The Failure of Leibniz's Infinite Analysis view of Contingency

Joel Velasco

Stanford University

Abstract: In this paper, it is argued that Leibniz's view that necessity is grounded in the availability of a demonstration is incorrect and furthermore, can be shown to be so by using Leibniz's own examples of infinite analyses. First, I show that modern mathematical logic makes clear that Leibniz's "infinite analysis" view of contingency is incorrect. It is then argued that Leibniz's own examples of incommensurable lines and convergent series undermine, rather than bolster his view by providing examples of necessary mathematical truths that are not demonstrable. Finally, it is argued that a more modern view on convergent series would, in certain respects, help support some claims he makes about the necessity of mathematical truths, but would still not yield a viable theory of necessity due to remaining problems with other logical, mathematical, and modal claims.

From his early metaphysical writings, such as "On Freedom and Possibility" to his later writings such as "The Monadology", Leibniz distinguished between those propositions which are necessary and those that are contingent. A central problem for Leibniz is to explain how there could be such a distinction. Since all truths necessarily follow from God's choice to actualize this world, (truths that Leibniz calls hypothetically or morally necessary), it seems that since God necessarily exists and necessarily chooses the best, then God must necessarily have chosen to actualize this world and so all truths would be absolutely necessary. Leibniz cannot simply ignore this charge as he says, "Above all, I hold a notion of possibility and necessity according to which there are some things that are possible, but yet not necessary…" (FP 20). This distinction is extremely important to his overall philosophy as well for he bases his accounts of human freedom on the contingency of human action. He also rests at least one of his proofs of God's existence on the existence of contingent truths (Theodicy §7).

Leibniz initially attempted to solve this problem by denying that it can be demonstrated that God makes that which is most perfect (FP 20), but this seems inconsistent with other things he says. In writings after "Freedom and Possibility" (1680-1682?), Leibniz avoids the problem of "blind necessity" by denying that it is necessary that this world is the best of all possible worlds.[1] To many readers this seems equally implausible, (Curly 1974: 92) yet Leibniz denies this seemingly obvious claim with appeal to a distinction that he claims to have discovered between the necessary and the contingent. As he says in "On Freedom", "…Derivative truths are, in turn, of two sorts, for some can be resolved to basic truths, and others, in their resolution, give rise to a series of steps that go to infinity. The former are necessary, the latter contingent" (OF 96) And even more suggestively,

> And here is discovered the inner distinction between necessary and contingent truths, which no one will easily understand unless he has some tincture of Mathematics – namely, that in necessary propositions one arrives, by an analysis continued to some point, at an identical equation (and this very thing is to demonstrate a truth in geometrical rigor); but in contingent propositions the analysis proceeds to infinity by reasons of reasons, so that indeed one never has a full demonstration, although there is always, underneath, a reason for the truth, even if, it is perfectly understood only by God, who along goes through an infinite series in one act of the mind" (Gr 303).

Thus the answer to the problem of blind necessity is that there could be no finite demonstration that this world is the best since that would require showing that it was better than all other possible worlds – clearly an infinite task. Since the fact that this world is the best is contingent, then our conclusion that God necessarily actualizes this world is blocked. It is this "infinite analysis" view of necessity and contingency that I will show in inconsistent with Leibniz's own views on the necessity of claims in mathematics and logic as well as claims about modality.

DEMONSTRATION

Before we move on, we need to know just what Leibniz has in mind when he says that there is a "demonstration" available. Leibniz claims that "Demonstration is nothing but displaying a certain equality or coincidence of the predicate with the subject (in the case of a reciprocal proposition) by resolving the terms of a proposition and substituting a definition or part of one for that which is defined" (OF 96). He also says, "The analysis [of some proposition] is either finite or infinite. If it is finite, it is said to be a demonstration and the truth is necessary" (SC 98) and "in contingent propositions one continues the analysis to infinity through reasons for reasons, so that one never has a complete demonstration" (OC 28). This makes clear that for Leibniz, demonstrations are necessarily finite. Yet the matter is confused by his repeated claim that every truth, whether necessary or contingent, has an apriori proof (FP 19, LA 141). Hacking (1973, 1974) insists that Leibniz has a well-defined notion of infinite proof. We can understand what is going on by acknowledging that Leibniz treats demonstrations differently than proofs. Demonstrations are completed finite analyses consisting of propositions reduced to identicals through substitutions. While these demonstrations are a type of proof, "proof" is used much more generally (Sleigh 1990: 85). For example, in many places, Leibniz argues that pointing out that one thing has more reason to exist than another is proving that it exists; yet this would not count as a demonstration. Incidentally, Sleigh points out that in one of his unpublished texts, Leibniz wrote in a margin that no contingent truth has an apriori proof because no contingent truth can be demonstrated (Sleigh 1990: footnote 86). The most charitable view is that this is a misstatement on Leibniz's part. The correct interpretation is just that in the case of contingent truths, while there exists a proof of them, no demonstration is possible and any attempted analysis will proceed to infinity without resolution.

