in ∂D^2 from z_1 to, say, z_2 such that each component of $D^2-\beta$ contains some points of K. Then

$$w(f(a)) = w[\varrho([z_0', z_2']) * \beta^{-1} * \beta * \varrho([z_2', z_0'])] = w[\varrho([z_0', z_2']) * \beta^{-1}] w[\beta * \varrho([z_2', z_0'])]$$

is the product of words in R. This completes the proof of the theorem. ∎

As an example of this theorem, consider the crosscap; this is the projection, in general position, of an embedding of the projective plane in R^4. To find its knot group we note that there is only one region, Σ_1, and only one arc of double points. Thus we have one generator, σ_1, and two relations; the first is $\sigma_1 = \sigma_1^{-1}$, the second is $\sigma_1\sigma_1^{-1}\sigma_1\sigma_1^{-1} = 1$; thus the knot group is isomorphic to Z_2. Similarly one may consider the embedding of the Klein bottle in R^4 whose projection in R^3 is a surface with a single circle of self-intersection and find the knot group of this embedding to be isomorphic to Z_2.

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Accepté par la Rédaction le 15. 10. 1973

Two notes on abstract model theory
II. Languages for which the set of valid sentences is semi-invariantly implicitly definable

by

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Abstract. It is shown that if L is an abstract model-theoretic language, the syntax of L is represented in a structure $\mathfrak{A} = (A, \ldots)$, the Löwenheim-Skolem property holds down to card (A) and $\models_L$ is uniformly i.i.d. in L then the set of L-valid consequences of a set S of sentences is s.i.i.d. in L whenever S itself is so definable. This generalizes a theorem of Kunen for admissible fragments of $L_{\infty,\omega}$. The final part of the paper relates this to a program of study of good properties of model-theoretic languages.

Introduction. The aim of this note is quite different from that of the preceding [F2], though it makes use of the same general preliminaries. In content, it is a sequel to [F1], § 3 where some connections were studied, for arbitrary languages L, between implicit definability of the satisfaction relation $\models_L$ and logical properties of L. The basic relevant notions of [F1] are recalled below, in particular that of the *syntax* of L being *represented in a structure* $\mathfrak{A} = (A, \ldots)$ and (relative to any such representation) that of $\models_L$ being *uniformly invariantly implicitly definable* (uiid) *in* L. We add here a related notion of a subset S of A being *semi-invariantly implicitly definable* (siid) *in* L (1). This includes Kunen's definition in [Ku] of siid for admissible structures $\mathfrak{A} = (A; \varepsilon, R_1, \ldots, R_k)$ as a special case, and in the same line extends model-theoretic generalizations of recursion theory. Kunen showed that being siid is equivalent to other proposed

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(1) To be more precise, several variants of the notions siid are introduced and compared in § 2. In accordance with one of these, uiid is rewritten $\#$-uiid$_x$.

generalizations of being *recursively enumerable*, at least for countable admissible $\mathfrak{A}$.

The main result obtained here (§ 3) is that for L represented in $\mathfrak{A} = (A; \dots)$ (and satisfying some elementary conditions): *if the Löwenheim-Skolem property holds down to* $\operatorname{card}(A)$ *and* $\models_L$ *is uiid in L then the set* S-Vd_L *of L-valid consequences of a set S of sentences is siid in L whenever S is siid in L. In particular, the set* Vd_L *of L-valid sentences is siid in L.* These generalize the corresponding conclusions of [Ku] for arbitrary admissible fragments of $L_{\infty,\omega}$. It will be seen that the proofs isolate the ideas of Kunen's work in abstract terms. § 4 situates the results within an extended discussion of *good properties of model-theoretic languages*.

§ 1. Further general preliminaries. It is assumed throughout, as in [F2], that *L is any regular, $L_{\omega,\omega}$-closed language.* $\mathrm{Fm}_L(\tau)$ denotes the class of *formulas* of $L(\tau)$, explained as in [F2], 1.2. This determines the meaning of $\preccurlyeq_L$, the *elementary substructure relation for L.*

1.1. Löwenheim-Skolem properties. By the cardinality $\operatorname{card}(\mathfrak{M})$ of a structure $\mathfrak{M} = (\langle M_j\rangle_{j\in J}; \dots)$ is meant $\operatorname{card}(\bigcup_{j\in J} M_j)$. Let $\varkappa$ be any infinite cardinal.

DEFINITION 1. L is said to have the *global L-S property down to $\varkappa$* if whenever $\mathfrak{M}_0 \subseteq \mathfrak{M}$ and $\operatorname{card}(\mathfrak{M}_0) \leqslant \varkappa \leqslant \operatorname{card}(\mathfrak{M})$ then there exists $\mathfrak{M}_1$ such that

(i) $\mathfrak{M}_0 \subseteq \mathfrak{M}_1 \subseteq \mathfrak{M}$, $\operatorname{card}(\mathfrak{M}_1) \leqslant \varkappa$, and

(ii) $\mathfrak{M}_1 \preccurlyeq_L \mathfrak{M}$.

DEFINITION 2. L is said to have the *local L-S property down to $\varkappa$* if whenever $\mathfrak{M}_0 \subseteq \mathfrak{M}$, $\operatorname{card}(\mathfrak{M}_0) \leqslant \varkappa \leqslant \operatorname{card}(\mathfrak{M})$, $\varphi \in \mathrm{Stc}_L$ and $\mathfrak{M} \models \varphi$ then there exists $\mathfrak{M}_1$ with

(i) $\mathfrak{M}_0 \subseteq \mathfrak{M}_1 \subseteq \mathfrak{M}$, $\operatorname{card}(\mathfrak{M}_1) \leqslant \varkappa$, and

(ii) $\mathfrak{M}_1 \models \varphi$.

EXAMPLES. These are with reference to the list E1-E5 of [F2], 1.3. We are particularly concerned with whether L-S holds down to $\operatorname{card}(A)$ for L represented in A. (The general notion of representation, in 2.2 below, is not needed here.) Verifications of the statements are standard.

E1. L_A (A admissible). Global L-S holds down to $\operatorname{card}(A)$. For $A \subseteq H(\varkappa^+)$, local L-S holds down to $\varkappa$.

E2. $L_A(Q_\varkappa)$. When $\operatorname{card}(A) < \varkappa$, local L-S does not hold down to $\operatorname{card}(A)$. Global L-S holds down to $\operatorname{card}(A)$ for admissible A with $\operatorname{card}(A) \geqslant \varkappa$.

E3. $L_{\omega,\omega}(\omega)$ (ω-logic). Global L-S holds down to ω.

E4. $L^2_{\omega,\omega}$ (2nd order logic). Local L-S does not hold down to ω.

E5. $L_{\varkappa,\varkappa}$, $\varkappa$ inaccessible. Global L-S holds down to $\varkappa$.

1.2. Reducing the number of sorts. It suffices to deal with single-sorted structures in the case of L_A, by means of the method of *unification of domains* which replaces consideration of $(\langle M_j\rangle_{0\leqslant j\leqslant n}; \dots)$ by that of $(\bigcup_{j=0}^{n} M_j; M_0, M_1, \dots, M_n, \dots)$. This method can also be applied to other languages, such as the $L_A(Q_\varkappa)$; but it cannot be applied, to 2nd-order logic in the treatment followed here, according to which each domain M_j of "individuals" must have an associated domain $M_{j*} = \mathfrak{P}(M_j)$. A slightly weaker requirement is that one can trade any domain M_j for a unary predicate when M_j happens to be a subset of some other M_i.

DEFINITION. L is said to have the *sort-reduction property* if for each $\tau \in \mathrm{Typ}_L$ with

(a) $$\mathrm{Sort}(\tau) = \{\boldsymbol{M}_0, \boldsymbol{M}_1, \dots, \boldsymbol{M}_n\}, \quad n \geqslant 1,$$

and with each $\varphi \in \mathrm{Fm}(\tau)$ and each $\mathfrak{M} \in \mathrm{Str}_L(\tau)$, for which

(b) $$\mathfrak{M} = (M_0, M_1, \dots, M_n; R_0, \dots) \quad \text{and} \quad M_0 \subseteq M_1,$$

are associated respectively, a new $\tau^0 \in \mathrm{Typ}_L$, a formula φ^0 of $L(\tau^0)$, and an L-structure $\mathfrak{M}^0$ of type τ^0 satisfying the following conditions:

(i) $\mathrm{Sort}(\tau^0) = \mathrm{Sort}(\tau) - \{\boldsymbol{M}_0\}$, $\mathrm{Symb}(\tau^0) = \mathrm{Symb}(\tau) \cup \{\boldsymbol{M}_0\}$ (considering $\boldsymbol{M}_0$ as a unary relation symbol).

