CDD: 510.1 http://dx.doi.org/10.1590/0100-6045.2015.V38N2.GV

HILBERT BETWEEN THE FORMAL AND THE INFORMAL SIDE OF MATHEMATICS

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Received: 19.11.2014; Revised:08.09.2015; Accepted: 21.09.2015

Abstract: *In this article we analyze the key concept of Hilbert's axiomatic method, namely that of axiom. We will find two different concepts: the first one from the period of Hilbert's foundation of geometry and the second one at the time of the development of his proof theory. Both conceptions are linked to two different notions of intuition and show how Hilbert's ideas are far from a purely formalist conception of mathematics. The principal thesis of this article is that one of the main problems that Hilbert encountered in his foundational studies consisted in securing a link between formalization and intuition. We will also analyze a related problem, that we will call "Frege's Problem", form the time of the foundation of geometry and investigate the role of the Axiom of Completeness in its solution.*

keywords: *Hilbert, Axioms, Intuitions, Axiom of Completeness, Frege, Reference of axioms, Cesar's problem, Foundations of geometry, Proof theory, Grundlagen der Geometrie.*

1. Two concepts

Hilbert's foundational papers can be divided into two periods though not neatly separated. The first one centers around the foundation of geometry and *a sketched attempt to prove the consistency of a weak form of arithmetic*[1]*, while the second one deals with the foundations of mathematics. In what follows we will argue that to this periodization corresponds a related distinction on the general notion of axiom that underlies Hilbert's work. Moreover, the recognition of these two different notions is meant to undermine the formalist label, often attributed to Hilbert, because of their link to two different forms of intuition. We do not claim here to give an exhaustive account of Hilbert's conception of axioms; our aim consists in showing that it changes through time and that, although in a continuous manner, the aims of Hilbert's foundational work determines at least two different notions of axioms. We believe that focusing on the notion of axiom, instead of reflecting on the role of axiomatization in Hilbert's thought, is very instructive in order to understand the philosophical ideas that lay behind Hilbert's proposals.*

Before a brief presentation of the two notion of axiom, it is important to clarify that the methods, scopes and notions linked to the foundations of geometry and to the foundations of mathematics different significantly. The first trivial observation is that geometry is a a particular mathematical theory; hence a local opposed to a global perspective characterizes the foundational work of the first period. Consequently, the aims of formalization are clearly different: while in the case of geometry the foundational work aimed at systematizing and operating a conceptual clarification of the basic notions a theory that was already largely formal but that had intuitive roots in the physical world, on the contrary, at the time of the foundations of mathematics Hilbert's aim consists in reducing a large part of mathematics to a contentual one, that was then formalized by means of his new proof theory. Without entering here a general discussion about these terms, it is important to stress that words like "formalization" and "formal" have different meanings when placed in the right theoretical – and historical – context. In the case of the foundation of geometry a formal presentation is meant to cut the link with spatial intuition, while during the foundation of mathematics the formalization of a theory was considered only the first step toward the application of proof theory.

*Coming back to the notion of axiom, in the first period Hilbert linked this notion to a "*deepening of the foundations *of the individual domains of knowledge"*[2]*. Indeed he believed that the axiomatization of a theory aimed at identifying the principles able to give a precise description of a theory and at codifying their*

[1] See [Hilbert, 1967a].

[2] [Hilbert, 2005a], p. 1109 in [Ewald, 2005].

meaning by means of axioms. In 1917 we can still find traces of this attitude in Hilbert's talk Axiomatichen Denken.

> *When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science*[3].

The second period starts in the early Twenties, after Hilbert resumed the study of the foundations of mathematics. In this period's works we can see an effort to build the whole of mathematics on few axioms. These axioms were supposed to gain their legitimacy from Hilbert's proof theory*, that was intended to "make a protocol of the rules according to which our thinking actually proceeds"*[4]*. These rules were meant to be the* a priori *component of any form of mathematical knowledge. In this second period the axioms of a particular formal system are assimilated to other mathematical propositions, while the axioms of proof theory - arithmetical and logical in character - were meant to describe the way our mathematical thinking proceeds.*

> *also [...] mathematical knowledge in the end rests on a kind of intuitive insight [*anschaulicher Einsicht*] of this sort, and even that we need a certain intuitive* a priori *outlook for the construction of number theory*[5].

As a first attempt to differentiate the two periods, we can say that the evolution of Hilbert's conception of axiom reflects a different emphasis on two different aspects of mathematical work: first the possibility to clarify the fundamental principle of a mathematical theory, while later the elucidation of mathematical thinking in general. Of course these two aspects are mixed and it is possible to find continuity in Hilbert's reflections. However, it is useful to stress that in addition to his theorization, the foundational problems Hilbert encountered influenced pragmatically his conception of axiom. Another theme that we will discuss is also the presence of prescriptive features of axioms, besides their descriptive character.

[3] [Hilbert, 2000], p. 1104 in [Ewald, 2005].

[4] [Hilbert, 1967c], p. 475 in [van Heijenoort, 1967].

[5] [Hilbert, 2005d], p. 1161 in [Ewald, 2005].

1.1. The first period

At the time of the foundations of geometry the axiomatization of a theory is seen as a way to delve into the logical relationship of its theorems. Here 'theory' is not used in the formal sense, but it refers to any mathematical field of research that features a unique subject of enquiry and homogeneous methods.

> *On the contrary I think that wherever, from the side of the theory of knowledge or in geometry, or from the theories of natural or physical science, mathematical ideas come up, the problem arises for mathematical science to investigate the principles underlying these ideas and so to establish them upon a simple and complete system of axioms, that the exactness of the new ideas and their applicability to deduction shall be in no respect inferior to those of the old arithmetical concepts*[6]*.*

The analysis of the basic principles of a theory on the one hand leads to the choice of the axioms, and on the other hand defines the concepts and relations in play. In this first period Hilbert has a precise idea of what axioms are: they are implicit definitions. *Axioms define basic concepts and relations of a theory*[7] *and the process of formalization is complete*[8] *only when no other characteristic note can be added. Even if the idea of implicit definitions keeps axioms and concept within the realm of formal mathematics, however it is important to notice that in order to determine when a definition of a concept is complete we need to have a pre-formal grasp of it; we need to know what we are defining. Indeed, even if one of the central novelties of Hilbert's use of the axiomatic method is the separation between the logical-mathematical sphere of the axioms and the epistemological one, it should be explained how it is possible to match axioms and meaning.*

[6] [Hilbert, 2000], p. 1100 in [Ewald, 2005].

[7] We see here an implicit use of a sort of principle of comprehension that Hilbert states in this form: "the fundamental principle that a concept (a set) is defined and immediately usable if only it is determined for every object whether the object is subsumed under the concept or not." in [Hilbert, 1967a] p. 130.

[8] At this point, Hilbert has not handled the problem of the formalization of logic yet, nor Russell and Whitehead have written the *Principia mathematica*. For this reason it is clear that this 'completeness' is neither the completeness of logic, nor of the deductive methods. Hence this notion remains at an intuitive level; we will argue conceptual.

