A COURSE IN BIMODAL PROVABILITY LOGIC

by

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1 Introduction

The aim of the present paper is twofold: first I am somewhat dissatisfied with current treatments of Bimodal Provability Logic: the models employed there are singled out by certain syntactical conditions, moreover they validate the logics under consideration only locally. In this paper I give a decent model- & frame-theory for these logics.

Secondly I study the modal logic of subsystems of Peano Arithmetic whose axiom sets are bounded by non-standard numbers (to be specific: non-standard numbers specifiable as the smallest number satifying some Δ_0-formula). These systems play a role in certain arguments concerning Relative Interpretability. Moreover the Arithmetical Completeness Theorem for these systems can be applied to characterize which formulas of the language of ordinary, 'unimodal', provability logic are Σ_1 (modulo provable equivalence in Peano Arithmetic) under all arithmetical interpretations (where the $\Box$ is interpreted as provability in Peano Arithmetic).

Why go bimodal? Why study the logic of the provability predicate of a system in combination with the provability predicate of a familiar system like Peano Arithmetic as opposed to simply studying the logic of the new provability predicate alone? One possible answer is: because there is a 'coupling' effect between the 'new' predicate and the familiar one. The familiar predicate functions as an auxiliary to prove and express facts about the 'new' predicate. For an elaboration of this theme, see: Smoryński[1985], chapter 4. This answer will not do however for the systems studied in this paper, for in each case there is provably a complete decoupling between the predicates considered. In fact the logic of the 'new' predicate, taken alone, is in each case Löb's Logic L. Our answer should rather be (i) that only in combination with the familiar predicate do the specific properties of the new one become visible at all (or perhaps: that we are simply interested in the interaction between the two predicates) and (ii) that in some cases results about the bimodal system can be applied to the traditional 'unimodal' system (see section 11 of this paper).

2 Prerequisites

Knowledge of Smoryński[1985] should be amply sufficient.

3 Acknowledgements

I thank Dick de Jongh, Franco Montagna, Craig Smoryński & Frank Veltman for

stimulating discussions.

4 Contents of the paper

In section 5 some notations & simple or known facts are introduced. Section 6 describes the modal systems studied in this paper. Section 7 gives the arithmetical interpretations that motivate the study of these systems. Section 8 is an extensive treatment of the Kripke model semantics of the systems under consideration. Section 9 studies the closed fragment of one of these systems. In section 10 I prove Arithmetical Completeness of the systems and section 11 contains an application of these completeness results: I give a characterization of the formulas of the usual modal language for Provability Logic that are Σ_1 (modulo provable equivalence in Peano Arithmetic) under all arithmetical interpretations (where the $\Box$ is interpreted as provability in Peano Arithmetic). Section 11 briefly dwells on the connection between the work in this paper and Relative Interpretability.

5 Conventions, notions & elementary facts

In this paper we restrict our attention (mainly) to RE theories T extending PA in the language of PA. This restriction is not at all essential: many results go through for RE theories T into which PRA can be interpreted. For certain results we use that the theories considered are essentially reflexive. These results evidently cannot be claimed for e.g. PRA.

5.1 Terms

We will employ 'terms' for any definable function that is provably total in PA. For our purposes we may remain neutral as to whether these 'terms' are really in the language or just function as abbreviations. It is convenient to make a terminological distinction between 'terms' for provably recursive functions and others. If we have a 'term' for a provably recursive function we will simply call it a term, otherwise we will speak of a semiterm.

5.2 Formulas

At certain points in this paper the precise form of formulas will be relevant, so we need some slightly idiosyncratic conventions.

A formula A of the language of PA is Δ_0 if all quantifiers of A are bounded (i.e. bounded by terms, where the variable of quantification does not occur in the bounding term).

A formula A is Σ if it is of the form $\exists x_1 \ldots \exists x_n A_0(x_1,\ldots,x_n)$, where $A_0 \in \Delta_0$.

A formula A is Π if it is of the form $\forall x_1 \ldots \forall x_n A_0(x_1,\ldots,x_n)$, where $A_0 \in \Delta_0$.

A formula A is Δ_1 if it is provably equivalent in PA both to a Σ- and to a Π-formula.

A formula A is Σ_1 if it is of the form $\exists x A_0(x)$, where $A_0 \in \Delta_1$. (It is essential that we have one existential quantifier here!)

Clearly the difference between Σ and Σ_1 disappears modulo provable equivalence in PA.

5.3 Provability

Let $Proof_T(x,y)$ be the Δ_1 arithmetical formula representing the relation: x is the Gödelnumber of a T-proof of the formula with Gödelnumber y. We assume that every theory comes equipped with a Δ_1-formula α_T representing the set of (non-predicate-logical) axioms. So identity of theories simply is not identity of the sets of theorems. $Proof_T$ will be built in some standard way from α_T. If we want to stress that we are looking at the Proof-relation based at a certain specific formula β we write: $Proof_\beta$.

We assume for convenience that: $PA \vdash \forall x \exists ! y\, Proof_T(x,y)$. Let $Prov_T(y) := \exists x Proof_T(x,y)$.

We write par abus de langage '$Proof_T(x, A(x_1,\ldots,x_n))$' for:
$Proof(x, \ulcorner A(\dot{x}_1,\ldots,\dot{x}_n)\urcorner)$, here:

i) all free variables of A are among those shown.
ii) $\ulcorner A(\dot{x}_1,\ldots,\dot{x}_n)\urcorner$ is the "Gödelterm" for $A(x_1,\ldots,x_n)$ as defined in Smoryński [1985], p43.

'$\Box_T A(x_1,\ldots,x_n)$' or '$\Delta_T A(x_1,\ldots,x_n)$' will stand for: $Prov_T(\ulcorner A(\dot{x}_1,\ldots,\dot{x}_n)\urcorner)$. The choice whether to use $\Box$ or Δ will depend on extra-arithmetical considerations, namely the modal system we are studying.

If t is a term (by our convention: for a provably recursive function) we will have (supposing that t is substitutable for x in A):

$$PA \vdash (\Box_T A(x))[t/x] \leftrightarrow \Box_T A(t).$$

So as long as we only consider terms we may indeed treat $x_1,\ldots,x_n$ in $\Box_T A(x_1,\ldots,x_n)$ as free variables. Occurrences of semiterms within 'modal' context should always be read with the smallest possible scope. Similarly for Δ_T.

'$\Diamond_T$' will stand for: $\neg\Box_T\neg$, and '∇_T' for: $\neg\Delta_T\neg$.

5.3 $\Box_T\restriction x$ and $\Box_T{}^*$

Suppose T is given by α.

Define:

$$\begin{aligned} \alpha\restriction x(y) &:\Leftrightarrow (\alpha(y)\wedge y<x), \\ \Box_T\restriction x\, A &:\Leftrightarrow \Box_{\alpha\restriction x} A, \\ \Diamond_T\restriction x\, A &:\Leftrightarrow \neg\Box_T\restriction x\neg A, \\ \Box_T{}^*A &:\Leftrightarrow \exists x\Box_T\restriction x\, A. \end{aligned}$$

Of course PA$\vdash$ $\Box_T A \leftrightarrow \Box_T{}^*A$, but the difference in form will be of some importance when Rosser-orderings come into play. (The usefulness of $\Box_T{}^*$ in this connection was discovered by Švejdar, see Švejdar[1983].)

5.4 Witnessing and the Rosser-ordering

Let A be of the form $\exists x A_0(x)$. Define for terms t: t wit A :$\Leftrightarrow$ $A_0(t)$. Here we assume that bound variables in A_0 are renamed -if necessary- to make t substitutable for x in A_0.

Let A be of the form $\exists x A_0(x)$ and B of the form $\exists x B_0(x)$. The Rosser-orderings between A and B are defined as follows:

$$\begin{aligned} A\leq B &:\Leftrightarrow \exists x\,(A_0(x)\wedge \forall y<x\neg B_0(y)), \\ A<B &:\Leftrightarrow \exists x\,(A_0(x)\wedge \forall y\leq x\neg B_0(y)). \end{aligned}$$

We will always apply witnessing and the Rosser-ordering to the precise forms in which the relevant arithmetical formulas are introduced.

In connection with the NB-systems we will consider formulas of the form $\Box_{PA}{}^*C<S$, where S is a Σ_1-sentence. It is easily seen that $\Box_{PA}{}^*C<S$ is itself Σ_1. (This depends crucially on the fact that S consists of just one existential quantifier followed by a Δ_1-formula.) On the other hand $S\leq\Box_{PA}{}^*C$ need not even be provably equivalent to a Σ_1-sentence. ($S\leq\Box_{PA}{}^*C$ happens to be Δ_2.)

5.5 Relative Interpretability

Let T be an RE theory (verifiably) extending PA in the language of PA. In this case T will be essentially reflexive.

'$A\lhd_T B$' stands for the arithmetization of: T+A is relatively interpretable in T+B. We write '$A\equiv_T B$' for: $A\lhd_T B$ and $B\lhd_T A$.

By a result of Orey and Hájek:

$$\text{PA}\vdash A\lhd_T B \leftrightarrow \forall x\Box_T(B\rightarrow\Diamond_T\restriction x A).$$

We list a number of principles valid for $\Box_T$ and $\lhd_T$:

I1	PA⊢ $\Box_T(B\rightarrow A) \rightarrow A\lhd_T B$
I2	PA⊢ $(A\lhd_T B\wedge B\lhd_T C) \rightarrow A\lhd_T C$
I3	PA⊢ $(A\lhd_T B\wedge A\lhd_T C) \rightarrow A\lhd_T(B\vee C)$
I4	PA⊢ $A\lhd_T B \rightarrow (\Diamond_T B\rightarrow \Diamond_T A)$
I5	PA⊢ $\Diamond_T A\lhd_T B \rightarrow \Box_T(B\rightarrow \Diamond_T A)$
I6	PA⊢ $A\lhd_T \Diamond_T A$
I7	PA⊢ $A\lhd_T B \rightarrow (A\wedge\Box_T C)\lhd_T(B\wedge\Box_T C)$

The principle I7 is due to Franco Montagna. An additional useful principle of which I4 and I6 are consequences is:

J for all P in Π_1: PA⊢ $P\lhd_T B \rightarrow \Box_T(B\rightarrow P)$

For further information see: Švejdar[1983] and Visser[1986].

6 The Modal Systems

For the record I first describe the usual Löb's logic L.

Let L_0 be the language of modal propositional logic. The truthfunctional connectives are: $\bot,\wedge,\vee,\neg,\rightarrow,\leftrightarrow$. The modal operator is $\Box$. Löb's logic L is given as the minimal set of L_0-formulas containing the following axioms and closed under the following rules:

L0	All tautologies of propositional logic
L1	$\vdash \Box(\phi\rightarrow\psi) \rightarrow (\Box\phi\rightarrow\Box\psi)$
L2	$\vdash \Box\phi \rightarrow \Box\Box\phi$
L3	$\vdash \Box(\Box\phi\rightarrow\phi) \rightarrow \Box\phi$
L4	$\vdash\phi$ and $\vdash(\phi\rightarrow\psi) \Rightarrow \vdash\psi$
L5	$\vdash\phi \Rightarrow \vdash\Box\phi$

We turn to the bimodal systems that are the subject of this paper.

Let L be the language of bimodal propositional logic. L is the result of adding the modal operator Δ to L_0. The logic CSM_0 is given as the minimal set of L-formulas containing the following axioms and closed under the following rules:

A1	All tautologies of propositional logic
A2	$\vdash \Delta(\phi\rightarrow\psi) \rightarrow (\Delta\phi\rightarrow\Delta\psi)$
A3	$\vdash \Delta(\Delta\phi\rightarrow\phi) \rightarrow\Delta\phi$
A4	$\vdash \Box(\phi\rightarrow\psi) \rightarrow (\Box\phi\rightarrow\Box\psi)$

A5 $\vdash \Delta\phi \rightarrow \Box\phi$

A6 $\vdash \Box\phi \rightarrow \Delta\Box\phi$

R1 $\vdash\phi$ and $\vdash(\phi\rightarrow\psi) \Rightarrow \vdash\psi$

R2 $\vdash\phi \Rightarrow \vdash \Delta\phi$

Some theorems of CSM_0 are :

B1 $\vdash \Delta\phi \rightarrow \Delta\Delta\phi$

B2 $\vdash \Delta\phi \rightarrow \Box\Delta\phi$

B3 $\vdash \Box(\Box\phi\rightarrow\phi) \rightarrow \Box\phi$

B4 $\vdash \Box\phi \rightarrow \Box\Box\phi$

The logic CSM_1 is given as the minimal set of *L*-formulas closed under R1, R2 and containing A1-6, plus:

A7 $\vdash \Box(\Delta\phi\rightarrow\phi)$

The logic CSM_2 is given as the minimal set of *L*-formulas closed under R1 and containing the following axioms:

A8 All theorems of CSM_1

A9 $\vdash \Delta\phi \rightarrow \phi$

The logic CSM_3 is given as the minimal set of *L*-formulas closed under R1 and containing A8 and:

A10 $\vdash \Box\phi \rightarrow \phi$

Clearly A9 is derived in CSM_3.

The logic NB_1 is given as the minimal set of *L*-formulas closed under R1, R2 and containing A1-6, plus:

A11 $\vdash (\neg\Delta\phi\wedge\Box\phi) \rightarrow \Box(\Delta\psi\rightarrow\psi)$

Finally the logic NB_2 is the minimal set of *L*-formulas closed under R1 and containing A10, plus:

A12 All theorems of NB_1

A13 $\vdash \Box\phi \rightarrow \Delta\phi$

'CSM' stands for: Carlson-Smoryn®ski-Montagna. CSM_0 is PRL_1 of Smoryński[1985] and F^- of Montagna[1984]. CSM_1 is PRL_{ZF} of Smoryński[1985] and F of Montagna[1984]. CSM_2 is PRL_{ZF}+Reflection$_\triangle$ of Smoryński[1985] (Smoryński writes 'Reflection$_\Box$', but his $\Box$ is my $\triangle$, and his $\triangle$ my $\Box$) and it is F_1 of Montagna[1984]. CSM_3 is PRL_{ZF}+Reflection$_\Box$ of Smoryński[1985]. The NB-systems are new on the scene. NB stands for: Non-standardly Bounded.

It is easily seen that each of the CSM and NB systems extends L.

7 Arithmetical Interpretations

An *interpretation function* f is a function from the propositional variables of *L* to the sentences of the language of PA. We will also consider functions from the propositional variables of *L* to *formulas* of the language of PA. In this case we will speak of *open interpretation functions.*

7.1 CS-style Interpretations for the CSM theories

Let T and U be RE extensions of PA in the language of PA. We will write '$\triangle_T$' for provability in T, '$\Box_U$' for provability in U. We assume that $U \vdash \triangle_T A \to A$, for all sentences A of the language of arithmetic. (The restriction to the language of PA is not at all essential here: all results of this section could be stated for RE T and U such that PRA can be interpreted in T (say via $(.)^+$), T can be interpreted in U (say via $(.)^*$) and $U \vdash ((\triangle_T A)^+)^* \to (A)^*$ for all sentences A in the language of T.)

7.1.1 Definition

Let f be an interpretation function. We define (.)(f,U,T) from *L* to the sentences of the language of PA as follows

- (p)(f,U,T) := fp
- (.)(f,U,T) commutes with the propositional logical constants
- $(\Box\phi)(f,U,T) := \Box_U(\phi)(f,U,T)$
- $(\triangle\phi)(f,U,T) := \triangle_T(\phi)(f,U,T)$

7.1.2 Soundness for CSM_1

$$CSM_1 \vdash \phi \;\Rightarrow\; T \vdash (\phi)(f,U,T)$$

Proof: entirely routine. □

7.1.3 Soundness for CSM_2

$$CSM_2 \vdash \phi \;\Rightarrow\; U \vdash (\phi)(f,U,T)$$

Proof: entirely routine. □

7.1.3 Soundness for CSM_3

Suppose U is true, then:

$$CSM_3 \vdash \phi \Rightarrow \mathbb{N} \models (\phi)(f,U,T)$$

Proof: entirely routine. □

7.2 M-style Interpretations for the CSM theories

Let f be an *open* interpretation function. Let v be a fixed variable. We write "$\Delta_{PA,v}A$" for: $\Box_{PA}\restriction vA$.

We set our definition of interpretation up in a way slightly different from Montagna's (see Montagna[1984]). We take it to be known that some finite subtheory of PA implies (translations of) all the arithmetical axioms of PRA. (This uses the existence of a Σ_1-truthpredicate for Σ_1-sentences.) We fix a number N such that N is bigger than all the Gödelnumbers of the arithmetical axioms of this finite subtheory. Because $PRA \vdash \forall A \forall x \Box_{PA}(\Box_{PA}\restriction xA \rightarrow A)$, we have $PA\restriction N \vdash \forall A \forall x \Box_{PA}(\Box_{PA}\restriction xA \rightarrow A)$.

7.2.1 Definition

Let t be a term in the language of PA. We define (.)(f,PA,t) from *L* to the sentences of the language of PA as follows

- $(p)(f,PA,t) := fp[t/v]$
- (.)(f,PA,t) commutes with the propositional logical constants
- $(\Box\phi)(f,PA,t) := \Box_{PA}(\phi)(f,PA,t)$
- $(\Delta\phi)(f,PA,t) := \Delta_{PA,t}(\phi)(f,PA,t)$

Note that for this definition to work it is necessary that t is substitutable for v in fp for all atoms p.

7.2.2 Soundness for CSM_1

$$CSM_1 \vdash \phi \Rightarrow PA\restriction N \vdash \forall v \geq \underline{N}(\phi)(f,PA,v)$$
$$\Rightarrow PA \vdash \forall v \geq \underline{N}(\phi)(f,PA,v)$$

Proof: One shows that $\{\phi | PA\restriction N \vdash \forall v \geq \underline{N}(\phi)(f,PA,v)\}$ contains the axioms of CSM_1 and is closed under R1,R2. As usual we just treat a sample.

First we check R2. Clearly it is sufficient to show:

$$PA\restriction N \vdash \forall v \geq \underline{N}Bv \Rightarrow PA\restriction N \vdash \forall v \geq \underline{N}\Delta_{PA,v}Bv.$$

Suppose $PA\restriction N\vdash \forall v\geq \underline{N}Bv$. It follows that $PA\restriction N\vdash \Box_{PA}\ulcorner\underline{N}\forall v\geq\underline{N}Bv$ and hence that $PA\restriction N\vdash \forall v\geq\underline{N}\Box_{PA}\ulcorner vBv$.

Next we check A7. Clearly it is sufficient to show:

$$PA\restriction N\vdash \forall v\geq\underline{N}\Box_{PA}(\Delta_{PA,v}Bv\rightarrow Bv).$$

But this last principle is an immediate consequence of the fact that the Reflexiveness of PA is verifiable in $PA\restriction N$.

The other axioms and rules are routine. □

7.2.3 Soundness for CSM_2

$$CSM_2\vdash\phi \Rightarrow \text{ for all } k\geq N\ PA\vdash(\phi)(f,PA,\underline{k})$$

Proof: The fact that $\{\phi|\text{for all } k\geq N\ PA\vdash(\phi)(f,PA,\underline{k})\}$ contains A8 is an immediate consequence of 7.2.2. . A9 is in because of the Reflexiveness of PA. Closure under R1 is trivial. □

7.2.4 Soundness for CSM_3

$$CSM_3\vdash\phi \Rightarrow \text{ for all } k\geq N\ \mathbb{N}\models(\phi)(f,PA,\underline{k})$$
$$\Leftrightarrow \mathbb{N}\models\forall v\geq\underline{N}(\phi)(f,PA,v)$$

Proof: entirely routine. □

7.3 Interpretations for the NB theories

Let S be a false Σ_1-sentence. Say $S=\exists xS_0x$, where $S_0\in\Delta_1$. Define:

$$\Delta_{PA,S}A := \Box_{PA}{}^*A<S.$$

Clearly $\Delta_{PA,S}A\in\Sigma_1$.