When discussing demonstrations, Leibniz seems to want to allow substitution of identicals as the only proof rule. Yet this would obviously be too restrictive to prove much of interest. When giving an actual example of demonstration (OF 96) Leibniz freely uses the Aristotelian syllogisms, as well he should. When suitably adjusted by adding more proof rules (or Modus Ponens plus more axioms) Leibniz's view of demonstration becomes remarkably like our modern notion of a mathematical proof (Hacking 1973).

Several commentators on Leibniz have mentioned this infinite analysis view of contingency and tried to understand what Leibniz could have been saying. It has been widely argued that the infinite analysis view will not help Leibniz understand free will or other central topics in his philosophy (Curly 1974, Ishiguro 1972). While I agree, I will set this issue aside and attempt to deal with the many logical problems that arise. I will argue that this view has consequences that can be shown to be inconsistent with Leibniz's own views about the necessity of mathematics and logic as well as his views about modality. First, I will show how 20th century mathematics and logic has shown that Leibniz view would lead to the conclusion that many truths of mathematics and logic would be contingent. In later sections, I will argue that Leibniz's own mathematical examples can be shown to have these same unacceptable consequences.

20th CENTURY LOGIC

Recall Leibniz's view that truth consists in the concept of the predicate being contained in the concept of the subject. Given this, how it is possible that a true proposition, which is apparently analytic, could be only contingently true? The proposed solution is that while it is still true that the predicate is contained in the subject, this containment is not demonstrable. As an example that is supposed to help the reader understand how this is possible, Adams (1977) cites the logical notion of *ω*-consistency and then Blumenfeld (1985: 498), who refers to Adams, cites the same example (though misquotes it – some changes are trivial, but one sentence is completely changed which significantly alters the meaning). Adams later repeats the same passage with only minor, unimportant changes in an updated version of his earlier paper (Adams 1994). However, there are a few important things to note about this example that apply to all three versions:

> 1) The definition of *ω*-consistency is not exactly what Adams and Blumenfeld think it is.
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> 2) The correct definition does not allow Adams and Blumenfeld to make the claims that they do about this notion.
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> 3) There are related examples of the sort that Adams and Blumenfeld surely want to discuss, but they are in fact counterexamples to Leibniz's view, not examples of it.

As for the example, Adams says:

> It may be that there is a property, ϕ, such that for every natural number *n*, it can be proved that *n* has ϕ, but that the universal generalization that every natural number has ϕ cannot be proved except by proving first that 7 has ϕ, then that 4 has ϕ, and so on until every natural number has been accounted for - a task that can never be completed. In this case it is a purely mathematical truth that every natural number has ϕ but it cannot be demonstrated. And it is a purely mathematical falsehood that some natural number lacks ϕ, but no contradiction can be derived from it in a finite number of steps. Tarski decided to say that a system of which these conditions hold, but in which "Some natural number lacks ϕ" can be proved, is *consistent* but not ω-*consistent*. He thus reserved the use of "consistent" and "inconsistent," without qualification, to express proof-theoretical notions rather than notions of mathematical possibility and mathematical falsity. Similarly, Leibniz reserves "implies a contradiction" to express a proof-theoretical notion rather than the notion of conceptual falsity or being false purely by the relations of concepts. He thinks, of course, the latter notion is expressed simply by "false" (Adams 1977: 15-16).