We choose to keep the notion of meaning at an intuitive level, since it is not the center of our work, and also because of a certain ambiguity in the role that this concept plays in Hilbert's foundational work. Indeed, even though it may not be understood in term of reference, because of the schematic character of mathematical entities given by the implicit definitions, nevertheless, once a formal system is interpreted questions of meaning rise in trying to connect the kinds defined by a formal system and their interpretations. In other terms, although for Hilbert the formal presentation of a mathematical theory consists in a scaffolding for concept, it is not clear how to associate formal axioms and their content, once we allow the possibility to interpret them. This explanation is needed also because the axiomatic method is intended to analyze the meaning of the axioms.

Hence, where does the meaning of the axioms come from if not from their correctness (or truth)?[9]*. Even if we accept the idea that correctness (or truth) is not a precondition for the meaning of an axiom, we need to accept - even granting the possibility of different applications of an axiom system - that axioms allow a correct formalization of the basic concepts of the subject matter of a particular theory; and so they can be hold true in that context. In other words, one of the main problems that Hilbert's ideas encounter in the formal treatment of a theory is the explanation of why an axiomatic system could be considered a good formalization of – among others – its intended interpretation.*

A mixture of concerns about meaning and the possibility to have different interpretations of Hilbert's axioms for geometry is the content of an objection raised by Frege.

> *Your system of definitions is like a system of equations with several unknowns, where there remains a doubt whether the equations are soluble and, especially, whether the unknown quantities are uniquely determined. If they were uniquely determined, it would be better to give the solutions, i.e. to explain each of the expressions 'point', 'line', 'between' individually through something that was already known. Given your definitions, I do not know how to decide the question whether my pocket watch is a point. The very first axiom deals with two points; thus if I wanted to know whether it held for my watch, I should first have to know of some other object*

[9] A similar point has been rised by W.Tait in the notes from his talk "Dialectic and logic: the truth of axioms".

> *that is was a point. But even if I knew this, e.g. of my penholder, I still could not decide whether my watch and my penholder determined a line, because I would not know what a line was*[10].

The objection is justified on the basis of Frege's understanding of a formal theory and on the basis of his work on sense and reference. Indeed, from Frege's perspective the fact that the basic concepts of a theory are open for re-interpretation generates a major problem for their univocal identification.

Frege's critic, however, is easily rebutted by Hilbert[11]*. In fact he argues that that was exactly the strength of his method: to establish a formal system able to define abstract concepts, granting consistency, which would respond only to the requirements imposed by the axioms.*

> *This is apparently where the cardinal point of the misunderstanding lies. I do not want to assume anything as known in advance; I regard my explanation in sec. 1 as the definition of the concepts point, line, plane - if one ads again all the axioms of groups I to V as characteristic marks. If one is looking for another definitions of a 'point', e.g. through paraphrase in terms of extensionless, etc., then I must indeed oppose such attempts in the most decisive way; one is looking for something one can never find because there is nothing there*[12].

The problem with Hilbert's reply is that it just points at a distinction of levels (the formal one against the one where formulas are interpreted) but does not give an explanation to another problem implicit in Frege's objection. We will call it Frege's problem and we formulate it as follows: "why is the axiomatic system presented by Hilbert in the Grundlagen der Geometrie *to be considered a correct formal presentation of the basic geometrical concepts?" In other words, what is the definition of geometry and of its basic concepts once his axiomatic method has cut off the link between formalization and spatial intuition? In the next section we will try to explain a possible answer to this objection.*

[10] Letter from Frege to Hilbert January 6th, 1900; in [Frege, 1980], p. 45.

[11] Or at least this is what Hilbert would have answered, because he chose not to reply to Frege's letter of January 6th, 1900. However, next quoted passage is from Hilbert's previous letter; and we can assume that if Hilbert did not reply to Frege's letter is because he had already made his point.

[12] Letter from Hilbert to Frege December 29th, 1899; in [Frege, 1980], p. 39.

Coming back to the definitional features of axioms, it is also interesting to see how they act in this process of definition. Hilbert acknowledges that if the axiomatization of a theory is complete, then the entities implicitly defined are uniquely determined by the axioms. But, given a strong link between consistency and existence, any non-contradictory set of axiom defines something. Hence, the process of extending a set of axioms gives rise to a sequence of different definitions: "every axiom contributes something to the definition [of the concept], and hence every new axiom changes the concept"[13]*. However, since this notion of completeness is teleologically oriented, we may assume the existence of what we would call an intended interpretation, able to keep together different approximations of the same concept: "[...] the definition of the concept point is not complete till the structure of the system of axioms is complete*[14]*". Even more interestingly Hilbert acknowledges that the question whether different axiomatic systems can both be legitimate is theoretically interesting. However he does not explain how it is possible that two different theories can talk about the same things, since different axioms define different concepts. Consequently Hilbert seems to admit that we have a semantic grasp of a concept whose formalization can be seen as a process.*

There is also another interesting question in connection to the definitional character of the axioms: what kind of entities are defined by the implicit definitions of the axioms? This problem was again explicitly raised by Frege in correspondence with Hilbert.

> *The characteristic marks you give in your axioms are apparently all higher than first-level; i.e., they do not answer to the question "What properties must an object have in order to be a point (a line, a plane, etc.)?", but they contain, e.g., second-order relations, e.g., between the concept* point *and the concept* line. *It seems to me that you really want to define second-level concepts but do not clearly distinguish them from first-level ones*[15]*.*

Indeed Hilbert is not precise in saying what axioms define, sometimes, as next quotation show, they seem to define mathematical concepts.

> *I regard my explanation in sec. 1 as the definition of the concepts*

[13]Letter from Hilbert to Frege December 29th, 1899; in [Frege, 1980], p. 40.
[14]Letter from Hilbert to Frege December 29th, 1899; in [Frege, 1980], p. 42.
[15]Letter from Frege to Hilbert January 6th, 1900; in [Frege, 1980], p.46.

> *point, line, plane - if one adds again all the axioms of groups I to V as characteristic marks.*[16].

Instead, sometimes Hilbert maintains that axioms define relations between mathematical concepts. As a matter of fact, in the same letter to Frege, he maintains that the axioms of the Grundlagen der Geometrie *can also define the concept of "between". This is a hint that, even though we can find, in Hilbert, a weak form of realism about mathematical concepts, this is far enough from a platonic position, able to distinguish between objects and properties of objects. In what follows we will see that Hilbert will abandon this careless ontological commitments, considering axioms just as definitions of relations between mathematical concepts, or as "images of thoughts". Even if Hilbert chose not to reply to the objections of Frege, we could argue that he indeed carefully considered them.*

During the first period Hilbert conncets explicitly intuition and axioms.

> *These axioms may be arranged in five groups. Each of these groups expresses, by itself, certain related fundamental facts of our intuition*[17].

Moreover he considers the axiomatic method as a tool for performing an analysis of their conceptual content.

> *This problem [the choice of the axioms of geometry] is tantamount to the logical analysis of our intuition of space.* [18].

These ideas are clearly far from a formalist conception of mathematics[19], *since formalization is not meaningless, but it is able to represent the basic concepts of a mathematical theory by means of symbols that have intuitive content. This content is intrinsic to the theory and not to our way of formalizing it. Before formalization Hilbert sees an* historical *development*[20], *after which an*

[16]Letter from Hilbert to Frege December 29th, 1899; in [Frege, 1980], p. 39.

[17][Hilbert, 1902], p. 3.

[18][Hilbert, 1902], p. 1.

[19]Against a formalist interpretation of Hilbert's philosophy see [Hintikka, 1997], [Kreisel, 2007] and [Sieg, 1999].

