. The *homomorphism relation* is strictly projective in $L_{\omega,\omega}$. This is the relation: $\mathcal{R}(\mathfrak{M}, \mathfrak{N}) \Leftrightarrow$ there is a homomorphism H of $\mathfrak{M}$ onto $\mathfrak{N}$.

R4. The relation $\mathcal{R}(\mathfrak{M}, \mathfrak{N})$ of $\mathfrak{N}$ being a *vector space over the field* $\mathfrak{M}$ is strictly projective in $L_{\omega,\omega}$. (The scalar product which connects $\mathfrak{M}, \mathfrak{N}$ must be adjoined.) The subrelation $\mathcal{R}_1(\mathfrak{M}, \mathfrak{N})$ for which $\mathfrak{N}$ is a *finite-dimensional vector space* over $\mathfrak{M}$ is strictly projective in L_A, where A is the least admissible containing ω.

The subrelation $\mathcal{R}_2$ of $\mathcal{R}$ for which $\mathfrak{N}$ is an *infinite-dimensional vector space* over $\mathfrak{M}$ is (strictly) PC^δ in $L_{\omega,\omega}$.

R5. The relation $\mathcal{R}(\mathfrak{M}, \mathfrak{N})$ which holds when $\mathfrak{M}$ is *isomorphic to a subdirect power* of $\mathfrak{N}$ is projective in $L_{\omega,\omega}$. It is not (prima-facie) strictly projective, since we must add an index set I as a new sort. (This example is due to Mal'cev [Ma].)

2.2. Elementary and projective operations on structures. An algebraic (i.e. $\cong$ — preserving) operation $\mathcal{F}$ on structures is said to be in $\mathrm{EC}(\mathrm{PC}, \mathrm{PC}^0, \mathrm{EC}^\delta, \mathrm{PC}^\delta)$ for L if there is a relation $\mathcal{R}$ in the same class such that:

(1) (i) $\mathcal{R}(\mathfrak{M}, \mathfrak{N}_1) \mathbin{\&} \mathcal{R}(\mathfrak{M}, \mathfrak{N}_2) \Rightarrow \mathfrak{N}_1 \cong \mathfrak{N}_2$,
(ii) $\mathrm{Dom}(\mathcal{F}) = \mathrm{Dom}(\mathcal{R})$ and $\mathcal{R}(\mathfrak{M}, \mathcal{F}(\mathfrak{M}))$ for each $\mathfrak{M} \in \mathrm{Dom}(\mathcal{F})$.

(7) Projective relations in this sense were first studied by Mal'cev [Ma]; he later changed terminology, using reductive and projective in the way projective and strictly projective, resp. are used here.

Any relation $\mathcal{R}$ satisfying (i) can be considered to determine an operation satisfying (ii).

Operations $\mathcal{F}$ with finitely many arguments, $\mathrm{Dom}(\mathcal{F}) \subseteq \mathrm{Str}(\tau_0) \times \ldots \times \mathrm{Str}(\tau_n)$, can be identified with operations of one argument in $\mathrm{Str}([\tau_0, \ldots, \tau_n])$ by renaming types.

The notion of projectively definable operation is closely related to Gaifman's notion in [G] of single-valued elementary definition. The following are some simple examples of definable operation; cf. [G] for further examples.

F1. The operation $\mathcal{F}$ which associates with each integral domain $\mathfrak{M}$ its *field of quotients* $\mathfrak{N}$ is elementary in $L_{\omega,\omega}$. (Simply describe $\mathfrak{N}$ as a field with $\mathfrak{M} \subseteq \mathfrak{N}$ such that every element c of $\mathfrak{N}$ can be written as $c = a/b$ for some a, b in $\mathfrak{M}$ with $b \neq 0$.) The restriction of $\mathcal{F}$ to fields of characteristic 0 is in EC^δ. If we regard $\mathcal{F}$ as only giving an embedding of $\mathfrak{M}$ in a suitable field then $\mathcal{F}$ is strictly projective.

F2. Let $p(x)$ be a polynomial with integer coefficients. The operation which associates with each $\mathfrak{M}$, over which p is irreducible, the *root field* $\mathfrak{M}(a)$ is elementary in $L_{\omega,\omega}$.

F3. The operation which associates with each field $\mathfrak{M}$ its *algebraic closure* is elementary in L_A for the least admissible A containing ω.

F4. The operations of *ordered sum* $\mathfrak{M}_0 + \mathfrak{M}_1$ and *ordered product* $\mathfrak{M}_0 \cdot \mathfrak{M}_1$ on pairs $[\mathfrak{M}_0, \mathfrak{M}_1]$ of ordered structures are projective in $L_{\omega,\omega}$. The operation $\mathfrak{M}_0^{\mathfrak{M}_1}$ of *ordered power* is projective in L_A for the least admissible containing ω.

