Arch. Math. Logic 44, 473–491 (2005)
Digital Object Identifier (DOI):
10.1007/s00153-004-0266-7 **Mathematical Logic**

George Voutsadakis

Categorical abstract algebraic logic categorical algebraization of first-order logic without terms

Received: 23 March 2004 /
Published online: 8 September 2004 –

Abstract. An algebraization of multi-signature first-order logic without terms is presented. Rather than following the traditional method of choosing a type of algebras and constructing an appropriate variety, as is done in the case of cylindric and polyadic algebras, a new categorical algebraization method is used: The substitutions of formulas of one signature for relation symbols in another are treated in the object language. This enables the automatic generation via an adjunction of an algebraic theory. The algebras of this theory are then used to algebraize first-order logic.

1. Introduction

An algebraization of a system of first-order logic without terms is presented. The purpose is three-fold. First, instead of using either traditional concrete algebraic logic methods, as is done in the case of cylindric [11] and polyadic algebras [10], or the universal abstract algebraic logic method, as is done in Appendix C of [3], a novel categorical algebraization method (see [15] and [17]) which has proven more effective in the case of multi-signature logics is used. Second, the gap of the absence of a treatment of the algebraization of specific multi-signature logics via the modern categorical method, filled by [18] for equational logic, is now being filled for first-order logic without terms. In this sense, this paper is continuing that work and follows the same structure in presentation. The hope is that they will provide an impetus for investigating the algebraization of an increasing number of multi-signature systems using the modern method. This will lead to a better understanding of both the method and of the logical systems and the algebraizing systems of algebras. Finally, this paper fulfills a promise, given in [17], where a very similar

G. Voutsadakis: School of Mathematics and Computer Science, Lake Superior State University, Sault Sainte Marie, MI 49783, USA

Partially supported by National Science Foundation grant CCR - 9593168

Mathematics Subject Classification (1991): Primary: 03Gxx, 18Cxx, Secondary: 08Bxx, 08Cxx, 68N05

Keywords or phrases: Algebraic logic – Equivalent deductive systems – Algebraizable logics – Institutions – Equivalent institutions – Algebraizable institutions – Algebraic theories – Monads – Triples – Adjunctions – First-order logic – Cylindric algebras – Polyadic algebras

algebraization process was sketched without details. It was promised that details would be given elsewhere. Despite following a slightly different presentation than the one in [17], the present paper, hopefully, fulfills that promise, since the two developments are completely parallel.

One of the first and best known treatments of the algebraization of first-order logic was the creation of cylindric algebras [11]. They consist, roughly speaking, of a Boolean reduct, unary operations corresponding to existential quantifications and nullary operations corresponding to equalities between individual variables. Later, polyadic algebras were invented by Halmos [10] and constitute an alternative attempt at the algebraization of first-order logic. The extra Boolean operations here correspond to substitutions and existential quantifications. There are two characteristics of the algebraizations of first-order logic via cylindric and polyadic algebras that become apparent by observing their definitions. First, the algebraic signatures of these classes of algebras are chosen in an ad-hoc, artificial way, based on the perception of the importance of different aspects of the logical system to be algebraized. In contrast, the method presented here, exactly because it is categorically-based, generates an entire clone of operations. Second, based on the choice of signatures, the collection of identities that are postulated for these systems, as chosen among the complete set of identities corresponding to properties of the logical system, is an artificial, tedious and, rather unsatisfactory task. After choosing these set of identities, one has to prove representation theorems to explore how well the identities chosen capture the essence of the logical system. In the case of cylindric algebras, this task occupies a good part of [11] and [12].

These characteristics naturally led to the investigation of more methodical and natural ways of algebraizing logical systems. Blok and Pigozzi [3], based on work of Czelakowski [5] and their own previous work [2], started a systematic investigation of the process of algebraizability of logical systems. The bulk of work that has appeared since and has been influenced by their monograph has come to be collectively known under the term *abstract algebraic logic*. It uses universal algebraic techniques to explore the algebraization of classes of sentential logics and provides methods for automatically associating classes of algebras, constituting the algebraic counterparts, to sentential logics that are *algebraizable*, i.e., amenable to these algebraization techniques. In Appendix C of [3], an attempt at the algebraization of first-order logic in this new context is made. Although the algebraization presented there is much more natural than the ones summarized above, it has a serious drawback that stems from an inherent restriction in the framework of [3]. This framework is geared towards dealing with sentential structural logics. First-order logic is neither sentential nor structural. Therefore, before its algebraization via the techniques of [3], it has to be transformed to a sentential structural counterpart. It is this transformation, based on cylindric algebras, that is now ad-hoc. The entire process is still unsatisfactory; only the shortcomings have shifted from the actual algebraization process to the process of adapting the logical system to make it amenable to the general algebraization method.