[20]Remember Hilbert's description of the development of science: "The edifice of science is not raised like a dwelling, in which the foundations are first firmly laid and only then one proceeds to construct and to enlarge the rooms. Science prefers to secure

axiomatization is possible. Moreover the knowledge conveyed by a developed theory is the source of the meaning of signs.

In this first period Hilbert lists three requisite for the axioms: simple, complete and independent[21]*. We have already discussed completeness and we will not discuss here the problem of simplicity, because it would lead us too far from the focus of this paper*[22]*.*

For what concerns independence, we underline that this requirement is strongly linked to the idea of deepening the foundations of a science. Indeed if it possible to show that the axioms are formally independent, then it is possible to argue in favor of the understanding of the intuitive content of the axioms. Indeed if the search for axioms is an analysis of the basic principles of a theory, an independence proof would mean that the analysis has been accurate enough to single out the real meaning of an axiom: a basic principle of the theory. In other words, independence is deeply linked with the notion of intuitive content of an axiom, related to a specific domain of knowledge.

This need for an independence proof, unlike that of a consistency proof, will be dropped in the second period. This is a hint of a change in the concept of axiom.

1.2. The second period

Hilbert's second period begins, publicly, in the early Twenties, but its roots can be traced back to the last years of the previous decade[23]*. Different reasons concur in indicating a change of opinion and attitude towards the foundations of mathematics. Hilbert abandons the confidence in the systematization of logic proposed by Russell and Whitehead and expounded in the* Prinicipia mathematica. *Hilbert then starts his most original contribution to the study of logic, in*

as soon as possible comfortable spaces to wander around and only subsequently, when signs appear here and there that the loose foundations are not able to sustain the expansion of the rooms, it sets about supporting and fortifying them. This is not a weakness, but rather the right and healthy path of development." In [Hilbert], p. 102. Traslation by Leo Corry in Corry [2006a].

[21]See for example the introduction of the *Festschrift* (the first edition of the *Grundlagen der Geometrie*), or the introduction of the 1902 lectures notes *Grundlagen der Geometrie*.

[22]However we acknowledge its importance, as shown in Thiele [2003].

[23]For a detailed description of this transition period see [Sieg, 1999].

order to improve its formalization. Moreover, in that period the debate around intuitionism became more and more controversial.

In the lectures Neubegründung der Mathematik. Erste Mitteilung *(1922) and* Die logischen Grundlagen der Mathematik *(1923) Hilbert outlines a new analysis of the concept of axiom. These two works can be regarded as belonging to a transition period, not only chronologically but also from a conceptual point of view. The work that clearly marks a more explicit change is* Über das Unendliche *(1925).*

In 1922, the concept of axiom is defined in the following ways:

> *The continuum of real numbers is a system of things which are linked to one another by determinate relations, the so-called axioms*[24]*.*

Next definition can also be found in Die Grundlagen der Mathematik *(1928) and in* Über das Unendliche*,*

> *Certain formulas which serve as building blocks for the formal structure of mathematics are called* axioms[25]*.*

First of all we need to notice that the axioms do not define mathematical objects, but just their relations[26]*. On the other hand the difference between axioms and other formulas begins to be less marked.*

> *The axioms and provable theorems [...] are the images of the thoughts that make up the usual procedure of traditional mathematics; but they are not themselves the truth in any absolute sense. Rather, the absolute truths are the insights that my proof theory furnishes into the provability and the consistency of these formal systems*[27]*.*

Here one can see not only that axioms and provable propositions have a similar status, but also that axioms are deprived of their independent meaning:

[24] [Hilbert, 2005b], p. 1118 in [Ewald, 2005].
[25] [Hilbert, 2005b], p. 1125 in [Ewald, 2005] and [Hilbert, 1967b].
[26] As we noted in the previous section this change may be traced back to his correspondence with Frege.
[27] [Hilbert, 2005c], p. 1138 in [Ewald, 2005].

they are "images of the thoughts"[28]. *They keep just an operational character for the "usual procedure of traditional mathematics". It is also possible to read the beginning of a separation between the concepts of consistency and truth, as it will then become apparent later, considering the further development of logic in the Thirties. The particular status - with respect to meaning and truth - that axioms had at the time of the axiomatization of geometry is now assimilated to that of other propositions of ordinary mathematics. Hilbert is now interested in a more general point of view: the foundation of mathematics. This shift is responsible for the change of the notion of axiom, or more properly to the birth of a new more fundamental notion of axiom; not specific anymore, but global in character. In the search for the content of these new axioms, Hilbert then turns his attention towards the working mathematician; if mathematics cannot be defined what rests is its practice.*

During the Twenties Hilbert's proof theory was born. In this new perspective Hilbert defines a new kind of axiom. The only ones that truly deserve such a name.

> *This program already affects the choice of axioms for our proof theory*[29].

These new axioms are not of the same nature of the ones mentioned in the previous quotations. They are the axioms on which the mathematical edifice rests.

> *Certain of the formulas correspond to mathematical axioms. The rules whereby the formula are derived from one another correspond to material deduction. Material deduction is thus replaced by a formal procedure governed by rules. The rigorous transition from a naïve to a formal treatment is effected, therefore, both for the axioms (which, though originally viewed naïvely as basic truth, have been long treated in modern axiomatics as mere relations between concepts) and for the logical calculus (which originally was supposed to be merely a different language)*[30].

[28] Not that Hilbert considered axioms as absolutely meaningful in the first period, but previously he assigned them the function of defining concepts, independently of our way to think them.

[29] [Hilbert, 2005c], p. 1138 in [Ewald, 2005].

[30] [Hilbert, 1967b], p. 381 in [van Heijenoort, 1967]. Notice how it is possible here

Hilbert thought that his proof theory brought to an end the process of formalization of mathematics. Consequently local axioms do not have a privileged epistemological status with respect to other mathematical propositions. These ideas mirror an old idea of Hilbert that seems to be constant over time: "Usually, in the story of a mathematical theory we can easily and clearly distinguish three stages of development: naïve, formal and critical[31]*".*

These axioms are logical and arithmetical in characters and they are true axioms in an absolute sense, since they draw their certitude and evidence from the way Hilbert is now setting the problem of the foundation of mathematics: a proof theory that tries to justify ideal elements with finitary tools.

> *This circumstance corresponds to a conviction I have long maintained, namely, that a simultaneous construction of arithmetic and formal logic is necessary because of the close connection and inseparability of arithmetical and logical truth*[32]

*However, since - at least finitary - mathematical statements have a content (*Inhalt*), intuition cannot be ignored in the foundation of mathematics. In the next section we will propose an analysis of the concept of intuition. For now it is enough to say that intuition is the source of certainty and evidence for mathematics and is capable of making mathematical truths absolute. Intuition is the origin of certainty in the finitary setting. So, considering the whole mathematics as a complex of formal propositions, Hilbert founded certainty in the intuitive relationship between the thinking subject and the symbols, that are "immediately clear and understandable".*

Let us now analyze the concept of intuition. We anticipate that we will find two different concepts of intuition and that this difference is responsible for two different concepts of axiom.

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to mark a change in the role of logic with respect to axiomatization: before it was a tool for calculation, while now it also has an epistemological role.

[31]In [Hilbert, 1970b], p. 383 in [Hilbert, 1970a]. In German: *In der Geschichte einer mathematischen Theorie lassen sich meist 3 Entwicklungsperioden leicht und deutlich unterscheiden: Die naive, die formale und die kritische.* My translation.

