Edited by Peter Gobets
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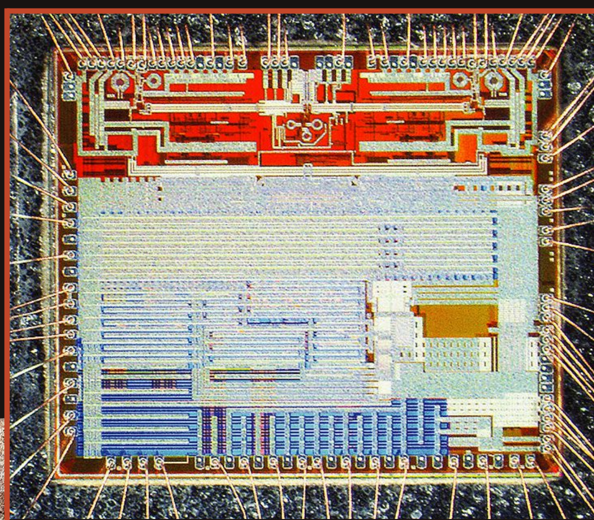
The Origin and Significance of Zero

An Interdisciplinary Perspective



VALUE INQUIRY BOOK SERIES

BRILL

CHAPTER 22

Category Theory and the Ontology of *Śūnyata*

Sisir Roy and Rayudu Posina

Abstract

Notions such as *śūnyata*, *catuskoti*, and Indra's net, which figure prominently in Buddhist philosophy, are difficult to readily accommodate within our ordinary thinking about everyday objects. Famous Buddhist scholar Nāgārjuna considered two levels of reality: one called conventional reality, and the other ultimate reality. Within this framework, *śūnyata* refers to the claim that at the ultimate level objects are devoid of essence or 'intrinsic properties', but are interdependent by virtue of their relations to other objects. *Catuskoti* refers to the claim that four truth values, along with contradiction, are admissible in reasoning. Indra's net refers to the claim that every part of a whole is reflective of the whole. Here we present category theoretic constructions that are reminiscent of these Buddhist concepts. The universal mapping property definition of mathematical objects, wherein objects of a universe of discourse are defined not in terms of their content, but in terms of their relations to all objects of the universe is reminiscent of *śūnyata*. The objective logic of perception, with perception modeled as [a category of] two sequential processes (sensation followed by interpretation), and with its truth value object of four truth values, is reminiscent of the Buddhist logic of *catuskoti*. The category of categories, wherein every category has a subcategory of sets with zero structure within which every category can be modeled, is reminiscent of Indra's net. Our thorough elaboration of the parallels between Buddhist philosophy and category theory can facilitate better understanding of Buddhist philosophy, and bring out the broader philosophical import of category theory beyond mathematics.

Keywords

Cantor – contradiction – emptiness – essence – figure – functor – Nāgārjuna – natural transformation – object – property – reality – relation – set – shape – structure – structure-respecting morphism – truth value – Yoneda – zero

 | DOI:10.1163/9789004691568_026

1 Introduction

Buddhist philosophy, especially Nāgārjuna's Middle Way (Garfield, 1995; Siderits and Katsura, 2013), is intellectually demanding (Priest, 2013). The sources of the difficulties are many. First it argues for two realities: conventional and ultimate (Priest, 2010). Next, ultimate reality is characterized by śūnyata or emptiness, which is understood as the absence of a fundamental essence underlying reality (Priest, 2009). Equally importantly, contradictions are readily deployed, especially in catuskoti, as part of the characterization of reality (Deguchi, Garfield, and Priest, 2008; Priest, 2014). Lastly, reality is depicted as Indra's net – a whole, whose parts are reflective of the whole (Priest, 2015). The ideas of relational existence, admission of contradictions, and parts reflecting the whole are seemingly incompatible with our everyday experiences and the attendant conceptual reasoning used to make sense of reality. However, notions analogous to these ancient Buddhist ideas are also encountered in the course of the modern mathematical conceptualization of reality. These parallels may be, in large part, due to 'experience' and 'reason' that are treated as the final authority in both mathematical sciences and Buddhist philosophy. Here, we highlight the similarities between Buddhist philosophy and mathematical philosophy, especially category theory (Lawvere and Schanuel, 2009). The resultant cross-cultural philosophy can facilitate a proper understanding of reality – a noble goal to which both Buddhist philosophy and mathematical practice are unequivocally committed.

2 Two Realities

There are, according to Buddhist thought, two realities: the conventional reality of our everyday experiences and the ultimate reality (Priest, 2010; Priest and Garfield, 2003). In our conventional reality, things appear to have intrinsic essences. It is sensible, at the level of conventional reality, to speak of essences of objects, but at the level of ultimate reality there are no essences, and everything exists but only relationally. There is an analogous situation in mathematics. On one hand, mathematical objects can be characterized in terms of their relations to all objects, in which case the nature of an object is determined by the nature of its relationship to all objects. In a sense, there is nothing inside the object; an object is what it is by virtue of its relations to all objects. The objects of mathematics are, as Resnik (1981, p. 530) notes, 'positions in structures', which is in accord with the Buddhist understanding of things as 'loci in a field of relations' (Priest, 2009, p. 468). However, there is another level of mathematical reality, wherein we can speak of essences of objects (e.g., theories of objects; Lawvere and Rosebrugh, 2003, pp. 154–155). For example, one can characterize a set as a collection of elements or 'sum' of basic-shaped figures (1-shaped figures, where $1 = \{\bullet\}$), with basic shapes understood as essences (Lawvere, 1972, p. 135; Lawvere and Schanuel, 2009, p. 245; Reyes, Reyes, and Zolfaghari, 2004, p. 30). Similarly, every graph is made up of figures of two basic shapes (arrow- and dot-shaped figures; Lawvere and Schanuel, 2009, pp. 150, 215). This characterization of an object in terms of its contents, i.e., basic shapes or essences (Lawvere, 2003, pp. 217–219; Lawvere, 2004, pp. 11–13), can be contrasted with the relational characterization, wherein each and every object of a universe of discourse (a mathematical category; Lawvere and Schanuel, 2009, p. 17) is characterized in terms of its relationship to all objects of the universe or category (see Appendix 7.1 below). The relational nature of mathematical objects, as elaborated below, is reminiscent of the Buddhist notion of emptiness – an assertion that objects are what they are not by virtue of some intrinsic essences but by virtue of their mutual relationships.