(ii) $\mathfrak{M}^0 = (M_1, \dots, M_n; M_0, R_0, \dots)$.

(iii) The free variables of φ^0 are the same as those of φ, except that each $x^{(0)}$ is replaced by a variable $x^{(1)}$.

(iv) If s is any assignment to the free variables of φ then

$$\mathfrak{M} \models \varphi(s) \Leftrightarrow \mathfrak{M}^0 \models \varphi^0(s).$$

It is easily seen that *the languages* E1-E3 (*among others*) *have the sort-reduction property*. Whenever this property holds for L and we have local L-S down to $\operatorname{card}(A)$ then from existence of expansions $\mathfrak{A}'$ of $\mathfrak{A}$ having certain properties we can obtain existence of such $\mathfrak{A}'$ without new sorts. This idea is applied in Theorem 3 of Section 2.3 below.

1.3. Joins of indexed structures.

DEFINITION. L is said to have the *join property* if with each $\tau \in \mathrm{Typ}_L$ is associated $\tau^\Sigma \in \mathrm{Typ}_L$ having at least one new sort $\boldsymbol{I}$, such that we have operations $\varphi \mapsto \varphi^\Sigma$, $\langle \mathfrak{M}_i\rangle_{i\in I} \mapsto \sum_{i\in I} \mathfrak{M}_i$, and $\mathfrak{M} \mapsto \mathfrak{M}_{(i)}$ satisfying the following conditions:

(i) (*Join*) $(\sum_{i\in I} \mathfrak{M}_i) \in \mathrm{Str}_L(\tau^\Sigma)$ whenever $\mathfrak{M}_i \in \mathrm{Str}_L(\tau)$ for each $i \in I$, and $\sum_{i\in I} \mathfrak{M}_i = (I, \dots)$.

(ii) (*Projection*) If $\mathfrak{M} \in \mathrm{Str}_L(\tau^\Sigma)$ and $\mathfrak{M} = (I, \ldots)$ then $\mathfrak{M}_{(i)} \in \mathrm{Str}_L(\tau)$ for each $i \in I$ and $(\sum_{i \in I} \mathfrak{M}_i)_{(k)} = \mathfrak{M}_k$ for each $k \in I$.

(iii) If $\varphi \in \mathrm{Fm}_L(\tau)$ then $\varphi^\Sigma \in \mathrm{Fm}_L(\tau^\Sigma)$ and φ^Σ has the free variables of φ together with a new free variable u of sort $\boldsymbol{I}$.

(iv) For any $\mathfrak{M} = (I, \ldots) \in \mathrm{Str}_L(\tau^\Sigma)$, $i \in I$, $\varphi \in \mathrm{Fm}_L(\tau)$ and assignment s to the free variables of φ in $\mathfrak{M}_{(i)}$,

$$\mathfrak{M} \models \varphi^\Sigma(i, s) \Leftrightarrow \mathfrak{M}_{(i)} \models \varphi(s)\,.$$

All familiar languages (including E1-E5) *have the join property.* We illustrate the verification of this for L_A and for τ single-sorted with one binary relation symbol $\boldsymbol{R}$. Given $\mathfrak{M}_i = (M_i; R_i)$ for $i \in I$, take

$$\sum_{i \in I} \mathfrak{M}_i = (I, \bigcup_{i \in I} M_i; \bigcup_{i \in I} \{i\} \times M_i, \bigcup_{i \in I} \{i\} \times R_i)\,.$$

Thus $\mathrm{Sort}(\tau^\Sigma) = \{\boldsymbol{I}, \boldsymbol{M}\}$ and $\mathrm{Symb}(\tau^\Sigma) = \{\boldsymbol{M}^*, \boldsymbol{R}^*\}$ with $\boldsymbol{M}^*$ binary, $\boldsymbol{R}^*$ ternary. Inversely, given

$$\mathfrak{M} = (I, M; M^*, R^*)$$

of type τ^Σ, define $\mathfrak{M}_{(i)} = (M_{(i)}; R_{(i)})$ by

$$x \in M_{(i)} \Leftrightarrow \langle i, x\rangle \in M^* \quad \text{and} \quad \langle x, y\rangle \in R_{(i)} \Leftrightarrow \langle i, x, y\rangle \in R^*.$$

$\varphi^\Sigma(u, \ldots)$ is obtained from $\varphi(\ldots)$ by replacing each atomic $\boldsymbol{R}(x, y)$ by $\boldsymbol{R}^*(u, x, y)$ and replacing each quantifier $\bigwedge x(\ldots)$ by $\bigwedge x[M^*(u, x, y) \to \ldots]$. The join is thus a kind of *disjoint union operation* displaying the index set as a new sort. This particular method of realizing the join property must be modified slightly in special cases like 2nd order logic.

§ 2. Representation of syntax; implicit definability of syntactic and semantic notions.

2.1. Valid consequence.

Definition. Given $S \subseteq \mathrm{Stc}(\tau)$ with $\mathrm{Mod}(S) \neq 0$ and $\varphi(x_1, \ldots, x_n)$ in $\mathrm{Fm}_L(\tau)$ write

$$S \vdash \varphi$$

if $\mathrm{Mod}(S) \subset \mathrm{Mod}(\bigwedge x_1 \ldots \bigwedge x_n \varphi)$.

$$S\text{—}\mathrm{Vd}(\tau) \underset{\mathrm{Def}}{=} \{\varphi\colon \varphi \in \mathrm{Stc}(\tau) \text{ and } S \vdash \varphi\}\,.$$

when $\mathrm{Mod}(S) = 0$ we write $S\text{—}\mathrm{Vd}(\tau) = A$. S is omitted in these notations when S is empty.

2.2. Representation of syntax in a structure.

Definition 1. L is said to be *represented in* $\mathfrak{A}$ *relative to* $\langle \pi_a(x)\rangle_{a \in A}$ if $\mathfrak{A} = (A, \ldots)$ is an L-structure and each $\pi_a(x)$ is an L-formula such that:

(i) $\bigcup \mathrm{Stc}(\tau)[\tau \in \mathrm{Typ}_L] \subseteq A$,

(ii) $\mathfrak{A} \models \pi_a(a)$ for each $a \in A$,

(iii) $\vdash [\pi_a(x) \wedge \pi_a(y) \to x = y]$ for each $a \in A$.

By (ii), (iii), not only does π_a define a in $\mathfrak{A}$, but it defines a unique element a' in each structure satisfying $\exists x \pi_a(x)$.

It is not required that $\mathfrak{A}$ be single-sorted; A is just a distinguished domain of $\mathfrak{A}$. But *we shall assume that* A *is of maximum cardinality among the domains of* $\mathfrak{A}$, i.e.

$$\mathrm{card}(A) = \mathrm{card}(\mathfrak{A})\,. \tag{1}$$

The principal example is that of *set-theoretical representation*, i.e. where $\mathfrak{A} = (A; \varepsilon)$ and A is a transitive set. Namely, if L contains L_A, conditions (ii) and (iii) are satisfied by use of the following (inductively defined) L_A-formulas:

$$\mu_a(x) = \bigwedge y[y \in x \leftrightarrow \bigvee_{b \in a} \bigvee \mu_b(y)]\,, \tag{2}$$

$$\pi_a(x) = \mu_a(x) \wedge \bigwedge_{b \in \mathrm{TC}(\{a\})} \bigwedge \bigvee!\, y \mu_b(y) \text{ (2)}. \tag{3}$$

Roughly speaking, μ_a describes a with respect to its *members*, while π_a describes it with respect to its *entire set of predecessors*, taking $b \lesssim_{\mathrm{TC}} a \underset{\mathrm{Def}}{\Leftrightarrow} b \in \mathrm{TC}(\{a\})$.

Every familiar language L *(including those in* E1-E5), *for which* Stc_L *is a set, has a representation in a transitive* A *such that* L *contains* L_A. For in every case formulas are *built up from* atomic formulas by repeated application of syntactic operations and constitute certain *coded well-founded trees*; these can be identified in a canonical manner with certain sets in the cumulative hierarchy.

Remark. It may be expected that all "natural" languages which will ever be used will have such a set-theoretical representation. One might therefore think of including this as part of the abstract definition of model-theoretical language. That has not been done here (or elsewhere) since many (if not most) results of abstract model theory do not depend on such a build-up. (Even the results below require only special aspects of such a build up, just those given in the preceding definition.) It is also possible that languages with some kind of non-well-founded formulas could be used for counter examples.