We note some equivalents of $\Delta_{PA,S}A$.

Suppose p is the Gödelnumber of a PA-proof π. Let lp be the supremum of the Gödelnumbers of the arithmetical axioms occurring in π. If p is not a Gödelnumber of a PA-proof, let lp be 0. We make the reasonable assumption that the Gödelnumbers of formulas occurring in a proof π are smaller then the Gödelnumber p of π. Hence for $p\neq 0$: $lp<p$. We have:

$$PA\vdash\ \Delta_{PA,S}A \leftrightarrow \exists y\ Proof_{PA}(y,A)\wedge\forall z\leq ly\neg S_0z.$$

Suppose πx is the Δ_1-formula associated with PA. Define:

$$\pi\restriction\mu S(y) := \pi y\wedge\forall z\leq y\neg S_0z.$$

We have:

$$PA\vdash \Delta_{PA,S}A \leftrightarrow \Box_{\pi\upharpoonright\mu S}A.$$

7.3.1 Definition

Let f be an interpretation function. We define (.)(f,PA,S) from *L* to the sentences of the language of PA, as follows:

- (p)(f,PA,S) := fp
- (.)(f,PA,S) commutes with the propositional logical constants
- $(\Box\phi)$(f,PA,S) := $\Box_{PA}(\phi)$(f,PA,S)
- $(\Delta\phi)$(f,PA,S) := $\Delta_{PA,S}(\phi)$(f,PA,S)

7.3.2 Soundness for NB_1

$NB_1\vdash\phi \Rightarrow$ for all false Σ_1-sentences S and for all interpretation functions f: $PA\vdash(\phi)$(f,PA,S).

Proof: One shows that $\{\phi\in L|PA\vdash(\phi)(f,PA,S)\}$ contains the axioms of NB_1 and is closed under the rules of NB_1. Most of this is routine. Closure under R2 essentially uses the falsity of S. We check A11.

Suppose S is a false Σ_1-sentence. It is clearly sufficient to show:

$$PA\vdash (\neg\Delta_{PA,S}A\wedge\Box_{PA}A) \rightarrow \Box_{PA}(\Delta_{PA,S}B\rightarrow B).$$

Reason in PA:

Suppose $\neg\Delta_{PA,S}A$ and $\Box_{PA}A$. It clearly follows that $S\leq\Box_{PA}{}^*A$, and hence that S. Let u be the smallest witness of S. Clearly $\Box_{PA}(\Delta_{PA,S}B\leftrightarrow\Box_{PA}\upharpoonright uB)$. Hence (by the essential reflexiveness of PA): $\Box_{PA}(\Delta_{PA,S}B\rightarrow B)$. $\Box$PA

Note that we cannot go from $S\leq\Box_{PA}{}^*A$ to $\Box_{PA}(S\leq\Box_{PA}{}^*A)$, $S\leq\Box_{PA}{}^*A$ not being in general provably equivalent to a Σ_1-sentence. If we could, the principle:

$$\vdash(\neg\Delta\phi\wedge\Box\phi)\rightarrow\Delta(\neg\Delta\phi\wedge\Box\phi),$$

would be valid, but it isn't: the arithmetical completeness theorem for NB_1 provides a counterexample (see §10).

7.3.3 Soundness for NB_2

$NB_2\vdash\phi \Rightarrow$ for all false Σ_1-sentences S and for all interpretation functions f: $\mathbb{N}\models(\phi)$(f,PA,S).

Proof: entirely routine . $\Box$

7.3.4 $\Delta_{PA,S}$ and $\lhd_{PA}$

One of the reasons to be interested in $\Delta_{PA,S}$ is the fact that it interacts with relative interpretability in an interesting way. Let S be a Σ_1-sentence (not necessarily false). We have:

$$PA \vdash S \to (A \lhd_{PA} B \to \Box_{PA}(B \to \nabla_{PA,S} A)),$$

and:

$$PA \vdash \neg S \to A \lhd_{PA} \nabla_{PA,S} A.$$

Proof: Reason in PA:

To show the first principle assume S and $A \lhd_{PA} B$. It follows that $\forall x \Box_{PA}(B \to \Diamond_{PA} \upharpoonright x A)$. Let u be the smallest witness of S. Clearly for any C: $\Box_{PA}(\Delta_{PA,S} C \leftrightarrow \Box_{PA} \upharpoonright u C)$, hence $\Box_{PA}(\nabla_{PA,S} A \leftrightarrow \Diamond_{PA} \upharpoonright u A)$. Conclude: $\Box_{PA}(B \to \nabla_{PA,S} A)$.

To show the second principle, assume $\neg S$. We have: $\forall x \neg S_0 x$. Hence by Σ-completeness $\forall x \Box_{PA} \neg S_0 x$. Thus: $\forall x \Box_{PA}(\Box_{PA} \upharpoonright x C \to \Delta_{PA,S} C)$ for any C. It follows that: $\forall x \Box_{PA}(\nabla_{PA,S} A \to \Diamond_{PA} \upharpoonright x A)$ and hence $A \lhd_{PA} \nabla_{PA,S} A$. $\Box$PA

We can use the above principles to produce a variant of an argument due to Per Lindström to show that sentences of the form $A \lhd_{PA} B$ are not always provably equivalent to a Σ_1-sentence. Pick by the Gödel Fixed Point Lemma a Σ_1-sentence J such that $PA \vdash J \leftrightarrow \Delta_{PA,S} \neg J$. We claim:

$$PA \vdash J \lhd_{PA} \top \leftrightarrow (S \to \Box_{PA} \bot).$$

Proof: Reason in PA:

"$\to$": Assume $J \lhd_{PA} \top$ and S. It follows that $\Box_{PA} \nabla_{PA,S} J$ and hence $\Box_{PA} \neg J$. On the other hand $J \lhd_{PA} \top$ implies by I4: $\Diamond_{PA} \top \to \Diamond_{PA} J$. In other words: $\Box_{PA} \neg J \to \Box_{PA} \bot$. Conclude $\Box_{PA} \bot$.

"$\leftarrow$": First assume $\neg S$. It follows that $J \lhd_{PA} \nabla_{PA,S} J$. Hence by I1,I2: $J \lhd_{PA} \neg J$. On the other hand (by I1) $J \lhd_{PA} J$. Hence (by I3) $J \lhd_{PA} (J \vee \neg J)$, i.e. (by I1,I2) $J \lhd_{PA} \top$. Secondly assume $\Box_{PA} \bot$. It follows immediately that: $J \lhd_{PA} \top$. Hence from $(S \to \Box_{PA} \bot)$ i.e. $(\neg S \vee \Box_{PA} \bot)$ we have $J \lhd_{PA} \top$. $\Box$PA

If we take e.g $S := \Box_{PA} \Box_{PA} \bot$, then it is easily seen that $J \lhd_{PA} \top$ is not provably equivalent to a Σ_1-sentence.

Note that if S is false (or even just if the smallest witness of S is 'big enough'), Löb's Logic will be valid for $\Delta_{PA,S}$ and hence J will be provably equivalent in PA to $\Delta_{PA,S} \bot$.

In §12 we will look at the above argument from the point of view of embedding a countermodel to $NB_1 \vdash p \lhd \top \rightarrow \Box(p \lhd \top)$ into PA.

8 Kripke Semantics

Our aim in this section is to provide a decent semantics for the CSM- and NB-systems. The usual treatment of CSM-style systems in the literature suffers from three disadvantages: in the first place the models considered are partly specified 'syntactically': there should be nodes that force such and such formulas. Consequently the usual treatment does not admit a frame-theory as opposed to a model-theory. This is surely inelegant, but, what is more, some of the practical ease of constructing models using the geometrical intuition is lost. The second disadvantage is that the models employed in the literature only locally (i.e. for certain restricted classes of formulas) satisfy the principles of the theory under consideration. Thirdly and lastly under the usual approach Solovay-style arithmetical interpretations do not provide an embedding of the diagonalizable algebras associated with the Kripke models into the diagonalizable algebra of Peano Arithmetic. All three disadvantages are absent in the present approach. One sacrifice has to be made however: our models cannot be finite anymore. I think this is only a seeming disadvantage: the models considered are not finite but -as will be explained in due course- compact, which means that their 'propositions' can be specified in a direct and simple way. Compactness guarantees that the sentences used in the arihmetical embeddings have a particularly simple form. Moreover we have the additional advantage that certain natural models like the Henkin model of the closed fragments of the theory under consideration are in the relevant class of models used in both the Kripke-model and the arithmetical completeness theorem for that theory.

The main problem we are facing is how to deal with the Reflection Principle in irreflexive Kripke-style frames. The constraint we want to put on solutions is that the nodes satisfying the Reflection Principle should be precisely the nodes satisfying some simple condition that is given in terms of structure. Our solution is to have our frames equipped with a certain topology. The nodes satisfying Reflection will be precisely the limit points in the sense of this topology.

First we introduce frames.

8.1 Definition

A *preframe* *F* is a structure <K,R,S>, where:

i) K is a non empty set.
ii) R and S are binary, irreflexive relations on K.
iii) $R \subseteq S$.
iv) R and S are transitive, and: $xSyRz \Rightarrow xRz$.

R is the accessibility relation for the $\Box$; S is the accessibility relation for Δ.

Per abus de langage we will ascribe relational properties to *F* while intending to convey that S satisfies these properties. E.g. we say "*F* is upwards wellfounded" meaning that S is upwards wellfounded.

8.2 Some notations

xWSy :⇔ xSy or x=y
xWRy :⇔ xRy or x=y
xS := {y∈K|xSy}
Sx := {y∈K|ySx}
etcetera ...

8.3 Definition

i) Consider a preframe *F*=<K,S,R>. Define a topology O_F by taking the sets of the form xWS and Sx as subbasis.
ii) *F* is a *frame* if *F* is treelike, i.e. xSz and ySz ⇒ xWSy or ySx, and the sets xR are open in O_F. The clopens of O_F are the *propositions* of the frame *F*.

At this point we give some information about the topology O_F . A.o. we characterize what it is for a frame to be compact. Compactness is important for us because the models we are going to embed into arithmetic will be compact models. The crucial consequence of compactness is that the clopens are finite unions of finite intersections of the elements of our subbasis.

8.4 Fact

Let *F* be a frame, then:
i) xWS, Sx and Rx are clopen.
ii) O_F is Hausdorff.
iii) If K is finite, then O_F is discrete.

Proof: Left as an excercise. $\Box$

8.5 Definition

Let *F*=<K,R,S> be a frame. Consider a subset X of K. x in K is a *limit of X* if for all opens Y in O_F with x∈Y there is a y in K such that y≠x and y∈Y∩X. We say that x is a *limit* if x is a limit of K.

8.6 Fact

Let F=<K,R,S> be a frame and let $X \subseteq K$. x is a limit of X iff for some u uSx and for every u,v with xSu and xSv there is a $z \in X$ such that xSz, zSu and zSv.

Proof: left as an excercise to the reader. □

8.7 Definition

Let F=<K,R,S> be a frame. u is an *antidirect successor* of x if xSu and $xS \cap WSu$ does not have a minimum.

8.8 Theorem

Let F=<K,R,S> be a frame. F is compact iff:

i) For every x there is an S-minimal y with yWSx.
ii) S is upwards wellfounded.
iii) Every S-antichain is finite.
iv) If x has an antidirect successor, then x is a limit.

Proof: Consider a frame F.

"⇒"

Suppose F is compact.

i) Consider $Y:=\{uWS|u \in K\}$. Clearly Y is an open cover of K. Consider a finite subcover Y_0. Let y be S-minimal such that $x \in yWS \in Y_0$. It is easily seen that y is S-minimal in K.
ii) Suppose S is not upwards wellfounded. There is an ascending sequence $x_1Sx_2Sx_3$... . Consider $Y:=\{Sx_n|n \in \omega\} \cup \{zWS|\text{for no } n\ zSx_n\}$. It is easily seen that Y is an open cover of K that has no finite subcover.
iii) Suppose there is an infinite S-antichain. Then by Zorn's Lemma there is a maximal infinite S-antichain, say X. Consider $Y:=\{xWS|x \in X\} \cup \{Sx|x \in X\}$. It is easily seen that Y is an open cover of K that has no finite subcover.
iv) Consider an x with antidirect successor y. Suppose x is not a limit. Clearly {x} is open. Let $Y:=\{WSx\} \cup \{uWS|\text{not } uWSx\}$. Clearly Y is an open cover of K. If Y had a finite subcover, $xS \cap WSy$ would be covered by a finite number of sets $u_0WS,\dots,u_kWS$ for u_i with xSu_iWSy. Clearly the u_i are linearly ordered by S, so there is a minimum u_j. It follows that u_jWS covers $xS \cap WSy$, and hence that u_j is the minimum of $xS \cap WSy$.

"⇐"

Suppose F satisfies (i), (ii), (iii) and (iv). By (i) the set of S-minimal elements is non empty, by (iii) it is finite. So without loss of generality we may assume that F has a bottom b.

Let Y be an open cover of K. To find a finite subcover we construct a finitely branching tree as follows: the nodes of the tree will be of the form $\langle x,O\rangle$ where $O\in Y$. Moreover if $\langle y,O'\rangle$ lies above $\langle x,O\rangle$ in the tree, then xSy. The O such that $\langle x,O\rangle$ is in the tree will form a finite open subcover.

As bottom of the tree take $\langle b,O_0\rangle$, where O_0 is some element of Y containing b. Suppose $\langle x,O\rangle$ is a node we already created. We choose its direct successors as follows. Let $X:=\{y\in xWS \mid xWS\cap Sy\subseteq O,\ y\notin O\}$. The elements of X are pairwise incomparable, hence X is finite. For each $y\in X$ pick some $O'\in Y$ such that $y\in O'$ and take $\langle y,O'\rangle$ as immediate successor of $\langle x,O\rangle$ in the tree.

Our tree is finitely branching. Moreover if $\langle y,O'\rangle$ lies above $\langle x,O\rangle$ we have xSy, hence by the upwards wellfoundedness of S the tree has no infinite paths. Conclude by König's Lemma that the tree is finite.

Let $Y_0:=\{O \mid \langle x,O\rangle \text{ is in the tree}\}$. We claim: $\bigcup Y_0=K$. Suppose z is not in $\bigcup Y_0$. Let $Z:=\{x \mid \text{for some } O\ \langle x,O\rangle \text{ is in the tree}\}$. Clearly for some x (e.g. b) xSz and $x\in Z$. Pick x maximal such that xSz and $x\in Z$. Consider the node $\langle x,O\rangle$ in the tree. Let y be maximal such that xWSySz and $xWS\cap WSy\subseteq O$. Suppose y is a limit. Clearly for some $x_1,\ldots,x_k$:

$$y\in yWS\cap Sx_1\cap\ldots\cap Sx_k\cap Sz\subseteq O,$$

hence there is an y' with $y'\neq y$ and $y'\in yWS\cap Sx_1\cap\ldots\cap Sx_k\cap Sz\subseteq O$. It follows that $yWS\cap WSy'\subseteq O$ and thus that $xWS\cap WSy'=(xWS\cap WSy)\cup(yWS\cap WSy')\subseteq O$. Moreover xWSy'Sz. But ySy', contradicting the maximality of y. Conclude that y is not a limit. By (iv) $yS\cap WSz$ has a minimum, say, u. If u were in O, then $u\neq z$, hence xWSuSz and $xWS\cap WSu=(xWS\cap WSy)\cup\{u\}\subseteq O$, contradicting the maximality of y. So $u\notin O$. Clearly $xWS\cap Su=xWS\cap WSy\subseteq O$. By the construction of the tree u will be in Z and thus $u\neq z$ and uSz, contradicting the maximality of x.

Conclude that Y_0 is a finite open cover. □

We turn to models.

8.9 Definition

Consider a preframe $F=\langle K,R,S\rangle$.

i) We define the following operations on $P(K)$:

$$\begin{aligned}
\bot &:= \emptyset\\
\neg X &:= X^c\\
X\wedge Y &:= X\cap Y\\
X\vee Y &:= X\cup Y\\
X\rightarrow Y &:= \neg X\vee Y\\
X\leftrightarrow Y &:= (X\rightarrow Y)\wedge(Y\rightarrow X)
\end{aligned}$$

$\triangle X \quad := \{x \in K | xS \subseteq X\}$
$\Box X \quad := \{x \in K | xR \subseteq X\}$

ii) A *preassignment* f on *F* is a function from the propositional atoms p_0, p_1, p_2, ... to the subsets of K. We define the interpretation ⟦.⟧ from our bimodal language and preassignments to subsets of K as follows:
- $[\![p_i]\!]f := fp_i$
- ⟦.⟧f "commutes" with the logical constants, i.e. $[\![\phi \wedge \psi]\!]f = [\![\phi]\!]f \wedge [\![\psi]\!]f$, etc.

If f is an preassignment on a preframe *F*, we say that *G*:=<*F*,f> is a *premodel*. Define: $x \Vdash \phi(f) :\Leftrightarrow x \in [\![\phi]\!]f$.

iii) Suppose *F* is a frame. A preassignment f on *F* is an *assignment* on *F* if fp_i is clopen for all i.

iv) A premodel *G*=<*F*,f> is a *model* if *F* is a frame and f is an assignment.

v) On frames we define:

$x \Vdash \phi \quad :\Leftrightarrow$ for all assignments f on *F*: $x \Vdash \phi(f)$
$F \Vdash \phi \quad :\Leftrightarrow$ for all x in K: $x \Vdash \phi$

8.10 Fact

Let *F* be a frame. The propositions are closed under $\bot$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\triangle$ and $\Box$. And thus: $[\![\phi]\!]f$ is clopen for any assignment f on *F*.

Proof: We treat the cases of $\triangle$ and $\Box$. Let X be any subset of X. We show that $\triangle X$ and $\Box X$ are clopen.

Suppose $x \in \triangle X$. Consider y with xSy. If ySz, then xSz and hence $z \in X$. It follows that $y \in \triangle X$. Ergo $x \in xWS \subseteq \triangle X$.

Suppose $x \notin \triangle X$. For some y xSy and $y \notin X$. Clearly $x \in Sy$. Moreover if zSy, then $z \notin \triangle X$. Ergo $x \in Sy \subseteq (\triangle X)^c$.

Suppose $x \in \Box X$. Consider y with xSy. If yRz then xSyRz, hence xRz and thus $z \in X$. It follows that $y \in \Box X$. Ergo $x \in xWS \subseteq \Box X$.

Suppose $x \notin \Box X$. For some y xRy and $y \notin X$. Clearly $x \in Ry$. Moreover if zRy, then $z \notin \Box X$. Ergo $x \in Ry \subseteq (\Box X)^c$. □

8.11 Theorem

Let *F* be a frame. $F \Vdash \triangle(\triangle p \rightarrow p) \rightarrow \triangle p$ iff S is upwards wellfounded.

Proof:

"⇐"

Entirely routine.

"⇒"

Let $X := \{x \mid xWS$ is upwards wellfounded w.r.t. $S\}$. It is easily seen that X is clopen. Set $fp := X$. Clearly every x in K forces $\Delta(\Delta p \to p)$ under f but any x *not* in the upwards wellfounded part would not force Δp under f. □

8.12 Theorem

Let F be a frame, $x \in K$. We have:

i) $x \Vdash \Delta p \to p \Leftrightarrow$ x is a limit point

ii) $x \Vdash \Box p \to p \Leftrightarrow$ x is a limit point of xR

Proof:

i)

"⇒"

Suppose $x \Vdash \Delta p \to p$. If x were not a limit point, $\{x\}$ would be open and hence clopen (because our topology is Hausdorff). Thus $\{x\}^c$ would be clopen. Clearly $x \in \Delta\{x\}^c$ and $x \notin \{x\}^c$. To arrive at a contradiction set $fp := \{x\}^c$.