While the concept that Adams is discussing might seem to be coherent, in fact, it is not what ω-consistency is, nor is it even possible for this situation to arise. Let us note the common usage of ω-consistency as first used in Gödel (1931) (not in Tarski[2]): An ω-inconsistent theory is one in which there is a formula $\phi(x)$ with one free variable such that for each numeral n, $\phi(n)$ is a theorem, but that $\neg\forall x\phi(x)$ [or equivalently $(\exists x)\neg\phi(x)$] is also a theorem. ω-consistent theories are just those that are not ω-inconsistent.[3] Importantly, we note that Adams states that the theory proves that "'Some natural number lacks ϕ' can be proved", but the correct definition merely states that $(\exists x)\neg\phi(x)$ is a theorem. This is an extremely important difference.

"Some natural number lacks ϕ" cannot be represented as a formula of first order logic since concepts like 'is a natural number', 'is finite', 'is infinite', 'is a finite number of steps away from zero', etc. are all impossible to write in first order logic thus we can never say that "Some natural number lacks ϕ" in any theory written in first order logic therefore it could not possibly be a theorem. This fact is exactly what makes ω-consistency a useful concept. If we instead merely make the claim that $(\exists x)\neg\phi(x)$ is a theorem, this might be correct. How could this be? This could be true if there were elements in the domain of our model that were not natural numbers. By the soundness theorem we already know that if a theory is consistent and $(\exists x)\neg\phi(x)$ is a theorem, then there really is some element x which doesn't have the property in question. From the description of the case, we know it can't be a natural number since every natural number has the relevant property.

This mistaken understanding is very important as it leads Adams to say that "He [Tarski] thus reserved the use of 'consistent' and 'inconsistent,' without qualification, to express proof-theoretical notions rather than notions of mathematical possibility and mathematical falsity." This is very misleading. Given the completeness theorem, proof theoretic inconsistency is equivalent to model-theoretic inconsistency (called unsatisfiability). If we are talking about theories in first order logic,[4] then when we say the word "inconsistent" it doesn't matter whether we mean "can be proved to lead to a contradiction" or we mean "does not have any models" (is mathematically false) since they are true or false in exactly the same circumstances. Everything forced to be true by any set of sentences can be proved (finitely) to follow from those axioms. Any set of sentences that can't possibly be true all at the same time can be shown to lead to contradiction in a finite number of steps. Importantly, it follows that the restriction on proofs being finite is not really important in first order logic. That is, even if we allowed our proofs to be infinitely long we would not be able to prove anything that we would not have been able to prove in a finite number of steps. The soundness and completeness theorems together tell us that our proof theory is as powerful as we could possibly want it to be without being inconsistent.[5]

The goal of Adams' example is to explain how it is conceptually possible to separate mathematical truths from those that are provable. Although the particular example used by Adams and Blumenfeld doesn't do this, it is not important since there are other examples that do just as well. Gödel's second incompleteness theorem shows us is that sentences such as those equivalent to saying that something is not provable in some theory are themselves not provable in that theory. So for example, the claim that Peano arithmetic (PA) is consistent can be formulated in PA, but is not a theorem of PA (assuming PA is consistent). This is not good news for the infinite analysis theory for surely it is not merely a contingent fact that PA is consistent. In fact, it is rather bizarre that Adams and Blumenfeld go searching for examples of mathematical truths that require infinite analysis to help understand Leibniz in the first place. If they were to discover such truths (which do exist), that would show that on Leibniz's view, these truths are contingent. But it is exactly these kinds of truths that Leibniz is claiming do not exist.

It seems a conceptual possibility that something could be impossible even if this were not provably so. Leibniz's "discovery" would thus be that this apparent possibility is an illusion. But we now know that Leibniz was wrong; necessary truth and provability are separable. Gödel's incompleteness theorems give us at least one way of showing this.

If Leibniz had 20th century logic under his belt he wouldn't have believed necessary truth and demonstrability were coextensive. But are there any examples from his own time that Leibniz was aware of that would have tipped him off that the link between necessity and provability is not as tight as he thought? There certainly are. Leibniz's own examples to help his readers understand the notion of an infinite analysis lead to equally grave problems.