[32][Hilbert, 2005b], pp. 1131-1132 in [Ewald, 2005]. In [Hintikka, 1997] can be found an interesting account of the importance of combinatorial aspects of arithmetic in particular, and calculation, in general. See also [Ogawa, 2004] in this respect. A discussion of Hilbert's logicism can be found in [Ferreirós, 2009].

2. Hibert and intuition

We will try now to understand the many references to intuition that we can find in Hilbert's work in both periods. Our thesis is that it is exactly the intuitive character of axioms that marks the main difference between the two periods[33]*.*

Fist of all we need to stress the difference between "intuitive" and "evident", since the confusion between these two concepts has always been source of ambiguity. By "evident sentence"we mean a sentence that does not need an active work of the mind to exhibit its truth. By "intuitive sentence"we mean a sentence whose truth, in a given context and due to a given background knowledge, is immediately perceived, so that it is possible to skip some step of reasoning that, in other cases, would be necessary. This distinction pertains to the difference between the level of validity and that of justification. Unlike evidence, which is innate within our mind, intuition can be educated thanks to training and mathematical practice. The intuition we are referring to, that we could call a contextual intuition, is not an intuition that depends on a specific faculty of the mind different from intellect. In other words it is not a Kantian-style intuition, i.e. a faculty whose structure depends on pure forms, that are given once and for all, like space and time, and that governs sensible knowledge. On the contrary the intuition we are considering here can be refined by the same knowledge that it helps to create.

We also distinguish two modes of intuition; following [Parsons, 1995] we call them intuition of *and intuition* that*, to stress the difference between the conception of intuition as a kind of perception -* à la *Gödel*[34] *- and the idea that intuition can be a propositional attitude. In neither cases intuition is, in principle, a form of knowledge. What any kind of intuition lacks to become knowledge is the evident character that makes clear the intuition of an objective reality and true the propositions intuited. What is important to stress here is that intuition, once distinguished from evidence, may become knowledge thanks*

[33]For a study of the role of intuition in the second period of Hilbert'd foundational work see [Kitcher, 1967], [Legris, 2005], [Parsons, 2008] and [Parsons, 1998]. For a more comprehensive study see [Majer, 2006] and [Corry, 2006b].

[34]In [Parsons, 1995] Parsons shows that this kind of intuition, although is explicitly defended by Gödel in [Gödel, 1964], is not the only one that can be found in Gödel's works. Starting from this right remark, it would be interesting to analyze the analogies and differences between the - at least - two different conception on intuition in Gödel's thought in comparison with Hilbert's.

to a rational process.

Moreover, in what follows we will also consider intuition as a faculty of the mind, à la Kant. *However, we have to recall that this type of intuition is not intellectual, since it acts in perception and makes perception possible.*

2.1. First period

In Hilbert's works at the time of the foundation of geometry, the context of a mathematical theory plays a fundamental role in the choice of the axioms. Indeed the "axiomatic investigation of their [i.e. of the signs] conceptual content" is relative to a theory and allows the "use of geometrical signs as a means of strict proof". Moreover, since "the use of geometrical signs is determined by the axioms", intuition and mathematical practice are connected.

A precise account of mathematical signs is then outlined. Mathematical signs, including geometrical figures, can be used in a proof as far as their conceptual content is adequate to the context; that is when signs formalize principles that are coherent with the basic concepts of the underling theory. Then they can be used as demonstrative tools, in the ways allowed by the axioms. So, the "conceptual content" is just the meaning of signs in the context of use. This meaning, however, depends on the axioms that concur, as implicit definition, in determining the basic principles of a theory. As we see the formal and the pre-formal[35] *sides of mathematics mutually influence each other. Axioms determine the meaning of signs and their demonstrative use, but where the axioms come from and how can they match with ideas and use? A link and a correspondence then must be found between these two sides of mathematical knowledge.*

Hilbert's solution appeals to intuition, as he states clearly at the beginning of the Grundlagen der Geometrie*: axioms express "certain [...] fundamental facts of our intuition". This fact could sound at odd with the position expressed by Hilbert in correspondence with Frege, where he marks clearly a distance from an old conception of geometry; one that sees in the spatial intuition the source of legitimacy of its axioms. Indeed the* incipit *of the* Grundlagen der Geometrie *and its reference to intuition is partly the result of an immature reflection on the sources of knowledge in geometry*[36]*, but it also springs from a notion of*

[35] By pre-formal here we mean "before the axiomatical presentation of an intuitive theory". Indeed such an intuitive theory gathers both formal and informal components.

[36] For a detailed study of the origins and the early influences on Hilbert's concep-

intuition that is not only the empirical intuition of space, as in the Euclidean formulation. Although Hilbert recognizes that intuition of space is the starting point of any geometrical reflection, he maintains that it is not the ultimate source of meaning and truth of geometrical propositions. A different notion of intuition leads Hilbert to argue that the analysis of the foundations of geometry consists of "a rigorous axiomatic investigation of their [of the geometrical signs] conceptual content"[37]*. As a matter of fact Hilbert is explicit in recognizing that the axioms of geometry have different degrees of intuitiveness.*

> A general remark on the character of our axioms I-V *might be pertinent here. The axioms I-III [incidence, order, congruence] state very simple, one could even say, original facts; their validity in nature can easily be demonstrated through experiment. Against this, however, the validity of IV and V [parallels and continuity in the form of the Archimedean Axiom] is not so immediately clear. The experimental confirmation of these demands a greater number of experiments.*[38]*.*

*In order to clear this intricate connection between intuition and formalization, it could be useful to see in details how the axiomatic method works, as it is described by Hilbert. The process of axiomatization starts from an intuition concerning a domain of facts (*Tatsachen*), then, in the process of formalizing the latter, it clears the logical relationships within the concepts of the theory. This process, as Hilbert describes it leads from the subject matter of a theory to a conceptual level*[39]*.*

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tion of geometry see [Toepell, 1986a], [Toepell, 1986b] and [Toepell, 2000]. In [Corry, 2006b] Leo Corry argues that the progression of Hilbert's works marks the shift, for what concerns geometrical knowledge, from intuition to experience, due to his work on general relativity theory. We generally agree with Corry about the presence of such a shift. Indeed, as we suggested in the beginning, we believe that the descriptive character of the axioms changes from mathematical reality to the transcendental structure of our mathematical knowledge; hence in the case of geometry from spatial intuition to a wider form of geometrical experience. However, we think that the intuition Hilbert's refers to during the period of the foundations of geometry is not a Kantian-style intuition as Corry's article seems to presuppose.

[37][Hilbert, 2000], p. 1101 in [Ewald, 2005].

[38][Hilbert, 2004d], p. 380 in [Hallet and Majer, 2004].

[39]Recall that at the beginning of [Hilbert, 1902] Hilbert quotes Kants' *Critique of*

> *The method of the axiomatic construction of a theory presents itself as the procedure of the mapping [*Abbildung*] of a domain of knowledge onto a framework of concepts, which is carried out in such a way that to the object of the domain of knowledge there now correspond the concepts, and to statements about the objects there correspond the logical relations between the concepts*[40].