"⇐"

Suppose x is a limit point, X is clopen and $x \in \Delta X$. For a reductio assume $x \notin X$. We have $x \in xWS \cap X^c$ and $xWS \cap X^c$ is open. x is a limit point, so there is a y with xSy and $y \notin X$. Ergo $x \notin \Delta X$, contradiction.

ii)

"⇒"

Suppose $x \Vdash \Box p \to p$, suppose for a reductio that x is not a limit point of xR. Then there is an open X such that $x \in X$ and $X \cap xR = \emptyset$. Clearly for certain $y_1, \ldots, y_k$: $x \in Z := xWS \cap Sy_1 \cap \ldots \cap Sy_k \subseteq X$. Z is clopen by 8.4(i). If xRx', then $x' \notin Z$. Hence $x \in \Box Z^c$, but $x \notin Z^c$. To arrive at a contradiction put $fp := Z^c$.

"⇐"

Suppose x is a limit point of xR, X is clopen and $x \in \Box X$. Suppose for a reductio $x \notin X$. We have: $x \in X^c$ and X^c is open. x is a limit point of xR so there is a y with xRy and $y \notin X$. Contradiction. □

We present the frames needed for our treatment of the various logics. It is pleasant to present these frames in the kind of format discovered by Timothy Carlson.

8.13 Definition

i) A *set-preframe* F is a structure <K,K_0,K_1,S>, here:
- K is a non empty set.
- $K_0 \subseteq K_1 \subseteq K$
- S is transitive and treelike.
- $x \in K_1$ and $xSy \Rightarrow y \in K_1$.

Define $xRy :\Leftrightarrow xSy$ and ($x \notin K_1$ or $y \in K_0$).

It is easily verified that <K,R,S> is a preframe.

If $K=K_1$ we speak of a *Carlson-preframe*. We will write <K,K_0,S> for <K,K_0,K,S>.

ii) A set-preframe F is a *set-frame* if K_1 is closed (and hence clopen) in the topology generated by the xWS and the Sx. It is easily seen that a set-frame can be viewed as a frame, defining R as in (i).

K_1 is the Kripke model counterpart of the Σ_1-sentence S in the arithmetical interpretation of the NB systems. The fact that S is Σ_1 is reflected by the fact that K_1 is upwards closed. The fact that K_1 corresponds to an arithmetical sentence is shown by K_1's clopenness.

If $K=K_1$ we speak of a *Carlson-frame*. Carlson-frames are frames for Carlson-models as introduced in Smoryński[1985], p196. Note that Carlson-frames have the property: $xSz, yRz \Rightarrow xRz$. Conversely: every frame with this property can be presented as a Carlson-frame by taking K_0:=range R.

iii) A *Carlson-1-frame* is a Carlson-frame, in which every x in K_0 is a limit point.

iv) A *Carlson-3-frame* is a Carlson-1-frame with a bottom b which is a limit point of K_0.

v) A *set-1-frame* is a set-frame in which every element of K_0 is a limit point.

vi) A *set-2-frame* is a set-1-frame with bottom b, which is not in K_1 and which is a limit point. (The fact that $b \notin K_1$ reflects the falsity of the Σ_1-sentence S in the interpretation of the NB systems.)

Set-premodels, set-models etc. are defined in the obvious way.

8.14 Little fact

Suppose F is a Carlson-frame. Then x is a limit point of xR iff x is a limit point of K_0. Consequently: $x \Vdash \Box p \rightarrow p$ iff x is a limit point of K_0.

Proof: Left to the industrious reader. □

8.15 Soundness Theorem

i) $CSM_0 \vdash \phi \Rightarrow$ for all upwards wellfounded frames F $F \Vdash \phi$
ii) $CSM_1 \vdash \phi \Rightarrow$ for all upwards wellfounded Carlson-1-frames F $F \Vdash \phi$
iii) $CSM_2 \vdash \phi \Rightarrow$ for all upwards wellfounded Carlson-1-frames $F=\langle K,K_0,S\rangle$, for all x in K_0 $x \Vdash \phi$
iv) $CSM_3 \vdash \phi \Rightarrow$ for all upwards wellfounded Carlson-3-frames F with bottom b $b \Vdash \phi$
v) $NB_1 \vdash \phi \Rightarrow$ for all upwards wellfounded set-1-frames F $F \Vdash \phi$
vi) $NB_2 \vdash \phi \Rightarrow$ for all upwards wellfounded set-2-frames F with bottom b $b \Vdash \phi$

Proof: Mostly routine using 8.11 and 8.12. We sample two cases.

First we show that A11 is valid on upwards wellfounded set-1-frames. Suppose F is a upwards wellfounded set-1-frame and let f be an assignment on F. Suppose $x \Vdash \neg\Delta\phi \wedge \Box\phi(f)$. Clearly $xS \neq xR$ and thus $x \in K_1$. Consider y with xRy. y is in K_0 by the definition of R, hence y is a limit point. It follows that $x \Vdash \Box(\Delta\psi \rightarrow \psi)(f)$.

Secondly we show that A13 is forced at the bottom b of a set-2-frame. Suppose F is a set-2-frame with bottom b. Let f be an assignment on F. Suppose $b \Vdash \Box\phi(f)$. Suppose bSy. $b \notin K_1$, hence by the definition of R: bRy and thus $y \Vdash \phi(f)$. Conclude $b \Vdash \Delta\phi(f)$.

□

Of course we want to reverse the arrows of 8.15. To arrive at the desired completeness theorems we need two procedures to transform premodels into set-premodels, two procedures to add certain limit points to set-models and the Henkin construction for CSM_0.

8.16 Definition

We define two procedures to transform premodels into set-premodels.

Let $F=\langle K,R,S\rangle$ be a preframe and let f be a preassignment on F. $G:=\langle F,f\rangle$.

i) First we transform G into a Carlson-premodel. Define $XG:=G'=\langle F',f'\rangle$, where $F'=\langle K',K'_0,S'\rangle$, as follows:
   - $K' := \{\langle x_1,x_2,\dots,x_n\rangle | x_1Sx_2S\dots Sx_n, n=1,2,\dots\}$.
   - $K'_0 := \{\langle x_1,x_2,\dots,x_n,y\rangle \in K' | x_nRy\}$
   - $\langle x_1,x_2,\dots,x_n\rangle S' \langle y_1,y_2,\dots,y_k\rangle :\Leftrightarrow n<k$ and $x_i=y_i$ for $i=1,\dots,n$.

   The resulting structure is clearly treelike and irreflexive.

   Define $f'p_i := \{\langle x_1,x_2,\dots,x_n\rangle \in K' | x_n \in fp_i\}$. We claim: $\langle x_1,x_2,\dots,x_n\rangle \Vdash \phi(f') \Leftrightarrow x_n \Vdash \phi(f)$.

Proof: By induction on ϕ. The cases of the propositional atoms and the truthfunctional connectives are trivial.

Suppose $\phi=\Delta\psi$. Suppose moreover: $<x_1,x_2,...,x_n>\Vdash\Delta\psi(f')$ and x_nSy. Clearly $<x_1,x_2,...,x_n>S'<x_1,x_2,...,x_n,y>$, so $<x_1,x_2,...,x_n,y>\Vdash\psi(f')$. Hence by the Induction Hypothesis: $y\Vdash\psi(f)$. Conversely suppose $x_n\Vdash\Delta\psi(f)$ and $<x_1,x_2,...,x_n>S'<y_1,y_2,...,y_k>$. By the transitivity of S clearly x_nSy_k, hence $y_k\Vdash\psi(f)$ and so by the Induction Hypothesis: $<y_1,y_2,...,y_k>\Vdash\psi(f')$.

The case that $\phi=\Box\psi$ is analogous. □

ii) We transform G into a set-model.

Consider a set of formulas Γ that is closed under subformulas and such that $\Box p\in\Gamma \Rightarrow \Delta p\in\Gamma$. Define $\Lambda(G,\Gamma):=G'=<F',f'>$, where $F'=<K',K'_0,K'_1,S'>$ and:

- $K' := \{<x_1,x_2,...,x_n>|x_1Sx_2S...Sx_n,\ n=1,2,...\}$.
- $K'_1 := \{<x_1,x_2,...,x_n>\in K'|$for some $i\leq n$ and for some $\Box\psi\in\Gamma$: $x_i\Vdash\Box\psi$ and $x_i\nVdash\Delta\psi\}$.
- $K'_0 := \{<x_1,x_2,...,x_n,y>\in K'|<x_1,x_2,...,x_n>\in K'_1$ and $x_nRy\}$.
- $<x_1,x_2,...,x_n>S'<y_1,y_2,...,y_k> :\Leftrightarrow n<k$ and $x_i=y_i$ for $i=1,...,n$.

Clearly S' is transitive, treelike and irreflexive. Moreover $K'_0\subseteq K'_1\subseteq K'$ and $x\in K'_1$ and $xS'y \Rightarrow y\in K'_1$.

Define $f'p_i := \{<x_1,x_2,...,x_n>\in K'|x_n\in fp_i\}$. We claim:

for all $\phi\in\Gamma$: $<x_1,x_2,...,x_n>\Vdash\phi(f') \Leftrightarrow x_n\Vdash\phi(f)$.

Proof: By induction on ϕ in Γ. The cases of the propositional atoms and the truthfunctional connectives are trivial. The case of Δ is as in (i).

Suppose $\phi=\Box\psi$. And suppose: $<x_1,x_2,...,x_n>\Vdash\Box\psi(f')$ and x_nRy. Clearly if $<x_1,x_2,...,x_n>\notin K'_1$, then $<x_1,x_2,...,x_n>R'<x_1,x_2,...,x_n,y>$; if $<x_1,x_2,...,x_n>\in K'_1$, then $<x_1,x_2,...,x_n,y>\in K'_0$ and hence $<x_1,x_2,...,x_n>R'<x_1,x_2,...,x_n,y>$; Conclude: $<x_1,x_2,...,x_n,y>\Vdash\psi(f')$. Hence by the Induction Hypothesis: $y\Vdash\psi(f)$. Suppose $x_n\Vdash\Box\psi(f)$ and suppose $<x_1,x_2,...,x_n>R'<y_1,y_2,...,y_k>$. In case $<x_1,x_2,...,x_n>\notin K'_1$ it follows that $x_n\Vdash\Delta\psi(f)$ and hence $y_k\Vdash\psi(f)$. In case $<x_1,x_2,...,x_n>\in K'_1$ we have: x_nRy_k and thus $y_k\Vdash\psi(f)$. In both cases: $y_k\Vdash\psi(f)$ and so by the Induction Hypothesis: $<y_1,y_2,...,y_k>\Vdash\psi(f')$. □

8.17 Definition

We define a transformation Φ_S of set-frames. Φ_S has the effect of "expanding" the elements of K_0 in such a way that the downmost element of the expansion (which will be in the new K_0) is a limit point. Suppose F is a set-frame.

$\Phi_S F := F'$, where $F'=\langle K',K'_0,K'_1,S'\rangle$, with:

$K' := \{\langle x,i\rangle | (x\in K_0 \text{ and } i\in\omega) \text{ or } (x\notin K_0 \text{ and } i=0)\}$

$K'_0 := \{\langle x,0\rangle \in K' | x\in K_0\}$

$K'_1 := \{\langle x,i\rangle \in K' | x\in K_1\}$

$i \prec j :\Leftrightarrow (i=0 \text{ and } j\neq 0) \text{ or } (i\neq 0 \text{ and } j<i)$ (here $<$ is the usual ordering of ω)

$\langle x,i\rangle S' \langle y,j\rangle :\Leftrightarrow xSy \text{ or } (x=y \text{ and } i\prec j)$

We define two functions F and G respectively from K to K' and from K' to K by: $Fx:=\langle x,0\rangle$ if $x\notin K_0$, $Fx:=\langle x,1\rangle$ if $x\in K_0$, $G\langle x,i\rangle:=x$.

8.18 Fact

Under the conditions of 8.17:

i) F' is a set frame.

ii) F and G are continuous.

Proof: Ad (i): the first four conditions of the definition of set-frame are easily verified. The satisfaction of the fifth immediately follows from the continuity of G, seeing that $K'_1=G^{-1}K_1$.

Ad (ii): it is sufficient to observe: $F^{-1}\langle x,i\rangle WS'=xWS$, $F^{-1}S'\langle x,i\rangle=Sx$, $G^{-1}xWS=\langle x,0\rangle WS'$, $G^{-1}Sx=S'\langle x,0\rangle$. □

8.19 Fact

Under the conditions of 8.17:

i) $\langle x,i\rangle$ is a limit point in F' $\Leftrightarrow$ $\langle x,i\rangle\in K'_0$ or $\langle x,i\rangle=Fx$ and x is a limit point of F.

ii) $\langle x,i\rangle$ is a limit point of $\langle x,i\rangle R'$ $\Leftrightarrow$ $\langle x,i\rangle=Fx$ and x is a limit point of xR.

Proof:

i)

"⇒"

Suppose $\langle x,i\rangle$ is a limit point. If $\langle x,i\rangle\in K_0'$ we are done. So suppose $x\notin K'_0$. Clearly if $i=2,3,...$ $\langle x,i\rangle$ is not a limit. Hence i is 0 or 1. Conclude $\langle x,i\rangle=Fx$. Suppose $x\in O$. Then $\langle x,i\rangle\in G^{-1}O\cap\langle x,i\rangle WS'$. It follows that there is an $\langle y,j\rangle$ with $\langle y,j\rangle\in G^{-1}O\cap\langle x,i\rangle WS'$ and $\langle y,j\rangle\neq\langle x,i\rangle$. Clearly $\langle x,i\rangle S'\langle y,j\rangle$, hence because $\langle x,i\rangle=Fx$: $y\neq x$. Moreover $y\in O$.

"⇐"

The simple verification is left to the reader.

ii)

"⇒"

Suppose $\langle x,i\rangle$ is a limit point of $\langle x,i\rangle R'$. Suppose $\langle x,i\rangle \neq Fx$. Then $x \in K_0$ and $i \neq 1$. Hence $S'\langle x,1\rangle \cap \langle x,i\rangle R' = S'\langle x,1\rangle \cap \langle x,i\rangle S' \cap K'_0 = \emptyset$. This is impossible. So $\langle x,i\rangle = Fx$. Suppose $x \in O$. Then $\langle x,i\rangle \in G^{-1}O$. Let $\langle y,j\rangle$ be in $G^{-1}O \cap \langle x,i\rangle R'$. As is easily seen it follows that xRy. Also $y \in O$ and we are done.

"⇐"

Suppose x is a limit point of xR and $\langle x,i\rangle = Fx$. Suppose $\langle x,i\rangle \in O'$. Then for some $\langle z_1,j_1\rangle,\ldots,\langle z_n,j_n\rangle$ $\langle x,i\rangle \in \langle x,i\rangle WS' \cap S'\langle z_1,j_1\rangle \cap \ldots \cap S'\langle z_n,j_n\rangle =: O'' \subseteq O'$. Clearly $x \in F^{-1}O''$. Let y be in $F^{-1}O'' \cap xR$. In case $x \notin K_1$ we have $\langle x,i\rangle \notin K'_1$, hence $\langle x,i\rangle R'Fy$ and $Fy \in O''$. In case $x \in K_1$ we find $y \in K_0$. It follows that: $\langle x,i\rangle R'\langle y,0\rangle$. Also $\langle y,0\rangle S'Fy$ and $Fy \in O''$, hence by our choice of O'': $\langle y,0\rangle \in O''$. □

8.20 Fact

Suppose F is a compact set-frame. Then $\Phi_S F$ is compact.

Proof: To show that compactness is preserved, it is sufficient to show that each of the properties (i), (ii), (iii) and (iv) of 8.8 is preserved. Preservation of (i), (ii) and (iii) is easy. We treat (iv). Suppose that in F every element that has an antidirect successor is a limit point. Consider $\langle x,i\rangle$ in F and suppose $\langle x,i\rangle$ has an antidirect succesor $\langle y,j\rangle$. Clearly $i \notin \{2,3,\ldots\}$. Moreover if $i=0$ and $x \in K_0$ then $\langle x,i\rangle$ is a limit point and we are done. So we may assume that $\langle x,i\rangle = Fx$. By 8.19(i) we only need to show that x is a limit point, hence it is sufficient to see that y is an antidirect sucessor of x. Suppose $xS \cap WSy$ has a minimum z. Clearly $\langle z,0\rangle \in \langle x,i\rangle S' \cap WS'\langle y,j\rangle$, so there is a $\langle u,s\rangle$ with $\langle x,i\rangle S'\langle u,s\rangle$ and $\langle u,s\rangle S'\langle z,0\rangle$. Because $\langle x,i\rangle = Fx$ it follows that xSu. Moreover clearly uSz. Contradiction. □

8.21 Definition

We define an operation Ψ_S on set-models as follows:

$$\Psi_S\langle F,f\rangle := \langle \Phi_S F, G^{-1}\circ f\rangle.$$

Let's call $\Psi_S\langle F,f\rangle$: $\langle F',f'\rangle$. Obviously $\langle F',f'\rangle$ is a set-model.

8.22 Theorem

Let Γ be a set of formulas that is closed under subformulas. Let $G=\langle F,f\rangle$ be a set-model. $\langle F',f'\rangle := \Psi_S\langle F,f\rangle$. Suppose that for every $\Delta\phi$ in Γ and for every x in K_0: $x \Vdash \Delta\phi \rightarrow \phi(f)$. Then for every ψ in Γ and every $\langle y,j\rangle$ in K': $\langle y,j\rangle \Vdash \psi(f') \Leftrightarrow y \Vdash \psi(f)$.

Proof: By induction on ψ in Γ. The cases of the atoms, $\bot$, $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$ are trivial. We treat Δ and $\Box$.

Suppose ψ is $\Delta\chi$. Suppose $y \Vdash \Delta\chi(f)$ and $\langle y,j\rangle S'\langle u,s\rangle$. In case ySu we have: $u \Vdash \chi(f)$ and thus by the Induction Hypothesis: $\langle u,s\rangle \Vdash \chi(f')$. In case not ySu we have: $y \in K_0$, y=u and j<s. Because $y \in K_0$ it follows that $y \Vdash \chi(f)$. Hence by the Induction Hypothesis: $\langle y,s\rangle \Vdash \chi(f')$. Conversely assume $\langle y,j\rangle \Vdash \Delta\chi(f')$ and ySu. Clearly $\langle y,j\rangle S'\langle u,0\rangle$, hence $\langle u,0\rangle \Vdash \chi(f')$ and so by the Induction Hypothesis: $u \Vdash \chi(f)$.

Suppose ψ is $\Box\chi$. Suppose $y \Vdash \Box\chi(f)$ and $\langle y,j\rangle R'\langle u,s\rangle$. As is easily seen it follows that yRu, hence $u \Vdash \chi(f)$. By Induction Hypothesis: $\langle u,s\rangle \Vdash \chi(f')$. Conversely assume $\langle y,j\rangle \Vdash \Box\chi(f')$ and yRu. Clearly $\langle y,j\rangle R'\langle u,0\rangle$, hence $\langle u,0\rangle \Vdash \chi(f')$. By the Induction Hypothesis: $u \Vdash \chi(f)$. □

Note that we could trivially strengthen 8.22 by replacing Γ by the closure of Γ under the truthfunctional connectives.

8.23 Definition

We define an operation Φ_R on set-frames with bottom as follows:

$\Phi_R F := F$, where $F = \langle K', K'_0, K'_1, S'\rangle$, with:
$K' := \{\langle x,i\rangle \mid x \in K \text{ and } ((x=b \text{ and } i \in \omega) \text{ or } (x \neq b \text{ and } i=0))\}$
$K'_0 := \{\langle x,i\rangle \in K' \mid (b \in K_1 \text{ and } ((x=b \text{ and } i \neq 0) \text{ or } (x \neq b \text{ and } x \in K_0))) \text{ or } (b \notin K_1 \text{ and } x \in K_0)\}$
$K'_1 := \{\langle x,i\rangle \in K' \mid x \in K_1\}$
$\langle x,i\rangle S'\langle y,j\rangle :\Leftrightarrow xSy \text{ or } (x=y=b \text{ and } i<j)$

We define two functions H and J respectively from K to K' and from K' to K as follows: $Hx := \langle x,0\rangle$ if $x \neq b$, $Hb := \langle b,1\rangle$, $J\langle x,i\rangle := x$.