EUCLID'S ALGORITHM

Perhaps the central example that Leibniz uses to help us think about the difference between the necessary and contingent is to think about the difference between rational and 'surd', or irrational, numbers. In several places (e.g. OC, OF, SC) Leibniz points out that if we attempt to find a common measure between incommensurable lines the resolution will proceed to infinity. Leibniz even wrote a paper in which the analogy between contingency and incommensurable proportions is displayed in parallel columns. Its title is "The Origin of Contingent Truths from an Infinite Process, Compared with the Example of Proportions between Incommensurable Quantities".

Parkinson (1995) argues that when Leibniz claims that mathematical considerations of the infinite sheds light on the problem of contingency, he meant no more than that they "suggested" a solution without actually providing one. Parkinson points out that not only does the mathematics of the infinite suggest a solution, but that it brings up problems for Leibniz as well. However, the problem that Parkinson seems most concerned with is the idea that “human beings also will be able to comprehend contingent truths with certainty” (C388: PLP 78). Parkinson then devotes a great deal of effort to considerations of how Leibniz can solve this problem. However, there is a serious problem with any proposed solution. These claims should not be contingent in the first place, though Leibniz’s view entails that they are.

Let’s look more carefully at Leibniz example of incommensurable ratios. The goal of Euclid’s algorithm, which Leibniz specifically refers to in this context (in *De Contingentia* Gr: 302-306) is to construct a line that was a common measure of two given lines. What Leibniz refers to is the phenomenon that if we try to construct a line that is a common measure of two incommensurable lines using the method that Euclid uses for commensurable lines, we will find ourselves constructing smaller and smaller lines to no end. It could be said that we are approaching the common measure of “infinitesimal” length. Now just what is this example supposed to show us? The attempted construction of a common measure proceeds to infinity. Does that mean that some claim in this area is contingent? It would have to be a true claim that you would attempt to prove using the Euclidean algorithm. But there is no such true claim in this neighborhood. It seems that this is just supposed to be an example of a non-terminating sequence in which case the example seems to backfire. It now appears that “lines of length A and B are incommensurable” counts as contingent since that is the truth we are analyzing. Leibniz certainly doesn’t want this to be an example of a non-terminating analysis and thus an example of a contingent truth since he thinks all mathematical truths are necessary.

It may seem that there is a way out of this; namely, we can prove that the analysis will never end without actually carrying out the analysis. For example, if we have lines of length 1 and √2, we could prove that √2 was irrational and then show that there will be no common measure and thus that Euclid's algorithm would never terminate. But this assumes that we know the lengths of the lines. If we were simply presented with two lines or we just drew two random lines and asked if they were commensurable, there is no way to measure a line to reveal that it has an irrational length. Any measurement in this context uses a known length as a benchmark and would just be repeated uses of the Euclid's algorithm to see how many times this known length fits into the length in question. If the lines are incommensurable, this process would never terminate. There is no way to demonstrate that two lines are incommensurable and thus if they are, the fact that they are appears to be a contingent truth. This is an unacceptable conclusion.

CONVERGENT SERIES

A second example that Leibniz often uses is that of a convergent series. For example, the sum of the infinite series 1\1, 1\2, 1\4, 1\8, 1\16, etc. equals $\sum 1/2^n$ equals 2. If we attempt to compute the sum one step at a time, we get the following series: 1, 1.5, 1.75, 1.875, 1.9375, … The limit of this series is two, so we say that the sum of the infinite series *is* 2. But how can we prove that the sum is 2? The obvious method of simply adding up each of the numbers results in something parallel to an infinite analysis.

Again, there is the obvious problem that Leibniz does not want the fact that the sum of the series is 2 to be a contingent truth. Today, we solve these kinds of problems by giving proofs of convergence. But it is unclear whether these can be included as part of demonstrations in the official Leibnizian sense. What Leibniz says on the matter is unclear. He says that "as with asymptotes and incommensurables, so with contingent things we can see many things with certainty… But we can no more give the full reason for contingent things than we can constantly follow asymptotes and run through infinite progressions of numbers" (C389, P78).