The task of the axiomatic method is not exhausted by the formalization of a pre-formal theory, because one of its tasks is to analyze the meaning of signs, by means of formal methods. Indeed, by deepening the foundations of a domain of knowledge one elucidates, at once, the logical structure of the theory and the intuitions about the subject matter of the theory. Therefore axioms have a double role with respect to signs. On the one hand axioms, through the axiomatic enquiry, are used to elucidate the meaning to signs, on the other hand they grant the demonstrative power of signs, linking intuition to mathematical practice in the act of justification of their use. Indeed intuition both precedes axiomatization and guides the work of a mathematician.

> *[O]ne should always be guided by intuition when laying things down axiomatically, and one always has intuition before oneself as a goal [*Zielpunkt*]. Therefore, it is no defect if the names always recall, and even make easier to recall, the content of the axioms, the more so as one can avoid very easily any involvement of intuition in the logical investigations, at least with some care and practice*[41].

The kind of intuition that allows to give meaning to mathematical propo-

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pure reason and writes "All human knowledge begins with intuitions, thence passes to concepts and ends with ideas". This quotation, though not Kantian in spirit, explains how Hilbert wanted to use the axiomatic method in his researches. Indeed for Hilbert the mathematical concepts - or relations, recalling Frege's criticism - defined by the axioms of the *Grundlagen der Geometrie* are not strictly speaking geometrical objects but conceptual entities that can be interpreted as geometrical objects. The intended interpretation is of course that of geometry, but this does not narrow the range of possible interpretations that can be given to formulas of the system. We then can see three distinct levels of things: 1) empirical entities 2) formal objects 3) elementary ideas of Geometry. This distinction also mirrors the evolutive steps of a theory: naïve, formal and critical.

[40] [Hilbert, 2015b], p. 3. Translation in [Hallet, 2008].

[41] [Hilbert, 2015a], pp. 87-88. Translation in [Hallet, 2008].

sitions, is not evidence, but a contextual intuition that is developed in parallel with the demonstrative techniques. It is the same intuition according to which mathematicians isolate and choose the axioms of a theory[42]*. It is the intuition that one develops when working within a theory. The axiomatic method then consists in formalizing, by means of signs - figures, symbols or diagrams - a* modus operandi *acquired by habit*[43]*. Indeed, in 1901, Hilbert, in discussing the primacy of his work with respect to Kleine's program, maintains that the concepts of Euclidean geometry are more familiar, not because of our outer intuition of the world, but due to our elementary study of this subject at school.*

> *On the basis of Riemann and Helmholtz Lie set up a system of axioms which differs fundamentally from those systems that are developed according to the Euclidean model. Lie's axioms contain function-theoretic parts since he requires motion to be expressed by differentiable functions. [...] The question arises whether the function-theoretic components are only necessary because of the desire to apply this (group-theoretic) method, or whether they are foreign to the subject matter itself and are thus superfluous. It turns out that in fact they are. Thereby we once again draw closer*

[42]When working on the foundation of geometry Hlibert explains his goal in the following way: "we can outline our task as constituting *a logical analysis of our intuition [Anschauungsvermögens]*" ([Hilbert, 2004a], p. 2), i.e. an analysis of the most fundamental principles of geometry, conducted with formal means. Among these principles there are of course also our spatial intuitions, but "the question of whether spatial intuition has an *a priori* or empirical character is not hereby elucidated" ([Hilbert, 2004a], p. 2). As a matter of fact, in these years, there is no philosophical analysis of the faculty of intuition. Nevertheless the quotation above (from [Hilbert, 2015a]) shows that the notion of intuition involved is not just a faculty of sensation.

[43]We can find an antecedent of this kind of intuition in Klein's words: "Mechanical experiences, such as we have in the manipulation of solid bodies, contribute to forming our ordinary metric intuition, while optical experiences with light-rays and shadows are responsible for the developement of a 'projective' intuition" (in [Klein, 1897], p. 593). However a different conception of the axiomatic method and of a formalistic treatment of mathematics (see [Torretti, 1984]) will lead Klein to a different approach to geometry. Indeed Klein's geometrical enquires and Erlangen's Programme presupposed an uncritical treatment of the intuitive treatment of the notion of continuity, contrary to the basic principle that aims Hilbert's axiomatic method. Indeed, while Klein tried to analyze and classify the different kind of spaces, Hilbert dealt with intuitions prior to the concept of space.

> *to the old Euclid, insofar as we don't need to impose the additional infinitesimal properties on the concept of motion which Lie still thought necessary. Instead, the elementary postulates which are already contained in the Euclidean concept of congruence suffice, a concept with which we are all familiar, due to the theorems about the congruence of triangles known from school*[44]*.*

Following the terminology fixed before it is an intuition that[45]*: a propositional attitude towards mathematics, that can be formalized and gains certainty, once a consistency proof is given for the formal system that embodies its syntactic counterpart: the signs. It is not an innate intuition, but it is sufficiently reliable to be used as an heuristic criterion and that can be formalized. Obviously this criterion is not always safe:*

> *[. . .] we do not habitually follow the chain of reasoning back to the axioms in arithmetical, any more than in geometrical discussions. On the contrary we apply, especially in first attacking a problem, a rapid, unconscious, not absolutely sure combination, trusting to a certain arithmetical feeling for the behavior of the arithmetical symbols, which we could dispense with as little in arithmetic as with the geometrical imagination in geometry*[46]*.*

All these remarks show that at the beginning of Hilbert's reflections there is no coincidence between the notion of intuition and the notion of evidence. Indeed, Hilbert's explicit purpose, in the Grundlagen der Geometrie*, was to give a safe basis to geometry different from space intuition, unlike the Euclidean axiomatic setting. Hilbert wanted to justify also non-Euclidean geometries and so, after refuting evidence as a criterion for truth, he looked for a sufficiently general and comprehensive principle to give foundation to geometry, i.e. the axiomatic method*[47]*. Nevertheless signs need meaning, in order to avoid a meaningless discourse. This is the "conceptual content" mentioned by Hilbert, where the intuition that gives meaning to signs is not the pure intuition of space - in the case of geometry - but it is the intuition of the basic concepts of*

[44]From a lecture before the Royal Academy of Science in Göttingen, 1901. In [Majer, 2006], p. 61.

[45]This observation rules out Kantian's intuition, as far as intuition *of*, active in the act of perception.

[46][Hilbert, 2000], p. 1101 in [Ewald, 2005].

[47]Even if, at the time, Hilbert lacked the proper logical tools.

the theory that are formalized by means of axioms. This intuition is contextual to the formal system, it is the intuition that allows to determine, by means of the implicit definitions of the axioms, what points, lines and space are, as geometrical entities, i.e. part of a geometrical formal theory.

Now, assuming consistency, on which basic can we considered Hilbert's system a good formalization of geometry? In other words, is the notion of geometry that we can find in the Grundlagen der Geometrie *good enough to formalize our geometrical intuition in the wider sense we just described? In other words, is Hilbert's axiomatization of geometry sufficient for answering Frege's Problem?*

A preliminary question could consists in asking if there is a notion of geometry that for Hilbert proceeds formalization. In his lectures on projective geometry we can find the following statement, which still suffers from a conception that shortly thereafter would radically change.