These problems have been dealt with in the categorical abstract algebraic logic approach which has been developed by the author under the supervision of Pigozzi (see [15], [16] and [17]). It generalizes the framework of [3], using categorical algebraic techniques, to specifically address the problems mentioned above. The main innovation lies in using the institution framework of Goguen and Burstall [8], [9] instead of that of sentential logics. Institutions have proven to be very effective in handling logics with multiple signatures and quantifiers. One of their key features that is used in the theory of algebraization is the inclusion of the substitution operators in the object language in the description of a logical system rather than their delegation to the metalanguage, as is done in the sentential logic context. Including the substitution operations in the object language allows the generation of an algebraic theory via an adjunction and algebras of the theory are then used to carry out the final stage of the algebraization.

It has to be pointed out that the algebraization of first-order logic that is presented in this paper is not purely categorical but rather a hybrid between a universal algebraic and a categorical algebraic treatment. The contrast between the two treatments is more transparent and more clearly seen in the algebraization of equational logic, which has been carried out in [18].

For the categorical language used in the paper and the terminology concerning categorical algebra, the reader is referred to [1], [13] and [14]. By **Set** will be denoted throughout the category of small sets.

Recall the definition of an institution [8], [9], and that of a π-institution [7]. Given an institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$, define, for all $\Sigma \in |\mathbf{Sign}|$, $\Gamma \subseteq \mathrm{SEN}(\Sigma)$, the collection of models $M \subseteq |\mathrm{MOD}(\Sigma)|$ of Γ by

$$\Gamma^* = \{m \in |\mathrm{MOD}(\Sigma)| : m \models_\Sigma \Gamma\}$$

and the theory of M by

$$M^* = \{\phi \in \mathrm{SEN}(\Sigma) : M \models_\Sigma \phi\}$$

and let $C_\Sigma(\Gamma) = \Gamma^{**}$, for all $\Sigma \in |\mathbf{Sign}|$, $\Gamma \subseteq \mathrm{SEN}(\Sigma)$, be the deductive closure of Γ. Then $\pi(\mathcal{I}) = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ is a π-institution, called the **π-institution associated with** the institution $\mathcal{I}$ and denoted by $\pi(\mathcal{I})$, or, sometimes, also by $\mathcal{I}$, for simplicity. In the sequel, instead of $C_\Sigma(\Gamma)$ to denote the closure of a set Γ of Σ-sentences of an institution or of a π-institution, the simplifying notation Γ^c will be used. Since the signature Σ is usually clear from context, this notation will not cause any confusion.

Let $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle$ and $\mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ be two π-institutions. A **translation** of $\mathcal{I}_1$ in $\mathcal{I}_2$ is a pair $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ consisting of a functor $F : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ and a natural transformation $\alpha : \mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_2 F$.

A translation is called an **interpretation** if, in addition, for all $\Sigma_1 \in |\mathbf{Sign}_1|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}_1(\Sigma_1)$,

$$\phi \in C_{\Sigma_1}(\Phi) \quad \text{if and only if} \quad \alpha_{\Sigma_1}(\phi) \subseteq C_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\Phi)).$$

$\mathcal{I}_1$ and $\mathcal{I}_2$ are called **deductively equivalent** if there exist interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ and $\langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$, such that

1. $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ is an adjoint equivalence, for some natural transformations η, ϵ,

2. for all $\Sigma_1 \in |\mathbf{Sign}_1|$, $\Sigma_2 \in |\mathbf{Sign}_2|$, $\phi \in \mathrm{SEN}_1(\Sigma_1)$, $\psi \in \mathrm{SEN}_2(\Sigma_2)$,

$$C_{G(F(\Sigma_1))}(\mathrm{SEN}_1(\eta_{\Sigma_1})(\phi)) = C_{G(F(\Sigma_1))}(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi)))$$

$$C_{\Sigma_2}(\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\psi)))) = C_{\Sigma_2}(\psi).$$