I read this as saying that proofs of convergence are not relevant to demonstrations in the official sense. But regardless, Leibniz is caught in a serious dilemma. If he does deny that proofs of convergence count as showing that the claims in question are necessary, then it is hard to see how any other kind of proof could be possible in the example of an infinite series. If there is no demonstration possible, then these truths will count as contingent. This is extremely problematic for then many of the arguments he gives in other works for truths about calculus will be merely arguments for contingent truths. On the other hand, he could say the more natural thing which is that giving a proof that a sequence has a sum of x in the limit in the rigorous way that we do today does count as giving a demonstration of its truth. This is much clearer and certainly has advantages. But this way out is no solution at all.

First, we should point out that it doesn't actually solve the problems with mathematical claims in the first place since many mathematical facts that have no finite proofs don't have infinite proofs that converge to the truth either. For example, the question of whether some powerful mathematical theory (say the full strength ZFC Set theory) is consistent is a metaphysically settled question. The answer is determined by whether there is a possible proof of a contradiction allowed by the rules of the system from the axioms. Now there are proofs that (assuming it is consistent) we could never prove that it was. But surely God would know the answer. If there were a proof of a contradiction, he would know and thus he knows if the theory is consistent. Yet there is no sense in which there is a converging infinite proof of this. One could attempt to "check all of the theorems" and see if 0=1 was among them, but it would seem that in fact the more you check the less sure you should be that the theory is consistent. We seem to have more reason to doubt a theory's consistency the more things we know that we can prove in it (since inconsistent theories can prove anything). And even if we were to examine the full infinite list of all the theorems, in order to count as a proof we would need to prove that these are in fact all of the theorems and that there aren't any more. But to prove this is equivalent to proving its consistency. Thus the example of the non-converging series seems to be exactly the example that Leibniz would want to discuss in order to separate out our mental abilities from those of God. But while this does lead to Leibniz's view that God can see through to the end of the infinite analysis and thus know that contingent truths are true but that humans lack this power, the fact that it arises in mathematical examples is not a solution that Leibniz can take.

Perhaps Leibniz has a loose conception of what counts as "convergence" with respect to an infinite analysis such that all mathematical truths have analyses that converge. This makes sense, however, again, it will not help Leibniz. The problem here is that he argues that all contingent truths have analyses that converge in this manner. If these count as demonstrations, we would not have any contingent truths. So it must be that in certain mathematical cases, proofs of convergence count as demonstrations, but in non-mathematical cases, we have convergence without the proofs of convergence. This move seems ad hoc – especially given that Leibniz gives arguments that various contingent truths converge to identical propositions. These arguments would now have to lack the force of necessity. Set this aside. In this model, there are still facts about logic and mathematics that are not provably true even accepting convergence proofs as genuine demonstrations and allowing a very loose conception of convergence for proofs that are "about mathematics". As was pointed out earlier, the consistency of a given mathematical theory cannot be proved to be consistent in that theory. These kinds of claims are part of a general class that we can think of as truths *about* provability. In general, there may not be proofs about whether any given thing is provable or not. And in some cases such as those mentioned above, we can actually prove that there are no possible proofs of some given claim.

In this case, it is harder to argue that Leibniz should have easily foreseen this conclusion, but I will argue that by reflecting on his views about modality, it would not be a difficult leap for Leibniz to see inconsistencies in his own views.

THE NECESSITY OF MODAL CLAIMS

Given Leibniz's link between provable and necessary, we can expect that now we have exactly the same problem with propositions about the modality of various claims. For example, there is no finite proof available that Adam will eat the apple. Is this claim itself, that is the claim "there is no finite proof available that Adam will eat the apple" itself finitely provable? Given Leibniz's link, this is equivalent to asking whether "it is contingent that Adam will eat the apple" is itself contingent.

Many philosophers (e.g. Plantinga 1974) claim that S5 is the 'correct' theory of modality in that all propositions have their modalities necessarily. That is, if it is necessary that x, then it is necessary that x is necessary. And if it is contingent that x, then it is necessarily contingent that x. The way that Leibniz sets up his scheme of possible worlds certainly makes it seem like this is how he thinks of necessity and Adams agrees that this is the familiar conception of Leibniz, though he would dispute it (Adams 1994: 9). But given that Leibniz says that all truths about possibles and essences are necessary (FP 19) this is the view that he has to take.