The language $L_{\infty,\omega}$ is not represented in a set. One could extend the notion to that for representability in classes, but it seems preferable to deal with $L_{\varkappa,\omega}$ where $V_\varkappa$ satisfies explicit closure properties (such as satisfying ZF) in place of $L_{\infty,\omega}$.

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(2) Correction to [F1], § 3.3 (2): write $\bigvee! y$ for $\bigvee y$, as in formula (3) of the text here

Before languages with infinite formulas were taken up systematically, one considered representations in the natural numbers, or in the set of all finite sequences from a finite alphabet. For representation in $\mathfrak{A} = (\omega; <)$ we may take each $\mu_n(x)$ $(n \in \omega)$ to express that x has exactly $n-1$ predecessors, and $\pi_n(x) = \mu_n(x) \wedge \bigwedge_{m \leqslant n} \bigwedge \bigvee ! \, y \, \mu_m(y)$.

We assume throughout the following that $\mathfrak{A} = (A, \ldots)$ *is any* L*-structure and* $\langle \pi_a(x) \rangle_{a \in A}$ *any sequence relative to which* L *is representable in* $\mathfrak{A}$. $\tau_{\mathfrak{A}}$ denotes the type of $\mathfrak{A}$.

Definition 2. For $\mathfrak{A}' = (A', \ldots)$ of type $\tau_{\mathfrak{A}}$, we write $\mathfrak{A}_0 \subseteq_\pi \mathfrak{A}'$, and call $\mathfrak{A}'$ a *π-extension* of $\mathfrak{A}_0$ if

(i) $\mathfrak{A}_0 \subseteq \mathfrak{A}'$ and

(ii) $\mathfrak{A}' \models \pi_a(a)$ for each $a \in A_0$.

More generally we write $\nu \colon \mathfrak{A}_0 \underset{\pi}{\to} \mathfrak{A}'$ if

(i) $\nu \colon \mathfrak{A}_0 \to \mathfrak{A}'$ is an embedding and

(ii)′ $\mathfrak{A}' \models \pi_a(\nu(a))$ for each $a \in \mathfrak{A}_0$.

In the case of set-theoretical representation for $\mathfrak{A} = (A, \varepsilon)$ $A_0 \subseteq A$ $\mathfrak{A}' = (A', E')$, then $\mathfrak{A}_0 \subseteq_\pi \mathfrak{A}'$ just in case $\mathfrak{A}'$ is an *end-extension of* $\mathfrak{A}_0$. Thus, more generally, there exists ν with $\nu \colon \mathfrak{A}_0 \underset{\pi}{\to} \mathfrak{A}'$ just in case $\mathfrak{A}'$ is *isomorphic to an end-extension of* $\mathfrak{A}_0$; in that case ν is unique. Given $a \in A_0$, $a' \in A'$ we have

$$\mathfrak{A}' \models \pi_a(a')$$

if and only if $a' = \nu(a)$. Then $\nu_1 \colon \mathrm{TC}(\{a\}) \underset{\pi}{\to} \mathfrak{A}'$ for the restriction ν_1 of ν.

2.3. Invariant implicit definability of subsets of A. We shall have to make considerable use in the remainder of the paper of projections of L-elementary classes. Following [F2], 1.1(1), we write:

(1) $$\mathfrak{M} \models \exists_{(\tau'-\tau)} \varphi \underset{\mathrm{Def}}{\Leftrightarrow} \exists \mathfrak{M}' \in \tau' - \mathrm{Exp}(\mathfrak{M}) \quad (\mathfrak{M}' \models \varphi),$$

whenever $\tau, \tau' \in \mathrm{Typ}_L$, $\tau \subseteq \tau'$ and $\varphi' \in \mathrm{Stc}_L(\tau')$. When there is no ambiguity, we simply write $\mathfrak{M} \models \exists \varphi$ if this holds.

Some motivation for the following definitions and comparison of the terminology with previous usage will be found in the next section 2.4. We abbreviate (*semi-*) *invariantly implicitly definable* by (s)iid.

The subscript x abbreviates *extra* (for the case of possibly new sorts) and # abbreviates *sharply*.

L_1 is assumed to be any language contained in L ([F1], § 3.1); for simplicity this can be taken so that

(2) $$\mathrm{Stc}_{L_1} \subseteq \mathrm{Stc}_L .$$

In particular, when L is $L_{\omega,\omega}$-closed we assume that $L_{\omega,\omega}$ is contained in L in this sense (of course with types and structures restricted as for L).

Definition 1. Suppose $S \subseteq A$. Then S is said to be *siid$_x$* (resp. *iid$_x$*) in L_1 if there are τ^+ and φ satisfying:

(i) $\tau^+ \in \mathrm{Typ}_L$, $\tau^+ \supseteq \tau_{\mathfrak{A}} \cup \{S\}$, $\varphi \in \mathrm{Stc}_{L_1}(\tau^+)$,

(ii) $[\mathfrak{A}, S] \models \exists \varphi$, and

(iii) if $[\mathfrak{A}', S'] \models \exists \varphi$ and $\mathfrak{A} \subseteq_\pi \mathfrak{A}'$ then $S \subseteq S'$ (resp. $S = S' \cap A$). S is said to be *siid* (resp. *iid*) in L_1 if, in addition

(iv) $\mathrm{Sort}(\tau^+) = \mathrm{Sort}(\tau)$.

Note that equivalently in (iii) we can write $\nu \colon \mathfrak{A} \underset{\pi}{\to} \mathfrak{A}'$ in place of $\mathfrak{A} \subseteq_\pi \mathfrak{A}'$ and $\nu(S) \subseteq S'$ (resp. $\nu(S) = S' \cap \nu(\mathfrak{A})$) in the conclusion.

Definition 2. S is said to be *#-siid$_x$* (resp. *#-iid$_x$*) in L_1 if there are τ^+, φ satisfying (i), (ii) of the preceding and in addition: (iii) if $[\mathfrak{A}', S'] \models \exists \varphi$ and $\mathfrak{A}' \models \pi_a(a')$ then $a \in S \Rightarrow a' \in S'$ (resp. $a \in S \Leftrightarrow a' \in S'$). Again we drop x if $\mathrm{Sort}(\tau^+) = \mathrm{Sort}(\tau)$.

Of course all of these definitions may be given just as well for $S \subseteq A^n$.

The notions with x are called *unrestricted*; each notion obviously implies its unrestricted version.

Theorem 3. *Suppose*

(i) L_1 *satisfies the sort-reduction property and*

(ii) L_1 *satisfies the local* L*-*S *property down to* $\mathrm{card}(A)$.

Then each of the notions siid, iid, #-siid, #-iid (in L_1*) is equivalent to its unrestricted version.*

Proof. Suppose S is siid by φ in L_1. By local L-S down to $\mathrm{card}(A)$ $(= \mathrm{card}(\mathfrak{A}))$ we can assume there exists an expansion $\mathfrak{A}^+$ of $[\mathfrak{A}, S]$ which satisfies φ and is of the form:

$$\mathfrak{A}^+ = [\mathfrak{A}, S, (M_1, \ldots, M_n; R_0, \ldots)]$$

where $M_1, \ldots, M_n$ are all the new sorts and

$$M_1 \cup \ldots \cup M_n \subseteq A .$$

Then by successive reduction of sorts, we obtain φ^0 in L_1 and $\mathfrak{A}^{+0}$ where

$$\mathfrak{A}^{+0} \models \varphi^0, \quad \mathfrak{A}^{+0} = [\mathfrak{A}, S; M_1, \ldots, M_n, R_0, \ldots]$$

but with $M_1, \ldots, M_n$ now considered as unary predicates in A. Suppose $[\mathfrak{A}', S'] \models \exists \varphi^0$ with $\mathfrak{A} \subseteq_\pi \mathfrak{A}'$. Then we have some $\mathfrak{B} = [\mathfrak{A}', S', \ldots]$ of type τ^{+0}, which satisfies φ^0. Then $\mathfrak{B}$ is of the form $\mathfrak{S}^0$ where $\mathfrak{S}$ is an expansion of $[\mathfrak{A}', S']$ of type τ^+ which satisfies φ. It follows that $S \subseteq S'$ and hence that τ^{+0} and φ^0 fulfill the conditions to make S siid in L_1. The argument proceeds in the same way for #-siid, as well as for iid and

$\#$-iid. For the latter two we can also make use of the following, which is easily proved.