Let $\mathbf{C}$ be a category, $\mathbf{T} = \langle T, \eta, \mu \rangle$ an algebraic theory in monoid form in $\mathbf{C}$, $\mathbf{L}$ a full subcategory of the Kleisli category $\mathbf{C_T}$ of $\mathbf{T}$ in $\mathbf{C}$, $\Xi : \mathbf{C} \to \mathbf{Set}$ a functor and $\mathbf{Q}$ a subcategory of the Eilenberg-Moore category $\mathbf{C^T}$ of $\mathbf{T}$ in $\mathbf{C}$. Also denote by $\langle F_\mathbf{T}, U_\mathbf{T}, \eta_\mathbf{T}, \epsilon_\mathbf{T} \rangle : \mathbf{C} \to \mathbf{C_T}$ and by $\langle F^\mathbf{T}, U^\mathbf{T}, \eta^\mathbf{T}, \epsilon^\mathbf{T} \rangle : \mathbf{C} \to \mathbf{C^T}$ the Kleisli and the Eilenberg-Moore adjunctions of $\mathbf{T}$ in $\mathbf{C}$, respectively. Define the $\langle \mathbf{L}, \Xi, \mathbf{Q} \rangle$**-algebraic institution** [17] $\mathcal{I}_{\langle \mathbf{L}, \Xi, \mathbf{Q} \rangle} = \langle \mathbf{L}, \mathrm{EQ}, \mathrm{ALG}, \models \rangle$ as follows:

(i) $\mathrm{EQ} : \mathbf{L} \to \mathbf{Set}$ is given by $\mathrm{EQ} = ((\Xi \circ U_\mathbf{T}) \restriction_\mathbf{L})^2$, i.e.,

$$\mathrm{EQ}(L) = \Xi(T(L))^2, \quad \text{for every } L \in |\mathbf{L}|,$$

and, given $f : L \rightharpoonup K \in \mathrm{Mor}(\mathbf{L})$,

$$\mathrm{EQ}(f)(\langle s, t \rangle) = (\Xi(\mu_K T(f))(s), \Xi(\mu_K T(f))(t)),$$

for all $\langle s, t \rangle \in \Xi(T(L))^2$.

$$\Xi(T(L)) \xrightarrow[\Xi(T(f))]{} \Xi(T(T(K))) \xrightarrow[\Xi(\mu_K)]{} \Xi(T(K))$$

(ii) $\mathrm{ALG} : \mathbf{L} \to \mathbf{CAT}^{\mathrm{op}}$ is the functor that sends an object $L \in |\mathbf{L}|$ to the category $\mathrm{ALG}(L)$ with objects triples of the form $\langle \langle X, \xi \rangle, f \rangle$, $\langle X, \xi \rangle \in |\mathbf{Q}|$, $f : L \rightharpoonup X \in \mathrm{Mor}(\mathbf{C_T})$, and morphisms $h : \langle \langle X, \xi \rangle, f \rangle \to \langle \langle Y, \zeta \rangle, g \rangle$ $\mathbf{Q}$-morphisms $h : \langle X, \xi \rangle \to \langle Y, \zeta \rangle$, such that $g = T(h)f$.

$$\begin{array}{ccc} & L & \\ {}^{f}\swarrow & & \searrow^{g} \\ T(X) & \xrightarrow[T(h)]{} & T(Y) \end{array}$$

Moreover, given $k : L \rightharpoonup K \in \mathrm{Mor}(\mathbf{L})$, $\mathrm{ALG}(k) : \mathrm{ALG}(K) \to \mathrm{ALG}(L)$ is the functor that sends $\langle \langle X, \xi \rangle, f \rangle \in |\mathrm{ALG}(K)|$ to $\langle \langle X, \xi \rangle, f \circ k \rangle \in |\mathrm{ALG}(L)|$ and $h : \langle \langle X, \xi \rangle, f \rangle \to \langle \langle Y, \zeta \rangle, g \rangle \in \mathrm{Mor}(\mathrm{ALG}(K))$ to

$$\mathrm{ALG}(k)(h) = h : \langle \langle X, \xi \rangle, f \circ k \rangle \to \langle \langle Y, \zeta \rangle, g \circ k \rangle \in \mathrm{Mor}(\mathrm{ALG}(L)).$$

(iii) $\models_L \subseteq |\mathrm{ALG}(L)| \times \mathrm{EQ}(L)$ is defined by

$$\langle \langle X, \xi \rangle, f \rangle \models_L \langle s, t \rangle \text{ iff } \Xi(\xi \mu_X T(f))(s) = \Xi(\xi \mu_X T(f))(t),$$

$$T(L) \xrightarrow[T(f)]{} T(T(X)) \xrightarrow[\mu_X]{} T(X) \xrightarrow[\xi]{} X$$

for all $\langle \langle X, \xi \rangle, f \rangle \in |\mathrm{ALG}(L)|$, $\langle s, t \rangle \in \mathrm{EQ}(L)$.