8.24 Fact

Under the conditions of 8.23:
i) F is a set-frame with bottom $\langle b,0\rangle$.
ii) H and J are continuous.

Proof: a trivial variation on the proof of 8.18. □

8.25 Fact

Under the conditions of 8.23:
i) $\langle x,i\rangle$ is a limit point in F $\Leftrightarrow$ $\langle x,i\rangle = \langle b,0\rangle$ or $\langle x,i\rangle = Hx$ and x is a limitpoint of F.
ii) $\langle x,i\rangle$ is a limit point of $\langle x,i\rangle R'$ $\Leftrightarrow$ $\langle x,i\rangle = \langle b,0\rangle$ or $\langle x,i\rangle = Hx$ and x is a limit point of xR.

Proof: trivial. □

8.26 Fact

Suppose F is a compact set-frame with bottom. Then $\Phi_R F$ is compact with bottom.

Proof: easy. □

8.27 Definition

We define an operation Ψ_R on set-models with bottom as follows:

$$\Psi_R\langle F,f\rangle := \langle \Phi_R F, J^{-1} \circ f\rangle.$$

Obviously $\Psi_R\langle F,f\rangle$ is a set-model with bottom.

8.28 Theorem

Let $\langle F,f\rangle$ be a set-model with bottom; let $\langle F',f'\rangle := \Psi_R\langle F,f\rangle$. Let Γ be a set of formulas that is closed under subformulas. Suppose that for every $\Box\phi$ in Γ: $b \Vdash \Box\phi \to \phi(f)$ and that for every $\Delta\phi$ in Γ: $b \Vdash \Delta\phi \to \phi$. Then for every ψ in Γ and for every $\langle y,j\rangle$ in K':

$$\langle y,j\rangle \Vdash \psi(f') \Leftrightarrow y \Vdash \psi(f).$$

Proof: Trivially for $y \neq b$: $\langle y,0\rangle \Vdash \psi(f') \Leftrightarrow y \Vdash \psi(f)$. So it is sufficient to show: $\langle b,i\rangle \Vdash \psi(f') \Leftrightarrow b \Vdash \psi(f)$. This is done by an easy induction on ψ which we leave to the reader. □

Note that we could strengthen 8.28 by replacing Γ by the closure of Γ under the truthfunctional connectives.

We turn to the Henkin construction. This construction is essentially the same as the one in Smoryński[1985] and as the one in Montagna[1984].

8.29 The Henkin Construction

Let X be a set of formulas of L. We write $X \vdash \phi$ for: there is a finite $X_0 \subseteq X$ such that $CSM_0 \vdash (\bigwedge X_0) \to \phi$.

Fix a set Γ of formulas that is closed under subformulas. A set X is *Γ-saturated* iff $X \subseteq \Gamma$, $X \nvdash \bot$ and for all ϕ and ψ in Γ: $X \vdash \phi \vee \psi \Rightarrow \phi \in X$ or $\psi \in X$. Note that if X is Γ-saturated then: $X \vdash \phi \Rightarrow \phi \in X$.

By an entirely routine argument one may show: for every $Y \subseteq \Gamma$ such that $Y \nvdash \phi$ there is a Γ-saturated X such that $Y \subseteq X$ and $X \nvdash \phi$.

We define a premodel G_Γ:=<F,f>, where F=<K,R,S>, as follows:

- K := $\{X \subseteq \Gamma | X$ is Γ-saturated$\}$ (Note that K is non-empty.)
- XRY :⇔ $(\Box\phi \in X \Rightarrow \phi, \Box\phi \in Y)$ and
  $(\Delta\phi \in X \Rightarrow \phi, \Delta\phi \in Y)$ and
  for some ψ: $(\Delta\psi \in Y$ and $\Delta\psi \notin X)$ or $(\Box\psi \in Y$ and $\Box\psi \notin X)$.
- XSY :⇔ $(\Box\phi \in X \Rightarrow \Box\phi \in Y)$ and
  $(\Delta\phi \in X \Rightarrow \phi, \Delta\phi \in Y)$ and
  for some ψ: $(\Delta\psi \in Y$ and $\Delta\psi \notin X)$ or $(\Box\psi \in Y$ and $\Box\psi \notin X)$.

Clearly R and S are irreflexive and transitive. Moreover XSYRZ ⇒ XRZ, and $R \subseteq S$.

Define $fp_i := \{X \in K | p_i \in X\}$. Clearly G_Γ is a premodel.

We have for ϕ in Γ and X in K: $X \Vdash \phi(f) \Leftrightarrow \phi \in X$.

Proof: By induction on ϕ in Γ. The cases of atoms and the truth functional connectives are trivial.

Suppose $\phi = \Box\psi$.

Suppose $\Box\psi \in X$ and XRY. Clearly $\psi \in Y$ so by the Induction Hypothesis: $Y \Vdash \psi(f)$.

Conversely suppose that $\Box\psi \notin X$. Define:

$$X_\Box := \{\chi, \Box\chi | \Box\chi \in X\} \cup \{\chi, \Delta\chi | \Delta\chi \in X\} \cup \{\Box\psi\}.$$

We claim: $X_\Box \nvdash \psi$. If it did we would have:

$$\{\chi, \Box\chi | \Box\chi \in X\} \cup \{\chi, \Delta\chi | \Delta\chi \in X\} \vdash \Box\psi \rightarrow \psi,$$

and hence: $X \vdash \Box(\Box\psi \rightarrow \psi)$. Thus it would follow that: $X \vdash \Box\psi$, quod non.

Let Y be Γ-saturated such that $X_\Box \subseteq Y$ and $Y \nvdash \psi$. It is easily seen that XRY and $\psi \notin Y$. By the Induction Hypothesis: $Y \nVdash \psi(f)$.

Suppose $\phi = \Delta\psi$.

Suppose $\Delta\psi \in X$ and XSY. Clearly $\psi \in Y$. Thus by the Induction Hypothesis: $Y \Vdash \psi(f)$.

Conversely suppose $\Delta\psi \notin X$. Define:

$$X_\Delta := \{\Box\chi | \Box\chi \in X\} \cup \{\chi, \Delta\chi | \Delta\chi \in X\} \cup \{\Delta\psi\}.$$

We claim: $X_\Delta \nvdash \psi$. If it did we would have:

$$\{\Box\chi | \Box\chi \in X\} \cup \{\chi, \Delta\chi | \Delta\chi \in X\} \vdash \Delta\psi \rightarrow \psi,$$

and hence $X \vdash \Delta(\Delta\psi \rightarrow \psi)$. Thus it would follow that $X \vdash \Delta\psi$, quod non.

Let Y be saturated such that $X_\Delta \subseteq Y$ and $Y \nvdash \psi$. It is easily seen that XSY and $\psi \notin Y$. By the Induction Hypothesis: $Y \nVdash \psi(f)$. □

8.30 Completeness for CSM_0

Suppose $CSM_0 \nvdash \phi$. Then there is a compact Carlson-frame F_a such that $F_a \nVdash \phi$.

Proof: Let Γ:=the set of subformulas of ϕ. Construct $G_a := XG_\Gamma$. Say $G_a = <F_a, f_a>$ and $F_a = <K_a, K_{a0}, S_a>$. Because Γ is finite, G_a is finite, O_{Ga} trivializes to the discrete topology and G_a is compact. Hence G_a is a model. There is a Γ-saturated set X such that $X \nvdash \phi$. By 8.29 and 8.16 we have $<X> \nVdash \phi(f_a)$. It follows that $F_a \nVdash \phi$. □

8.31 Definition

Let $F = <K,R,S>$ be a frame and let $x \in K$. Define $F[x] := <K',R',S'>$, where $K' := \{y \in K | xWSy\}$, $R' := R \upharpoonright K'$, $S' := S \upharpoonright K'$. If $G = <F,f>$ is a model, define $G[x] := <F[x], f[x]>$, where $f[x]p_i := fp_i \cap K'$. It is easily seen that $F[x]$ and $G[x]$ are again frame respectively model. Clearly for $x \in K'$: $x \Vdash \phi(f) \Leftrightarrow x \Vdash \phi(f[x])$. This definition is trivially adapted to Carlson-frames, etc. .

8.32 Completeness for CSM_1

Suppose $CSM_1 \nvdash \phi$. Then there is an compact Carlson-1-frame F such that $F \nVdash \phi$.

Proof: Suppose $CSM_1 \nvdash \phi$. Define:

$$\chi := (\bigwedge \{\Box(\Delta\psi \to \psi) | \Delta\psi \text{ is a subformula of } \phi\} \to \phi).$$

Clearly $CSM_0 \nvdash \chi$. Let $\Gamma := \{\psi | \psi \text{ is a subformula of } \chi\}$. Consider the treeification of the Henkin premodel for Γ, i.e. consider G_b with $G_b := XG_\Gamma$ (see 8.29 and 8.16(i)). G_b is finite so O_{Gb} is discrete. Thus G_b is a compact model. There is a node x of K_b such that: $x \Vdash \Box(\Delta\psi \to \psi)(f_b)$, for all subformulas $\Delta\psi$ of ϕ, and: $x \nVdash \phi(f_b)$. We may assume that x is of the form $<X>$, so that $x \notin K_{b0}$.

Consider $G_b[x]$. Say $G_a := G_b[x]$. Clearly G_a is compact. Let Γ_0 be the set of subformulas of ϕ. Trivially for all $y \in K_{a0}$ and for all $\Delta\psi$ in Γ_0 : $y \Vdash \Delta\psi \to \psi(f_a)$. We change the elements of K_{a0} into limits by transforming G_a into G with $G := \Psi_S G_a$. Let $b := <x,0>$. Clearly b is the bottom of G. Say $G = <F,f>$. By 8.19 and 8.20 G is a compact Carlson-1-model and by 8.22 and the fact that ϕ is in Γ_0: $b \nVdash \phi(f)$. □

8.33 Definition

Let $F = <K, K_0, S>$ be a Carlson-frame and let $x \in K$. Define $F<x> := <K', K'_0, S'>$, where $K' := \{y \in K | xWSy\}$, $K'_0 := (K_0 \cap K') \cup \{x\}$, $S' := S \upharpoonright K'$. If $G = <F,f>$ is a Carlson-model we define: $G<x> := <F<x>, f<x>>$, where $f<x>p_i := fp_i \cap K'$. Clearly $F<x>$ and $G<x>$ are a Carlson-frame respectively a Carlson-model. For $y \in K'$ we have: $y \Vdash \phi(f) \Leftrightarrow y \Vdash \phi(f<x>)$.

8.34 Completeness for CSM_2

Suppose $CSM_2 \nvdash f$. Then there is a compact Carlson-1-frame $F=\langle K,K_0,S\rangle$ and an $x \in K_0$ such that $x \nVdash \phi$.

Proof: Suppose $CSM_2 \nvdash \phi$. Let $\Gamma_0 := \{\psi \mid \psi$ is a subformula of $\phi\}$. Define:

$$\chi := (\bigwedge\{\Delta\psi \to \psi, \Box(\Delta\psi \to \psi) \mid \Delta\psi \in \Gamma_0\} \to \phi)$$

and $\Gamma := \{\psi \mid \psi$ is a subformula of $\chi\}$. Clearly $CSM_0 \nvdash \chi$. Let $G_b := XG_\Gamma$. G_b is a compact Carlson-model. There is an x in K_b such that: $x \Vdash \Delta\psi \to \psi(f_b)$ and $x \Vdash \Box(\Delta\psi \to \psi)(f_b)$ for all $\Delta\psi$ in Γ_0 and $x \nVdash \phi(f_b)$.

Consider $G_b\langle x\rangle$. Say $G_a := G_b\langle x\rangle$. Clearly for all $y \in K_{a0}$ and $\Delta\psi \in \Gamma_0$: $y \Vdash \Delta\psi \to \psi(f_a)$. Define $G := \Psi_S G_a$ and $b := \langle x,0\rangle$. Say $G = \langle F,f\rangle$. G is a compact Carlson-1-model, $b \in K_0$ and by 8.22 : $b \nVdash \phi(f)$. □

8.35 Completeness for CSM_3

Suppose $CSM_3 \nvdash \phi$. Then there is a Carlson-3-frame F with bottom b such that $b \nVdash \phi$.

Proof: Suppose $CSM_3 \nvdash \phi$. Let $\Gamma_0 := \{\psi \mid \psi$ is a subformula of $\phi\}$. Define:

$$\chi := \bigwedge(\{\Box\psi \to \psi \mid \Box\psi \in \Gamma_0\} \cup \{\Delta\psi \to \psi, \Box(\Delta\psi \to \psi) \mid \Delta\psi \in \Gamma_0\}) \to \phi.$$

Let $\Gamma := \{\psi \mid \psi$ is a subformula of $\chi\}$. Clearly $CSM_0 \nvdash \chi$. Consider $G_b := XG_\Gamma$. Clearly G_b is a compact Carlson-model. There is an x such that $x \Vdash \Box\psi \to \psi(f_b)$ for $\Box\psi \in \Gamma_0$, $x \Vdash \Delta\psi \to \psi(f_b)$ and $x \Vdash \Box(\Delta\psi \to \psi)(f_b)$ for $\Delta\psi \in \Gamma_0$ and $x \nVdash \phi(f_b)$.

Consider $G_b[x]$. Say $G_a := G_b[x]$. Clearly for all $y \in K_{a0}$ and all $\Delta\psi \in \Gamma_0$: $y \Vdash \Delta\psi \to \psi(f_a)$. Moreover: $x \Vdash \Delta\psi \to \psi(f_a)$ for all $\Delta\psi \in \Gamma_0$ and $x \Vdash \Box\psi \to \psi(f_a)$ for all $\Box\psi \in \Gamma_0$. Let $G := \Psi_S \Psi_R G_a$ and $b := \langle\langle x,0\rangle,0\rangle$. Say $G = \langle F,f\rangle$. By 8.19 and 8.20 we find that G is is a compact Carlson-3-model. By 8.22, 8.28 and the fact that $\phi \in \Gamma_0$ we find that $b \nVdash \phi(f)$. □

8.36 Completeness of NB_1

Suppose $NB_1 \nvdash \phi$. Then there is a compact set-1-frame F with $F \nVdash \phi$.

Proof: Suppose $NB_1 \nvdash \phi$. Let $\Gamma_0 := \{\psi \mid \psi$ is a subformula of $\phi\}$. Define: $\Delta^+ p := p \wedge \Delta p$. $\chi := ((\Delta^+ \bigwedge\{(\neg\Delta\sigma \wedge \Box\sigma) \to \Box(\Delta\psi \to \psi) \mid \Box\sigma, \Delta\psi \in \Gamma_0\}) \to \phi)$ and $\Gamma := \{\psi \mid \psi$ is a subformula of $\chi\}$. Clearly $CSM_0 \nvdash \chi$. Take $G_d := G_\Gamma$. There is an $x \in K_d$ with:

$$x \Vdash \Delta^+((\neg\Delta\sigma \wedge \Box\sigma) \to \Box(\Delta\psi \to \psi))(f_d)$$

for all $\Box\sigma$ and $\Delta\psi$ in Γ_0 and $x \nVdash \phi(f_d)$. Let $\Gamma_1 := \Gamma_0 \cup \{\Delta p \mid \Box p \in \Gamma_0\}$ and let $G_c := \Lambda(G_d, \Gamma_1)$. G_c is a compact set-model. Let $x_c := \langle x\rangle$. Clearly $x_c \nVdash \phi(f_c)$. Suppose $y_c \in K_{c0}$ and $x_c WS_c y_c$. We have: $y_c = \langle y_1, \ldots y_k\rangle$, where $y_1 = x$, $y_{k-1} R_d y_k$ and there is an i with $1 \le i < k$ and there is a $\Box\sigma \in \Gamma_0$ such that $y_i \Vdash \Box\sigma(f_d)$ and $y_i \nVdash \Delta\sigma(f_d)$. $xWSy_i$ and thus $y_i \Vdash \Box(\Delta\psi \to \psi)(f_d)$ for all

$\Delta\psi\in\Gamma_0$. Finally $y_i R_d y_k$ so $y_k \Vdash \Delta\psi\rightarrow\psi(f_d)$ for all $\Delta\psi\in\Gamma_0$. Conclude: $y_c \Vdash \Delta\psi\rightarrow\psi(f_c)$ for all $\Delta\psi\in\Gamma_0$.

Construct $G_b := G_c[x_c]$. By the above: for all $y_b \in K_{b0}$: $y_b \Vdash \Delta\psi\rightarrow\psi(f_b)$ for all $\Delta\psi\in\Gamma_0$. Let $G := \Psi_S G_b$, say $G = \langle F, f\rangle$. Clearly G is a compact set-1-model and $\langle x_c, 0\rangle \nVdash \phi(f)$.

□

8.37 Completeness of NB_2

Suppose $NB_2 \nvdash \phi$. Then there is a compact set-2-frame F with bottom b such that $b \nVdash \phi$.

Proof: Suppose $NB_2 \nvdash \phi$. Let $\Gamma_0 := \{\psi \mid \psi$ is a subformula of $\phi\}$. Define:

$$\chi := ((\bigwedge\!\!\!\bigwedge\{\Delta^+((\neg\Delta\sigma\wedge\Box\sigma)\rightarrow\Box(\Delta\psi\rightarrow\psi)), \Delta\psi\rightarrow\psi, \Box\sigma\rightarrow\sigma, \Box\sigma\rightarrow\Delta\sigma \mid \Box\sigma, \Delta\psi\in\Gamma_0\})\rightarrow\phi).$$

$\Gamma := \{\psi \mid \psi$ is a subformula of $\chi\}$. Clearly $CSM_0 \nvdash \chi$.

Take $G_c := G_\Gamma$. There is an $x\in K_c$ with $x \Vdash \Delta^+((\neg\Delta\sigma\wedge\Box\sigma)\rightarrow\Box(\Delta\psi\rightarrow\psi))(f_c)$, $x \Vdash \Delta\psi\rightarrow\psi(f_c)$, $x \Vdash \Box\sigma\rightarrow\sigma(f_c)$, $x \Vdash \Box\sigma\rightarrow\Delta\sigma(f_c)$ for all $\Box\sigma$ and $\Delta\psi$ in Γ_0 and $x \nVdash \phi(f_c)$.

Let $\Gamma_1 := \Gamma_0 \cup \{\Delta\sigma \mid \Box\sigma\in\Gamma_0\}$ and let $G_b := \Lambda(G_c, \Gamma_1)$. G_b is a compact set-model. Let $x_b := \langle x\rangle$. Clearly $x_b \nVdash \phi(f_b)$. By the same reasoning as in 8.36 we have that for $y_b \in K_{b0}$ with $x_b W S_b y_b$: $y_b \Vdash \Delta\psi\rightarrow\psi(f_b)$ for all $\Delta\psi\in\Gamma_0$. Moreover $x_b \Vdash \Box\sigma\rightarrow\Delta\sigma(f_b)$ for all $\Box\sigma\in\Gamma_1$ and hence $x \notin K_{b1}$.

Construct $G_a := G_b[x_b]$. By the above: for all $y_a \in K_{a0}$: $y_a \Vdash \Delta\psi\rightarrow\psi(f_a)$ for all $\Delta\psi\in\Gamma_0$ and $x_a \Vdash \Delta\psi\rightarrow\psi(f_a)$, $x \Vdash \Box\sigma\rightarrow\sigma(f_a)$ for all $\Delta\psi, \Box\sigma\in\Gamma_0$. Let $G := \Psi_S \Psi_R G_a$, say $G = \langle F, f\rangle$. Clearly G is a compact set-2-model and $\langle\langle x_b, 0\rangle, 0\rangle \nVdash \phi(f)$.