> *Geometry is the theory about the properties of space*[48]*.*

However, in Hilbert's lectures for the summer semester, in 1894, entitled Die Grundlagen der Geometrie *there is no longer an explicit definition of geometry, but rather of geometrical facts. It is also worth noting that in the 1899* Grundlagen der Geometrie *we do not find a definition of space.*

> *Among the phenomena, or facts of experience that we take into account observing nature, there is a particular group, namely the group of those facts which determine the external form of things. Geometry concerns itself with these facts*[49]*.*

Here there is a subtle, but basic, shift in addressing the problem of a foundations for geometry. Hilbert is not trying to give an explicit definition of geometry, but his aim consists in finding a consistent system of axioms that allows a formalization of all geometrical facts. Moreover, already in 1894 Hilbert was explicit in describing the goals of his foundational studies.

> *Our colleague's problem is this: what are the* necessary *and* sufficient[50] *conditions, independent of each other, which one must posit for a system of things, so that every property of these things corresponds to a geometrical fact and vice versa, so that by means*

[48][Hilbert, 2004b], p. 5.
[49][Hilbert, 2004c], p. 7.
[50]My emphasis.

> *of such a system of things a complete description and ordering of all geometrical facts is possible*[51].

Hilbert's statement of intent is clear: find necessary and sufficient conditions to describe every geometrical fact. Then geometry is implicitly and extensionally defined by geometrical facts. This is precisely the purpose of an analysis conducted with the axiomatic method. Again in 1902, Hilbert is clear on this point.

> *I understand under the axiomatical exploration of a mathematical truth [or theorem] an investigation which does not aim at finding new or more general theorems being connected with this truth, but to determine the position of this theorem within the system of known truths in such a way that it can be clearly said which conditions are necessary and sufficient for giving a foundation of this truth*[52].

We believe that the search for necessary and sufficient conditions, not with respect to the truth of geometrical propositions, but with respect to the definitions of geometrical concepts, motivates also the adjunction of the Axiom of Completeness[53] *to the* Festschrift*; the first edition of the* Grundlagen der Geometrie*. And thanks to this axiom Hilbert could have been able to give an answer to Frege's Problem.*

Let us argue this point more in details. It was clear to Hilbert that continuity was not needed to prove the relevant geometrical fact; and this is the main reason for not including the Axiom of Completeness in the Festschrift *and to isolate the Archimedes's Axiom from the principles of continuity normally used to formalize analytic geometry. However, the proof-theoretic useless of the Axiom of Completeness does not undermine its centrality to Hilbert's eyes. Indeed it "forms the cornerstone of the entire system of axioms [although] the completeness axiom is in general not assumed*[54]*". The capital importance of*

[51] [Hilbert, 2004c], p. 8.

[52] [Hilbert, 1903], p. 50.

[53] V.2 (Axiom of Completeness) The elements (points, straight lines, planes) of geometry form a system of things that, compatibly with the other axioms, can not be extended; i.e. it is not possible to add to the system of points, straight lines, planes another system of things in such a way that in the resulting system all the axioms I-IV, V.1 are satisfied.

[54] [Hilbert, 1971], p. 28. From the seventh German edition onward.

this axiom is in terms of a notion of completeness that is conceptual more than proof-theoretic. It helps fixing the reference of the basic notions of the system, even though it is a reference up to isomorphism. Still, the concerns raised by Frege in his letters would not be satisfied since the references of points and lines are not specific mathematical objects univocally individuated by the axioms, but Frege's problem would find a solution, because what is fixed are the concepts of point and lines, up to isomorphism. We may find here the germ of a structuralist attitude in Hilbert's work, centered on the notion of isomorphism (and so also of cardinality). In fact, one of the main effect of the Axiom of Completeness is ruling out the countable models of analytic geometry. We could be tempted to rename Frege's problem as Skolem's problem, but this would be misleading because of the absence of a well codified model theory at that time. And indeed this is one of the standard interpretation of the Axiom of Completeness: an attempt to impose a categoricity result from inside the theory. On the contrary our proposal rests on on the idea that the aim of the axioms is to determine abstract concepts, that may be interpreted in many different ways, but that are sufficiently determined to be able, for example, to tell how many objects fall under that concept. Moreover, the remarkable importance of the notion of cardinality, in connection to the axiomatic presentation of a theory, fits perfectly with the relevance given to the question about the cardinality of the continuum[55] *in the list of mathematical problems proposed by Hilbert at the 1900 Paris conference.*

Of course we are not claiming that the answer to Frege's problem was the only reason that motivated the introduction of the Axiom of Completeness, but we believe that its role is not only mathematical but that it has deep roots in Hilbert's philosophy of logic.

There are indeed two distinct levels where the Axiom of Completeness acts: the first mathematical and the second metamathematical[56]*. On the mathematical side the axiom builds a bridge between the real line and the field of real numbers*[57]*. On the other hand at the metamathematical level we believe that*

[55] Recall that the axiomatization of the fields of the real numbers was introduced in the same period. Moreover, a similar axiom of completeness was given also for the axiomatization of the continuum.

[56] These two notions of completeness are well explained in Majer [2006]. We believe that also at the metamathematical level we find ontological considerations

[57] This problem was not new, since already Cantor in 1872 (but also Dedkind), felt the need for such an axiom "In order to complete the connection [...] with the geometry of the straight line, one must only add an axiom which simply says that

the axiom of completeness allows to define geometrical objects up to isomorphism, acting as a necessary conditions (with respect to the other axioms) fixing univocally the class of geometrical objects allowed in the system. Moreover, in both cases we can find a prescriptive character of this axiom.

With respect to the possibility to fix univocally the class of geometrical objects we can find Hilbert's solution to Frege's problem. Indeed the axiom of completeness acts, as it has been correctly proposed by Ferreiros in Ferreirós [2009], as a set-theoretical maximality principle similar, but opposite to the principle that Dedekind used to characterize the natural numbers. Here we can see in action the prescriptive character of the Axiom of Completeness at a meta-theoretical level. The geometrical objects are completely - and implicitly - defined in the following sense: "a concept (a set) is defined and immediately usable if only it is determined for every object whether the object is subsumed under the concept or not." (in [Hilbert, 1967a] p. 130). Hence the geometrical objects are all and only *those whose existence is granted by the axioms of the* Grundlagen der Geometrie*. Interestingly enough the reference to such a* Principle of Completeness *is explicit also in Frege: "A definition of a concept (a possible predicate) must be complete; it has to determine unambiguously for every object whether it falls under the concept or not (whether the predicate can be applied to it truly). [...] Figuratively, we can also express it like this: a concept must have sharp boundaries." (section 56 in Frege [1893]). However, what differs between the two thinkers is the underlying notion of mathematical object: for Hilbert something characterized up to isomorphism toward which the meaning of mathematical propositions points, while for Frege an entity existing in a strong platonic sense, that is thus susceptible of identification by means of a concrete reference.*

Holding this conceptual interpretation of the notion of completeness we may affirm that the Axiom of Completeness, although not enough for Frege's notion of reference, answers a problem of individuation (i.e. what we called Frege's problem) up to isomorphism fixing the cardinality of the collection of objects falling under a concept[58].

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conversely every numerical quantity also has a determined point on the straight line, whose coordinate is equal to that quantity [...] I call this proposition an *axiom* because by its nature it cannot be universally proved. A certain objectivity is then subsequently gained thereby for the quantities although they are quite independent of this". In [Cantor, 1872], p. 128.