By the $\langle \mathbf{L}, \Xi, \mathbf{Q} \rangle$-**algebraic** π-**institution**, we will understand the π-institution (also denoted by $\mathcal{I}_{\langle \mathbf{L}, \Xi, \mathbf{Q} \rangle}$) associated with the institution $\mathcal{I}_{\langle \mathbf{L}, \Xi, \mathbf{Q} \rangle}$.

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ is **algebraizable** if it is deductively equivalent to an $\langle \mathbf{L}, \Xi, \mathbf{Q} \rangle$-algebraic π-institution. Similarly, an institution $\mathcal{I}$ is **algebraizable** if it is deductively equivalent to an $\langle \mathbf{L}, \Xi, \mathbf{Q} \rangle$-algebraic institution.

In Section 2, the institution of first-order logic without terms is defined in detail. This differs from the institution of Goguen and Burstall [9] in several ways. First, it is restricted to single sorted structures rather than handling the general case of multiple sorts. On the other hand, the institution of Section 2 allows for the substitution of arbitrary formulas of one signature for basic relation symbols of another whereas the one presented in [9] is restricted to substitutions of basic relation symbols for basic relation symbols. The added generality, in this respect, is crucial for our algebraization framework. On the other hand, it has the negative effect that, in the model categories, only isomorphisms of models may be considered. First-order structure homomorphisms that fail to be either injective or surjective do not satisfy the properties required for the structure constructed in Section 2 to be an institution. The presentation is split into syntax, semantics and the interaction between them via the Tarski-style satisfaction relations. In Section 3, the algebraic institution of first-order algebras, corresponding to cylindric algebras in the classical framework, is constructed. The exact relationship between the categorical algebras of this institution and cylindric algebras was further investigated in [20]. The adjunction that gives rise to the algebraic theory is developed first. The theory is then described as it is naturally extracted from the adjunction in the usual way. They both form the basis of the algebraic institution of first-order algebra. Finally, in Section 4, the actual algebraization process is presented. The functors and the natural transformations are first constructed and, then, the conditions that show that the corresponding translations are inverse interpretations are proven in detail.

2. First-Order Logic Without Terms

The Underlying Category

Recall that by **Set** is denoted the category of all small sets. By ω will be denoted the set of natural numbers, $N \subset_{\mathrm{f}} \omega$ will mean that N is a finite subset of ω and $\mathcal{P}_{\mathrm{f}}(\omega)$ will denote the set of all finite subsets of ω.

Definition 1. *By a* **hierarchy of sets** *or, simply, an* **h-set** *A, we mean a family of sets* $A = \{A_N : N \in \mathcal{P}_{\mathrm{f}}(\omega)\}$, *such that* $A_N \cap A_M = A_{N \cap M}$, *for every* $N, M \subset_{\mathrm{f}} \omega$. *By a* **morphism of h-sets** *or, simply, an* **h-set morphism** $f : A \to B$, *we mean a family of set maps* $f = \{f_N : A_N \to B_N : N \in \mathcal{P}_{\mathrm{f}}(\omega)\}$, *such that the following diagram commutes, for every* $N \subseteq M \subset_{\mathrm{f}} \omega$,

$$\begin{array}{ccc} A_M & \xrightarrow{f_M} & B_M \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ A_N & \xrightarrow[f_N]{} & B_N \end{array}$$

where by $i : A_N \hookrightarrow A_M$ *and* $i : B_N \hookrightarrow B_M$ *we denote the inclusion maps, i.e.,* $f_M \restriction_{A_N} = f_N$, *for all* $N \subseteq M \subset_{\mathrm{f}} \omega$.

Given two chain set morphisms $f : A \to B$ and $g : B \to C$ we define their **composite** $gf : A \to C$ to be the collection of maps $gf = \{g_N f_N : A_N \to C_N : N \in \mathcal{P}_{\mathrm{f}}(\omega)\}$. With this composition the collection of h-sets with h-set morphisms between them forms a category. It is called the **category of h-sets** and denoted by **HSet**.