LEMMA 4. *S is iid in $L_1 \Leftrightarrow S$ and $A-S$ are siid in L_1. The same holds with the qualifications $\#$ on both sides, as well as for the unrestricted versions.*

All of the foregoing may be *relativized to parameters* $a_1, \ldots, a_n$ in A simply by replacing $\mathfrak{A}$ throughout by $[\mathfrak{A}, a_1, \ldots, a_n]$. This is unnecessary in the case $L_1 = L$ since there we have every π_{a_i} available to define a_i. All notions relativized to some choice of parameters are indicated in boldface.

2.4. Comparisons with previous notions. These were introduced successively by Fraissé [Fr], Kreisel [Krl] and Kunen [Ku] for model-theoretic generalizations of recursion theory to a variety of structures $\mathfrak{A}$. The generalized notion of *recursive subset* S of $\mathfrak{A} =: (A; R_1, \ldots, R_k)$ given in [Fr] agrees with that of being iid in $L_{\omega,\omega}$ if we replace $\mathfrak{A} \subseteq_\pi \mathfrak{A}'$ by $\mathfrak{A} \subseteq \mathfrak{A}'$. The definitions of [Ku] apply to structures $\mathfrak{A} = (A; E, R_1, \ldots, R_k)$ with E binary and make use of a relation of *transitive extension* $\subseteq_E$ generalizing that of end-extension. The notions siid (iid) of [Ku] are equivalent to those here of siid, (iid) in $L_{\omega,\omega}$ if we replace $\subseteq_\pi$ by $\subseteq_E$. This includes as a special case the definitions given for ordinals $(\alpha; <, R_1, \ldots, R_k)$ in [Kr]. The other cases of principal interest are the $\mathfrak{A} = (A; \varepsilon, R_1, \ldots, R_k)$ which are admissible (w.r. to $R_1, \ldots, R_k$). In these particular cases, being siid in $L_{\omega,\omega}$ relative to the set-theoretical representation $\langle \pi_a \rangle_{a \in A}$ agrees with Kunen's siid because $\subseteq_\pi$ is the same as end-extension. Kunen obtained various results on the *explicit definability* of siid sets. In particular he showed for countable admissible $\mathfrak{A}$, that siid agrees with Σ_1 (rel. to $\mathfrak{A}$) i.e. with the class of sets proposed as $\mathfrak{A}$-*recursively enumerable* by Kripke and Platek ([3]).

The motivations for these definitions are perhaps better understood if one considers the ideas for the proof in the particular case that $L_1 = L = L_{\omega,\omega}$, $\mathfrak{A} = (\omega; <, R_1, \ldots, R_k)$ and $S \subseteq \omega$ of:

(1) $\quad S$ is siid $\Leftrightarrow S$ is rec. enumerable in $R_1, \ldots, R_k$.

The forward direction makes use of the *completeness theorem for* $L_{\omega,\omega}$. It is the converse which interests us here. If S is r.e. in $R_1, \ldots, R_k$, it may be given by an elementary inductive definition using previously obtained $S_1, \ldots, S_m$ as auxiliaries. Take φ to be the conjunction of all the inductive clauses involved (with o', defined in terms of $<$), so that

(2) $$(\omega; <, R_1, \ldots, R_k, S, S_1, \ldots, S_m) \models \varphi .$$

([3]) Kunen also obtained characterizations for uncountable $\mathfrak{A}$ as well, in some cases as Σ_1 and in other cases as Π_1^1. An elegant refinement of these for arbitrary admissible $\mathfrak{A}$ was found by Barwise [B2].

Whenever it is verified on the basis of φ that $a \in S$ (where of course $a \in \omega$) we make use only of some finite segment of ω. Hence for each $a \in S$ there exists $b \in \omega$ with $b \geqslant a$ such that if

(3) $\quad (A'; <', R_1', \ldots, S', S_1', \ldots) \models \varphi$ and $(A'; <')$ is an end-extension of $(b, <)$ then $a \in S'$.

It follows that S is siid.

In fact we can obtain here the stronger conclusion that S is $\#$-siid simply by adjoining to φ the statements λ and σ that there is a least element and that every element has a unique successor (resp.). Once the position of $\boldsymbol{a}$ is fixed by π_a, the verification of $a \in S'$ holds equally in any structure (3) satisfying $\varphi \wedge \lambda \wedge \sigma$.

2.5. Invariant implicit definability of semantic notions. The following definition was given in [F1] ([4]). Here u abbreviates *uniform*.

DEFINITION 1. Suppose $T\colon \mathrm{Str}_L(\tau) \to \mathfrak{P}(A)$ where $\tau \in \mathrm{Typ}_L$, $\tau_{\mathfrak{A}} \cap \tau = 0$. T is $\#$-uiid$_x$ in L_1 if for some τ^+ and θ we have:

(i) $\tau^+ \supseteq \tau_{\mathfrak{A}} \cup \tau' \cup \{\boldsymbol{T}\}$ and $\theta \in \mathrm{Stc}_{L_1}(\tau^+)$,

(ii) for each $\mathfrak{M} \in \mathrm{Str}_L(\tau)$, $[\mathfrak{A}, \mathfrak{M}, T(\mathfrak{M})] \models \exists\theta$, and

(iii) if $a \in A$ and $[\mathfrak{A}', \mathfrak{M}, T'] \models \exists\theta$ and $\mathfrak{A}' \models \pi_a(a')$ then $a' \in T' \Leftrightarrow a \in T(\mathfrak{M})$.

It is clear how one would define notions of usiid and uiid with or without $\#$ or x. However, only the preceding will be used below.

The main application of interest is to the function Tr_L or more precisely $\mathrm{Tr}_{L,\tau}$ for any $\tau \in \mathrm{Typ}_L$ (renamed so as to be disjoint from $\tau_{\mathfrak{A}}$,) where

(1) $$\mathrm{Tr}_L(\mathfrak{M}) = \{\varphi\colon \varphi \in \mathrm{Stc}_L(\tau) \ \&\ \mathfrak{M} \models \varphi\} .$$

DEFINITION 2. L_1 is *adequate to truth in* L if Tr_L is $\#$-uiid$_x$ in L_1 for each $\tau \in \mathrm{Typ}_L$.

It was shown in [F1], § 3 ("Adequacy Theorem") that L_A *is adequate to truth in itself, for any admissible* A; the same method of proof also works for *all familiar languages* (including E1-E5). Roughly speaking, to prove this we first considered the usual implicit definition of the satisfaction relation

$$\mathrm{Sat}_L(\mathfrak{M}) = \{\langle s, \varphi\rangle\colon \varphi \in \mathrm{Fm}(\tau) \text{ and } s \in \mathfrak{M}^{(\omega)} \text{ and } \mathfrak{M} \models \varphi[s]\}$$

([4]) The symbol $\#$ and subscript x were not used in designating this notion in [F1]; they are added here to be in accord with Definitions 1 and 2 of § 2.5.

where $\mathfrak{M}^{(\omega)}$ is the set of all finite (or eventually constant) sequences from $\mathfrak{M}$. $\mathfrak{M}^{(\omega)}$ is introduced as an auxiliary sort. This is a sharp (#) implicit definition because the answer to each question: is $\langle s, \varphi\rangle \in \mathrm{Sat}(\mathfrak{M})$?, is determined simply by the answers to: is $\langle s', \psi\rangle \in \mathrm{Sat}(\mathfrak{M})$? for *proper subformulas* ψ *of* φ (so $\psi <_{\mathrm{TC}} \varphi$). The uniformity lies in using the same θ to express the inductive clauses independent of $\mathfrak{M}$. In contrast to the syntactic role of $\mathfrak{A}$, $\mathfrak{M}$ must be kept fixed in $[\mathfrak{A}', \mathfrak{M}, T', \ldots]$ in order that quantification be absolute.

It is seen from the proof that already $L_{\omega,\omega}$ *is adequate to truth in* L_A. In fact, *the same holds for the languages* E3-E5, i.e. the implicit definition of truth in these L can always be given by an ordinary finite formula, using of course the restriction of satisfaction to L-admitted structures. This gives interest to the following observation.

THEOREM 3. *Suppose L_1 is $L_{\omega,\omega}$-closed and that L_1 is adequate to truth in L. Then whenever S is siid$_x$ in L it is also* siid$_x$ *in* L_1.

Proof. Let $\varphi \in \mathrm{Stc}_L(\tau^+)$ satisfy the conditions to make S siid$_x$ in L, and θ the conditions to make Tr_{L,τ^+}#-uiid$_x$ in L_1. We add the parameter θ to $\mathfrak{A}$ considered as an element c of A. Then the sentence $\theta \wedge \boldsymbol{T}(c)$ of L_1 expresses that φ is true and serves to make S siid$_x$ in L_1.