3 Emptiness

According to Buddhist philosophy, everything is empty and the totality of empty things is empty. Here, emptiness is understood as the absence of essences. Things, in the ultimate analysis, are what they are and behave the way they do not because of [some] essences inherent in them, but by virtue of their mutual relationships (Priest, 2009). This idea of relational existence has parallels in mathematical practice. Mathematical objects of a given mathematical category (e.g., a category of sets) are what they are, not by virtue of their intrinsic essences but by virtue of their relations to all objects of the category. For example, a single-element set is a set to which there exists exactly one function from every set (Lawvere and Schanuel, 2009, pp. 213, 225). Note that the singleton set is characterized not in terms of what it contains (a single element), but in terms of how it relates to all sets of the category of sets. In a similar vein, the truth value set $\Omega = \{\text{false}, \text{true}\}$ is defined in terms of its relation to all sets of the category of sets. The truth value set, instead of being defined as a set of two elements 'false' and 'true', is defined as a set Ω such that functions from any set X to the set Ω are in one-to-one correspondence with the parts of X (ibid. pp. 339–344). To give one more example, a product of two sets is defined not by specifying the contents of the product set (pairs of elements), but by characterizing its relationship to all sets. More explicitly, the product of two sets A and B is a set A × B along with two functions (projections to the factors)

$$p_A: A \times B \to A, p_B: A \times B \to B$$

such that for every set Q and any pair of functions $q_A: Q \to A$, $q_b: Q \to B$, there is exactly one function $q: Q \to A \times B$ satisfying both the equations: $q_A = p_A \circ q$ and $q_B = p_B \circ q$, where '∘' denotes composition of functions (ibid. pp. 339–344). The universal mapping property definition of mathematical constructions brought to sharp focus the relational nature of mathematical objects. It conclusively established that 'the substance of mathematics resides not in substance (as it is made to seem when ∈[membership] is the irreducible predicate, with the accompanying necessity of defining all concepts in terms of a rigid elementhood relation) but in form (as is clear when the guiding notion is an isomorphism-invariant structure, as defined, for example, by universal mapping properties)' (Lawvere, 2005, p. 7). More broadly, Yoneda lemma (Lawvere and Rosebrugh, 2003, pp. 249–250; Appendix 7.1 below), according to which a mathematical object of a given universe of discourse (i.e., category) is completely characterized by the totality of its relations to all objects of the universe (category), is an unequivocal assertion of the relational nature of mathematical objects. Yoneda lemma, as pointed out by Barry Mazur, establishes that 'an object X of a category C is determined by the network of relationships that the object X has with all the other objects in C' (Mazur, 2008). Thus the Buddhist idea of emptiness or relational existence finds resonance in mathematical practice, especially in terms of universal mapping properties and the Yoneda lemma.

However, note that according to the Buddhist doctrine of emptiness, not only is everything empty, but the totality of empty things is also empty (Priest, 2009). In other words, even the notion of relational existence is empty, i.e., emptiness is not the essence of existence; emptiness is also empty. This idea of emptiness being empty is much more challenging to comprehend. When we say that objects are empty, we are saying that objects are mere locations in a network of relations. But when we say that the totality of empty things is empty, we are asserting that the existence of totality is also relational just like that of the objects in the totality. What is not immediately clear is how are we to think of relations especially when all we have is the totality, i.e., one object. Within mathematics, note that the totality of all objects (along with their mutual relations) forms a category. More importantly, categories are objects in the category of categories (Lawvere, 1966), and hence the totality of objects, i.e., category, is also empty or relational as much as the objects of a category. Thus the idea of śūnyata (everything is empty) resonates with the relational nature of objects and of the totality of objects (within the mathematical framework of the category of categories).

Equally importantly, Nāgārjuna's Middle Way, having gone to great lengths to distinguish two realities (conventional essences vs. ultimate emptiness) identifies the two: 'There is no distinction between conventional reality and ultimate reality' (Deguchi, Garfield, and Priest, 2008, p. 399). Contradictions (such as these) within Buddhist philosophy, on a superficial reading, are diagnostic of irrational mysticism. However, as we point out in the following, contradictions also figure prominently in the foundations of mathematical modeling of reality. In light of these parallels, 'contradiction' may be intrinsic to the nature of reality, which is the common subject of both Buddhist and mathematical investigations, and not a sign of faulty Buddhist reasoning.

4 Contradiction

Within the Buddhist philosophical discourse, one often encounters contradictions and these contradictions are treated as meaningful (Deguchi, Garfield, and Priest, 2008; Priest, 2014). There is an analogous situation in mathematics. Although not every contradiction is sensible, there are sensible contradictions such as the boundary of an object A formalized as 'A and not A' (Lawvere, 1991, 1994a, p. 48; Lawvere and Rosebrugh, 2003, p. 201). More importantly, within mathematical practice, it is now recognized that contradictions do not necessarily lead to inconsistency (an inconsistent system, according to Tarski, is where everything can be proved; Lawvere, 2003, p. 214). Of course, admitting a contradiction invariably leads to inconsistency in classical Boolean logic. In logics more refined than Boolean logic contradiction does not necessarily lead to inconsistency. This recognition is very important, especially since contradiction plays a foundational role in mathematical practice. Briefly, Cantor's definition of SET is, as pointed out by F. William Lawvere, 'a strong contradiction: its points are completely distinct and yet indistinguishable' (ibid. p. 215; Lawvere, 1994a, pp. 50–51). Zermelo, and most mathematicians following him, concluded that Cantor's account of sets is 'incorrigibly inconsistent' (Lawvere, 1994b, p. 6). Lawvere, using adjoint functors, showed that Cantor's definition is 'not a conceptual inconsistency but a productive dialectical contradiction' (Lawvere and Rosebrugh, 2003, pp. 245–246), which is summed up as the unity and identity of adjoint opposites (Lawvere, 1992, pp. 28–30; Lawvere, 1996).