F5. The *ultrapower operation* $\mathfrak{M}_0^I/U$ is projective in 2nd order logic in the following sense. It operates on pairs $[\mathfrak{M}_0, \mathfrak{M}_1]$ where $\mathfrak{M}_1 = (I, \mathfrak{S}(I); \epsilon, U)$ and U is an ultrafilter on I.

2.3. The principal notions and theorem. Let τ_0, τ_1 be disjoint and $\mathcal{R} \subseteq \mathrm{Str}(\tau_0) \times \mathrm{Str}(\tau_1)$. A sentence φ for type τ_1 structures is said to be *invariant on the range* of $\mathcal{R}$ if for all $\mathfrak{M}, \mathfrak{N}_1, \mathfrak{N}_2$:

(1) $$\mathcal{R}(\mathfrak{M}, \mathfrak{N}_1) \mathbin{\&} \mathcal{R}(\mathfrak{M}, \mathfrak{N}_2) \Rightarrow [\mathfrak{N}_1 \models \varphi \Leftrightarrow \mathfrak{N}_2 \models \varphi].$$

$\psi \in \mathrm{Stc}(\tau_2)$ is said to be an *associate for φ on the domain of* $\mathcal{R}$ if for all $\mathfrak{M}, \mathfrak{N}$:

(2) $$\mathcal{R}(\mathfrak{M}, \mathfrak{N}) \Rightarrow [\mathfrak{N} \models \varphi \Leftrightarrow \mathfrak{M} \models \psi].$$

When this happens, ψ provides a *uniform reduction* of the property φ of $\mathfrak{N}$ to a property of $\mathfrak{M}$. Obviously (2) implies (1).

UNIFORM REDUCTION THEOREM. *Suppose L-Interpolation holds and that $\mathcal{R} \in \mathrm{PC}^\delta$. Then each φ which is invariant on the range of $\mathcal{R}$ has an associate ψ on its domain.*

Proof. $\mathcal{R}$ is defined by projection of a set $S \in \boldsymbol{I}$, $S \subseteq \mathrm{Stc}(\tau)$ for some $\tau \supseteq [\tau_0, \tau_1]$:

(a) $$[\mathfrak{M}, \mathfrak{N}] \in \mathcal{R} \Leftrightarrow \exists Q[\mathfrak{M}, \mathfrak{N}, Q] \models S .$$

By regularity of L, since $\varphi \in \mathrm{Stc}(\tau_1)$ also $\varphi \in \mathrm{Stc}(\tau)$ and

(b) $$\mathcal{R}(\mathfrak{M}, \mathfrak{N}) \Rightarrow (\mathfrak{N} \models \varphi \Leftrightarrow [\mathfrak{M}, \mathfrak{N}, Q] \models \varphi) .$$

Further since L is $L_{\omega,\omega}$-closed there exists $(\sim\varphi) \in \mathrm{Stc}(\tau)$ for which

$$\mathfrak{N} \models \sim\varphi \Leftrightarrow \mathfrak{N} \not\models \varphi .$$

Let

(c) (i) $S_1 = S \cup \{\varphi\}$, $S_2 = S \cup \{\sim\varphi\}$ and
(ii) $\mathcal{K}_i = \{\mathfrak{M}\colon \exists \mathfrak{N}, Q([\mathfrak{M}, \mathfrak{N}, Q] \in \mathrm{Mod}(S_i))\}$.

This puts $\mathcal{K}_1, \mathcal{K}_2 \in \mathrm{PC}^\delta(\tau_0)$. Also $\mathcal{K}_1 \cap \mathcal{K}_2 = 0$; for if we had $\mathfrak{M} \in \mathcal{K}_1 \cap \mathcal{K}_2$ there would exist $\mathfrak{N}_1, \mathfrak{N}_2, Q_1, Q_2$ with $[\mathfrak{M}, \mathfrak{N}_i, Q_i] \models S_i$ from which would follow $[\mathfrak{M}, \mathfrak{N}_1], [\mathfrak{M}, \mathfrak{N}_2] \in \mathcal{R}$ and $\mathfrak{N}_1 \models \varphi$ while $\mathfrak{N}_2 \not\models \varphi$. Since L-Interpolation implies L^δ-Interpolation, there is some $\mathcal{E} \in \mathrm{EC}(\tau_0)$ which separates $\mathcal{K}_1, \mathcal{K}_2$. Take $\psi \in \mathrm{Stc}(\tau_0)$ with $\mathcal{E} = \mathrm{Mod}(\psi)$. Whenever $\mathcal{R}(\mathfrak{M}, \mathfrak{N})$ holds we have

$$(\mathfrak{N} \models \varphi \Rightarrow \mathfrak{M} \models \psi)$$

by $\mathcal{K}_1 \subseteq \mathcal{E}$ and we have

$$(\mathfrak{M} \models \psi \Rightarrow \mathfrak{N} \models \varphi)$$

by $\mathcal{E} \cap \mathcal{K}_2 = 0$.