However, even if we allow Leibniz a large scope for demonstrability, why think that it can be demonstrated that there is no demonstration available that Adam will eat the apple? Leibniz wants to say that in demonstrating truth, one demonstrates that the predicate is in the subject. Here the problem might be pushed back into asking what exactly is to be demonstrated in this case? It certainly seems unlikely that this fact about demonstrability is reducible to identities.

Just as Leibniz has a problem with demonstrating that a truth is contingent, he also will have a problem showing that certain falsities are possible. For example, what about the truth (if it is one) that there is a possible world where gravity exerts twice the force it does in our world? If the analysis of the claim reaches a contradiction, then it is necessarily false which we are assuming is incorrect. The other option is that the analysis will proceed to infinity where in the limit, the weight of reasons favor something other than this being true. Here the claim is free from contradiction; but how could it be demonstrated that this is free from contradiction? This is asking whether there is a maximal set of compossible truths of which it is a part. On the face of it, this is impossible to demonstrate in the Leibnizian sense. This problem seems relevantly similar to the question of whether this world is the best of all possible worlds which Leibniz claims is contingent. This is a problem since the fact that "it is possible that the strength of gravity is half of what it actually is" is a necessary fact – it does not depend in any way on what world God chooses to actualize.

We have argued that Leibniz has a problem with iterated modality: all contingent claims are necessarily contingent, but in some cases it is impossible to demonstrate this contingency. The above argument relies on Leibniz's modal logic to be the modal system S5. Adams points out that this is the traditional understanding, but argues that contrary to popular belief, Leibniz's view that necessity is equivalent to demonstrability is instead equivalent to S4. This is a mistake. In the first place, Adams only argues that the accessibility relation between possible worlds is reflexive and transitive, but not symmetric. Even if this is entirely correct, it only shows that Leibniz's modal system is at least as strong as S4 but not equal to S5. This leaves a very large number of possibilities open. But the deeper flaw is that he fails to see an obvious problem with his argument which leads directly to a proof that Leibniz's system is in fact S5. While it might be true that the infinite analysis view of contingency is not consistent with S5, which is how Adams argues, other central doctrines from Leibniz guarantee that Leibniz is committed to modal facts which are only true in S5. Therefore it is misleading to say that Leibniz's modal system is S4. Rather, I would argue that it is better to say that Leibniz's modal system is S5 and this shows that there is a problem with his infinite analysis view. There are, after all, quite strong reasons for the more traditional understanding of Leibniz's modal views. But perhaps to avoid making judgments about which of Leibniz's views are problematic, perhaps we should just say that there are some inconsistencies in Leibniz's views about modality.

First, it is easy to see that Leibniz's system must be at least as strong as S4. S4 contains two axioms: $Np \rightarrow p$ and $Np \rightarrow NNp$. The first guarantees that the accessibility relation is reflexive and simply asserts that whatever is demonstrably true is in fact true. The second guarantees that the accessibility relation is transitive and says that if some statement is demonstrable, then there is a demonstration that it is demonstrable. This seems entirely reasonable since a demonstration of *p* just is a demonstration of *p*'s demonstrability. Now Adams argues that the relation is not symmetric since he claims, "a proposition may be indemonstrable without being demonstrably indemonstrable." Later he says "there is, as Leibniz supposes, at least one proposition *p* which is possible, and actually true, but not demonstrably possible" which amounts to the same thing when we understand "possible" as "not demonstrably false" (Adams 1994: 47,48). These claims certainly seem reasonable, but denying symmetry has unacceptable consequences.

If *p* is not demonstrably possible, then there is another possible world w_2 accessible from the actual world, where *p* is demonstrably false. Now we have a situation where *p* is actually true, but at another possible world, it is demonstrably false. Surely this is unacceptable for the Leibnizian understanding of "demonstration". If *p* is demonstrably true in one world, then it is demonstrably true in any world since the same demonstration is available. Demonstrations of propositions cannot have anything to do with which world God has actualized.