A related notion is catuskoti, which is routinely employed in Buddhist reasoning (Priest, 2014; Westerhoff, 2006). To place it in perspective, in the familiar Boolean logic, any proposition is either true or false. Put differently, there are only two possible truth values, and they are mutually exclusive and jointly exhaustive. Unlike Boolean logic, in Buddhist reasoning more than two truth values are admissible. In the Buddhist logic of catuskoti, a proposition can possibly take, in addition to the familiar truth values of 'true' or 'false', the truth values of 'true and false', or 'not true and not false'. Given a proposition A, there are four possibilities: 1. A; 2. not A; 3. A and not A; 4. not A and not not A. Here contradiction is admissible, i.e., 'A and not A' is a possible state of affairs, which is reminiscent of the boundary operation and the unity and identity of adjoint opposites in mathematics, alluded to earlier. Moreover, double negation is not the same as identity operation as in the case of, to give one example, the non-Boolean logic of graphs (Lawvere and Schanuel, 2009, p. 355). Note that if not not A = A, then the fourth truth value of catuskoti is equal to the third.

As an illustration of how the four truth values of catuskoti could be a reflection [of an aspect] of reality, we consider the category of percepts. Perception involves two sequential processes of sensation followed by interpretation (Albright, 2015; Croner and Albright, 1999). So, we define the category of percepts as a category of two sequential functions of decoding after coding. The truth value object of the category of percepts has four truth values (Appendix 7.2 below). Thus the objective logic of perception, with its truth value object of four truth values, is reminiscent of the Buddhist logic of catuskoti (see Linton, 2005).

5 Indra's Net and Zero Structure

Another important concept in Buddhist philosophy is the idea of Indra's net, wherein reality is compared to a vast network of jewels such that every jewel is reflective of the entire net (Priest, 2015). In abstract terms, reality is characterized as a whole wherein every part is reflective of the whole. Admittedly, this Buddhist characterization of reality sounds mystifying, but there is an analogous situation, involving part-whole relations, in mathematics.

How can a part of a whole reflect the whole? First, note that mathematical structures of all sorts can be modeled in the category of sets (Lawvere and Schanuel, 2009, pp. 133–151). Sets have zero structure (Lawvere, 1972, p. 1; Lawvere and Rosebrugh, 2003, pp. 1, 57; Lawvere and Schanuel, 2009, p. 146). Negating the structure (cohesion, variation) inherent in mathematical objects, Cantor created sets: mathematical structures with zero structure (Lawvere, 2003, 2016; Lawvere and Rosebrugh, 2003, pp. 245–246). In comparing his abstraction of sets with zero structure to the invention of number zero, Cantor considered sets as his most profound contribution to mathematics (Lawvere, 2006). Sets, by virtue of having zero structure, serve as a blank page – an ideal background to model any category of mathematical objects (Lawvere, 1994b; Lawvere and Rosebrugh, 2003, pp. 154–155). However, structureless sets are a small part – the only part – of the mathematical universe that reflects all of mathematics. It seemed so until Lawvere axiomatized the category of categories (Lawvere, 1966; Lawvere and Schanuel, 2009, pp. 369–370). Along the lines of Cantor's invention of structureless sets, Lawvere defined a subcategory of structureless (discrete, constant) objects within a category by negating its structure (cohesion, variation; Lawvere, 2004, p. 12; Lawvere and Schanuel, 2009, pp. 358–360, 372–377). Thus, within any category of mathematical objects, there is a part, a structureless subcategory, that is like the category of sets in having zero structure, and hence serves as a background to model all categories of mathematical objects (Lawvere, 2003; Lawvere and Menni, 2015; Picado, 2008, p. 21). Modeling a category of mathematical objects requires, in addition to the subcategory with zero structure, another subcategory objectifying the structural essence(s) of the objects of the category, i.e., the theory of the given category of mathematical objects. Finding the theory subcategory also depends on the structureless subcategory, by way of contrasting or negating the structureless subcategory (Lawvere, 2007). Once we have the subcategory with zero structure and the subcategory objectifying the essence (theory) of a given category, interpreting the theory subcategory into the structureless subcategory gives us models of the given category of mathematical objects. Thus, thanks to the recognition of significance of Cantor's zero structure, every mathematical category can be modeled in any category of the category of categories.

If we compare the category of categories to Indra's net, then categories within the category of categories would correspond to jewels in Indra's net. Just as in the case of Indra's net, wherein every jewel in the network of jewels is reflective of the entire network, in the category of categories every category (part) of the category of categories (whole) reflects the whole. For example, the category of dynamical systems is a part of the category of categories. Within the category of dynamical systems, we have the constant subcategory (obtained by negating the variation) of dynamical systems (wherein every state is a fixed point), which is like the category of sets, and within which any category can be modeled. Similarly, the category of graphs is another part of the category of categories. Within the category of graphs there is the discrete subcategory (obtained by negating the cohesion) of graphs (with one loop on each dot), which is also like the category of sets, and hence can model every category. Thus, we find that within the category of categories, every part is reflective of the whole, which is reminiscent of the Buddhist depiction of reality as Indra's net: a whole with parts reflective of the whole.

6 Conclusion

There are similarities between Buddhist philosophy and mathematical practice, especially with regard to essence vs. emptiness, contradictions, and part-whole relations. These similarities might be a natural consequence of identical objectives – understanding reality and commitment to truth– and identical means – experience and reason – employed toward those ends. It is in this respect that the practices of the two – mathematicians and Buddhists – can be compared. Our exercise, on that score, can help better appreciate the rationality of Buddhist reasoning. Oftentimes, admission of contradiction (as in *catuskoti*) tends to be equated with irrational mysticism.

However, as we have seen, contradictions are also an integral and indispensable part of the mathematical understanding of reality. On the other hand, in drawing parallels between Buddhist thought and mathematical practice, we hope to have brought out the broad philosophical import of category theory beyond mathematics.

7 Appendices

7.1 *Yoneda Lemma*

We begin with an intuitive introduction to the mathematical content of Yoneda lemma (Lawvere and Rosebrugh, 2003, pp. 175–176, 249). With simple illustrations of figures-and-incidences (along with [its dual] properties-and-determinations) interpretations of mathematical objects, we prove the Yoneda lemma (Lawvere and Schanuel, 2009, pp. 361, 370–371). Broadly speaking, Yoneda lemma is about [properties of] objects [of categories] and their mutual determination.