COROLLARY 4. *If, in addition to the hypotheses of the preceding theorem:* (i) *L_1 satisfies the sort-reduction property,* (ii) *L_1 satisfies local L-S down to* card(A), *then S siid in L_1 implies S* siid *in* L_1.

COROLLARY 5. *If further, L has the set-theoretical representation in $\mathfrak{A} = (A; \varepsilon)$ and $L_A \subseteq L$ then S siid in $L \Leftrightarrow S$* siid *in* L_1.

In particular, as we have seen, all these hypotheses apply to $L = L_A$ (A admissible) and $L_1 = L_{\omega,\omega}$.

§ 3. The principal theorems.

THEOREM 1. *Suppose that L satisfies the join property and that L is adequate to truth in itself. Then for each $\tau \in \mathrm{Typ}_L$, $\mathrm{Vd}_L(\tau)$ is #-siid$_x$ in L.*

THEOREM 2. *Under the same conditions as Theorem* 1, *if $S_0 \subseteq \mathrm{Stc}_L(\tau)$ and S_0 is siid$_x$ in L then $S_0 - \mathrm{Vd}_L(\tau)$ is siid$_x$ in L.*

THEOREM 3. *Under the same conditions as Theorem* 1, *if in addition*

(i) *the local L-S property holds down to* card(A) *and*

(ii) *L satisfies the sort-reduction property then we can conclude that $\mathrm{Vd}_L(\tau)$ is #-siid in L and that $S_0 - \mathrm{Vd}_L(\tau)$ is siid in L whenever S_0 is siid in L.*

Proofs. We begin with a proof of Theorem 2. We then see how to get the stronger conclusion of Theorem 1 in case $S_0 = 0$.

Let $S = S_0 - \mathrm{Vd}_L(\tau)$ be the class of valid consequences of S_0. Since by definition $S = A$ when $\mathrm{Mod}(S_0) = 0$, we may assume $\mathrm{Mod}(S_0) \neq 0$. Thus for each $a \in A$,

(1) $$a \in S \Leftrightarrow a \in \mathrm{Stc}_L(\tau)\ \&\ \forall \mathfrak{M}[\mathfrak{M} \in \mathrm{Mod}(S_0) \Rightarrow \mathfrak{M} \models a] \Leftrightarrow \forall \mathfrak{M}[\mathfrak{M} \models S_0 \Rightarrow \mathfrak{M} \models a].$$

Then $a \notin S \Leftrightarrow \exists \mathfrak{M}[\mathfrak{M} \models S_0$ and $\mathfrak{M} \not\models a]$. For each $a \in A$ choose $\mathfrak{M}_a$ in such a way that

(2) $$\text{if} \quad a \notin S \quad \text{then} \quad \mathfrak{M}_a \models S_0 \quad \text{and} \quad \mathfrak{M}_a \not\models a .$$

Take $\mathfrak{M}_a$ to be arbitrary of type τ when $a \in S$. Thus

(3) $$a \in S \Leftrightarrow \{\forall b (b \in S_0 \Rightarrow \mathfrak{M}_a \models b) \Rightarrow \mathfrak{M}_a \models a\} .$$

Let θ make $\mathrm{Tr}_{L,\tau}$ #-uiid$_x$ in type τ^+. In particular,

(4) (i) for each $a \in A$, there is a τ^+-expansion $\mathfrak{M}_a^+$ of $[\mathfrak{A}, \mathfrak{M}_a, \mathrm{Tr}(\mathfrak{M}_a)]$ such that $\mathfrak{M}_a^+ \models \theta$ and

(ii) if $[\mathfrak{A}', \mathfrak{M}', T'] \models \exists\theta$ and $\mathfrak{A}' \models \pi_b(b')$ then
$$b' \in T' \Leftrightarrow b \in \mathrm{Tr}(\mathfrak{M}') .$$

We write $\boldsymbol{T}$ as the unary relation symbol for Tr in θ. Using the join property, take

(5) $$\mathfrak{M}^\Sigma = \sum_{a \in A} \mathfrak{M}_a^+, \qquad \tau^\Sigma = \text{type of } \mathfrak{M}^\Sigma.$$

$\mathfrak{M}^\Sigma$ may be considered to be an expansion of $\mathfrak{A}$, $\mathfrak{M}^\Sigma = [\mathfrak{A}, \ldots]$. For each formula $\varphi(x_1, \ldots, x_n)$ of type τ^+ we have a formula $\varphi^\Sigma(u, x_1, \ldots, x_n)$ of type τ^Σ such that for $a \in A$, s in $\mathfrak{M}_a$

(6) $$\mathfrak{M}^\Sigma \models \varphi^\Sigma(a, s) \Leftrightarrow \mathfrak{M}_a \models \varphi(s) .$$

More generally, given any $\mathfrak{N}$ of type τ^Σ with $\mathfrak{N} = [\mathfrak{A}', \ldots]$, $\mathfrak{A}' = (A', \ldots)$ and given any $a' \in A'$ we can form $\mathfrak{N}_{(a')}$ so that for each assignment s in $\mathfrak{N}_{(a')}$

(7) $$\mathfrak{N} \models \varphi^\Sigma(a', s) \Leftrightarrow \mathfrak{N}_{(a')} \models \varphi(s) .$$

Note for the following that $(\boldsymbol{T}(v))^\Sigma$ has the form $\boldsymbol{T}^\Sigma(u, v)$.

Let ψ give the siid$_x$ for S_0 in type τ^*. We may assume $\tau^* \cap \tau^\Sigma = \tau_{\mathfrak{A}}$. Thus there exists

(8) $\mathfrak{M}^* = [\mathfrak{A}, S_0, \_]$ with $\mathfrak{M}^* \models \psi$, such that whenever $[\mathfrak{A}', S'] \models \exists\psi$ and $\mathfrak{A} \subseteq_{\pi} \mathfrak{A}'$ then $S_0 \subseteq S'$.

ψ has a symbol $\boldsymbol{S}_0$ for S_0. Both $\mathfrak{M}^*$, $\mathfrak{M}^\Sigma$ have $\mathfrak{A}$ as their $\tau_{\mathfrak{A}}$-retract, and their union forms a structure that we denote $\mathfrak{M}^\#$:

(9) $$\mathfrak{M}^\# = \mathfrak{M}^* \cup \mathfrak{M}^\Sigma = [\mathfrak{A}, S_0, \_, \ldots] .$$

The following sentence Γ will be shown to provide an siid$_x$ of S; it contains the siid$_x$ of S_0, together with the statement expressing that θ holds in each $\mathfrak{M}_{a}^{+}$ together with a statement expressing (3). This uses a new symbol $\boldsymbol{S}$.

(10) $\Gamma = \psi \wedge \bigwedge u\theta^{\Sigma}(u) \wedge \bigwedge u\{\boldsymbol{S}(u) \leftrightarrow [\bigwedge v(\boldsymbol{S}_0(v) \to \boldsymbol{T}^{\Sigma}(u, v)) \to \boldsymbol{T}^{\Sigma}(u, u)]\}$.

Here u, v range over the sort $\boldsymbol{A}$. By all of the preceding, $[\mathfrak{M}^{\#}, S] = [\mathfrak{A}, S, S_0, \_, \ldots] \models \Gamma$, i.e.

(11) $$[\mathfrak{A}, S, S_0] \models \exists\Gamma.$$

Suppose now that

(12) $$[\mathfrak{A}', S', S_0'] \models \exists\Gamma \text{ with } \mathfrak{A} \subseteq_\pi \mathfrak{A}'.$$

Thus $[\mathfrak{A}', S', S_0']$ has an expansion in type $\tau^* \cup \tau^{\Sigma}$ which satisfies Γ; this expansion is a union of two structures $[\mathfrak{A}', S_0', \_]$ and $[\mathfrak{A}', \ldots]$ of types τ^*, τ^{Σ} resp. We have $A \subseteq A'$, $\mathfrak{A}' \models \pi_a(a)$ for each $a \in A$ and

(13) (i) $[\mathfrak{A}', S_0'] \models \exists\psi$,
(ii) $[\mathfrak{A}', \ldots] \models \bigwedge u\theta^{\Sigma}(u)$.