□

8.38 Corollary

The following facts are rather obvious using completeness for the appropriate arithmetical interpretations. They have however also purely Kripke model proofs.

i)	$CSM_1 \vdash \phi$	$\Leftrightarrow$	$CSM_1 \vdash \Delta\phi$
		$\Leftrightarrow$	$CSM_2 \vdash \Delta\phi$
		$\Leftrightarrow$	$CSM_3 \vdash \Delta\phi$
ii)	$CSM_2 \vdash \phi$	$\Leftrightarrow$	$CSM_1 \vdash \Box\phi$
		$\Leftrightarrow$	$CSM_2 \vdash \Box\phi$
		$\Leftrightarrow$	$CSM_3 \vdash \Box\phi$
iii)	$NB_1 \vdash \phi$	$\Leftrightarrow$	$NB_1 \vdash \Delta\phi$
		$\Leftrightarrow$	$NB_1 \vdash \Box\phi$
		$\Leftrightarrow$	$NB_2 \vdash \Delta\phi$
		$\Leftrightarrow$	$NB_2 \vdash \Box\phi$

iv) Let $\psi \in L_0$, then:

$$\begin{aligned} L \vdash \psi &\Leftrightarrow CSM_1 \vdash \psi \\ &\Leftrightarrow CSM_2 \vdash \psi \\ &\Leftrightarrow NB_1 \vdash \psi \end{aligned}$$

Proof: Left as an excercise to the reader. □

9 The closed fragment of CSM_1

Let *CL* be the set of closed formulas (i.e. of formulas not containing propositional variables) of *L*. We describe the behaviour of the *CL* formulas in CSL_1.

Define for ϕ in *L*: $\Box^0\phi:=\phi$, $\Box^{n+1}\phi:=\Box\Box^n\phi$ and $\Delta^0\phi:=\phi$, $\Delta^{n+1}\phi:=\Delta\Delta^n\phi$. Define further: $\bot_{\omega.m+n}:=\Delta^n\Box^m\bot$ and $\bot_\infty:=\top$. Let "α" and "β" range over $\omega^2\cup\{\infty\}$. We stipulate that for all α $\alpha\leq\infty$, that $\infty+\alpha=\alpha+\infty=\infty$ and that: $\infty.\alpha=\alpha.\infty=\infty$ for $\alpha\neq 0$.

9.1 Fact

$$\alpha\leq\beta \Leftrightarrow CSM_1\vdash\bot_\alpha\rightarrow\bot_\beta.$$

Proof:

"$\Rightarrow$"

The only interesting subcase is:

$$CSM_1\vdash\Delta^n\Box^m\bot\rightarrow\Delta^k\Box^l\bot, \text{ for } m<l,$$

We have:

$$\begin{aligned} CSM_1\vdash\ \Delta^n\Box^m\bot &\rightarrow \Box\Delta^n\Box^m\bot \\ &\rightarrow \Box^{m+1}\bot \\ &\rightarrow \Box^l\bot \\ &\rightarrow \Delta^k\Box^l\bot \end{aligned}$$

"$\Leftarrow$"

Suppose $\alpha>\beta$ we have:

$$\begin{aligned} CSM_1\vdash\ \bot_\alpha\rightarrow\bot_\beta &\Rightarrow CSM_1\vdash\bot_{\beta+1}\rightarrow\bot_\beta \\ &\Rightarrow CSM_1\vdash\bot_\beta \qquad \text{(Löb's Rule for } \Delta) \end{aligned}$$

But by an easy Kripke model argument: $CSM_1\nvdash\bot_\beta$ (β being below ∞). □

9.2 Convention

If $\Psi_1,\ldots,\Psi_n$ are forms for formulas in certain variables (e.g. $\Psi_1=\bot_\gamma\rightarrow\Box\sigma$), we write: $\bigvee[\Psi_1,\ldots,\Psi_n]$ for a finite disjunction of formulas, where any of these formulas is of one of the forms $\Psi_1,\ldots,\Psi_n$. If n=1 we drop "[" and "]". Similarly for $\bigwedge$.

9.3 Lemma

Let ϕ be a Boolean combination of $\bot_\alpha$'s, then for some β: $CSM_1\vdash\Delta\phi\leftrightarrow\bot_\beta$. Moreover $\beta=\gamma+1$ and $CSM_1\vdash(\phi\wedge\Delta\phi)\leftrightarrow\bot_\gamma$.

Proof: note that $CSM_1\vdash(\bot_\gamma\vee\bot_\delta)\leftrightarrow\bot_{max(\gamma,\delta)}$, and $CSM_1\vdash(\bot_\gamma\wedge\bot_\delta)\leftrightarrow\bot_{min(\gamma,\delta)}$.

Consider a Boolean combination of $\bot_\alpha$'s ϕ. Clearly ϕ can be brought in the form

$\bigwedge\!\!\!\bigwedge \bigvee\!\!\!\bigvee [\bot_\gamma, \neg\bot_\delta]$. In the disjunction we can contract (modulo CSM_1-provable equivalence) the $\bot_\gamma$'s to $\bot_{\gamma^*}$, where γ^* is the maximum of the γ's, and we can contract the $\bot_\delta$'s to $\bot_{\delta^*}$, where δ^* is the minimum of the δ's. Hence $CSM_1 \vdash \phi \leftrightarrow \bigwedge\!\!\!\bigwedge(\bot_{\delta^*} \rightarrow \bot_{\gamma^*})$ and so $CSM_1 \vdash \Delta\phi \leftrightarrow \Delta \bigwedge\!\!\!\bigwedge(\bot_{\delta^*} \rightarrow \bot_{\gamma^*})$ and thus $CSM_1 \vdash \Delta\phi \leftrightarrow \bigwedge\!\!\!\bigwedge \Delta(\bot_{\delta^*} \rightarrow \bot_{\gamma^*})$. In case $\delta^* \leq \gamma^*$ $CSM_1 \vdash \Delta(\bot_{\delta^*} \rightarrow \bot_{\gamma^*}) \leftrightarrow \bot_\infty$. In case $\delta^* > \gamma^*$:

$$\begin{aligned} CSM_1 \vdash\ \Delta(\bot_{\delta^*} \rightarrow \bot_{\gamma^*}) &\rightarrow \Delta(\bot_{\gamma^*+1} \rightarrow \bot_{\gamma^*}) \\ &\rightarrow \Delta\bot_{\gamma^*} \qquad \text{(Löb's Principle for } \Delta) \\ &\rightarrow \Delta(\bot_{\delta^*} \rightarrow \bot_{\gamma^*}) \end{aligned}$$

Hence $CSM_1 \vdash \Delta(\bot_{\delta^*} \rightarrow \bot_{\gamma^*}) \leftrightarrow \bot_{\gamma^*+1}$. So $CSM_1 \vdash \Delta\phi \leftrightarrow \bot_{\gamma^{**}+1}$, where γ^{**} is the minimum of the γ^*'s.

The second claim is an easy consequence of the above. □

9.4 Lemma

Let ϕ be a Boolean combination of $\bot_\alpha$'s, then for some $\beta \in \omega \cup \{\infty\}$: $CSM_1 \vdash \Box\phi \leftrightarrow \bot_{\omega.\beta}$.

Proof: the proof is essentially similar to the proof of 9.3 using $CSM_1 \vdash \Box(\Delta^k\sigma \leftrightarrow \sigma)$.

□

9.5 Theorem

Consider any $\psi \in CL$. We have: ψ is provably equivalent in CSM_1 to a Boolean combination of $\bot_\alpha$'s.

Proof: By a simple induction on ψ in CL using 9.3 and 9.4. □

Consider the following Carlson-frame: F_0:=<{0},Ø,Ø>. Consider: $\Phi_S\Phi_R F_0$. We claim that this last frame corresponds precisely with the closed fragment of CSM_1, i.e.: (i) for every proposition X of $\Phi_S\Phi_R F_0$ there is a $\phi \in CL$ such that X=$[\![\phi]\!]$f, where fp_i=Ø for all i, and (ii) for all $\phi \in CL$: $CSM_1 \vdash \phi \Leftrightarrow \Phi_S\Phi_R F_0 \Vdash \phi$ (thus: for all $\psi,\sigma \in CL$ $CSM_1 \vdash \psi \leftrightarrow \sigma \Leftrightarrow [\![\psi]\!]f = [\![\sigma]\!]f$).

Before we verify this claim it seems appropriate to replace the frame $\Phi_S\Phi_R F_0$ by an isomorphic one. Define: $\Omega^2 := \langle(\omega^2 \backslash \{0\}) \cup \{\infty\}, \{\omega.n \mid n \in \omega\}, \Sigma\rangle$, where $\alpha\Sigma\beta :\Leftrightarrow \alpha > \beta$. It is easily seen that Ω^2 is isomorphic to $\Phi_S\Phi_R F_0$. Note that $\alpha \Vdash \bot_\beta(f) \Leftrightarrow \alpha \leq \beta$. We prove our claim for Ω^2.

Proof: To see that the first part of our claim is correct it is sufficient to note that our frame is compact and that $\alpha WS = [\![\bot_\alpha]\!]f$ and $S\alpha = [\![\neg\bot_\alpha]\!]f$.

We turn to the second part. the "⇒" side is immediate, because Ω^2 is a Carlson-2-frame.

For the "⇐" side Suppose $\phi \in CL$ and $CSM_1 \nvdash \phi$. Let $X_0 := \{\psi \in CL | CSM_1 \vdash \psi\}$. Clearly $X_0 \nvdash \phi$. Let X be CL-saturated such that $X_0 \subseteq X$ and $X \nvdash \phi$. Clearly $\bot_\infty \in X$. Let O(X) be the smallest element α of $(\omega^2 \setminus \{0\}) \cup \{\infty\}$ such that $\bot_\alpha$ is in X. We have: $\bot_\beta \in X$ if $\beta \geq O(X)$ and $(\neg \bot_\beta) \in X$ if $\beta < O(X)$. By 9.5 X is uniquely determined by O(X) (among the CS-saturated extensions of X_0). Let $\alpha := O(X)$. Consider $Y := \{\psi \in CL | \alpha \Vdash \psi(f)\}$. Clearly $X_0 \subseteq Y$, Y is CS-saturated and $O(Y) = \alpha$. Conclude $X = Y, \phi \notin Y$ and thus $\alpha \nVdash \phi(f)$. □

9.6 Remarks

i) The above reasoning could be elaborated by showing that Ω^2 with the empty assignment is precisely the Henkin model of CSM_1 w.r.t. CL.

ii) As is easily seen for $\phi \in CL$:

$$CSM_2 \vdash \phi \Leftrightarrow \text{for all n } \omega.n \Vdash \phi(f),$$

$$CSM_3 \vdash \phi \Leftrightarrow \infty \Vdash \phi(f).$$

9.7 Question

Clearly NB_1 does not have such a well behaved closed fragment. This is plausible also from the arithmatical point of view: NB_1 is 'about' a number of different arithmetical interpretations of Δ at the same time. Many (all?) of these interpretations taken by themselves would yield a different provability logic and thus a different closed fragment. Is there some interesting description of the closed fragment of NB_1?

10 Arithmetical Completeness Results

Our aim in this section is to embed certain frames employed in the various modal completeness theorems into arithmetic. The precise nature of these embeddings will depend on the chosen interpretation of the bimodal logic involved. All embeddings map the propositions of the frames on equivalence classes of the relevant sort (e.g. w.r.t. provable equivalence in PA) of arithmetical sentences and 'commute' with the corresponding connectives and operators.

The neatest way to build the embeddings is in two stages. The first stage is common to all embeddings: we go from clopens to (equivalence classes of) arithmetical formulas in one variable that represent the clopens as sets of numbers in a canonical way. In the second stage we go from these formulas to sentences by something like substituting a term for the one free variable. We proceed to describe the first stage.

The frames we are going to embed all have the form $\Phi_S \Phi_R F_a$, where F_a is a *finite* set-frame. We assume that the domain K_a of F_a consists of numbers 0,...,n and that '<...>' is a standard numerical coding of sequences. Thus the elements of the domain of $F := \Phi_S \Phi_R F_a$ will be numbers and simple arithmetical descriptions of K_0, K_1, S (and hence of R) can be read of from our definitions of Φ_S and Φ_R. In arithmetical contexts we will simply confuse K, K_0, etc. with their arithmetical descriptions. All kind of simple facts about K, K_0, etc. can be formalized in PA -like: S is transitive, treelike, upwards well-founded. The facts we need will be collected along the way.

Let A be an arithmetical formula with just x free. Instead of the usual notation 'Ax' to exhibit the free occurrence of x in A, we will use '$x\in A$' to show our intention that A stands for a set.

Let A and B be arithmetical formulas with just x free. Define:

$$A\cong B :\Leftrightarrow PA\vdash \forall x(x\in A\leftrightarrow x\in B)$$

What we are going to do is map the propositions X of frames of the form $\Phi_S\Phi_R F_a$, where F_a is a *finite* set-frame with bottom, on $\cong$-equivalence classes in such a way that each element of the equivalence class assigned to X represents X as a set. (Clearly there are also inequivalent formulas that represent X as a set.) Even if it not strictly necessary the most pleasant way to give the representation is via a normal form theorem for clopens in compact frames. A normal form for a clopen X is going to be a designated finite set of clopens N_X such that $X=\bigcup N_X$.

10.1 Normal forms for clopens

Consider a compact frame $F=\langle K,R,S\rangle$. Let x,y be elements of K. We call $xWS\cap WSy$ an *interval* just in case xWSy and y is not a limit. If $xWS\cap WSy$ is an interval we use the notation: [x,y].

As is easily seen intervals are clopen.

Let X be clopen (and thus compact).

10.1.1 Prenormal forms for clopens

There is a finite collection of intervals M such that $X=\bigcup M$.

Proof: By compactness it is sufficient to produce a collection of intervals P such that $X=\bigcup P$. If $x\in X$ is not a limit put I_x:=[x,x]. Suppose x is a limit. $xWS\cap X$ is open, hence there is a y such that $y\neq x$ and $y\in xWS\cap X$. Pick a maximal such y, say y_0. y_0 cannot be a limit, otherwise there would be a z with $z\neq y$ and $z\in y_0WS\cap X$, hence y_0Sz and $z\in xWS\cap X$, contradicting the maximality of y_0. Put $I_x:=[x,y_0]$. Define $P:=\{I_x|x\in X\}$. □

10.1.2 Normal form theorem

There is a unique finite collection of intervals N_X such that:

i) $X=\bigcup N_X$.

ii) $[x,y] \in N_X$, $[u,v] \in N_X$ and $[x,y] \subseteq [u,v] \Rightarrow [x,y]=[u,v]$.

iii) $[z,w] \subseteq X \Rightarrow \exists [x,y] \in N_X\ [z,w] \subseteq [x,y]$.

Proof: It is easily verified that conditions (i),(ii),(iii) imply uniqueness. To prove existence, consider a finite set of intervals M such that $X=\bigcup M$. We convert M into N_X.

Clearly by compactness and treelikeness every x in K which is not a bottom element has an S-predecessor. We write 'y=pd(x)', for: y is the S-predecessor of x. We define a relation between pairs $<[x,y],q>$, where $q \in [x,y]$, as follows:

$$<[x,y],q>Q<[u,v],r> :\Leftrightarrow pd(r) \in [q,y].$$

We collect two simple facts about Q:

a) $<[x,y],q>Q<[u,v],r> \Rightarrow [x,v]$ is an interval and $[x,v]=[x,pd(r)] \cup [r,v] \subseteq [x,y] \cup [u,v]$.

b) $<[x,y],q>Q<[u,v],r>Q<[z,w],s> \Rightarrow <[x,v],r>Q<[z,w],s>$.

Define:

$$P:=\{[x,y] \mid \text{there is a Q-chain } <[x_1,y_1],q_1>Q\ldots Q<[x_n,y_n],q_n>, \text{ such that: } [x_i,y_i] \in M,\ x=x_1,\ y=y_n\}.$$

Clearly P is finite and by (a),(b) P is a set of intervals satisfying $\bigcup P=X$.

Let N_X be the set of $\subseteq$-maximal elements of P. It is immediate that N_X as defined satisfies (i) and (ii). We verify (iii): consider $[z,w] \subseteq X$. It is sufficient to produce $[x,y] \in P$ with $[z,w] \subseteq [x,y]$. We produce a Q-chain of pairs $<[x_i,y_i],q_i>$ with $q_i \in [z,w]$ as follows:

step 1) Pick $[x_1,y_1] \in M$ such that $z \in [x_1,y_1]$. Let q_1 be z.

step i+1) Suppose we have produced $<[x_i,y_i],q_i>$. Let p_i be the maximum of $[x_i,y_i] \cap [z,w]$. As is easily seen p_i is not a limit. If $p_i=w$ we stop and put $n:=i$. If $p_i \neq w$, let q_{i+1} be the immediate S-successor of p_i in [z,w] and pick $[x_{i+1},y_{i+1}] \in M$ such that q_{i+1} is in $[x_{i+1},y_{i+1}]$.

It is immediate that $<[x_i,y_i],q_i>Q<[x_{i+1},y_{i+1}],q_{i+1}>$. Moreover $q_i S q_{i+1}$, hence our procedure stops by the upwards wellfoundedness of S. Finally define $[x,y]:=[x_1,y_n]$. Clearly $x_1 WSzWSwWSy_n$, so $[z,w] \subseteq [x,y]$. □

Consider a frame *F* of the form $\Phi_S\Phi_R F_a$, where F_a is a finite set-model with bottom. Let X be a proposition of *F*. We represent X by $\ulcorner X \urcorner$ in the language of arithmetic with just x free, where:

$$x \in \ulcorner X \urcorner := \bigvee \{x \in [\underline{m},\underline{n}] \mid [m,n] \in N_X\}.$$

Note that:

$$m \in X \Rightarrow PA \vdash \underline{m} \in \ulcorner X \urcorner$$
$$m \notin X \Rightarrow PA \vdash \neg \underline{m} \in \ulcorner X \urcorner$$

It follows that if $X \neq Y$, then $PA \vdash \exists y((y \in \ulcorner X \urcorner \wedge y \notin \ulcorner Y \urcorner) \vee (y \notin \ulcorner X \urcorner \wedge y \in \ulcorner Y \urcorner))$, hence *not* $\ulcorner X \urcorner \cong \ulcorner Y \urcorner$.

Define for arithmetical formulas A with just x free, $\triangle A$ and $\Box A$ by:

$$x \in \triangle A \;:=\; \forall y\; xSy \Rightarrow y \in A.$$
$$x \in \Box A \;:=\; \forall y\; xRy \Rightarrow y \in A.$$

10.2 Lemma

Let X be a proposition of *F*. Suppose M is a prenormal form for X. Then:

$$\bigvee\{x \in [m,n] \mid [m,n] \in M\} \cong \ulcorner X \urcorner.$$

Proof: If $[m,n] \in M$, then there is a $[p,q] \in N_X$ such that $[m,n] \subseteq [p,q]$. The (formalization of the) fact that $[m,n] \subseteq [p,q]$ can easily be verified in PA. Conversely for $[p,q]$ in N_X there is a Q-chain $<[m_1,n_1],r_1>Q\ldots Q<[m_k,n_k],r_k>$ in M such that $[p,q]=[m_1,n_k]$. The facts about Q-chains can be verified in PA and hence also the fact that $[p,q] \subseteq \bigcup M$.

□

10.3 First Commutation Theorem

$\ulcorner . \urcorner$ commutes with the propositional constants, $\triangle$ and $\Box$ modulo $\cong$.

Proof: The proof is long, boring and trivial. We just sketch it.

Let X,Y be propositions of *F*.

We show: $\ulcorner \bot \urcorner \cong \bot$. This one is easy: $\ulcorner \bot \urcorner = \bot$!

We show: $\ulcorner \top \urcorner \cong \top$. $N_\top = \{[b,t] \mid t$ is a top element of *F*$\}$. The formalization of the fact that $\bigcup N_\top = K$ is easily verified in PA.

We show: $\ulcorner X \vee Y \urcorner \cong \ulcorner X \urcorner \vee \ulcorner Y \urcorner$. Clearly $\ulcorner X \urcorner \vee \ulcorner Y \urcorner \cong \bigvee\{x \in \underline{[m,n]} \mid [m,n] \in N_X \cup N_Y\}$. $N_X \cup N_Y$ is a prenormal form for $X \vee Y$. The desired result follows from 10.2.