First, let us consider a function

$$f\colon \mathrm{A} \to \mathrm{B}$$

We can think of the function f as (i) a figure of shape A in B, i.e., an A-shaped figure in B. For example, in the category of graphs, a map

$$d\colon \mathrm{D} \to \mathrm{G}$$

from a graph D (consisting of one dot) to any graph G is a D-shaped figure in G, i.e., a dot in the graph G. We can also think of the same function f as (ii) a property of A with values in B, i.e., a B-valued property of A (Lawvere and Schanuel, 2009, pp. 81–85).

For example, with sets, say, Fruits = {apple, grape} and Color = {red, green}, a function

$$c\colon \text{Fruits} \to \text{Color}$$

(with c (apple) = red and c (grape) = green) can be viewed as Color-valued property of Fruits.

Now let us consider two figures: an X-shaped figure in A

$$x_A\colon X \to A$$

and a Y-shaped figure in A

$$y_A\colon Y \to A$$

Given a transformation from the shape X to the shape Y, i.e., an X-shaped figure in Y

$$x_Y\colon X \to Y$$

we find that the X-shaped figure in Y(x_y) induces a transformation of a Y-shaped figure in A into an X-shaped figure in A via composition of maps

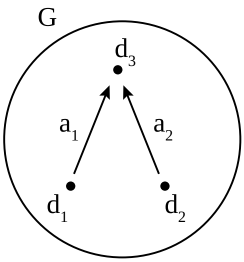
$$y_A \circ x_Y = x_A$$

(where '∘' denotes composition) displayed as a commutative diagram

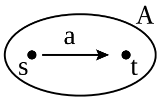
$$\begin{tikzcd}
\mathrm{X} \arrow[dr, "x_{\mathrm{A}} = y_{\mathrm{A}} \circ x_{\mathrm{Y}}"] \arrow[dd, "x_{\mathrm{Y}}"'] & \\
& \mathrm{A} \\
\mathrm{Y} \arrow[ur, "y_{\mathrm{A}}"'] &
\end{tikzcd}$$

showing the transformation of a Y-shaped figure in $A(y_A)$ into an X-shaped figure in $A(x_A)$ by an X-shaped figure in $Y(x_Y)$ via composition of maps.

As an illustration, consider an object (of the category of graphs), i.e., a graph G (shown below):



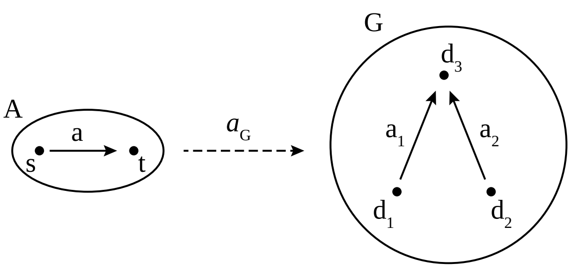
and a shape graph [arrow] A with exactly one arrow 'a', along with its source 's' and target 't', as shown



along with an A-shaped figure in G

$$a_G\colon A \to G$$

displayed as:

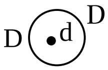


with, say,

$$a_G(a) = a_1$$

This A-shaped figure in G, i.e., the graph map a_G maps the [only] arrow 'a' in the shape graph A to the arrow 'a_1' in the graph G, while respecting the source (s) and target (t) structure of the arrow 'a', i.e., with arrow 'a' in shape A mapped to arrow 'a_1' in the graph G, the source 's' and target 't' of the arrow 'a' are mapped to the source 'd_1' and target 'd_3' of arrow 'a_1', respectively.

Next, consider another shape graph [dot] D with exactly one dot 'd'



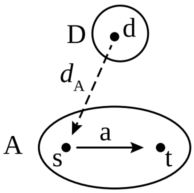
along with a D-shaped figure in A

$$d_A\colon D \to A$$

with

$$d_{\mathrm{A}}\,(\mathrm{d}) = \mathrm{s}$$

i.e., the graph map d_A maps the dot 'd' in the graph D to the dot 's' in the graph A, i.e., the source dot 's' of the arrow 'a', as shown below:



This graph map d_A from shape D to shape A induces a transformation of the (above) A-shaped figure in graph G

$$a_G\colon A \to G$$

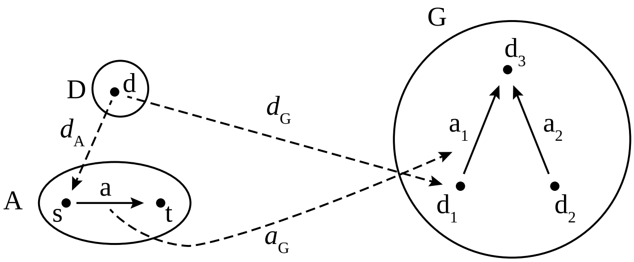
into a D-shaped figure in G

$$d_G\colon D \to G$$

via composition of graph maps

$$d_G = a_G \circ d_A$$

i.e., $d_G(d) = a_G \circ d_A(d) = a_G(s) = d_1$ as depicted below (Lawvere and Schanuel, 2009, pp. 149–150):



In general, every X-shaped figure in Y transforms a Y-shaped figure in A into an X-shaped figure in A, i.e., every map

$$x_Y\colon X \to Y$$

induces a map in the opposite direction (contravariant; Lawvere, 2017; Lawvere and Rosebrugh, 2003, p. 103; Lawvere and Schanuel, 2009, p. 338).

$$A^{x_Y} : A^Y \to A^X$$

where A^Y is the map object of the totality of all Y-shaped figures in A, A^X is the map object of the totality of all X-shaped figures in A, and with the map

$$A^{x_Y}$$

of map objects defined as

$$A^{x_Y}(y_A : Y \to A) = y_A \circ x_Y = x_A : X \to A$$

assigning a map x_A in the map object A^X to each map y_A in the map object A^Y. Thus, the figures in an object A of all shapes (all X-shaped figures in A for every object X of a category) along with their incidences

$$A^{x_Y} : A^Y \to A^X$$

induced by all changes of figure shapes

$$x_{\mathrm{Y}} = \mathrm{X} \to \mathrm{Y}$$

(i.e., every map in the category) together constitute the geometry of figures in A, i.e., a complete picture of the object A. Summing up, we have the complete characterization of the geometry of every object A of a category in terms of the figures of all shapes (objects of the category) and their incidences (induced by the maps of the category) in the object A (Lawvere and Schanuel, 2009, pp. 370–371).