[58]The effect of the Axiom of Completeness is actually much stronger, since it gives

Of course it is impossible to say whether there is a link between the Frege-Hilbert correspondence and the introduction of the axiom of completeness. We just underly that conceptually this axioms allows Hilbert to answer Frege's Problem, while chronologically it appears exactly after the Frege-Hilbert correspondence. Indeed the Festschrift *was presented on June 17th 1899, then in October 1899 there is the first appearance of a completeness principle for the real line in* Über den Zahlbegriff. *Between December 1899 and January 1900 takes place the Frege-Hilbert correspondence and finally in May 1900 appears the French translation of the* Grundlagen der Geometrie *where the Axiom of Completeness appears for the first time.*

Finally notice that both meanings of the Axioms of Completeness link intuition and formalization: the meta-theoretical meaning links the intuition about geometrical objects to the axiomatization of geometry, while the mathematical meaning links the intuition of the real line with the the axiomatization of the real numbers.

2.2. Second period

In order to stress and explain the difference between the first and the second period, and the corresponding two different kinds of intuition, there is a first question that needs to be answered: why Hilbert's solution of the first period does not have an analog - or an extension - in the second one? The easy answer is that, once it is the whole of mathematics that needs a foundation - and not a single theory - it is not possible to find a set of axioms that act as necessary and sufficient conditions, because mathematics is incompletable, due to Gödel's results. Although correct, this answer is not acceptable before 1931. However, keeping in mind that in this second period it is mathematics that needs a foundation, it is clear that if axioms are meant to describe a well fixed reality (or practice) Hilbert should have solved the philosophical problem of defining mathematics. We will see that Hilbert avoids this task rephrasing this question from a theoretical point of view to a practical one: "in what consists doing mathematics?". It is here that we will find the major novelty in Hilbert's second period of foundational work.

This concern is not explicitly expressed in Hilbert's writings, however we can outline two theoretical reasons that explain why the conception of axioms outlined in the first period is at least problematic in the context of a foundation

categoricity to analytic geometry, hence fixing also the set of true propositions.

of mathematics.

The first one is related to the importance that the use of ideal elements has in mathematics, for Hilbert.

> *We come upon quite another, wholly different interpretation, or fundamental characterization, of the notion of infinity when we consider the method - so extremely important and fertile - of* ideal elements. *The method of ideal element has an application already in the elementary geometry of the plane.*[59].

Indeed the notion of ideal elements is connected to the mathematical meaning of the Axiom of Completeness; indeed it has the effect of introducing all irrational numbers that are considered ideal elements with respect to a countable model of the axioms of geometry. However, granting the freedom of the development of mathematics and the usefulness of the possibility to introduce ideal elements, how it is possible to formulate a completeness axiom for the entire mathematics? This question would have been an insurmountable obstacle in any attempt to generalize the method that Hilbert used in the first period of his foundational reflection.

Secondly there was the problem of the formalization of logic. As a matter of fact, in the Twenties, Hilbert thought that Russell's formalization of logic was not adequate for the proposed foundation. Hence a careful formalization of the logical tools was needed prior to their use.

New ideas were needed. Indeed, in the Twenties, when engaged in foundational works for the second time, Hilbert's new conception of axiom mirrors a deeper enquiry about the concept of intuition, in the direction of a Kantian-style notion. Thanks to that Hilbert thought to have solved the problem of a safe foundation for mathematics.

We start with two quotations which sound very Kantian.

> *Instead, as a precondition for the applications of logical inferences and for the activation of logical operations, something must already be given in representation [*in der Vorstellung*]: certain extra-logical discrete objects, which exist intuitively as immediate experience before thought. [...] Because I take this stand point, the objects of number theory are for me [...] the sign themselves, whose shape can be generally and certainly recognized by us – independently*

[59] [Hilbert, 1967b], p. 372 in [van Heijenoort, 1967].

> *of space and time, of space and time, of the special conditions of the production of the sign, and of insignificant differences in the finished product*[60].
>
> *Kant taught [...] that mathematics treats a subject matter which is given independently of logic. Mathematics therefore can never be grounded solely on logic. [...] As a further precondition for using logical deduction and carrying out logical operations, something must be give in conception, viz. certain extralogical concrete objects which are intuited as directly experienced prior to all thinking*[61].

In these quotations Hilbert's Kantism is clearly outlined. Like Kant before, Hilbert now tries to give a foundation to the certainty of mathematical truths, not by means of logic, but reflecting on the very possibility of any mathematical knowledge. For Kant the a priori *conditions for geometrical and arithmetical knowledge were the pure spatial and temporal intuitions. Hilbert, going further in the same direction, gives a foundation to the certainty of mathematical truths by means of a sensible pure intuition of mathematical signs. He thinks that the intuition of mathematical symbols - sensible intuition, since symbols are written on a physical support, and pure, since it does not depend on the particular shape of the signs - is necessary to produce knowledge within a formal framework possible. So, since every piece of mathematics is formalizable, symbols are preconditions of any form of mathematical knowledge.*

Hilbert's purposes are clearly Kantian. It remains to see how much Hilbert's intuition is also Kantian. As a matter of fact, the affinity of the two thinkers may be for Hilbert functional to philosophers' approval. Indeed at that time the forms of neo-Kantism were quite spread and often quite far from Kant's original ideas. Indeed we can describe Hilbert's work as a critical deduction of mathematical knowledge*: one of the tasks of the* Neue Fries'sche Schule[62] *founded by the neo-Kantian philosopher Leonard Nelson, who was a colleague of Hilbert in Göttingen, during the Twenties.*

The work Naturerkennen und Logik *(1930) is the opportunity for a deep reflection on the philosophical meaning of this new conception. First of all Hilbert explicitly says that the older conception of the axiomatic method, offered at the beginning of the century, is not sufficient:*

[60] [Hilbert, 2005b], p. 1121 in [Ewald, 2005].
[61] [Hilbert, 1967b], p. 376 in [van Heijenoort, 1967].
[62] See [Peckhaus, 1990] on this subject.

> *How do matters stand with this axiomatics, which is today on everybody's lips? Now, the basic idea rests on the fact that generally even in comprehensive fields of knowledge a few propositions – called axioms – suffice for the purely logical construction of the entire edifice of the theory. But their significance is not fully explained by this remark*[63].

It follows an analysis of the sources of human knowledge. Hilbert claims that there are not just intellect and sensation - in Kantian terms - but there is a third way: "besides logic and experience we have a certain a priori *knowledge of reality"*[64]*. The latter is possible, for Hilbert, thanks to the intuition of mathematical symbols - of "their properties, differences, sequences and contiguities*[65]*" - that formalize this knowledge. This intuition is pure and sensible, and deeply linked to the finitary method.*

William Tait, in [Tait, 2010], showed with clear arguments that Kant's intuition is intuition of*, since it is active in the process of perception. As far as this kind of intuition is concerned, also Hilbert's conception of intuition, in the second period, is sensible. Nevertheless there is an important difference here between the two thinkers for what concerns the aspects of evidence linked to this mode of intuition. Indeed for Hilbert this intuition has an evident character as far as it is a kind of knowledge; the one that is able to ground mathematical reasoning: "also [...] mathematical knowledge in the end rests on a kind of intuitive insight [*anshaulicher Einsicht*]"*[66]*. On the contrary intuition, for Kant, is not a kind of knowledge, because in the intuitive process we cannot find the concepts under which the objects of intuition fall.*

This conception of intuition and the way we handle mathematical symbols determine the foundational axioms to be assumed for proof theory[67]*. Indeed if we want to avoid an infinite regression, we must justify these axioms extra-*

[63][Hilbert, 2005d], p. 1158 in [Ewald, 2005].