By (i), $S_0 \subseteq S_0'$. By (ii) we can form structures $\mathfrak{N}_{(a')}$ for each $a' \in A'$, of the form

(14) $$\mathfrak{N}_{(a')} = [\mathfrak{A}', \mathfrak{M}'_{(a')}, T'_{(a')}, \ldots] \text{ and } \mathfrak{N}_{(a')} \models \theta.$$

From (4)(ii) we have for any $b \in A$ and $a' \in A'$,

(15) $$b \in T'_{(a')} \Leftrightarrow b \in \mathrm{Tr}(\mathfrak{M}'_{(a')}).$$

Note that $\boldsymbol{T}^{\Sigma}(a', b')$ is interpreted in $[\mathfrak{A}', \ldots]$ as $b' \in T'_{(a')}$. To complete the proof, we show for each $a \in A$ that

(16) $$a \in S \Rightarrow a \in S'.$$

Suppose $a \in S$. By (10), (12) it suffices to show that

(17) the assignment a to u in $[\mathfrak{A}', S', S_0', \_, \ldots]$ satisfies $\bigwedge v(S_0(v) \to \boldsymbol{T}^{\Sigma}(u, v)) \to \boldsymbol{T}^{\Sigma}(u, u)$.

Here the hypothesis is equivalent to

$$\forall b'[b' \in S_0' \Rightarrow b' \in T'_{(a)}].$$

Assuming this we have in particular $\forall b[b \in S_0 \Rightarrow b \in T'_{(a)}]$, so by (15) $\mathfrak{M}'_{(a)} \models S_0$. Then also $\mathfrak{M}'_{(a)} \models a$ since $a \in S = S_0 - \mathrm{Vd}_L(\tau)$. Again by (15), $a \in T'_{(a)}$, i.e. a satisfies $\boldsymbol{T}^{\Sigma}(u, u)$, q.e.d.

For the proof of Theorem 1 where $S_0 = 0$ we can simply take

(18) $$\Gamma = \bigwedge u\theta^{\Sigma}(u) \wedge \bigwedge u[\boldsymbol{S}(u) \leftrightarrow \boldsymbol{T}^{\Sigma}(u, u)].$$

This is satisfied in $[\mathfrak{M}^{\Sigma}, S] = [\mathfrak{A}, S, \ldots]$. To show that we have $\#$-siid$_x$ here, suppose

(19) $$[\mathfrak{A}', S'] \models \exists\Gamma \quad\text{and}\quad a' \in A' \quad\text{and}\quad \mathfrak{A}' \models \pi_a(a').$$

We still have (14) and (15) as above. Thus if $a \in S$, i.e. a is valid, then $\mathfrak{M}'_{(a')} \models a$ so $a' \in T_{(a')}$; hence $\boldsymbol{T}^{\Sigma}(a', a')$ is satisfied and $a' \in S'$ by (18), (19). Thus (19) $\Rightarrow$ $(a \in S \Rightarrow a' \in S')$ which is as required for $\#$-siid.

Finally, Theorem 3 is a direct corollary of Theorems 1,2 and § 2.3, Theorem 3. By the results of § 2.5 this generalizes [Ku] Theorem 3.2.

4. Good properties of model-theoretic languages.

4.1. Background. The following expands on the discussions of Barwise [B1], Introduction, and at length in Kreisel [Kr2], concerning

(i) *good properties of languages L*, and

(ii) *suggestions for finding new languages with such properties*. I use "good" to cover what they had variously described as: useful, simple, basic, pleasing, balanced, coherent, etc. As examples of such properties one particularly had in mind the holding of suitable generalizations of

(a) *the compactness theorem*,

(b) *recursive enumerability of the valid sentences, and*

(c) *the interpolation theorem.*

At the time of those discussions there was reason for optimism due to the achievements on (a)-(c) for countable admissible L_A of Barwise [B1], which at the same time showed the superiority of generalizations obtained by *definability criteria* in place of the crude *cardinality criteria* initially considered. This optimism was bolstered by the progress being made on (a), (b) for uncountable L_A in [Ku] and [B2]. It continued with Keisler's results [K] giving (a), (b) for fragments of $L_{\omega_1,\omega}(Q_{\aleph_1})$. However, the failure of interpolation in all these languages was annoying. Since then there have been no evident successes, let alone with the fairly specific scheme proposed by Kreisel [Kr2] (cf. 4.4(1) below). This has led to pessimism by some as to the prospects for further progress.

Abstract model theory should provide the proper setting in which to give precise formulations of the desired properties of languages.

It is conceivable one could then use such formulations to obtain definitive negative results, thereby justifying the current pessimism. Personally, I do not think such will be found, but rather that abstract negative results will be useful to remove wide classes of languages from consideration. An example of this kind is already to be found in Barwise [B3].

The following is an attempt to formulate good properties of languages in abstract terms. As emphasized by Kreisel, the progress with the L_A consisted in generalizing the notion of *finiteness* alongside generalizations of being *recursive* and *semi-recursive* (recursively enumerable) for *sets of syntactic objects*. This notion is to be chosen not only with the compactness theorem in mind, but also with the idea that *the syntactic objects themselves*, considered as sets in the cumulative hierarchy V, *should be generalized finite*. Since we are after syntactically natural languages here, it is appropriate to restrict attention to L having a set-theoretical representation in some $\mathfrak{A} = (A; \varepsilon, \ldots)$ with $A \in V$, A transitive.

Remarks. (i) As already stated, the syntactic objects in a natural language are canonically represented by coded well-founded trees, and hence may be identified with elements of V. It should not be expected, though, that every element of A corresponds to a syntactic object.

(ii) The aim here relates to the problem of finding good generalizations of recursion theory (g.r.t.), but *only for structures* $\mathfrak{A}$ *of sets*. It is expected that a good g.r.t. should also be able to provide suitable notions for *any structure* $\mathfrak{A}$.

4.2. Some good properties (preliminary). These are formulated in terms of L, $\mathfrak{A}$, and three abstract classes $\mathbf{F}$, $\mathbf{R}$ and $s\mathbf{R}$ of subsets of A called respectively the $\mathfrak{A}$-*finite*, $\mathfrak{A}$-*recursive* and $\mathfrak{A}$-*semi-recursive sets*. The first task is to set down desirable properties which interrelate all five; then the question will be how to choose $L, \ldots, s\mathbf{R}$ so as to satisfy them. This section will concern *recursion theoretic properties*, i.e. which do not involve L explicitly. (They could involve L implicitly in those cases where $\mathbf{F}$, etc. are defined in terms of L, as, for example, suggested in 4.4(2) below.)

It is assumed that

(1) (i) $A \in V$, $A \neq 0$, A is transitive, and
(ii) A is closed under $\{\,,\}$, $\bigcup$ and TC.

Then sub-relations of A are certain subsets of A; also functions are identified with their graphs. The following hypotheses are fairly standard.

(2) (i) $\mathbf{F} \subseteq \mathbf{R} \subseteq s\mathbf{R} \subseteq P(A)$.
(ii) $\mathbf{R} = s\mathbf{R} \cap \breve{s\mathbf{R}}$.
(iii) $\mathbf{F}$ is closed under $\cup$, $-(A)$ and $s\mathbf{R}$ is closed under $\cup$, $\cap$.
(iv) $s\mathbf{R}$ is closed under substitution by A-recursive functions.
(v) The functions $\{\,,\}$, $\bigcup$ and TC are $\mathfrak{A}$-recursive.
(vi) The ε-relation on A is $\mathfrak{A}$-recursive.

These are not intended to exhaust desirable recursion-theoretic properties.

Remark. In the search for stronger languages than presently known, A will likely satisfy very strong closure conditions, including being admissible. But languages represented in some A satisfying weak closure conditions could be useful in other ways, e.g. in proof theory. For this reason closure of $\mathbf{R}$ under bounded quantification is not listed as an hypothesis.

A principal condition which ought to be satisfied is that *inductive definitions with* $\mathfrak{A}$-*recursive clauses always lead to* $\mathfrak{A}$-*semi-recursive sets*. We formulate this rather strictly. By a *rule on* A we mean a relation $R \subseteq A^2$ such that each $a \in \mathrm{Dom}(R)$ is a function $a\colon \mathrm{Dom}(a) \to A$ with $\mathrm{Dom}(a) \neq 0$. Whenever $R(\langle a_i \rangle_{i \in I}, b)$ we write

$$\frac{\ldots a_i \ldots (i \in I)}{b}(R).$$

A *collection* $(R_c)_{c \in C}$ of *rules* is said to be $\mathfrak{A}$-*recursive* if we have $R \in \mathbf{R}$ where $R(a, b, c) \Leftrightarrow R_c(a, b) \;\&\; c \in C$. Given any $X \subseteq A$ there is a least set $\mathrm{Der}_R(X)$ of objects *derivable from* X by the rules $(R_c)_{c \in C}$. Then we take

$x \in \mathbf{F}\text{-}\mathrm{Der}_R(X) \Leftrightarrow \exists d\{d \in \mathbf{F} \;\&\; d$ is a derivation of x from X by means of the rules $(R_c)_{c \in C}\}$,

where the notion of derivation is explained as usual. We should then require:

(3) If X is $\mathfrak{A}$-finite and $(R_c)_{c \in C}$ is an $\mathfrak{A}$-recursive collection of rules then $\mathbf{F}\text{-}\mathrm{Der}_R(X)$ is $\mathfrak{A}$-semi-recursive.