Let us now examine how figures of a shape X in an object A are transformed into figures of the [same] shape X in an object B. We find that an A-shaped figure in B

$$a_{\mathrm{B}} = \mathrm{A} \to \mathrm{B}$$

induces a transformation of an X-shaped figure in A

$$x_{\mathrm{A}} = \mathrm{X} \to \mathrm{A}$$

into an X-shaped figure in B

$$x_{\mathrm{B}} = \mathrm{X} \to \mathrm{B}$$

via composition of maps

$$a_{\mathrm{B}} \circ x_{\mathrm{A}} = x_{\mathrm{B}}$$

displayed as a commutative diagram

$$\begin{tikzcd}
\mathrm{X} \arrow[dr, "x_A"'] \arrow[drr, "x_B = a_B \circ x_A"] & & \\
& \mathrm{A} \arrow[r, "a_B"'] & \mathrm{B}
\end{tikzcd}$$

showing the transformation of an X-shaped figure in A (x_A) into an X-shaped figure in B (x_B) by an A-shaped figure in B (a_B) via composition of maps. Thus, every map

$$a_B\colon A \to B$$

induces a map in the same direction (covariant; Lawvere and Rosebrugh, 2003, pp. 102–103, 109; Lawvere and Schanuel, 2009, p. 319)

$$a_B{}^X\!: A^X \to B^X$$

where A^X is the map object of all X-shaped figures in A, B^X is the map object of all X-shaped figures in B, and with the map $a_B{}^X$ defined as

$$a_B{}^X (x_A\!: X \to A) = a_B \circ x_A = x_B\!: \mathrm{X} \to \mathrm{B}$$

assigning a map x_B in the map object B^{X} to each map x_A in the map object A^X. Thus, the totality of maps $a_B{}^X$ of map objects (for all objects and maps of the category) induced by a map a_B from A to B constitutes a covariant transformation of the figure geometry of object A into that of B, i.e., specifies how figures-and-incidences in A are transformed into figures-and-incidences in B.

Putting together these two transformations: (i) the covariant transformation of X-shaped figures in A into X-shaped figures in B induced by an A-shaped figure in B, and (ii) the contravariant transformation of Y-shaped figures in A into X-shaped figures in A induced by an X-shaped figure in Y, we obtain the diagram (Lawvere and Schanuel, 2009, p. 370):

$$\begin{tikzcd}
\mathrm{X} \arrow[dd, "x_Y"'] \arrow[dr, "x_A"] \arrow[drr, "x_B"] & & \\
& \mathrm{A} \arrow[r, "a_B"'] & \mathrm{B} \\
\mathrm{Y} \arrow[ur, "y_A"] \arrow[urr, "y_B"'] & &
\end{tikzcd}$$

from which we notice that there are two paths to go from a Y-shaped figure in A (y_A) to an X-shaped figure in B (x_B):

Path 1. First we map the Y-shaped figure in $A(y_A)$ into an X-shaped figure in $A(x_A)$ along the X-shaped figure in $Y(x_Y)$ via composition of the maps

$$y_A \circ x_Y$$

and then map the composite X-shaped figure in $A(y_A \circ x_Y)$ into an X-shaped figure in B along the A-shaped figure in $B(a_B)$ via composition

$$a_B \circ (y_A \circ x_Y)$$

Path 2. First we map the Y-shaped figure in A (y_{A}) into a Y-shaped figure in $B(y_B)$ along the A-shaped figure in $B(a_B)$ via composition of the maps

$$a_B \circ y_A$$

and then map the composite Y-shaped figure in $B(a_B \circ y_A)$ into an X-shaped figure in B along the X-shaped figure in $Y(x_Y)$ via composition

$$(a_B \circ y_A) \circ x_Y$$

Based on the associativity of composition of maps (Lawvere and Schanuel, 2009, pp. 370–371), we find that

$$a_B \circ (y_A \circ x_Y) = (a_B \circ y_A) \circ x_Y$$

i.e., the two paths of transforming a Y-shaped figure in A

$$y_A \colon Y \to A$$

into an X-shaped figure in B give the same map

$$a_B \circ y_A \circ x_Y = x_B \colon X \to B$$

Since the associativity of composition of maps holds for all maps of any category (Lawvere and Schanuel, 2009, p. 17), we find that every A-shaped figure in B induces a covariant transformation of the figure geometry of A into the figure geometry of B.

More explicitly, each A-shaped figure in B

$$a_B \colon A \to B$$

induces a commutative diagram (of maps of map objects)

$$\begin{tikzcd} A^X \arrow[r, "a_B{}^X"] & B^X \\ A^Y \arrow[u, "A^{x_Y}"] \arrow[r, "a_B{}^Y"'] & B^Y \arrow[u, "B^{x_Y}"'] \end{tikzcd}$$

satisfying

$$a_B{}^X \circ A^{x_Y} = B^{x_Y} \circ a_B{}^Y$$

for every map in the category, and hence a natural transformation (compatible with the composition of maps) of the figure geometry of A into the figure geometry of B. To see the commutativity, consider a Y-shaped figure in A, i.e., a map y_A of the map object A^Y and evaluate the above two composites:

$$a_B{}^X \circ A^{x_Y}(y_A) = a_B{}^X(y_A \circ x_Y) = a_B \circ (y_A \circ x_Y)$$

$$B^{x_Y} \circ a_B{}^Y(y_A) = B^{x_Y}(a_B \circ y_A) = (a_B \circ y_A) \circ x_Y$$

Again, according to the associativity of the composition of maps

$$a_B \circ (y_A \circ x_Y) = (a_B \circ y_A) \circ x_Y = a_B \circ y_A \circ x_Y$$

and hence both composites map each Y-shaped figure in A (a map in the map object A^Y)

$$y_A : Y \rightarrow A$$

to the X-shaped figure in B (a map in the map object B^X)

$$a_B \circ y_A \circ x_Y = x_B : X \rightarrow B$$

Since we have the above commutativity for every shape (object) and figure (map), i.e., for all objects and maps of the category, we conclude that an A-shaped figure in B corresponds to a natural transformation (respectful of figures-and-incidences) of the figure geometry of A into the figure geometry of B.