We show: $\ulcorner \neg X \urcorner \cong \neg \ulcorner X \urcorner$. Consider $[m,n] \in N_X$ and $[p,q] \in N_{\neg X}$. The fact that $[m,n] \cap [p,q] = \emptyset$ is easily verified in PA. Consequently $\ulcorner X \urcorner \wedge \ulcorner \neg X \urcorner \cong \bot$ On the other hand $X \vee \neg X = \top$, thus by the above $\ulcorner X \urcorner \vee \ulcorner \neg X \urcorner \cong \ulcorner \top \urcorner \cong \top$. Conclude $\ulcorner \neg X \urcorner \cong \neg \ulcorner X \urcorner$.

We show: $\ulcorner \triangle X \urcorner \cong \triangle \ulcorner X \urcorner$. Clearly -by the above- it is sufficient to show:

$$PA \vdash \forall x\; x \in \ulcorner \neg \triangle X \urcorner \leftrightarrow \exists y (xSy \wedge y \in \ulcorner \neg X \urcorner).$$

Consider $M := \{[b, pd(n)] \mid [m,n] \in N_{\neg X}\}$. M is a prenormal form for $\neg \triangle X$. By 10.2: $\bigvee\{x \in \underline{[m,n]} \mid [m,n] \in M\} \cong \ulcorner \neg \triangle X \urcorner$. It is easily verified that:

$$PA \vdash \forall x \bigvee\{x \in \underline{[m,n]} \mid [m,n] \in M\} \leftrightarrow \exists y (xSy \wedge y \in \ulcorner \neg X \urcorner).$$

Finally we show: $\ulcorner \Box X \urcorner \cong \Box \ulcorner X \urcorner$. Clearly -by the above- it is sufficient to show:

$$PA \vdash \forall x\; x \in \ulcorner \neg \Box X \urcorner \leftrightarrow \exists y (xRy \wedge y \in \ulcorner \neg X \urcorner).$$

First note that for every interval $[p,q]$: $[p,q]\cap K_1^c$ is also an interval. Define:

$$\lambda(m,n):=\max(K_0\cap[m,n]).$$

Consider:

$$M:=\{[b,pd(n)]\cap K_1^c \mid [m,n]\in N_{\neg X}\}\cup\{[b,pd(\lambda(m,n))]\mid \lambda(m,n) \text{ exists, } [m,n]\in N_{\neg X}\}.$$

M is a prenormal form for $\neg\Box X$, so by 10.2: $\bigvee\{x\in[\underline{m},\underline{n}]\mid[m,n]\in M\}\equiv\ulcorner\neg\Box X\urcorner$. We show:

$$PA\vdash\ \forall x\ \bigvee\{x\in[\underline{m},\underline{n}]\mid[m,n]\in M\}\ \leftrightarrow\ \exists y(xRy\wedge y\in\ulcorner\neg X\urcorner).$$

Reason in PA (We will insert remarks that are best viewed as coming from outside PA; these will be in italics):

Suppose $x\in[\underline{b},\underline{n}]$, *for $[b,n]\in M$. Clearly for some k in $\neg X$:* $\underline{n}R\underline{k}$, hence $xR\underline{k}$. *Say $k\in[l,s]\in N_{\neg X}$. Then:* $\underline{k}\in[\underline{l},\underline{s}]$. It follows that $\exists y$ (xRy and $y\in\ulcorner\neg X\urcorner$).

Conversely suppose xRy and $y\in[\underline{l},\underline{s}]$, *for $[l,s]$ in $N_{\neg X}$.* Suppose $x\notin K_1$. In this case $x\in[\underline{b},pd(\underline{s})]\cap K_1^c=[\underline{b},\underline{r}]$ *for some r. (The reader should convince him/herself by inspecting the coding that PA indeed proves the identity: $[\underline{b},pd(\underline{s})]\cap K_1^c=[\underline{b},\underline{r}]$) Clearly $[b,r]\in M$.* Suppose $x\in K_1$. *In case $[l,s]\cap K_0=\emptyset$ this is verifiable in PA and hence: $\perp$. In case $[l,s]\cap K_0\neq\emptyset$, $\lambda(l,s)$ exists and hence: $\lambda(\underline{l},\underline{s})=\underline{q}$ for some q. (The reader should convince him/herself that PA verifies this last identity.)* Hence $yWS\underline{q}$ and thus $x\in[\underline{b},pd(\underline{q})]=[\underline{b},\underline{pd(q)}]$. *Clearly $[b,pd(q)]\in M$.* $\Box$PA

The other cases follow from the cases treated thus far. $\Box$

10.4 Fact

i) Remember that par abus de langage we write "$x\in K_1$" for the arithmetization of "$x\in K_1$" that can be read off from a description of F_a and the definitions of Φ_R and Φ_S. We have: $PA\vdash\ x\in K_1\leftrightarrow x\in\ulcorner K_1\urcorner$.

ii) $$PA\vdash\ (x\in K_0\wedge\forall y\in xS\ y\in\ulcorner X\urcorner)\rightarrow x\in\ulcorner X\urcorner$$

Proof: Left to the industrious reader. $\Box$

We proceed to the second stage. This stage splits into cases depending on the chosen interpretation of the logic involved. In all cases a primitive recursive function h will be introduced, with the property: $PA\vdash\forall x\forall y(x<y\rightarrow hxWShy)$. Let $Lx:=(\exists y\forall z\geq y\ hz=x)$. We have: $PA\vdash\exists!xLx$. We use L as a semiterm for the unique x such that Lx. Define $[X]:=(L\in\ulcorner X\urcorner)$. $[X]$ is the arithmetical image of X that we are after (modulo an equivalence relation like provable equivalence). Note that:

$$PA\vdash\ [X]\ \leftrightarrow\ \bigvee\{L\in[\underline{m},\underline{n}]\mid[m,n]\in N_X\}.$$

And that:

$$PA\vdash\ L\in[\underline{m},\underline{n}]\ \leftrightarrow\ ((\exists x\ \underline{m}WShx)\wedge(\forall y\ hyWS\underline{n})).$$

It follows that $[X]$ is provably equivalent in PA to a Boolean combination of Σ_1-formulas.

10.5 The NB Theories

Consider a set-2-frame F of the form $\Phi_S\Phi_R F_a$, where F_a is a set-frame with bottom b_a with $b_a \notin K_{a1}$.

Remember that if x is the Gödel number of a proof π, then l(x) is the largest of the Gödel numbers of arithmetical axioms occurring in π. We plausibly assume: $l(x)<x$.

Define both by the Recursion Theorem and by Course of Values Recursion:

$$h0 := b$$

$$h(x+1) := \begin{cases} y & \text{if } hxRy \text{ and } Proof_{PA}(x, L\neq y) \\ y & \text{if } hxSy \text{ and } Proof_{PA}(x, L\neq y) \text{ and } \forall y\leq l(x)\ hy\notin K_1 \\ hx & \text{otherwise} \end{cases}$$

$$L := \text{the unique x such that } \exists y \forall z\geq y\ hz=x$$

$$S := \exists y\ hy\in K_1$$

$$[X] := L\in \ulcorner X \urcorner$$

(Note that "$\forall y\leq l(x)\ hy\notin K_1$" could even be simplified to: $h(lx)\notin K_1$.)

10.5.1 Lemma

i) PA⊢ $x<y \rightarrow hxWShy$

ii) PA⊢ "L exists"

Proof: Entirely routine. □

10.5.2 Second Commutation Theorem

[.] commutes with the propositional logical constants (modulo provability in PA) and:

a) PA⊢ $[\Delta X] \leftrightarrow \Delta_{PA,S}[X]$,

b) PA⊢ $[\Box X] \leftrightarrow \Box_{PA}[X]$.

Proof: The cases of the propositional logical constants are trivial by 10.5.1 and the First Commutation Theorem (10.3).

Case (a) "→". Reason in PA:

Suppose $[\Delta X]$. Say $L=x\in \ulcorner \Delta X \urcorner$ and hence $L=x\in \Delta \ulcorner X \urcorner$.

First suppose $x=\underline{b}$. *If b is indeed in ΔX, we have $\Delta X=X=\top$ and hence:* $\Delta_{PA}L\in X$. *If b is not in ΔX we have: $\bot$, and hence:* $\Delta_{PA}L\in X$.

Suppose $x\neq\underline{b}$. Let u be the smallest number such that hu=x. Clearly u is a successor, say $u=v+\underline{1}$. By the definition of h: $Proof_{PA}(v,L\neq x)$. We distinguish two cases:

Case I: Suppose $hv\notin K_1$. Surely lv<v, hence for all y≤lv $hy\notin K_1$. Thus we have both $Proof_{PA}(v,L\neq x)$ and for all y≤lv $hy\notin K_1$, hence $\Delta_{PA,S}L\neq x$. By Σ-completeness we have: $\Delta_{PA,S}hu=x$. Ergo $\Delta_{PA,S}xSL$ and so (by the fact that $x\in\ulcorner\Delta X\urcorner$, so by Σ-completeness: $\Delta_{PA,S}x\in\ulcorner\Delta X\urcorner$, thus by the First Commutation Theorem: $\Delta_{PA,S}x\in\Delta\ulcorner X\urcorner$): $\Delta_{PA,S}L\in\ulcorner X\urcorner$.

Case II: Suppose $hv\in K_1$. We distinguish two subcases.

Subcase II(i): Suppose $x\in K_0$. By Σ-completeness we have $\Delta_{PA,S}hu=x$ and so $\Delta_{PA,S}xWSL$. Also we have: $\Delta_{PA,S}x\in K_0$, $\Delta_{PA,S}x\in\ulcorner\Delta X\urcorner$ and hence: $\Delta_{PA,S}x\in\Delta\ulcorner X\urcorner$. It follows by 10.4(ii) that: $\Delta_{PA,S}\forall y(xWSy\rightarrow y\in\ulcorner X\urcorner)$. Ergo: $\Delta_{PA,S}L\in\ulcorner X\urcorner$.

Subcase II(ii): Suppose $x\notin K_0$. We have: $\Delta_{PA,S}L\neq x$ (how else could h move up from hv, which is in K_1, to x, which is not in K_0?). Also $\Delta_{PA,S}hu=x$. Ergo $\Delta_{PA,S}xSL$ and hence: $\Delta_{PA,S}L\in\ulcorner X\urcorner$. □PA

Case (a) "←". Reason in PA:

Suppose $\Delta_{PA,S}[X]$. Suppose for a reductio: $L=x\notin\ulcorner\Delta X\urcorner$. Clearly by 10.3: $x\notin\Delta\ulcorner X\urcorner$ and hence for some y xSy and $y\notin\ulcorner X\urcorner$. By Σ-completeness: $\Delta_{PA,S}y\notin\ulcorner X\urcorner$. Hence $\Delta_{PA,S}L\neq y$. It follows that for some u $Proof_{PA}(u,L\neq y)$ and $\forall v\leq lu\ \neg S_0v$, i.e. $\forall v\leq lu$ $hv\notin K_1$. Because L=x, we have: huWSxSy, hence huSy. By the definition of h: $h(u+\underline{1})=y$. Contradiction.

Conclude: $L\in\ulcorner\Delta X\urcorner$, i.e. $[\Delta X]$. □PA

Case (b) "→". Reason in PA:

Suppose $[\Box X]$, say $L=x\in\ulcorner\Box X\urcorner$ and hence $x\in\Box\ulcorner X\urcorner$. In case $x=\underline{b}$, it is easily seen (by the same reasoning as under case (a) "→") that: $\Box_{PA}L\in\ulcorner X\urcorner$. Suppose $x\neq\underline{b}$. Say hu=x. We have $\Box_{PA}L\neq x$ (how else could h move up to x?) and by Σ-completeness: $\Box_{PA}hu=x$. Hence $\Box_{PA}xSL$. If $x\notin K_1$ it follows that: $\Box_{PA}xRL$, hence $\Box_{PA}L\in\ulcorner X\urcorner$ and thus $\Box_{PA}[X]$. Suppose $x\in K_1$. We have: $\exists y\ hy\in K_1$, i.e. S. Hence for any A:

$$\Box_{PA}(\Delta_{PA,S}A\rightarrow A)$$

We claim: $\Box_{PA}L\in K_0$.

Reason inside $\Box_{PA}$:

Suppose $L=y\notin K_0$. We have xSy and $x\in K_1$. Thus $\Delta_{PA,S}L\neq y$ (how else could h move up from an element of K_1 to y, y not being in K_0?). By $\Delta_{PA,S}$-reflection: $L\neq y$. Contradiction. Conclude: $L\in K_0$. $\Box(\Box_{PA})$

We have: $\Box_{PA}xSL$ and $\Box_{PA}L\in K_0$. Hence: $\Box_{PA}xRL$ and thus: $\Box_{PA}L\in \ulcorner X\urcorner$, i.e. $\Box_{PA}[X]$.
$\Box$PA

Case (b) "$\leftarrow$". Reason in PA:

Suppose $\Box_{PA}[X]$. Suppose for a reductio: $L=x\notin\ulcorner\Box X\urcorner$. By 10.3 there is a y with xRy and $y\notin\ulcorner X\urcorner$. By Σ-completeness: $\Box_{PA}y\notin\ulcorner X\urcorner$. Hence from $\Box_{PA}L\in\ulcorner X\urcorner$: $\Box_{PA}L\neq y$. Say $\text{Proof}_{PA}(u,L\neq y)$. From L=x: huWSxRy. Hence: huRy. Thus by the definition of h: $h(u+\underline{1})=y$. Contradiction. Conclude: $L\in\ulcorner\Box X\urcorner$, i.e. $[\Box X]$. $\Box$PA

This ends our proof. $\Box$

Note that [.] really has the character of an embedding: it is injective modulo provable equivalence in PA. For suppose $X\neq Y$. Inspection of the frame shows that: for some k $(((X\leftrightarrow Y)\wedge\Delta(X\leftrightarrow Y))\rightarrow\Box^k\bot)=\top$. Hence:

$$PA\vdash (([X]\leftrightarrow[Y])\wedge\Delta_{PA,S}([X]\leftrightarrow[Y]))\rightarrow\Box_{PA}{}^k\bot.$$

Suppose $PA\vdash[X]\leftrightarrow[Y]$. It follows that:

$$PA\vdash ([X]\leftrightarrow[Y])\wedge\Delta_{PA,S}([X]\leftrightarrow[Y]),$$

hence: $PA\vdash\Box_{PA}{}^k\bot$. Quod non.

10.5.3 Definition

Consider a set-2-model $G=\langle F,f\rangle$. Suppose G is of the form $\Psi_S\Psi_R G_a$, where $G_a=\langle F_a,f_a\rangle$ and G_a is a finite set-model (also finite in the sense that $f_a(p)=\emptyset$ for all but finitely many p), with bottom $b_a\notin K_{a1}$. Define for $\phi\in L$:

$$[\phi] := [[\![\phi]\!]f].$$

Let $f^*p:=[p]$, and define:

$$\langle\phi\rangle := (\phi)(f^*,PA,S),$$

where S is defined as in the beginning of 10.5.

10.5.4 Theorem

$$PA\vdash \langle\phi\rangle\leftrightarrow[\phi]$$

Proof: By a trivial induction on ϕ using 10.5.2. $\Box$

10.5.5 Arithmetical Completeness for NB_1

Suppose $NB_1 \nvdash \phi$. Then there is an interpretation function f* and a Σ_1-sentence S such that $PA \nvdash (\phi)(f^*,PA,S)$.

Proof: Suppose $NB_1 \nvdash \phi$. Consider the finite set-model G_b constructed in the proof of 8.36. The bottom, say z of G_b forces $\neg\phi$ (under f_b). Pick a b with $b \notin K_b$. Define G_a as follows: $K_a := K_b \cup \{b\}$; $K_{a0} := K_{b0}$; $K_{a1} := K_{b1}$; $S_a := S_b \cup \{<b,y> | y \in K_b\}$; $f_a y := f_b y$, if $y \in K_b$; $f_a b := f_b z$. Clearly $b \notin K_{a1}$. Consider $G_e := \Psi_S \Psi_R G_a$. Let u=<<z,0>,0>. Clearly the submodel with domain uWS_e will be isomorphic to G of the proof of 8.36. Hence, because uWS_e is upwards closed G_e the forcing relations of G_e and G of 8.36 will coincide on the nodes connected by the isomorphism. Ergo $u \nVdash \phi(f_e)$.

It follows that $[\![\phi]\!]f_e \neq \top$, and hence that $PA \nvdash [\phi]$ (where [.] is based on G_e). Thus: $PA \nvdash <\phi>$.

□

10.5.6 Arithmetical Completeness for NB_2

Suppose $NB_2 \nvdash \phi$. Then there is an interpretation function f* and a Σ_1-sentence S such that $\mathbb{N} \nvDash (\phi)(f^*,PA,S)$.

Proof: Suppose $NB_2 \nvdash \phi$. Consider the model G of the proof of 8.37. Let b be the bottom of G. We have: $b \nVdash \phi(f)$. Consider h, [.], etc. based on G. Clearly $\mathbb{N} \vDash L=\underline{b}$, hence $\mathbb{N} \vDash \neg[\phi]$. Thus $\mathbb{N} \nvDash <\phi>$. □

10.6 The CSM Theories under the CS-Interpretation

Consider a Carlson-2-frame F of the form $\Phi_S \Phi_R F_a$, where F_a is a Carlson-frame with bottom b_a. Let U and T be RE theories in the language of PA, extending PA, such that for all sentences A of the language of PA: $U \vdash \Delta_T A \rightarrow A$.

Define by the Recursion Theorem:

$$h0 := b$$

$$h(x+1) := \begin{cases} y & \text{if } hxRy \text{ and } Proof_U(x, L \neq y) \\ y & \text{if } hxSy \text{ and } Proof_T(x, L \neq y) \\ hx & \text{otherwise} \end{cases}$$

$$L := \text{the unique } x \text{ such that } \exists y \forall z \geq y \; hz = x$$

$$[X] := L \in \ulcorner X \urcorner$$

10.6.1 Lemma

i) PA⊢ $x<y \rightarrow hxWShy$
ii) PA⊢ "L exists"

Proof: Entirely routine. □

10.6.2 Second Commutation Theorem

[.] commutes with the propositional logical constants (modulo provability in PA) and:

a) PA⊢ $[\Delta X] \leftrightarrow \Delta_T[X]$,
b) PA⊢ $[\Box X] \leftrightarrow \Box_U[X]$.

Proof: The cases of the propositional logical constants are trivial by 10.6.1 and the First Commutation Theorem (10.3). The cases of Δ and $\Box$ are very much like the corresponding cases in 10.5.2:

Case (a) "→". Reason in PA:

Suppose $[\Delta X]$. Say $L=x\in\ulcorner\Delta X\urcorner$ and hence $L=x\in\Delta\ulcorner X\urcorner$.

The case that $x=\underline{b}$ is easy.

Suppose $x\neq\underline{b}$. Let u be the smallest number such that hu=x. Clearly u is a successor, say $u=v+\underline{1}$. We distinguish two cases:

Case I: Suppose $x\in K_0$. By Σ-completeness we have $\Delta_T hu=x$ and so $\Delta_T xWSL$. Also we have: $\Delta_T x\in K_0$, $\Delta_T x\in\ulcorner\Delta X\urcorner$ and hence: $\Delta_T x\in\Delta\ulcorner X\urcorner$. It follows by 10.4(ii) that: $\Delta_T\forall y(xWSy\rightarrow y\in\ulcorner X\urcorner)$. Ergo: $\Delta_T L\in\ulcorner X\urcorner$.