Remark. More generally, one should formulate a notion of *uniform* $\mathfrak{A}$-*recursive* (or $\mathfrak{A}$-semi-recursive) monotone operator $\Gamma(X)$ and require that $\bigcap X[\Gamma(X) \subseteq X]$ is always $\mathfrak{A}$-semi-recursive. This will not be done here.

4.3. Some good properties (cont.). We now turn to properties which explicitly concern L. These are given in groups which seem to me to correspond to some degree of reasonableness for demands on L, with those of group I being in the nature of minimum requirements. The later groups contain additional or stronger properties. (This ranking is to be considered as tentative.)

I. (i) $\mathrm{Stc}_L \subseteq A$.
(ii) $\mathrm{Stc}_L \in \mathbf{R}$ and $\mathrm{Stc}_L(\tau) \in \mathbf{R}$ for each $\tau \in \mathrm{Typ}_L$.
(iii) $\mathrm{Stc}_L \subseteq \mathbf{F}$, i.e. every L-sentence considered as a set is $\mathfrak{A}$-finite.
(iv) L is regular, has the join property and the sort-reduction property.
(v) $\mathrm{Stc}_{L_A}(\tau) \subseteq \mathrm{Stc}_L(\tau)$ for each $\tau \in \mathrm{Typ}_L$ and satisfaction for L_A agrees with that for L on admitted L structures.
(vi) The function $\pi\colon a \mapsto \pi_a$ is $\mathfrak{A}$-recursive where $\pi_a(x)$ is the definition of a in L_A given in § 2.2(3).
(vii) $\mathrm{Vd}_L(\tau) \in s\mathbf{R}$ for each $\tau \in \mathrm{Typ}_L$; i.e. the valid sentences form an $\mathfrak{A}$-semi-recursive set.
(viii) If $(\mathrm{TC}(\{a\}); \varepsilon)$ is L-categorical (as defined below) then $a \in \mathbf{F}$.

II. (i) L is L_A-closed.

(ii) $S \in s\mathbf{R}$ and $S \subseteq \mathrm{Stc}_L(\tau) \Rightarrow S - \mathrm{Vd}_L(\tau) \in s\mathbf{R}$; i.e. the set of consequences of an $\mathfrak{A}$-semi-recursive set is again $\mathfrak{A}$-semi-recursive.

(iii) If $S \in \mathbf{F}$ then S is aid in L (as defined below).

(iv) If $S \in \mathbf{R}$ then S is iid in L.

(v) If $S \in s\mathbf{R}$ then S is siid in L.

(vi) L is adequate to truth in itself, i.e. Tr_L is $\#$-uiid$_x$ in L.

(vii) L is truth-maximal; i.e. if Tr_{L^*} is $\#$-uiid$_x$ in L then $L^* \subseteq L$.

III. (i) $\mathrm{Vd}_L(\tau) = \mathbf{F} - \mathrm{Der}_R(X)$ for some $\mathfrak{A}$-finite X and $\mathfrak{A}$-recursive collection R of rules (completeness).

(ii) The local L-S property holds down to card(A).

(iii) $L_{\omega,\omega}$ is adequate to truth in L.

IV. (i) $(\mathbf{F}, s\mathbf{R})$ forms a compactness property, as defined in [F2], § 1.5; in particular if S is an $\mathfrak{A}$-semi-recursive set of L-sentences and every $\mathfrak{A}$-finite subset of S has a model then S has a model.

V. (i) $\mathbf{F} = A$.

(ii) For any $S \in s\mathbf{R}$, $S - \mathrm{Vd}_L(\tau) = \mathbf{F} - \mathrm{Der}_R(S)$, for some $\mathfrak{A}$-recursive collection R of rules (strong completeness).

With reference to I(viii), $\mathfrak{M}$ is said to be *L-categorical* if for some $\varphi \in \mathrm{Stc}_L$,

(i) $M \models \varphi$ and

(ii) $\mathfrak{M} \cong \mathfrak{M}'$ whenever $\mathfrak{M}' \models \varphi$ (5).

The definition of *aid* (*absolutely implicitly definable*) used in II(iii) is given exactly like that for siid above (Definition 1, § 2.3) except that we replace $S \subseteq S'$ by $S = S'$ in the conclusion of (iii). More generally, in the same way we define: S is *aid in* L_1. Being aid in the sense of [Ku] is then the same as being aid in $L_{\omega,\omega}$.

Under the hypotheses of Corollary 5 of § 2.5, we also get:

$$S \text{ is aid in } L \Leftrightarrow S \text{ is aid in } L_{\omega,\omega}.$$

These hypotheses are certainly met if all the properties I–III are satisfied.

4.4. The problem of choosing $L, \mathfrak{A}, \mathbf{F}, \mathbf{R}, s\mathbf{R}$. This is the difficult part, if the aim is to satisfy a substantial portion of I-V in a language stronger than $L_{\omega_1,\omega}$ or $L_{\omega,\omega}(Q_{\aleph_1})$. The following is a scattered collection of proposals, remarks, examples, and questions.

(1) *Kreisel's scheme* [Kr2]. The following identification was proposed:

(i) $\mathfrak{A}$-finite $=$ aid in $L_{\omega,\omega}$,

(5) We might call $\mathfrak{M}$ *rigidly L-categorical* if whenever $\mathfrak{M}' \models \varphi$ then there is a unique H: $\mathfrak{M} \cong \mathfrak{M}'$.

(ii) $\mathfrak{A}$-recursive $=$ iid in $L_{\omega,\omega}$,

(iii) $\mathfrak{A}$-semi-recursive $=$ siid in $L_{\omega,\omega}$.

In addition, Kreisel assumed that A satisfies elementary closure conditions guaranteeing at least representation of L_A in A, and formal derivations, if used at all, are to be $\mathfrak{A}$-finite. The following problem was stated (loc. cit.) p. 145: "What further conditions must A satisfy in order that the basic properties of PC generalize to L_A for the translation given [above]? For what extensions of L_A do these properties persist?"

This problem has been studied with reference principally to the properties of completeness in the form I(vii) and II(ii) and compactness, IV(i). The main results of § 3 of this paper are of interest with respect to the first of these. These and the stability result of § 2.5, Corollary 5 show that II(ii) is a consequence of the proposed definition of $s\mathbf{R}$ and, basically, the properties that local L-S holds down to card(A) and that $L_{\omega,\omega}$ is adequate to truth in L. But the results do not seem to yield, for example, completeness of an $\mathfrak{A}$-recursive collection of rules (III(i)).

The problem of [Kr2] with reference to the compactness property IV(i) has been studied only for the L_A. Here the examples of Gregory [G] show that the above scheme definitely fails to give compactness on some (necessarily) uncountable admissible A. (In addition, it is consistent with ZFC to assume $\mathbf{F} = A$.) Thus stronger closure conditions on $\mathfrak{A}$ than admissibility would certainly be necessary to insure IV(i). Kunen [Ku] and Barwise [B2] only found strong conditions which guarantee some partial compactness results.

Remark. Variants may be worth considering in place of the above scheme. For examples, one may restrict to those siid where the auxiliary relations are themselves already siid. Another notion intermediate between siid and $\#$-siid was suggested near the end of § 2.4, to correspond to the idea that if $a \in S$ then this fact can be verified using only an $\mathfrak{A}$-finite part of $\mathfrak{A}$. One simple way to interpret this is that there exists $b \in A$ such that whenever $[\mathfrak{A}', S'] \models \exists\varphi$ and $\mathfrak{A}' \models \pi_a(a') \wedge \pi_b(b')$ then $(a \in S \Rightarrow a' \in S')$. Another would use the proposed definition of $\mathbf{F}$ itself.

(2) *Generalized-finiteness from the L-categorical sets.* The preceding scheme gives a general explanation of $\mathbf{F}, \mathbf{R}, s\mathbf{R}$ which depends only on $\mathfrak{A}$; they may then be used in the search for good L. But one can well imagine alternative schemes where these notions depend essentially on L. As an example, the following is immediately suggested by I(viii). For any L, define L-Cat $= \{a\colon (\mathrm{TC}(\{a\}); \varepsilon)$ is L-categorical$\}$, and then take

(i) $\mathbf{F} = L$-Cat and $A = \mathrm{TC}(\mathbf{F})$.