It is a fact that any modal system where every possible world must have the same set of necessary truths in conjunction with reflexivity is equivalent to S5. To see this, it suffices to show that we can demonstrate the validity of the Euclidean axiom: Pp → NPp (If possibly p then it is necessarily possible that p). This axiom together with Np → p yields the system S5. To demonstrate that Pp → NPp is true in every world, we assume that it is false in some arbitrary w_1. That means that in w_1, Pp is true, but ¬NPp is also true. Since ¬NPp is true, PN¬p is true and so in some w_2 accessible from w_1, N¬p is true. But since this is true in w_2, by our assumption that the same truths are necessary in every world, we have that it is also true in w_1. Since we are assuming that N¬p → ¬p, we have ¬p true in w1 as well contradicting our assumption. Since w_1 was arbitrary, this is a reductio of the possibility that Pp → NPp could be false in any possible world. Therefore, Leibniz's system is equivalent to S5. Our previous argument that Leibniz has a problem with iterated modalities stands unaffected.

CONCLUSION:

In the end it is clear that Leibniz’s view that necessity is determined by the availability of a demonstration of truth is unsupportable. Leibniz would be forced to accept the claim that parts of mathematics and logic would be only contingently true. For example, if "ZFC set theory is consistent" is true, then it is only a contingent truth. This conflicts with Leibniz's own claims about the necessity of mathematical truths as well as our common sense usage of modal terms. Perhaps of more interest is that it can be shown that the very examples that Leibniz uses to shed light on his infinite analysis views lead to mathematical facts such as the fact that two lines are incommensurable or that the sum of an infinite series is 2 would also become contingent. Many of these examples could easily be fixed by altering his view on converging series to include proofs of convergence as acceptable demonstrations of truth, but now we run the risk of being able to demonstrate too much. And even in this stronger system, still, there would be claims about provability and claims about modality that Leibniz would have to accept as contingent.

The examples that I raise as problems for Leibniz have a familiar pattern. They are what we now call “semi-decidable” questions. An example of this is a claim such that if true, it is finitely provable that it is true, but if false, there is no corresponding proof of its falsity. “PA is inconsistent”, “lines A and B are commensurable”, and “There is a finite demonstration that Adam will eat the apple” are all of this variety. Translating these facts about demonstrations leads to the view that some propositions are such that they are necessary if true but only contingent if false. This is not consistent with the typical way that we think of modality nor is it consistent with Leibniz’s own views on other matters. Given his views that claims about essences and alternate possibilities as well as claims of mathematics and logic are all necessary, it is clear that he cannot accept his own doctrine of infinite analysis.

NOTES:

1) To see how Leibniz's views on the solution to the problem of absolute necessity changed throughout his writings, see Adams (1977) and Rescher (2002).

2) On a historical note, while Adams implies that the notion of *ω*-consistency was developed by Tarski, it was introduced by Gödel in what is certainly its most famous application. Gödel (1931) argued that if a set of axioms in a logical theory T is ω-consistent and meets some other criteria, then there are undecidable sentences expressible in T – that is, sentences that are not provable and such that their negations are not provable either. Later, Rosser (1936) proved that Gödel did not even need to introduce the notion at all as his incompleteness theorem follows even if we weaken ω-consistency to ordinary consistency by constructing a different undecidable sentence than Gödel originally used. Tarski (1933) also uses the term 'ω-consistency' but in a way entirely different (though provably equivalent) to Gödel. Despite Adams' citation, his usage is much closer to that of Gödel.

3) For definitions, see Gödel's original paper of 1931 or for a more full discussion, see Boolos (1993). Incidentally, we should note that this corrected definition does not mention anything about the provability status of $\forall x\phi(x)$. Adams seems to imply that this should not be provable, but if we add the clause that $\forall x\phi(x)$ is not provable, then ω-inconsistent theories are automatically consistent. If stated correctly, consistency is strictly weaker than ω-consistency, (so all ω-consistent theories are consistent as well, but the converse implication does not hold) as it was intended to be. This is a minor point that makes no real difference to the argument.

4) I stick to discussing first order logic here since in anything stronger, such as second order logic, it is impossible for cases of *ω*-consistency to arise. It should be noted that if we consider theorems in stronger systems like second order logic to represent real logical truths, then since the compactness theorem fails, we will get the separation between logically true and finitely demonstrable that Adams seems to want, but this counts against Leibniz since now we have logical truths (about which sentences follow from the axioms) which have no finite demonstrations.

4) See Enderton (2000) or any of a large number of logic texts for a clear(er) discussion of Soundness, Completeness and the corresponding Compactness Theorems for first order logic.