Case II: Suppose $x\notin K_0$. We have: $Proof_T(v,L\neq x)$ (how else could h move up from hv to x, which is not in K_0?). Hence $\Delta_T L\neq x$. Also $\Delta_T hu=x$. Ergo $\Delta_T xSL$ and hence: $\Delta_T L\in\ulcorner X\urcorner$. □PA

Case (a) "←". Reason in PA:

Suppose $\Delta_T[X]$. Suppose for a reductio: $L=x\notin\ulcorner\Delta X\urcorner$. Clearly by 10.3: $x\notin\Delta\ulcorner X\urcorner$ and hence for some y xSy and $y\notin\ulcorner X\urcorner$. By Σ-completeness: $\Delta_T y\notin\ulcorner X\urcorner$. Hence $\Delta_T L\neq y$. It follows that for some u $Proof_T(u,L\neq y)$. Because L=x, we have: huWSxSy, hence huSy. By the definition of h: $h(u+\underline{1})=y$. Contradiction.

Conclude: $L\in\ulcorner\Delta X\urcorner$, i.e. $[\Delta X]$. □PA

Case (b) "→". Reason in PA:

Suppose $[\Box X]$, say $L=x\in\ulcorner\Box X\urcorner$ and hence $x\in\Box\ulcorner X\urcorner$. The case $x=\underline{b}$ is easy. Suppose $x\neq\underline{b}$. Say $hu=x$. We have $\Box_U L\neq x$ (how else could h move up to x?) and by Σ-completeness: $\Box_U hu=x$. Hence $\Box_U xSL$. We claim: $\Box_U L\in K_0$.

Reason inside $\Box_U$:

Suppose $L=y\notin K_0$. We have xSy. Thus $\Delta_T L\neq y$ (how else could h move up to y, y not being in K_0?). By Δ_T-reflection: $L\neq y$. Contradiction. Conclude: $L\in K_0$.

$\Box(\Box_U)$

We have: $\Box_U xSL$ and $\Box_U L\in K_0$. Hence: $\Box_U xRL$ and thus: $\Box_U L\in\ulcorner X\urcorner$, i.e. $\Box_U[X]$.

$\Box U$

Case (b) "←". Reason in PA:

Suppose $\Box_U[X]$. Suppose for a reductio: $L=x\notin\ulcorner\Box X\urcorner$. By 10.3 there is a y with xRy and $y\notin\ulcorner X\urcorner$. By Σ-completeness: $\Box_U y\notin\ulcorner X\urcorner$. Hence from $\Box_U L\in\ulcorner X\urcorner$: $\Box_U L\neq y$. Say $\mathrm{Proof}_U(u, L\neq y)$. From $L=x$: huWSxRy. Hence: huRy. Thus by the definition of h: $h(u+\underline{1})=y$. Contradiction. Conclude: $L\in\ulcorner\Box X\urcorner$, i.e. $[\Box X]$.

$\Box PA$

This ends our proof. $\Box$

10.6.3 Definition

Consider a Carlson-2-model $G=\langle F,f\rangle$. Suppose G is of the form $\Psi_S\Psi_R G_a$, where $G_a=\langle F_a,f_a\rangle$ and G_a is a finite Carlson-model (also finite in the sense that $f_a(p)=\emptyset$ for all but finitely many p), with bottom b_a. Define for $\phi\in L$:

$$[\phi] := [[\![\phi]\!]f].$$

Let $f^*p:=[p]$, and define:

$$\langle\phi\rangle := (\phi)(f^*,U,T),$$

10.6.4 Theorem

$$PA\vdash \langle\phi\rangle\leftrightarrow[\phi]$$

Proof: By a trivial induction on ϕ using 10.6.2. $\Box$

10.6.5 Definition

Let $\mathcal{F}$ be a compact set-frame. Define d from K to the ordinals by:

$$dx := 1+\sup\{dy|xSy\}.$$

Note that if x is a top element, dx=1.

10.6.6 Fact

Let $\mathcal{F}$ be of the form $\Phi_S\Phi_R\mathcal{F}_a$, where $\mathcal{F}_a$ is a finite Carlson-frame with bottom. Let d*x:=dx if $dx<\omega^2$, $d^*x:=\infty$ otherwise. We have for $\alpha\in\omega^2\cup\{\infty\}$

$$d^*x\leq\alpha \Leftrightarrow x\Vdash\bot_\alpha.$$

Proof: Left to the reader. □

10.6.7 Definition

i) $L_1(U,T)$:= {ϕ|for all interpretation functions f $T\vdash(\phi)(f,U,T)$}

$L_2(U,T)$:= {ϕ|for all interpretation functions f $U\vdash(\phi)(f,U,T)$}

ii) $CSM_1(\alpha)$:= $CSM_1+\bot_\alpha$ $(\alpha\in\omega^2\cup\{\infty\})$

$CSM_2(\beta)$:= $CSM_2+\bot_{\omega.\beta}$ $(\beta\in\omega\cup\{\infty\})$

10.6.6 Theorem

$L_1(U,T)=CSM_1(\alpha)$ and $L_2(U,T)=CSM_2(\beta)$ for some α with $\alpha\in\omega^2\cup\{\infty\}$ and some $\beta\in\omega\cup\{\infty\}$, such that: $\omega.\beta\leq\alpha\leq\omega.(\beta+1)$. (Note that if either α or β are ∞ then $L_1(U,T)=CSM_1$ and $L_2(U,T)=CSM_2$.)

Proof: First we show: $L_1(U,T)=CSM_1(\alpha)$ for some $\alpha\in\omega^2\cup\{\infty\}$. If χ is in the closed fragment $(\chi)(f,U,T)$ is independent of f, so let's write "$(\chi)(U,T)$" instead. Evidently $T\vdash(\bot_\infty)(U,T)$. Let α be the smallest element of $\omega^2\cup\{\infty\}$ such that $T\vdash(\bot_\alpha)(U,T)$. Clearly $CSM_1(\alpha)\subseteq L_1(U,T)$.

Suppose $\psi\in L_1(U,T)$ and $CSM_1(\alpha)\nvdash\psi$. It follows that $CSM_1\nvdash\bot_\alpha\rightarrow\psi$. Consider the Carlson-model $\mathcal{G}_a$ constructed in 8.32 for $\phi:=(\bot_\alpha\rightarrow\psi)$ with bottom x. Construct $\mathcal{G}_e:=\Psi_S\Psi_R\mathcal{G}_a$. Let z:=<<x,1>,0>. Clearly the restriction of $\mathcal{G}_e$ to zWS_e is isomorphic to $\mathcal{G}$ of 8.32. Hence, because zWS_e is upwards closed, the forcing relations of $\mathcal{G}_e$ and $\mathcal{G}$ will coincide on the nodes connected by the isomorphism. Consequently $z\nVdash\phi(f_e)$, hence $z\Vdash\bot_\alpha(f_e)$ and $z\nVdash\psi(f_e)$. We claim that for some $\beta<\alpha$ and for all $u\in K_e$:

$$u\Vdash (\psi\wedge\Delta\psi) \rightarrow \bot_\beta(f_e)$$

Inspection of the model shows that below z there just is a long tail, so $v\Vdash(\psi\wedge\Delta\psi)(f_e)$ implies zS_ev. Let $v_1,\ldots,v_n$ be the minimal elements of $[\![\psi\wedge\Delta\psi]\!]f_e$ (these exist e.g. by the normal form theorem). We have $d^*z\leq\alpha$ and hence $d^*v_i<\alpha$ (i=1,...,n). Let β be the maximum of the d^*v_i. We have $\beta<\alpha$ and $v_i \Vdash \bot_\beta(f_e)$ (i=1,...,n). Conclude:

$$[\![(\psi\wedge\Delta\psi) \rightarrow \bot_\beta]\!](f_e)=\top.$$

Let [.] and <.> be based on G_e. We have: $PA\vdash <(\psi\wedge\Delta\psi)\rightarrow\bot_\beta>$ and hence $T\vdash<(\psi\wedge\Delta\psi)\rightarrow\bot_\beta>$, so: $T\vdash (<\psi>\wedge\Delta_T<\psi>)\rightarrow<\bot_\beta>$. On the other hand $T\vdash<\psi>$, and hence $T\vdash<\psi>\wedge\Delta_T<\psi>$. Ergo $T\vdash<\bot_\beta>$, i.e. $T\vdash(\bot_\beta)(U,T)$. Quod non.

Conclude $L_1(U,T)=CSM_1(\alpha)$.

Secondly we show: $L_2(U,T)=CSM_2(\beta)$ for some $\beta\in\omega\cup\{\infty\}$. Evidently $U\vdash(\bot_{\omega.\infty})(U,T)$. Let β be the smallest element of $\omega\cup\{\infty\}$ such that $U\vdash(\bot_{\omega.\beta})(U,T)$. Note that, by Δ_T-Reflection, for no $\gamma<\omega.\beta$: $U\vdash(\bot_\gamma)(U,T)$. Clearly $CSM_2(\beta)\subseteq L_2(U,T)$.

Suppose $\psi\in L_2(U,T)$ and $CSM_2(\beta)\nvdash\psi$. It follows that $CSM_2\nvdash\bot_{\omega.\beta}\rightarrow\psi$. Consider the Carlson-model G_a constructed in 8.34 for $\phi:=(\bot_\alpha\rightarrow\psi)$ with bottom x. Construct $G_e:=\Psi_S\Psi_R G_a$. Let z:=<<x,1>,0>. Note that $z\in K_{e0}$. Clearly the restriction of G_e to zWS_e is isomorphic to G of 8.34. Hence, because zWS_e is upwards closed, the forcing relations of G_e and G coincide on the nodes connected by the isomorphism. Consequently $z\nVdash\phi(f_e)$, hence $z\Vdash\bot_{\omega.\beta}(f_e)$ and $z\nVdash\psi(f_e)$. We claim that for some $\gamma<\omega.\beta$ and for all $u\in K_e$:

$$u\Vdash (\psi\wedge\Box\psi) \rightarrow \bot_\gamma(f_e)$$

Below z there just is a long tail, so, because $z\in K_{e0}$: $v\Vdash(\psi\wedge\Box\psi)(f_e)$ implies zS_ev. Let $v_1,...,v_n$ be the minimal elements of $[\![\psi\wedge\Box\psi]\!]f_e$. We have $d^*z\leq\omega.\beta$ and hence $d^*v_i<\omega.\beta$ (i=1,...,n). Let γ be the maximum of the d^*v_i. We have $\gamma<\omega.\beta$ and $v_i\Vdash\bot_\gamma(f_e)$ (i=1,...,n). Conclude: $[\![(\psi\wedge\Box\psi) \rightarrow \bot_\gamma]\!](f_e)=\top$.

Let [.] and <.> be based on G_e. We have: $PA\vdash<(\psi\wedge\Box\psi)\rightarrow\bot_\gamma>$ and hence $U\vdash<(\psi\wedge\Box\psi)\rightarrow\bot_\gamma>$, so: $U\vdash(<\psi>\wedge\Box_U<\psi>)\rightarrow<\bot_\gamma>$. On the other hand $U\vdash<\psi>$, and hence $U\vdash<\psi>\wedge\Box_U<\psi>$. Ergo $U\vdash<\bot_\gamma>$, i.e. $U\vdash(\bot_\gamma)(U,T)$. Quod non.

Conclude $L_2(U,T)=CSM_2(\beta)$.

Suppose $L_1(U,T)=CSM_1(\alpha)$ and $L_2(U,T)=CSM_2(\beta)$. Clearly $T\vdash\bot_\alpha(U,T)$, and hence by Σ-completeness $U\vdash\Delta_T(\bot_\alpha(U,T))$. By Δ_T-reflection: $U\vdash\bot_\alpha(U,T)$. Thus $\omega.\beta\leq\alpha$. Also $U\vdash\bot_{\omega.\beta}(U,T)$, hence by Σ-completeness: $T\vdash\Box_U\bot_{\omega.\beta}(U,T)$. In other words: $T\vdash\bot_{\omega(\beta+1)}(U,T)$. Ergo $\alpha\leq\omega.(\beta+1)$ □

10.6.7 Theorem

Let α be in $\omega^2\cup\{\infty\}$ and let β be in $\omega\cup\{\infty\}$. Suppose $\omega.\beta\leq\alpha\leq\omega.(\beta+1)$. Then there are RE extensions T and U of PA such that $L_1(U,T)=CSM_1(\alpha)$ and $L_2(U,T)=CSM_2(\beta)$.

Proof: Let's write "RT" for:

$T+\{\Delta_T A \to A | A$ a sentence of the language of PA$\}$.

Suppose $\omega.\beta \leq \alpha < \omega.(\beta+1)$. Let $T := PA + \bot_\alpha(RPA,PA)$ and $U := RPA + \bot_\alpha(RPA,PA)$. By Σ_1-completeness: $T \vdash \Delta_{PA} A \leftrightarrow \Delta_T A$ and hence also $T \vdash \Box_{RPA} A \leftrightarrow \Box_U A$. Also $U \vdash \Delta_T A \to A$. We find: $T \vdash \bot_\alpha(U,T)$ and $U \vdash \bot_\alpha(U,T)$. By Δ_T-reflection it follows that $U \vdash \bot_{\omega.\beta}(U,T)$.

Suppose $\gamma < \alpha$ and $T \vdash \bot_\gamma(U,T)$, then $PA \vdash \bot_\alpha(RPA,PA) \to \bot_\gamma(RPA,PA)$. Quod non. Suppose $\gamma < \omega.\beta$ and $U \vdash \bot_\gamma(U,T)$, then $T \vdash \Box_U \bot_\gamma(U,T)$, i.e. $T \vdash \bot_{\gamma+1}(U,T)$. But $\gamma+1 < \omega.\beta \leq \alpha$. Quod non.

Let $\alpha = \omega.(\beta+1)$. Take $T := PA$ and $U := RPA + \bot_{\omega.\beta}(RPA,PA)$. By Σ_1-completeness $U \vdash \Box_{RPA} A \leftrightarrow \Box_U A$. Hence $U \vdash \bot_{\omega.\beta}(U,T)$. Clearly $T \vdash \Box_U \bot_{\omega.\beta}(U,T)$, so $T \vdash \bot_{\omega.(\beta+1)}$.

We leave it to the reader to verify that for no $\gamma < \omega.\beta$ $U \vdash \bot_\gamma(U,T)$ and for no $\delta < \omega.(\beta+1)$ $T \vdash \bot_\delta(U,T)$. □

10.6.8 Consequence

$L_1(RPA,PA) = CSM_1$, $L_2(RPA,PA) = CSM_2$.

10.6.9 An Arithmetical Completeness Result for CSM_3

Suppose $CSM_3 \nvdash \phi$. Then there is an interpretation function f^* such that:

$$\mathbb{N} \nvDash (\phi)(f^*, RPA, PA).$$

Proof: Suppose $CSM_3 \nvdash \phi$. Consider the model *G* constructed in 8.35. *G* is of the form $\Psi_S \Psi_R G_a$, where G_a is finite. We have $b \nVdash [\![\phi]\!] f$. Clearly $\mathbb{N} \vDash L = \underline{b}$. Consider h, [.], <.> based on *G*, RPA and PA. We find $\mathbb{N} \nvDash [\phi]$, and so by 10.6.4 $\mathbb{N} \nvDash \langle\phi\rangle$. □

10.7 The CSM Theories under the M-Interpretation

Consider a Carlson-2-frame *F* of the form $\Phi_S \Phi_R F_a$, where F_a is a Carlson-frame with bottom b_a.

Define by the Recursion Theorem:

$$h(0,v) := b$$

$$h(x+1,v) := \begin{cases} y & \text{if } hxRy \text{ and } Proof_{PA}(x, Lv \neq y) \\ y & \text{if } hxSy \text{ and } Proof_{PA \restriction v}(x, Lv \neq y) \\ hx & \text{otherwise} \end{cases}$$

$$Lv := \text{the unique } x \text{ such that } \exists y \forall z \geq y\ h(z,v) = x$$

$[X]v := Lv \in \ulcorner X \urcorner$

10.7.1 Lemma

i) $PA{\restriction}N \vdash x<y \rightarrow h(x,v)WSh(y,v)$

ii) $PA{\restriction}N \vdash$ "Lv exists"

Proof: Entirely routine. □

10.7.2 Second Commutation Theorem

[.]v commutes with the propositional logical constants modulo provability in PA↾N and:

a) $PA{\restriction}N \vdash \forall v \geq \underline{N}\ ([\triangle X]v \leftrightarrow \triangle_{PA,v}[X]v)$,

b) $PA{\restriction}N \vdash \forall v \geq \underline{N}\ ([\Box X]v \leftrightarrow \Box_{PA}[X]v)$.

Proof: The proof is completely analogous to the one of 10.6.2. Note that we need to verify that 10.3 even holds for PA↾N. □

10.7.3 Definition

Consider a Carlson-2-model G=<F,f>. Suppose G is of the form $\Psi_S\Psi_R G_a$, where G_a=<F_a,f_a> and G_a is a finite Carlson-model (also finite in the sense that $f_a(p)=\emptyset$ for all but finitely many p), with bottom b_a. Define for $\phi \in L$:

$[\phi]v := [[\![\phi]\!]f]v.$

Let f*p:=[p]v, and define:

$<\phi>v := (\phi)(f^*,PA,v),$

10.7.4 Theorem

$PA{\restriction}N \vdash \forall v \geq \underline{N}\ (<\phi>v \leftrightarrow [\phi]v)$

Proof: By a trivial induction on ϕ using 10.7.2. □

10.7.5 Arithmetical Completeness for CSM_1

Suppose $CSM_1 \nvdash \phi$. Then there is an open interpretation function f* such that:

$PA \nvdash \forall v \geq \underline{N}\ (\phi)(f^*,PA,v).$

Proof: Suppose $CSM_1 \nvdash \phi$. Consider the finite set-model G_a constructed in the proof of 8.32. The bottom, say z of G_a forces $\neg\phi$ (under f_a). Consider $G_c := \Psi_S\Psi_R G_a$. Let u=<<z,0>,0>. Clearly the submodel with domain uWS_c will be isomorphic to G of the proof of 8.32 Hence, because uWS_c is upwards closed G_c the forcing relations of G_c

and G of 8.32 will coincide on the nodes connected by the isomorphism. Ergo $u \nVdash \phi(f_c)$. Hence for some $\alpha<\infty$ and for all $w\in K_c$: $w \Vdash (\phi\wedge\Delta\phi)\rightarrow\bot_\alpha$.

Let h, $[.]v$, $<.>v$ be based on G_c. We have: $PA\vdash \forall v\geq\underline{N}\ [(\phi\wedge\Delta\phi)\rightarrow\bot_\alpha]v$ and thus by 10.7.2: $PA\vdash \forall v\geq\underline{N}\ ([\phi]v\wedge\Delta_{PA,v}[\phi]v) \rightarrow [\bot_\alpha]v$. Let π be a proof of this fact.

Suppose $PA\vdash \forall v\geq\underline{N}\ <\phi>v$. Then by 10.7.4: $PA\vdash \forall v\geq\underline{N}\ [\phi]v$. Let π^* be a proof of this fact. Let M be bigger than both N and the codes of the arithmetical axioms in π and π^*. It follows that:

$$PA\restriction M\vdash ([\phi]\underline{M}\wedge\Delta_{PA,\underline{M}}[\phi]\underline{M}) \rightarrow [\bot_\alpha]\underline{M},$$

and:

$$PA\restriction M\vdash [\phi]\underline{M}\wedge\Delta_{PA,\underline{M}}[\phi]\underline{M}.$$

Ergo $PA\restriction M\vdash[\bot_\alpha]\underline{M}$, quod non. □

10.7.6 Arithmetical Completeness for CSM_2

Suppose $CSM_2\nvdash\phi$. Then there is an open interpretation function f* such that for some $k\geq N$ $PA\nvdash(\phi)(f^*,PA,\underline{k})$.