[64][Hilbert, 2005d], p. 1161 in [Ewald, 2005]

[65][Hilbert, 1967b], p 376 in [van Heijenoort, 1967].

[66][Hilbert, 2005d], p. 1161 in [Ewald, 2005].

[67]Recall Hilbert's quotation: "This program already affects the choice of axioms for our proof theory" ([Hilbert, 2005c], p. 1138 in [Ewald, 2005]). For an interesting study on how intuition is also used by Hilbert to justify the correctness of material deduction, i.e. the manipulation of symbols that takes place in intuitive arithmetic, see [Legris, 2005]. However there is no analysis of this use of intuition in Hilbert's work, nor textual evidence. We agree with Legris, but we attain to Hilbert's work.

mathematically. Then Hilbert's idea is to appeal to the similarity between formalization and the way we are used to think mathematically. On this ground manipulation and calculation become two sides of the same idea. Moreover, Hilbert wanted to give a foundation to all mathematics and so he maintained that the axioms of his proof theory formalized the "fundamental elements of mathematical discourse[68]", that are for Hilbert pre-conditions of any knowledge within a formalized discourse.

> *The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds[69].*

Hilbert then goes a step further: he claims that our intuitions of symbols have not only an a priori *character, but they also manifest typical features of evidence.*

> *The subject matter of mathematics is [...] the concrete symbols themselves whose structure is immediately clear recognizable[70].*

This is the main difference between the first and the second period, and also the main difference between Kant's and Hilbert's intuition. In the first period

[68] [Hilbert, 2005c].

[69] [Hilbert, 1967c], p. 475 in [van Heijenoort, 1967]. Hilbert's interest for an application of Kant's transcendental method to mathematics and the possibility to give an extra-logical foundation to mathematics can be dated back to the first period of his foundational work. In 1905 Hilbert defines the following principle and he calls it "axiom of reasoning" or "philosopher's *a priori*": "I have the ability to think things, and to designate them by simple signs $(a, b, \dots X, Y, \dots)$ in such a completely characteristic way that I an always recognize them again without doubt. My thinking operates with these designated things in certain ways, according to certain laws, and I am able to recognize these laws through self-observation, and to describe them perfectly" (in [Hilbert, 2015a], p. 219). In this early period Hilbert is interested in what we could call the transcendental aspect of our reasoning about mathematics, namely the *deepening the foundations* of the axiomatic method. On the contrary in the second period Hilbert's aim is to show how the axioms for his proof theory make a "protocol of the rules according to which our thinking actually proceeds"; that, in Kantian terms, amount in a deduction of the axioms from *a priori* principles. See [Peckhaus, 2003] for an interesting discussion of the philosophical background related to the search for an *a priori* foundation of mathematics among philosophers and mathematicians in Göttingen in the late Twenties.

[70] [Hilbert, 1967b], p. 376 in [van Heijenoort, 1967].

intuition and evidence are kept apart, in the second one they coincide, thanks to a Kantian-style - for Hilbert - notion of intuition, that is like Leibniz's intuitive knowledge, as described in [Leibniz, 1989]: clear, distinct and adequate.

Hilbert thinks that this kind of knowledge makes basic arithmetic safe and makes possible to extend knowledge to the transfinite domain, assuming consistency of the formal system that incorporates mathematics. Logical tools are just, as Kant said, harness of reason. Hence, in this second period symbolic logic is a tool used for a complete deployment and correct use of intuition. Moreover intuition, since it coincides with evidence, is able to give knowledge from and certainty to finitary arithmetic and logic. There is though an objection that easily arises: if Hilbert is right and he managed to give an extra-mathematical foundation of mathematical knowledge, thanks to this Kantian-style intuition, why do we need a consistency proof for arithmetic? The solution is to be find in the reasons that motivate the invention of the proof theory. Indeed Hilbert's goal was to secure infinitary mathematics - part of which is also general arithmetic - by means of a finitary consistency proof, that is to justify the use of mathematical symbols in a meaningless context, as a source of knowledge. Then a consistency proof for arithmetic is not meant to secure contentual mathematics, for which we already have intuition, but gives a safe foundations to all mathematics, since it allows a consistent use of the same symbols that are used in contenutual mathematics in a more abstract and meaningless way. Where intuition is not available, then manipulation of symbols replaces intuitive arithmetic, once we know that this manipulation cannot generate contradictions[71].

It is important to stress that the intuition underlying Hilbert's foundational studies, at the time of the discovery of the proof theory, even being an intuition of, *does not witness an evolution towards a stronger realism in Hilbert's thought. Indeed the intuition* of *in this later period is not a philosophical shelter from mathematical problems. It is not intuitions of the numbers, whatever they are. It is the intuition that witnesses the accordance between the formalization of arithmetical-logical concepts, by means of symbols, and the concepts themselves*[72]*. In other words Hilbert's foundational effort is not ontological, but*

[71] Recall also that Hilbert's objected to Husserl about the possibility to give a complete axiomatization of arithmetic in arithmetical terms. As a matter of fact Hilbert acknowledges that there is a gap between contentual and formal mathematics already in the field of number theory.

[72] This is the reason why Nelson's critic does not effect Hilbert's proposal. What is here at stake is not a "metaphysics of chalk", as Nelson maintained in [Nelson,

epistemological and transcendental in character. The match between formalization and intuition is found, in this second period, at the level of contentual mathematics, and then extended to abstract mathematics by means of a consistency proof.

To sum up, at the base of two different concepts of axiom there are two different conceptions of our intuitive relation with formal mathematics. Initially axioms define basic concepts of specific theories. So the content of symbols depended on the axioms not only because of their character of implicit definitions, but also because axioms determine the use of symbols, and so their meaning in mathematical practice. Then axioms have both a definitional and operational aspects and their choice depends on a contextual intuition that is used to isolate the basic principles of a theory. It is intuition that *(hence, not Kantian) lacking an evident character meant to describe an independent mathematical reality.*

Later, at the time of Hilbert's program, evidence and intuition are identified and this coincidence is made apparent in the perception of mathematical symbols. The finitary point of view, together with Hilbert's proof theory, is based on this intuition. Intuition of mathematical symbols determines the a priori *principles of mathematical reasoning, in its formalized framework, - through self-observation - and hence the choice of the logical-arithmetical axioms. In this second period the manipulation of the signs mirrors our combinatorial abilities and intuition allows to tie together the subjective and the objective sides of mathematical knowledge. The intuition described in this period is then intuition* of *and thanks to a consistency proof mathematical knowledge can be extended from the finitary to the transfinite domain.*

Acknowledgments

I would like to thanks few people that with their works and discussions helped me in understanding Hilbert's thought and consequently the elaboration of this article, that is a product of a long process of stratification of ideas. I am then grateful to Abel Lassalle Casanave, Davide Crippa, Vincenzo de Risi, José Ferreirós, Gabriele Lolli, Alberto Naibo and Marco Panza. Finally I would like

1928], but a more general accordance between our rules of thought and the way we formalize the basic arithmetical and logical concepts. On this epistemological aspects of Hilbert's critical deduction of mathematical knowledge, see [Peckhaus, 2003].

to thank two anonymous referees for their useful comments that improved, also conceptually, the final version of this work.

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