This leaves open the choice of $\mathbf{R}$ and $s\mathbf{R}$. A natural companion choice by 4.2(3) would be

(ii) $s\mathbf{R}$ = all sets of the form $\mathbf{F}-\mathrm{Der}_R(X)$ where $X \in \mathbf{F}$ and R is an $\mathfrak{A}$-finite collection of rules,

and

$$\mathbf{R} = s\mathbf{R} \cap s\widetilde{\mathbf{R}}\,.$$

But one could also consider taking siid or the L-representable sets as defined in (3) below, for $s\mathbf{R}$.

The problem here is how to choose new L so that if we take $\mathbf{F}$, A as in (i) then we have some good properties; at the very least we should want $\mathrm{Stc}_L \subseteq A$ and $L_A \subseteq L$. The following is an imprecise *conjecture*: for any given reasonable L^0 we can build up a least L^0-closed language with

$$L_{\mathrm{TC(L\text{-}Cat)}} \subseteq L \quad \text{and} \quad \mathrm{Stc}_L \subseteq \mathrm{TC(L\text{-}Cat)}\,.$$

Then we could start with any L^0 whose properties are not satisfactory and try to obtain a good L from it in this way.

Examples. It may be of interest to compare the definitions of $\mathbf{F}$ proposed in (1), (2) on familiar languages. In the case of L_A, A countable admissible, it can be shown under (1) that $A = \mathbf{F}$ and that the same holds under (2) when A is also locally countable. (I do not know whether that hypothesis is essential.) For $L_{\omega_1,\omega} = L_{\mathrm{HC}}$, Kunen [Ku] pointed out that

$$\mathrm{HC} \cup \{\mathrm{HC}\} \subseteq \mathbf{F}$$

on the definition $\mathbf{F}$ = aid in $L_{\omega,\omega}$. But definition (2) gives

$$\mathrm{HC} = \mathbf{F},$$

simply by applying local L-S down to $\aleph_0$. It should be noted with this definition of $\mathbf{F}$, and those just above of $s\mathbf{R}$ and $\mathbf{R}$, that $L_{\omega_1,\omega}$ has all the properties I-III as well as IV(ii) and V(i).

(3) *Generalized Gödel-Mostowski recursion theory* [M]. Let $T_{\mathfrak{A}}$ be the set of sentences $\bigvee x\pi_a(x)$ for $a \in A$ together with $\mathrm{Diag}(\mathfrak{A})$. The work in [M] suggests defining $\mathbf{R}$, $s\mathbf{R}$ in terms of L by:

(i) $S \in s\mathbf{R} \Leftrightarrow$ for some φ, ψ:

$$\forall a \in A\{a \in S \Leftrightarrow \varphi \cup T_{\mathfrak{A}} \vdash \bigwedge x[\pi_a(x) \to \psi(x)]\}$$

and

(ii) $S \in \mathbf{R} \Leftrightarrow$ for some φ, ψ: we have the above and also

$$\forall a \in A\{a \notin S \Leftrightarrow \varphi \cup T_{\mathfrak{A}} \vdash \bigwedge x[\pi_a(x) \to \sim\psi(x)]\}\,.$$

It is easily seen that every set which is siid_x (iid_x) in L is $s\mathbf{R}(\mathbf{R})$ in this sense. But one would hope to do better and show that a single φ could

be used independently of S (analogous to the system Q in arithmetic). This approach leaves completely open how to determine $\mathbf{F}$. Again the problem is how to choose new L so that it has good properties at least with $\mathfrak{A}$, $\mathbf{R}$, $s\mathbf{R}$.

(4) *Strengthening the $Q_{\aleph_1}$ languages*. These would have been examples of good new languages if one had interpolation. The counter-example to this (due to Keisler) suggests introducing a certain quantifier with stronger expressive power, which directly gives the missing interpolant.

We add to any L_A a new operator Q^E which is a quantifier binding pairs x, y of variables (E for Equivalence relation). Consider formulas built up using the operations of L_A and $Q^E x, y\varphi$ for any φ. Satisfaction is defined as follows:

$\mathfrak{M} \models Q^E x, y\varphi(x, y) \Leftrightarrow (\equiv_\varphi)$ has at least $\aleph_1$ distinct equivalence classes, where $(\equiv_\varphi)$ is the equivalence relation:

$$x_1 \equiv_\varphi x_2 \Leftrightarrow \mathfrak{M} \models \bigwedge y[\varphi(x_1, y) \leftrightarrow \varphi(x_2, y)]\,.$$

(We may read $x_1 \equiv_\varphi x_2$ as: x_1, x_2 are φ-indistinguishable).

Then for φ in $L_{\omega,\omega}$ $\{\mathfrak{M}\colon \mathfrak{M} \models Q^E x, y\varphi(x, y)\}$ is in $\mathrm{PC} \cap \widetilde{\mathrm{PC}}$ but is not in general in EC for $L_{\omega,\omega}(Q_{\aleph_1})$; this is the counter-example to (even Souslin-Kleene) interpolation. Note that $Q_{\aleph_1} x\varphi(x)$ is definable using Q^E.

Conjecture. $L_{\omega,\omega}(Q^E)$ has a complete axiomatization by recursive rules of inference, so its valid sentences are recursively enumerable.

Question. What are the good properties of $L_{\omega,\omega}(Q^E)$ and of the $L_A(Q^E)$ more generally?

Subsequent to the writing of this paper, I learned that the conjecture above for $L_{\omega,\omega}(Q^E)$ is correct. This was established independently by J. A. Makowsky and J. Stavi; they have also obtained axiomatization and compactness results for a number of related stronger languages. Their work is to appear in a paper jointly with S. Shelah.

Correction to [F1]. J. Stavi has brought to my attention that the argument for the Corollary (*) in [F1] § 3.4 is incorrect at one point. For, it follows from work of Cohen that there exist uncountable well-founded transitive models of ZF with only countably many ordinals; cf. Keisler, Ann. Math Logic 1 (1970), p. 42. Certainly then there exist admissible $A \not\subseteq H(\omega_1)$ for which $\omega_1 \notin A$. Thus only the "if" part of the Corollary is justified by the proof. However, Stavi has observed that if CH is assumed then the "only if" part is also correct. He shows that in this case for each $A \not\subseteq H(\omega_1)$ there exists $a_0 \in A$ with $\mathrm{card}(a_0) = \aleph_1$. Such a_0 may be used to produce a counter-example to Souslin-Kleene interpolation $(I)^{A}_{L_A}$. It is of interest to consider the status of both $\mathrm{card}(a)$ for $a \in A$ and $(I)^{A}_{L_A}$ in the case that CH is false.

(*) This reads: *For A admissible, L_A is truth-complete if and only if $A \subseteq H(\omega_1)$.*

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Accepté par la Rédaction le 24. 10. 1973

On some functional equations with a restricted domain

by

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Abstract. The functional equations considered are of the form (1) and (2) where f, g, h map an abelian group G into the other abelian group H. We assume their validity for almost all $(x, y) \in G\times G$ and investigate the question whether there exist functions F, G, H almost equal to f, g, h respectively and fulfilling our equations everywhere. The notion "almost all" ("almost everywhere") has been introduced in an axiomatic way.

§ 1. Recently there has been an increased interest in functional equations and inequalities whose validity is postulated "almost everywhere" (abbreviated to a.e. in the sequel). This a.e. is understood in various wans (see for instance [3], [7], [9], [8] and [6]). We shall be interested here in two functional equations,

(1) $$f(x+y)\big(f(x+y)-f(x)-f(y)\big)=0 \quad \text{(of Mikusiński)}$$

and

(2) $$f(x+y)=g(x)+h(y) \quad \text{(of Pexider)},$$

related to the well-known Cauchy equation (cf. [4] and [1]), assuming their a.e. validity in the sense described explicitly below. Roughly speaking, we are going to answer the following question: does there exist a function F (or: do there exist functions F_1, F_2, F_3) such that it satisfies (1) (or: they satisfy (2)) everywhere and $f=F$ a.e. (or: $f=F_1$, $g=F_2$, $h=F_3$ a.e.)? Such a problem was first raised by P. Erdös [5] in connection with Cauchy's functional equation. Positively solved by N. G. de Bruijn [3] and independently by W. B. Jurkat [8], this problem was then investigated by M. Kuczma [9] in connection with convex functions and by the present author for polynomial functions (also with positive answers).