Proof: In fact we have for *all* $k\geq N$ $PA\nvdash(\phi)(f^*,PA,\underline{k})$. The proof for the pair PA, $PA\restriction k$ is fully the same as the proof for the pair RPA, PA. □

10.7.7 Arithmetical Completeness for CSM_3

Suppose $CSM_3\nvdash\phi$. Then there is an interpretation function f* and a $k\geq N$ such that:

$$\mathbb{N}\nvDash (\phi)(f^*,PA,k).$$

Proof: Again this holds for all $k\geq N$. The proof is the same as the proof of 10.6.9. □

10.7.8 Montagna's Uniformization Theorem

Let Cx be a formula of the language of PA. Define f^C by: $f^C p_i := C\underline{i}$. There is a Δ_2-formula Bx with just x free such that for all ϕ:

i) $CSM_1\vdash\phi \Leftrightarrow PA\vdash \forall v\geq\underline{N}\ (\phi)(f^B,PA,v)$,

ii) $CSM_2\vdash\phi \Leftrightarrow$ for all $k\geq N$ $PA\vdash(\phi)(f^B,PA,\underline{k})$.

Note that we do not only obtain uniformization in this way but also *closed* counterexamples.

Proof: We first give a sketch of the proof of (ii). This part of the proof is taken from Montagna[1984].

Let for ϕ such that $CSM_2\nvdash\phi$, f_ϕ stand for the counterexample function constructed from an appropriate countermodel.

Let TR be a Δ_2 truthpredicate for Boolean combinations of Σ_1-formulas with just v free. This means that if Dv is such a Boolean combination we have:

$$PA \vdash \forall v(TR(\ulcorner \underline{Dv} \urcorner, v) \leftrightarrow Dv).$$

Consider Bx such that:

$PA \vdash \forall x($ Bx $\leftrightarrow$ if for any $<y,v,\phi>$, which is the first (coded) tripel such that $CSM_2 \nvdash \phi$, $v \geq \underline{N}$, $Proof_{PA}(y,(\phi)(f^B,PA,v))$ we have: $TR(f_\phi p_x,v)$).

It is easily seen that Bx is Δ_2.

Suppose $CSM_2 \nvdash \phi$, but for some $k \geq N$ $PA \vdash (\phi)(f^B,PA,\underline{k})$. It follows that there is a $<q,m,\psi>$ such that:

$PA \vdash <\underline{q},\underline{m},\ulcorner \underline{\psi} \urcorner>$ is the first triple such that $CSM_2 \nvdash \psi$, $\underline{m} \geq \underline{N}$ and $Proof_{PA}(\underline{q},(\phi)(f^B,PA,\underline{m}))$

Hence:

$$PA \vdash \forall x(\text{ Bx } \leftrightarrow TR(f_\psi p_x,\underline{m})).$$

Because $PA \vdash (\psi)(f^B,PA,\underline{m})$, we find: $PA \vdash (\psi)(f_\psi,PA,\underline{m})$. Quod non.

Finally we prove (i): remember that by a Kripke model argument:

$$CSM_1 \vdash \phi \Leftrightarrow CSM_2 \vdash \Delta\phi.$$

We have:

$$\begin{aligned} CSM_1 \vdash \phi &\Rightarrow PA \upharpoonright N \vdash \forall v \geq \underline{N}\ (\phi)(f^B,PA,v) \\ &\Rightarrow \text{for all } k \geq N\ PA \vdash \Delta_{PA,\underline{k}}(\phi)(f^B,PA,\underline{k}) \\ &\Rightarrow CSM_2 \vdash \Delta\phi \\ &\Rightarrow CSM_1 \vdash \phi. \end{aligned}$$

□

11 The essentially Σ-formulas of L_0

A formula ϕ of L_0 is called *essentially Σ w.r.t. T* if for all interpretations f $\phi(f,T)$ is provably equivalent in T to a Σ_1-sentence.

In this section we want to characterize the essentially Σ-formula's of L_0 w.r.t. PA. The first conjecture that comes to mind turns turns out to be correct: such ϕ are provably equivalent in L to $\top$ or to $\bot$ or to a finite disjunction of sentences of the form $\Box\sigma$.

How to prove this conjecture? A first idea is to look at those ψ in L_0 such that $L \vdash \psi \rightarrow \Box\psi$. Perhaps they are precisely the essentially Σ-formulas w.r.t. PA? This idea however does not work. Consider e.g. $p \wedge \Box p$. Clearly $L \vdash (p \wedge \Box p) \rightarrow \Box(p \wedge \Box p)$, but there is an arithmetical sentence A such that $(A \wedge \Box_{PA} A)$ is not provably equivalent to a Σ_1-sentence in PA. (This well known result is due to Kent, see Kent[1973].) A second idea is to use an operator Δ standing for provability in a theory U which is weaker than PA (e.g.

PRA) and to consider those ψ such that $\psi \rightarrow \Delta\psi$ is arithmetically valid. This idea does not quite work yet: one only gets a characterization of the ϕ such that for all interpretations f ϕ(f,PA) is provably equivalent to a Σ_1-sentence *in U*. The way in which the second attempt fails suggests that one should look at a theory or theories that is/are in some sense weaker than PA, in some other sense equal to PA. This third idea works. I found two ways to implement it. The first one is to use Montagna's interpretation of CSM_2 plus his uniformized completeness theorem. The growing sequence of finite subtheories is as it were in the limit (extensionally) equal to PA. A disadvantage of this approach compared with the one elaborated below is that the counterexamples it produces tend to be Δ_2 rather than Boole(Σ_1). The second way to work out the third idea is to consider the interpretation associated with NB_1. This way will be pursued here. (Both succesful strategies use -in different senses- infinitely many interpretations of Δ; whether this is a necessary feature I don't know.)

11.1 Theorem

Suppose $\phi \in L_0$ and ϕ is essentially Σ w.r.t. PA. Then $NB_1 \vdash \phi \rightarrow \Delta\phi$.

Proof: Let be an essentially Σ-formula w.r.t. PA of L_0. Consider any false Σ_1-sentence S and any interpretation function f. Clearly: $(\phi)(f,PA)=(\phi)(f,PA,S)$. Ex hypothesi there is a Σ_1-sentence A such that $PA \vdash (\phi)(f,PA) \leftrightarrow A$. It follows that $PA \vdash \Delta_{PA,S}((\phi)(f,PA) \leftrightarrow A)$ and hence $PA \vdash \Delta_{PA,S}(\phi)(f,PA) \leftrightarrow \Delta_{PA,S}A$. Thus:

$$\begin{aligned} PA \vdash (\phi)(f,PA,S) &\rightarrow A \\ &\rightarrow \Delta_{PA,S}A \\ &\rightarrow (\Delta\phi)(f,PA,S) \end{aligned}$$

By the arithmetical completeness theorem for NB_1 we may conclude: $NB_1 \vdash \phi \rightarrow \Delta\phi$.

□

11.2 Fact

Suppose $\phi \in L_0$. Then: $NB_1 \vdash \phi \Leftrightarrow L \vdash \phi$.

Proof: The proof is surprisingly trivial. The "⇐" side is as usual. For the "⇒" side note that substituting "□" for "Δ" in NB_1 axioms and rules yields theorems and rules of L.

□

11.3 Theorem

Suppose $\phi \in L_0$ and $NB_1 \vdash \phi \rightarrow \Delta\phi$, then $L \vdash \phi \leftrightarrow \bigvee \Box\sigma$.

(We use convention 9.2 here. Note that the disjunction may be empty, in which case it reduces to $\bot$, or one of the σ may be $\top$, in which case the disjunction reduces to $\top$.)

Proof: Let $\phi \in L_0$ and suppose $NB_1 \vdash \phi \to \Delta\phi$. Clearly ϕ can be written in the form $\bigvee\bigwedge[\Box\psi, \neg\Box\chi, p, \neg q]$, where $\psi, \chi \in L_0$. Consider any disjunct $C = \bigwedge[\Box\psi, \neg\Box\chi, p, \neg q]$. We may assume that $L \nvdash \bigwedge\Box\psi \to \Box\chi$, otherwise C would reduce to $\bot$ and could be dropped from our disjunction. Similarly we may assume that the p in C and the q in C are disjoint. Clearly $NB_1 \vdash \bigwedge[\Box\psi, \neg\Box\chi, p, \neg q] \to \Delta\phi$. We claim: $L \vdash \bigwedge\Box\psi \to \phi$. Suppose not. There is a finite L-model $K_\phi = \langle\langle K_\phi, R_\phi\rangle, f_\phi\rangle$ with bottom b_ϕ such that: $b_\phi \nVdash \bigwedge\Box\psi \to \phi(f_\phi)$. Moreover for each of the χ occurring in C there is a finite L-model $K_\chi = \langle\langle K_\chi, R_\chi\rangle, f_\chi\rangle$ with bottom b_χ such that $b_\chi \nVdash \bigwedge\Box\psi \to \Box\chi(f_\chi)$. Let "$\rho$" range over ϕ and the χ in C. Without loss of generality we may assume that the K_ρ are pairwise disjoint and do not contain 0.

We "glue" the K_ρ together to a Carlson-model (and hence a set-model) G in the following way: $G := \langle F, f\rangle$, where $F = \langle K, K_0, S\rangle$ and:

$K := \{0\} \cup \bigcup K_\rho$

$K_0 := \bigcup(K_\rho \backslash b_\rho)$

$xSy :\Leftrightarrow$ ($x=0$ and $y \neq 0$) or (for some ρ: $x, y \in K_\rho$ and $xR_\rho y$)

$fp_i := \bigcup f_\rho p_i \cup \{0 \mid p_i$ is a p in C$\}$

Clearly if $x \in K_\rho$ and $\sigma \in L_0$: $x \Vdash \sigma(f) \Leftrightarrow x \Vdash \sigma(f_\rho)$. $b_\rho \Vdash \Box\psi(f)$ for each of the ψ in C. Moreover: $0Rx \Leftrightarrow$ ($b_\rho Rx$ for some ρ). It follows that $0 \Vdash \Box\psi(f)$. For each χ of C $b_\chi \nVdash \Box\chi(f)$, so there is an x in $K_\chi \backslash b_\chi$ with $x \nVdash \chi(f)$. Hence for each χ in C: $0 \nVdash \Box\chi(f)$. Also $0 \Vdash p(f)$ for the p in C and $0 \nVdash q(f)$ for the q in C. Conclude $0 \Vdash \bigwedge[\Box\psi, \neg\Box\chi, p, \neg q](f)$. Finally $b_\phi \nVdash \phi(f)$.

Now consider $G' := \Psi_S G$. Say $G' = \langle F', f'\rangle$. G' is a Carlson-1-model (and hence a set-1-model). By 8.22 $\langle 0,0\rangle \Vdash C(f')$, because $C \in L_0$ and no formulas of the form $\Delta\sigma$ occur in L_0. Also $\langle b_\phi, 0\rangle \nVdash \phi(f')$ (because $\phi \in L_0$). Hence $\langle 0,0\rangle \nVdash \Delta\phi(f')$. Contradiction!

It follows that $L \vdash (\bigvee\bigwedge\Box\psi) \to \phi$. On the other hand clearly $L \vdash \phi \to \bigvee\bigwedge\Box\psi$. Finally: $L \vdash \bigwedge\Box\psi \leftrightarrow \Box\sigma$ for some σ (which may be taken $\top$ if the conjunction is empty). Hence $L \vdash \phi \leftrightarrow \bigvee\Box\sigma$. □

11.4 Theorem

Suppose ϕ is in L_0. The following are equivalent:

i) ϕ is essentially Σ w.r.t. PA

ii) $NB_1 \vdash \phi \to \Delta\phi$

iii) $L \vdash \phi \leftrightarrow \bigvee\Box\sigma$.

Proof: "(i)⇒(ii)" is 11.1; "(ii)⇒(iii)" is 11.3; "(iii)⇒(i)" is trivial. □

11.5 Kent's Theorem revisited

Clearly if $\phi \in L_0$ is not essentially Σ w.r.t. PA our method should provide us with counterexamples to that effect. Let's by way of example show that $p \wedge \Box p$ is not essentially Σ w.r.t. PA.

First we show that $NB_1 \nvdash (p \wedge \Box p) \rightarrow \Delta(p \wedge \Box p)$. This is easily verified by considering the following Carlson-1-model: $G := \langle\langle\{1,2\},\emptyset,S\rangle,f\rangle$, where 1S2 and fp={1}.

To find the desired arithmetical counterexample we must change G into a set-2-model and embed this into arithmetic.

Let $G' := \langle\langle\{0,1,2\},\emptyset,\{1,2\},S'\rangle,f'\rangle$, where 0S1S2 and f'p={0,1}. Let G'' be $\Psi_R G'$. Clearly G'' is set-2-model of the desired sort. Let [.] be the interpretation function associated with NB, PA and G''. B:=[p] is clearly a counterexample as desired. Note that f''p is downwards closed in G'', so B is (provably equivalent in PA to) a Π_1-sentence.

Inspection of the model G'' shows that B itself has the property: $PA \vdash B \rightarrow \Box_{PA} B$, but B is not provably equivalent in PA to a Σ_1-sentence.

11.6 Open problems

i) Our proof of theorem 11.4 uses the essential reflexiveness of PA. The proof would also work if we substituted any essentially reflexive extension of PRA for PA. What is the situation for PRA? Do we still have the equivalence between (i) and (iii) of 11.4 for PRA? (Conjecture: yes!).

ii) What are the essentially Σ-formulas w.r.t., say, PA in L_0 *extended with the Rosser orderings*, i.e. the language of the theory R of Solovay & Guaspari under the usual interpretation. (This problem was first posed by D. Guaspari, see Guaspari[1983].)

11.7 Remark

Let's extend the language L_0 to $L_0(\Sigma)$ by adding new propositional variables s. An interpretation function for the extended language assigns to the old variables p arithmetical sentences and to the new variables s Σ_1-sentences. Let $L_\Sigma := L + \vdash s \rightarrow \Box s$. It is easily seen that L_Σ is arithmetically sound and complete (interpreting $\Box$ as $\Box_{PA}$). The argument of this § can be extended to show that the essentially Σ-formulas of $L_0(\Sigma)$ are precisely those ϕ that are provably equivalent in L_Σ to formulas of the form $\bigvee\!\!\bigwedge[s,\Box\psi]$.

12 A reduction theorem for Relative Interpretability

Consider a set-2-frame F, of the form $\Phi_S \Phi_R F_a$, where F_a is a finite set-frame. Let [.] be the embedding of the propositions of F into arithmetic defined in 10.5. $S := \exists x\ hx \in K_1$. Let X be a proposition. Finally we write "$\nabla^+_{PA,S}C$" for: $C \vee \nabla_{PA,S}C$, and "$\Diamond^+_{PA}C$" for: $C \vee \Diamond_{PA}C$.

We have:

12.1 Reduction Theorem

$$PA\vdash [X]\lhd_{PA}B \leftrightarrow ((\neg S\wedge\Box_{PA}(B\rightarrow\nabla^{+}{}_{PA,S}[X]))\vee(S\wedge\Box_{PA}(B\rightarrow\Diamond^{+}{}_{PA}[X])))$$

Proof: Reason in PA:

Suppose $\neg S$. We have: $[X]\lhd_{PA}\nabla_{PA,S}[X]$. Hence by I1,I3: $[X]\equiv_{PA}\nabla^{+}{}_{PA,S}[X]$. By 10.5.2: $\Box_{PA}(\nabla^{+}{}_{PA,S}[X]\leftrightarrow[\nabla^{+}X])$, hence, because $\nabla^{+}X$ is downwards closed, $\Box_{PA}(\nabla^{+}{}_{PA,S}[X]\leftrightarrow\forall x\ hx\in\ulcorner\nabla^{+}X\urcorner)$. Hence:

$[X]\lhd_{PA}B$	$\leftrightarrow$	$(\nabla^{+}{}_{PA,S}[X])\lhd_{PA}B$	I2
	$\leftrightarrow$	$(\forall x\ hx\in\ulcorner\nabla^{+}X\urcorner)\lhd_{PA}B$	I1,I2
	$\leftrightarrow$	$\Box_{PA}(B\rightarrow\forall x\ hx\in\ulcorner\nabla^{+}X\urcorner)$	
	$\leftrightarrow$	$\Box_{PA}(B\rightarrow\nabla^{+}{}_{PA,S}[X])$	

Suppose S. By I6: $[X]\lhd_{PA}\Diamond_{PA}[X]$. Hence by I1,I3: $[X]\equiv_{PA}\Diamond^{+}{}_{PA}[X]$. By 10.5.2: $\Box_{PA}(\Diamond^{+}{}_{PA}[X]\leftrightarrow[\Diamond^{+}X])$. By the reasoning of 10.5.2 (case (b) "$\rightarrow$"): $\Box_{PA}L\in K_0$. We claim: $\Box_{PA}([\Diamond^{+}X]\leftrightarrow\forall x\ hx\in\ulcorner\Diamond^{+}X\urcorner)$.

Reason in $\Box$PA:

The "$\leftarrow$" side is trivial. For the "$\rightarrow$" side, suppose $L=z\in\ulcorner\Diamond^{+}X\urcorner$. $z\in K_0$, hence for every y with yWSz we have yRz and thus $y\in\ulcorner\Diamond^{+}X\urcorner$.

$\Box\Box_{PA}$

It follows that: $\Box_{PA}([\Diamond^{+}X]\leftrightarrow\forall x\ hx\in\ulcorner\Diamond^{+}X\urcorner)$. Hence:

$[X]\lhd_{PA}B$	$\leftrightarrow$	$(\Diamond^{+}{}_{PA}[X])\lhd_{PA}B$	I2
	$\leftrightarrow$	$(\forall x\ hx\in\ulcorner\Diamond^{+}X\urcorner)\lhd_{PA}B$	I1,I2
	$\leftrightarrow$	$\Box_{PA}(B\rightarrow\forall x\ hx\in\ulcorner\Diamond^{+}X\urcorner)$	J
	$\leftrightarrow$	$\Box_{PA}(B\rightarrow\Diamond^{+}{}_{PA}[X])$	

$\Box$PA

Translating the above result back to the frame we define:

$$X\lhd Y := (\neg K_1\wedge\Box(Y\rightarrow\nabla^{+}X))\vee(K_1\wedge\Box(Y\rightarrow\Diamond^{+}X)).$$

Alternatively define:

$x\in(X\lhd Y) :\Leftrightarrow$ for every y with xRy and $y\in Y$ there is a z with xRz, yWSz and $z\in X$.

As is easily seen both definitions amount to the same thing. The Reduction Theorem implies: $PA\vdash [X\lhd Y] \leftrightarrow [X]\lhd_{PA}[Y]$.

The Reduction Theorem can be used to produce arithmetical counterexamples to various principles for $\Box_{PA}$ and $\lhd_{PA}$. For example consider the following countermodel to $p\lhd\top\rightarrow\Box(p\lhd\top)$: first define G_a by: K_a:={1,2,3,4}, K_{a1}:={2,3,4}, K_{a0}:={3}, 1S2S3S4, $f_a p$={4}. G:= $\Psi_S\Psi_R G_a$. Clearly <<1,0>,0>$\Vdash p\lhd\top$(f), but <<1,0>,0>$\nVdash \Box(p\lhd\top)$(f). Hence:

$$PA\nvdash [p]\lhd_{PA}\top \rightarrow \Box_{PA}([p]\lhd_{PA}\top).$$

Note that [p] is provably equivalent to a Σ_1-sentence. So our present counterexample is as good as the one produced in 7.3.4.

Excercise (De Jongh): Show that there are A,B,C in the language of PA such that:

$$PA\nvdash (\ ((B\vee C)\lhd_{PA}A)\wedge\neg(B\lhd_{PA}A)\wedge\neg(C\lhd_{PA}A)\wedge\Box_{PA}\Box_{PA}\bot\wedge\Box_{PA}((A\wedge\neg B\wedge\neg C)\vee(B\wedge\neg A \wedge\neg C)\vee(C\wedge\neg A\wedge\neg B))\ )\rightarrow(B\lhd_{PA}A\vee C\lhd_{PA}A)\ .$$

Our present reduction theorem is to poor to produce counterexamples to all arithmetically non-valid principles for $\lhd_{PA}$, e.g. $\vdash p\lhd\top\rightarrow\neg(\neg p\lhd\top)$ is arithmetically non-valid, but in no set-2-frame is there a clopen X with $(X\lhd\top\wedge\neg X\lhd\top)=\top$.

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