2.4. Corollaries. The following have been stated in the literature for $L_{\omega,\omega}$ or in certain cases more generally for admissible $L_A \subseteq L_{\omega_1,\omega}$. We give only the most inclusive reference for previous formulations in each case. Assume here that L is any language satisfying the general hypotheses and L-Interpolation. τ, τ_0, τ_1 range over Typ_L.

Cor. 1. Beth's definability theorem ([B1] p. 238). Suppose $\tau_0' = \tau_0 \cup \{\underset{\sim}{R}\}$ where $\underset{\sim}{R}$ is n-ary and $S \in \boldsymbol{I} \cap \mathcal{P}(\mathrm{Stc}_{\tau_0'})$ and that $\underset{\sim}{R}$ is implicitly defined by S. Then we can find a formula $\psi(x_1, \ldots, x_n)$ of $L(\tau_0)$ such that

$$S \vdash \underset{\sim}{R}(x_1, \ldots, x_n) \Leftrightarrow \psi(x_1, \ldots, x_n) .$$

(To prove this from the uniform reduction theorem take

$$\mathcal{R}(\mathfrak{M}, \mathfrak{N}) \Leftrightarrow \mathfrak{M} = [\mathfrak{M}_0, a_1, \ldots, a_n] \;\&\; \mathfrak{N} = [\mathfrak{M}, R] \;\&\; [\mathfrak{M}_0, R] \models S ,$$

and take φ to be the sentence $\underset{\sim}{R}(\underset{\sim}{c}_1, \ldots, \underset{\sim}{c}_n)$ of $L(\tau_0' \cup \{\underset{\sim}{c}_1, \ldots, \underset{\sim}{c}_n\})$.)

Cor. 2. Robinson's Invariance Theorem ([F2], § 2 "Main Lemma"). Suppose S_0, $S \in \boldsymbol{I} \cap \mathcal{P}(\mathrm{Stc}(\tau_0))$ and that the formula $\varphi(x_1, \ldots, x_n)$ of $L(\tau_0)$ is invariant rel. to S/S_0, i.e. whenever $\mathfrak{M}_0 \models S_0$, $\mathfrak{N}_i \models S$ and $\mathfrak{M}_0 \subseteq \mathfrak{N}_i$ $(i = 1, 2)$ and $a_1, \ldots, a_n$ are in $\mathfrak{M}_0$ then

$$\mathfrak{N}_1 \models \varphi(a_1, \ldots, a_n) \Leftrightarrow \mathfrak{N}_2 \models \varphi(a_1, \ldots, a_n) .$$

Then for some $\psi(x_1, \ldots, x_n)$ of $L(\tau_0)$ we have

$$\mathfrak{M}_0 \models \psi(a_1, \ldots, a_n) \Leftrightarrow \mathfrak{N} \models \varphi(a_1, \ldots, a_n)$$

whenever $\mathfrak{M}_0 \models S_0$, $\mathfrak{N} \models S$, $\mathfrak{M}_0 \subseteq \mathfrak{N}$ and $a_1, \ldots, a_n$ are in $\mathfrak{M}_0$. (This is immediate from the main theorem, using again identification of φ with a sentence of $L(\tau_0 \cup \{\underset{\sim}{c}_1, \ldots, \underset{\sim}{c}_n\})$ and 1.2 (3).)

Cor. 3. Gaifman's theorem on definable operations. ([G], § 1). Suppose $\mathcal{F}\colon \mathrm{Str}(\tau_0) \to \mathrm{Str}(\tau_1)$ is a projective operation. Then for each $\varphi \in \mathrm{Stc}(\tau_1)$ there exists $\psi \in \mathrm{Stc}(\tau_0)$ such that for every $\mathfrak{M} \in \mathrm{Dom}(\mathcal{F})$,

$$\mathcal{F}(\mathfrak{M}) \models \varphi \Leftrightarrow \mathfrak{M} \models \psi .$$

Hence also $\mathcal{F}$ preserves $\equiv_L$. (Immediate Corollary.)

Cor. 4. Barwise's theorem on interpretations ([B3]). Suppose $S \in \boldsymbol{I} \cap \mathcal{P}(\mathrm{Stc}_{\tau_1})$ and that π is a relative interpretation of $\mathrm{Stc}(\tau_0)$ in S. If for every $\mathfrak{N}_1, \mathfrak{N}_2$ such that $\mathfrak{N}_1 \models S$ and $\mathfrak{N}_2 \models S$ and $(\mathfrak{N}_1)^\pi = (\mathfrak{N}_2)^\pi$ we have $\mathfrak{N}_1 \models \varphi \Leftrightarrow \mathfrak{N}_2 \models \varphi$ then for some $\psi \in \mathrm{Stc}(\tau_0)$,

$$S \vdash (\varphi \Leftrightarrow \psi^\pi) .$$

(All we need to know about the concept of a relative interpretation π is that it associates with each $\mathfrak{N} \models S$ a structure $(\mathfrak{N})^\pi$ of type τ_0, in such a way that the relation

$$\mathcal{R}(\mathfrak{M}, \mathfrak{N}) \Leftrightarrow \mathfrak{M} = (\mathfrak{N})^\pi \text{ and } \mathfrak{N} \models S$$

is projective.)