Now we formally show that every A-shaped figure in B

$$a_B : A \rightarrow B$$

of a category C can be represented as a natural transformation

$$n^{a_B} : C(-, A) \rightarrow (-, B)$$

from the domain functor $\boldsymbol{C}(-, A)$ constituting the figure geometry of the object A to the codomain functor $\boldsymbol{C}(-, B)$ constituting the figure geometry of the object B, which is the core mathematical content of the Yoneda lemma (Lawvere and Rosebrugh, 2003, p. 249): 'Maps in any category can be represented as natural transformations' (Lawvere and Schanuel, 2009, p. 378). Since natural transformations represent structure-preserving maps between objects, the domain (codomain) functor of a natural transformation represents the domain (codomain) object of the structure-preserving map.

Let us define the (domain) functor

$$\boldsymbol{C}(-, A)\colon \boldsymbol{C} \to \boldsymbol{C}$$

as: for each object X of the category $\boldsymbol{C}$

$$\boldsymbol{C}(-, A)(X) = A^X$$

where A^X is the map object of all X-shaped figures in A

$$\mathrm{x_A}\colon \mathrm{X} \to \mathrm{A}$$

and, for each map

$$\mathrm{x_Y}\colon \mathrm{X} \to \mathrm{Y}$$

of the category C

$$C(-, A)(x_Y : X \to Y) = A^{x_Y} : A^Y \to A^X$$

where A^Y is the map object of all Y-shaped figures in A, and with the map A^{x^Y} of map objects defined as

$$A^{x_Y}(y_A : Y \to A) = y_A \circ x_Y = x_A : X \to A$$

assigning a map x_A in the map object A^X to each map y_A in the map object A^Y. Thus the functor

$$\boldsymbol{C}(-, A)\colon \boldsymbol{C} \to \boldsymbol{C}$$

in assigning to each map

$$x_{\mathrm{Y}} = \mathrm{X} \to \mathrm{Y}$$

(of the domain category $\boldsymbol{C}$) its [induced] map [of map objects]

$$C(-,A)(x_Y : X \to Y) = C(-,A)(Y) \to C(-,A)(X) = A^{x_Y} : A^Y \to A^X$$

(of the codomain category $\boldsymbol{C}$) is contravariant, i.e., a transformation of a shape X into a shape Y induces a transformation (in the opposite direction) of Y-shaped figures in A into X-shaped figures in A (Lawvere and Rosebrugh, 2003, pp. 236–237).

Now, we check to see if $\boldsymbol{C}(-, A)$ preserves identities, i.e., whether

$$\boldsymbol{C}(-,A)(1_X: X \to X) = 1_{C(-,A)(X)}$$

for every object X. Evaluating

$$\boldsymbol{C}(-,A)(1_X: X \to X) = A^{1_X}: A^X \to A^X$$

at a map

$$x_A: X \to A$$

we find that

$$A^{1_X}(x_A : X \to A) = (x_A \circ 1_X) = x_A : X \to A$$

(for every map x_A in the map object A^X). Next, evaluating

$$1_{C(-,A)(X)} = 1_{A^X} : A^X \to A^X$$

at the map

$$x_A: X \to A$$

we find that

$$1_{A^X}(x_A : X \to A) = (x_A \circ 1_X) = x_A : X \to A$$

(for every map x_A in the map object A^X). Since

$$A^{1_X} = 1_{A^X}$$

i.e.

$$\boldsymbol{C}(-,A)(1_X: X \to X) = 1_{C(-,A)(X)}$$

for every object X of the category C, we say $\boldsymbol{C}(-, \mathrm{A})$ preserves identities.

Next, we check to see if $\boldsymbol{C}(-, \mathrm{A})$ preserves composition. Since $\boldsymbol{C}(-, \mathrm{A})$ is contravariant, we check whether

$$\boldsymbol{C}(-, A)(y_Z \circ x_Y) = \boldsymbol{C}(-, A)(x_Y) \circ \boldsymbol{C}(-, A)(y_Z)$$

where y_Z: $Y \to Z$. Evaluating

$$C(-, A)(y_Z \circ x_Y) = A^{(y_Z \circ x_Y)}$$

at any map z_A in the map object A^{Z}, we find that

$$A^{(y_Z \circ x_Y)}(z_A) = z_A \circ (y_Z \circ x_Y)$$

Next, we evaluate

$$C(-, A)(x_Y) \circ C(-, A)(y_Z) = (A^{x_Y} \circ A^{y_Z})$$

also at the map z_A

$$(A^{x_Y} \circ A^{y_Z})(z_A) = A^{x_Y}(z_A \circ y_Z) = (z_A \circ y_Z) \circ x_Y$$

Since

$$z_A \circ (y_Z \circ x_Y) = (z_A \circ y_Z) \circ x_Y$$

by the associativity of the composition of maps, we have composition preserved

$$\boldsymbol{C}(-, A)(y_Z \circ x_Y) = \boldsymbol{C}(-, A)(x_Y) \circ \boldsymbol{C}(-, A)(y_Z)$$

Having checked that

$$\boldsymbol{C}(-, A) \colon \boldsymbol{C} \to \boldsymbol{C}$$

with

$$\boldsymbol{C}(-, A)(X) \colon A^X$$
$$C(-, A)(x_Y : X \to Y) = A^{x_Y} : A^Y \to A^X$$

where $A^{x_Y}(y_A) = y_A \circ x_Y$, is a contravariant functor, we consider another contravariant functor

$$\boldsymbol{C}(-, B)\colon \boldsymbol{C} \to \boldsymbol{C}$$

with

$$\boldsymbol{C}(-, B)(X) = B^X$$
$$C(-,B)(x_Y : X \to Y) = B^{x_Y} : B^Y \to B^X$$

where $B^{x_Y}(y_B) = y_B \circ x_Y$.