References:

Selected Leibniz works:

(C) Couturat, L. (ed.) *Opuscules et fragments inédits de Leibniz*. Paris: Alcan, 1903; reprinted Hildesheim: Olms, 1961.

(Gr) Grua, G. (ed.) *G.W. Leibniz: Textes inédits*. Paris, 1948.

(LA) Leibniz-Arnauld correspondence. Cited by page numbers from *Die Philosophischen Schriften von G.W. Leibniz,* Volume 2. Edited by C.J. Gerhardt. Berlin, 1875-1890. Reprint. Hildesheim: Georg Olms, 1965.

(P) Parkinson, G.H.R. (trans. and ed.) *Leibniz: Philosophical Writings*. London: Dent, 1973.

(PLP) Parkinson, G.H.R. (trans. and ed.) *Leibniz: Logical Papers*. Oxford: Clarendon Press, 1966.

Philosophical Essays. Ed. and trans. by Roger Ariew and Daniel Garber. Indianapolis. Hackett, 1989. From Ariew and Garber:

(FP) On Freedom and Possibility (1680-82?)

(MO) The Principles of Philosophy, or, the Monadology (1714)

(OC) On Contingency (1686?)

(OF) On Freedom (1689?)

(SC) The Source of Contingent Truths (1685-89?)

Dialogue on Human Freedom and the Origin of Evil (1695)

Letter to Coste, On Human Freedom (1707)

Theodicy Ed. by Austin Ferrar and translated by E.M. Huggard. LaSalle, Illinois. Open Court Publishing. 1985.

Other References:

Adams, RM (1977) Leibniz's theories of contingency. *Rice University Studies* 63(4): 1-41.

Adams, RM (1994) *Leibniz: Determinist, Theist, Idealist*. Oxford University Press.

Blumenfeld, D (1985) Leibniz on Contingency and Infinite Analysis. *Philosophy and Phenomenological Research* 45(4): 483-514.

Boolos, G (1993). *The Logic of Provability*. Cambridge University Press.

Curly, EM (1972) The Root of Contingency. In Frankfurt (ed) *Leibniz: A Collection of Critical Essays*, pp 69-97.

Enderton, H (2000) *A Mathematical Introduction to Logic*. Harcourt: Academic Press.

Frankfurt, H (1972) *Leibniz: A Collection of Critical Essays*. Garden City, N.Y.: Doubleday Anchor.

Gödel, K (1931) On Formally Undecidable Propositions of Principia Mathematica and Related Systems I in Gödel, *Collected Works Vol I*, pp. 145-195.

Gödel, K (1936) *Kurt Gödel: Collected Works: Publications 1929-1936*. London and New York. Oxford University Press.

Hacking, I (1973) Leibniz and Descartes: Proof and Eternal Truths. *Dawes Hicks Lecture on Philosophy,British Academy.*

Hacking, I (1974) Infinite Analysis. *Studia Leibnitiana* 6: 126-130.

Ishiguro, H (1972) *Leibniz's Philosophy of Logic and Language*. London: Duckworth.

Jolley, N (1995) *The Cambridge Companion to Leibniz*. Cambridge University Press.

Parkinson, GHR (1995) Philosophy and Logic. In Jolley (ed). *The Cambridge Companion to Leibniz* pp. 199-223.

Plantinga, A (1974) *The Nature of Necessity*. Oxford: Oxford University Press.

Rosser, JB (1936) Extensions of Some Theorems of Gödel and Church. *Journal of Symbolic Logic* 1(3): 87-91

Sleigh, RC Jr (1990) *Leibniz and Arnauld: A Commentary on their Correspondence.* New Haven, Conn. Yale University Press.

Tarski, A (1933) Einige Betrachtungen über die Begriffe ω-Widerspruchsfreiheit und der ω-Vollständigkeit", *Monatshefte für Mathematik und Physik*, 40 pp. 97-112. Reprinted as "Some observations on the concepts of w-consistency and w-completeness" in Tarski, A. and Woodger, JH (ed and translated) (1983) *Logic, Semantics, Metamathematics: Papers from 1923-1938* (2nd edition), pp. 279-295. Hackett Publishing Company.