2.5. Remarks on applications.

(i) **Non-reducibility of properties.** As Barwise noted, his theorem of [B3] generalizes one of Rosenthal [R] (actually, due still earlier to Büchi [Bü]) which was used to get results of the following kind: the structure of the constructible sets $(L, \in)$ is not determined (up to $\cong$) by the structure $\Omega = (ON, <, \vec{F})$ of the ordinals together with certain additional operations $\vec{F}$ (such as ordinal addition, multiplication and higher critical functions). This rests on getting explicitly definable models for the theory $\mathrm{Th}(\Omega)$. Rosenthal's Lemma [R], p. 499 for this may be put more generally here as follows, for $L_{\omega,\omega}$.

Theorem. *Suppose $\mathcal{R} \subseteq \mathrm{Str}(\tau_0) \times \mathrm{Str}(\tau_1)$ is the projection of a set S of sentences in $L_{\omega,\omega}$:*

$$\mathcal{R}(\mathfrak{M}, \mathfrak{N}) \Leftrightarrow \exists Q([\mathfrak{M}, \mathfrak{N}, Q] \models S) .$$

Suppose further that there are $\mathfrak{M}, \mathfrak{N}$ with $\mathcal{R}(\mathfrak{M}, \mathfrak{N})$ for which $\mathrm{Th}(\mathfrak{N})$ is not recursive in the join of S and $\mathrm{Th}(\mathfrak{M})$. Then there is some $\varphi \in \mathrm{Stc}(\tau_1)$ which has no associate ψ on the domain of $\mathcal{R}$.

Proof. If for each φ there is such an associate ψ then

$$[\mathfrak{M}, \mathfrak{N}, Q] \models \mathcal{S} \Rightarrow [\mathfrak{N} \models \varphi \Leftrightarrow \mathfrak{M} \models \psi]$$

so by completeness

$$\mathcal{S} \vdash (\varphi \leftrightarrow \psi)$$

in the sense of syntactic consequence. Hence there is a function g recursive in $\mathcal{S}$ which gives such $\psi = g(\varphi)$. Then whenever $\mathcal{R}(\mathfrak{M}, \mathfrak{N})$,

$$\varphi \in \mathrm{Th}(\mathfrak{N}) \Leftrightarrow g(\varphi) \in \mathrm{Th}(\mathfrak{M}),$$

contradicting the hypothesis.

It should be noted with reference to Rosenthal's applications, that the results of [F1], § 5 give a systematic means of producing extensive $\vec{F}$ for which $\mathrm{Th}(\Omega)$ is even hyperarithmetic.

(ii) **Preservation of $\equiv$ by algebraic operations.** Gaifman's theorem in the form of Cor. 3 applies immediately to the examples F1-F4 of 2.2, in all cases for certain sublanguages L_A of $L_{\omega_1,\omega}$. (F5 does not enter because of the failure of interpolation for 2nd order logic). These applications are in interesting contrast to the general $\equiv$-preservation theorems of [F1] which apply to various $\mathcal{F}$ in $L_{\infty,\lambda}$ and fragments $L^{\alpha}_{\infty,\lambda}$ (α limiting the quantifier-rank). There one had to verify some *functoriality conditions* for $\mathcal{F}$, rather than *definability conditions* as here. Hodges [H] has obtained improvements of [F1] to some languages $L_{\varkappa,\lambda}$ for $\mathcal{F}$ given by certain *explicit definitions by generators and relations*. (There is no simple comparison of these results since each is an implication involving different languages.)

This connects with an interesting (imprecise) question raised by Gaifman [G]. Suppose that $\mathcal{R}(\mathfrak{M}, \mathfrak{N})$ is elementary or projective in L and that it determines an algebraic operation $\mathcal{F}$ such that

$$\mathcal{R}(\mathfrak{M}, \mathfrak{N}_1) \,\&\, \mathcal{R}(\mathfrak{M}, \mathfrak{N}_2) \Rightarrow \exists! H\,(H\colon \mathfrak{N}_1 \cong \mathfrak{N}_2).$$

Is then $\mathcal{F}(\mathfrak{M})$ explicitly definable from $\mathfrak{M}$ in some sense [8]?

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Reçu par la Rédaction le 10. 9. 1973