With the two functors $\boldsymbol{C}$(–, A) and $\boldsymbol{C}$(–, B) representing the [figure geometry of] objects A and B, respectively, we now show that every structure-preserving map

$$a_B\colon A \to B$$

is represented by a natural transformation

$$n^{a_B} : C(-,A) \to C(-,B)$$

More explicitly, given a map a_{B}, we can construct a natural transformation n^{a_B}. A natural transformation n^{a_B} from the functor $\boldsymbol{C}(-, A)\colon \boldsymbol{C} \to \boldsymbol{C}$ to the functor $\boldsymbol{C}(-, B)\colon \boldsymbol{C} \to \boldsymbol{C}$ assigns to each object X of the domain category C (of both domain and codomain functors) a map

$$a_B{}^X\colon A^X \to B^X$$

(in the common codomain category **C**) from the value of the domain functor at the object X, i.e., $\boldsymbol{C}(-, A)(X) = A^X$ to the value of the codomain functor at X, i.e., $\boldsymbol{C}(-, B)(X) = B^X$; and to each map $x_Y\colon X \to Y$ (in the common domain category $\boldsymbol{C}$), a commutative square (in the common codomain category $\boldsymbol{C}$) shown below:

$$\begin{tikzcd}
A^X \arrow[r, "a_B{}^X"] & B^X \\
A^Y \arrow[u, "A^{x_Y}"] \arrow[r, "a_B{}^Y"'] & B^Y \arrow[u, "B^{x_Y}"']
\end{tikzcd}$$

satisfying

$$a_B{}^X \circ A^{x_Y} = B^{x_Y} \circ a_B{}^Y$$

(Lawvere and Rosebrugh, 2003, p. 241; Lawvere and Schanuel, 2009, pp. 369–370). We have already seen that with the composition-induced maps (of map objects):

$$
\begin{aligned}
&A^{x_Y}(y_A) = y_A \circ x_Y \\
&{a_B}^X(x_A) = a_B \circ x_A \\
&{a_B}^Y(y_A) = a_B \circ y_A \\
&B^{x_Y}(y_B) = y_B \circ x_Y
\end{aligned}
$$

the required commutativity:

$$
\begin{aligned}
&{a_B}^X \circ A^{x_Y}(y_A) = {a_B}^X(y_A \circ x_Y) = a_B \circ (y_A \circ x_Y) \\
&B^{x_Y} \circ {a_B}^Y(y_A) = B^{x_Y}(a_B \circ y_A) = (a_B \circ y_A) \circ x_Y
\end{aligned}
$$

is given by the associativity of the composition of maps

$$a_B \circ (y_A \circ x_Y) = (a_B \circ y_A) \circ x_Y = a_B \circ y_A \circ x_Y$$

Thus, each A-shaped figure in $B(a_B)$ is a natural transformation (n^{a_B}; homogenous with respect to composition of maps) of the figure geometry $\boldsymbol{C}(-, A)$ of A into the figure geometry $\boldsymbol{C}(-, B)$ of B.

Furthermore, we can obtain the set $|B^A|$ of all A-shaped figures in B based on the 1–1 correspondence between A-shaped figures in B and the points (i.e., maps with terminal object T of the category C as domain; Lawvere and Schanuel, 2009, pp. 232–234) of the map object B^A. This 1–1 correspondence, which follows from the universal mapping property defining exponentiation, along with the fact that the terminal object T is a multiplicative identity (Lawvere and Schanuel, 2009, pp. 261–263, 313–314, 322–323), involves the following two 1–1 correspondences between three maps:

$$
\frac{\dfrac{T \to B^A}{T \times A \to B}}{A \to B}
$$

Yoneda lemma says, in terms of our figures-and-incidences characterization of objects, that the set $|B^A|$ of A-shaped figures in B

$$a_B\colon A \to B$$

is isomorphic to the set $|\boldsymbol{C}(-,B)^{\boldsymbol{C}(-,A)}|$ of natural transformations

$$n^{a_B} : C(-,A) \to C(-,B)$$

of the figure geometry of A into that of B. The required isomorphism of sets

$$|B^A| = |\boldsymbol{C}(-,B)^{\boldsymbol{C}(-,A)}|$$

follows from the 1–1 correspondence between A-shaped figures in B and the natural transformations (compatible with all figures and their incidences) of the figure geometry of A into that of B, which we have already shown (see also Lawvere and Rosebrugh, 2003, pp. 104, 174).

Dually, a map

$$A \to B$$

viewed as a B-valued property on A induces a natural transformation

$$\boldsymbol{C}(-,B) \to \boldsymbol{C}(-,A)$$

of the function algebra of B into that of A (Lawvere and Rosebrugh, 2003, p. 249). Here also the proof of Yoneda lemma involves two transformations: (i) Contravariant: a map from an object A to an object B induces a transformation of properties of B into properties of A, for each type(object) of the category, and (ii) Covariant: a map from a type T to a type R (of properties) induces a transformation of T-valued properties into R-valued properties, for every object of the category. The calculations involved in proving Yoneda lemma in this case of function algebras are same as in the case of figure geometries, except for the reversal of arrows due to the duality between function algebra and figure geometry (Lawvere and Rosebrugh, 2003, p. 174; Lawvere and Schanuel, 2009, pp. 370–371). More specifically, function algebras and figure geometries are related by adjoint functors (Lawvere, 2016).

7.2 *Four Truth Values of the Logic of Perception*

Conscious perception involves two sequential processes of sensation followed by interpretation:

Physical stimuli → Brain → Conscious Percepts

(Albright, 2015; Croner and Albright, 1999), which can be thought of as

$$X - \text{coding} \rightarrow Y - \text{decoding} \rightarrow Z$$

and objectified as two sequential processes:

$$A -f\rightarrow B - g\rightarrow C$$

Without discounting that the processes of sensation and interpretation are much more structured than mere functions, and with the objective of simplifying the calculation of truth value object, we model percept as an object made up of three [component] sets C, B, and A, which are sets of physical stimuli, their neural codes, and interpretations, respectively, and two [structural] functions f and g specifying for each interpretation in A the neural code in B (of which it is an interpretation) and for each neural code in B the physical stimulus in C (of which it is a measurement), respectively (see Lawvere and Rosebrugh, 2003, pp. 114–117). The logic of [the category of] perception, whose objects are two sequential functions is determined by its truth value object (Lawvere and Rosebrugh, 2003, pp. 193–212; Lawvere and Schanuel, 2009, pp. 335–357; Reyes, Reyes, and Zolfaghari, 2004, pp. 93–107).The truth value object of a category is an object Ω of the category such that parts of any object X are in 1–1 correspondence with maps from the object X to the truth value object Ω. Since parts of an object are monomorphisms with the object X as codomain, for each monomorphism with X as codomain there is a corresponding X-shaped figure in Ω.

In order to calculate the truth value object, first we need to define maps between objects of the category of percepts. A map from an object

$$A - f \rightarrow B - g \rightarrow C$$

to an object

$$A' - f' \rightarrow B' - g' - C'$$

is a triple of functions

$$p\colon A \rightarrow A',\ q\colon B \rightarrow B',\ r\colon C \rightarrow C'$$

satisfying two equations

$$q \circ f = f' \circ p,\ r \circ g = g' \circ q$$

which make the two squares in the diagram

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "q"'] \arrow[d, "g"'] & B' \arrow[d, "g'"] \\ C \arrow[r, "r"'] & C' \end{tikzcd}$$

commute, i.e., ensure that maps between objects preserve the structural essence of the category (Lawvere and Schanuel, 2009, pp. 149–150).

Now that we have maps of the category of percepts defined, we can calculate its truth value object. The truth value object of a category is calculated based on the parts of the basic shapes (essence) constituting the objects of the category. In the category of sets, one-element set **1** (= {•}) is the basic shape in the sense that any set is made up of elements (see Posina, Ghista, and Roy, 2017 for the details of the calculation of basic shapes, i.e., theory subcategories of various categories). Since the set **1** is also the terminal object (i.e., an object to which there is exactly one map from every object; Lawvere and Schanuel, 2009, pp. 213–214) of the category of sets, and since every set is completely determined by its points (terminal object-shaped figures), we can determine the truth value object of the category of sets by determining its points, i.e., maps from **1** to the (yet to be determined) truth value object. According to the definition of truth value object, **1**-shaped figures in the truth value object are in 1–1 correspondence with parts of 1. Since the terminal set **1** has two parts: **0** (= {}) and **1**,the truth value set has two points (elements). Thus, the truth value object of the category of sets is **2** (= {false, true}).

Along similar lines, let us calculate the terminal object of the category of percepts. Since there is only one map from any object (two sequential functions) to the object T (two sequential functions from one-element set to one-element set):

$$1 \to 1 \to 1$$

the terminal object of the category of percepts is T and since parts of the terminal object T correspond to the points of the truth value object, let's look at the parts of the terminal object. The terminal object T

$1 \to 1 \to 1$

has four parts:

Part 1 (0: $0 \to T$)

$$\begin{tikzcd}
0 & 1 \arrow[d] \\
0 & 1 \arrow[d] \\
0 & 1
\end{tikzcd}$$

Part 2 (0_1: $0_1 \to T$)

$$\begin{tikzcd}
0 & 1 \arrow[d] \\
0 & 1 \arrow[d] \\
1 \arrow[r] & 1
\end{tikzcd}$$

Part 3 (0_2: $0_2 \to T$)

$$\begin{tikzcd}
0 & 1 \arrow[d] \\
1 \arrow[r] \arrow[d] & 1 \arrow[d] \\
1 \arrow[r] & 1
\end{tikzcd}$$

Part 4 (1: $T \to T$)

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & 1 \arrow[d] \\
1 \arrow[r] \arrow[d] & 1 \arrow[d] \\
1 \arrow[r] & 1
\end{tikzcd}$$

These four parts correspond to the four points (global truth values) of the truth value object, which means that the component set (of the truth value object) corresponding to the stage of interpretations is a four-element set **4** = $\{0, 0_1, 0_2, 1\}$. Since objects in the category of perception (two sequential functions) are not completely determined by points, we look for all other basic shapes that are needed to completely characterize any object of two sequential functions. The other basic shapes, besides the terminal object T, are: domains of the parts 0_2 and 0_1 of the terminal object T, i.e., shape 0_2

$0 \quad 1 \to 1$

and shape 0_1

0 0 1

Since the basic shape object 0_2 has three parts (*0, 0_1, and 1*), there are three 0_2-shaped figures in the truth value object, and since the object 0_1 has two parts (*0* and *1*), there are two 0_1-shaped figures in the truth value object, which means that the component set (of the truth value object) corresponding to the stage of neural coding is a three-element set **3** = {*0, 0_1, 1*}, while the component set (of the truth value object) corresponding to the stage of physical stimuli is a two-element set **2** = {*0, 1*}. Putting it all together we find that the truth value object of the category of percepts is:

$$\mathbf{4} - \mathrm{j} \to \mathbf{3} - \mathrm{k} \to \mathbf{2}$$

We still have to determine the functions *j* and *k*, which can be done by examining the structural maps between the basic shapes

$$0_1 - c \to 0_2 - \mathrm{d} \to T$$

which as a subcategory constitutes the theory (abstract essence) of the category of two sequential functions. More explicitly, the incidence relations between the three basic-shaped figures in the truth value object are calculated from the inverse images of the parts of the basic shapes (*0_1, 0_2*, and *T*) along the structural maps (*d* and *c*). The inverse images of each one of the four points (*0, 0_1, 0_2* and *1* corresponding to the four parts of the terminal object T) along the structural maps decoding *d* and coding *c* give for each one of the four global truth values **4** = {*0, 0_1, 0_2, 1*} its value in the truth value sets **3** = {*0, 0_1, 1*} and **2** = {*0, 1*} of the previous stages of neural codes and physical stimuli.

For example, the global truth value 0_2 corresponds to the part 0_2 of the basic shape T, and its inverse image along the structural map *d*: $0_2 \to$ T is the entire basic shape 0_2, which corresponds to the truth value *1* (of stage **3**); and the inverse image of the entire object 0_2 along the structural map *c*: $0_1 \to 0_2$ is the entire basic shape 0_1, which corresponds to the truth value *1* (of stage **2**). Along these lines we find that

$$j(0) = 0, j(0_1) = 0_1, j(0_2) = 1, j(1) = 1$$

$$k(0) = 0, k(0_1) = 1, k(1) = 1$$

which completely characterizes the truth value object

$$\mathbf{4} - j \to \mathbf{3} - k \to \mathbf{2}$$

of the category of percepts.

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ISBN 978-90-04-69155-1
ISSN 0929-8436
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