The Signatures

To be able to represent relational structures in a satisfactory way, a formalization of a system of first-order logic without terms must handle efficiently signatures, consisting of relation symbols with prespecified arities, together with all possible interpretations of one signature in another. Use of the institution structure as the underlying formalism encourages viewing relational signatures as objects in a category and the interactions between them as morphisms in this category. This category, called **Sign**, will now be defined.

By $\mathcal{L}$ is denoted the set of symbols $\{\neg, \wedge\} \cup \{\exists_k : k \in \omega\}$, which will be used as connectives and quantifiers, respectively, in the construction of the formulas below. Given a set X, by $\overline{X}$ will be denoted an isomorphic copy of X constructed in some canonical way. $\overline{x}$ will denote the copy of $x \in X$ in the set $\overline{X}$.

Definition 2. *Let* $X \in |\mathbf{HSet}|$. *The* **h-set of** X**-formulas**

$$\mathrm{Fm}_{\mathcal{L}}(X) = \{\mathrm{Fm}_{\mathcal{L}}(X)_N : N \in \mathcal{P}_{\mathrm{f}}(\omega)\} \in |\mathbf{HSet}|$$

is defined by letting $\mathrm{Fm}_{\mathcal{L}}(X)_N$ *be the smallest set with*

– $\mathrm{v}_i \approx \mathrm{v}_j \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, *for all* $i, j \in N$,
– $\overline{x} \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, *for every* $x \in X_N$,
– $\neg\phi, \phi_1 \wedge \phi_2 \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, *for all* $\phi, \phi_1, \phi_2 \in \mathrm{Fm}_{\mathcal{L}}(X)_N$,
– $\exists_k \phi \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, *for every* $\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_{N \cup \{k\}}$.

By an easy induction on the structure of X-formulas, it may be verified that, in fact, $\mathrm{Fm}_{\mathcal{L}}(X) \in |\mathbf{HSet}|$.

Given two h-sets X and Y, any h-set morphism f from X into the h-set $\mathrm{Fm}_{\mathcal{L}}(Y)$ may be extended to an h-set morphism f^* from $\mathrm{Fm}_{\mathcal{L}}(X)$ into $\mathrm{Fm}_{\mathcal{L}}(Y)$. The definition of this extension is given next.

Definition 3. *Let* $X, Y \in |\mathbf{HSet}|$, $f : X \to \mathrm{Fm}_{\mathcal{L}}(Y) \in \mathrm{Mor}(\mathbf{HSet})$. *Define* $f^* : \mathrm{Fm}_{\mathcal{L}}(X) \to \mathrm{Fm}_{\mathcal{L}}(Y)$, *with* $f^*_N : \mathrm{Fm}_{\mathcal{L}}(X)_N \to \mathrm{Fm}_{\mathcal{L}}(Y)_N$, *for every* $N \subset_{\mathrm{f}} \omega$, *by recursion on the structure of* X*-formulas as follows:*

– $f^*_N(\mathrm{v}_i \approx \mathrm{v}_j) = \mathrm{v}_i \approx \mathrm{v}_j$, *for all* $i, j \in N$,
– $f^*_N(\overline{x}) = f_N(x)$, *for every* $x \in X_N$,
– $f^*_N(\neg\phi) = \neg f^*_N(\phi)$, $f^*_N(\phi_1 \wedge \phi_2) = f^*_N(\phi_1) \wedge f^*_N(\phi_2)$, *for all* $\phi, \phi_1, \phi_2 \in \mathrm{Fm}_{\mathcal{L}}(X)_N$,
– $f^*_N(\exists_k \phi) = \exists_k f^*_{N \cup \{k\}}(\phi)$, *for every* $\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_{N \cup \{k\}}$.

It is not hard to check that $f^* : \mathrm{Fm}_{\mathcal{L}}(X) \to \mathrm{Fm}_{\mathcal{L}}(Y)$, as defined in 3, is an h-set morphism. In the sequel, we write $f : X \rightharpoondown Y$ to denote an **HSet**-morphism $f : X \to \mathrm{Fm}_{\mathcal{L}}(Y)$. Given two such maps $f : X \rightharpoondown Y, g : Y \rightharpoondown Z$, their **composition** $g \circ f : X \rightharpoondown Z$ is defined to be the **HSet**-morphism

$$g \circ f = g^* f.$$

It will now be shown that the composition $\circ$ is associative, i.e., that given three morphisms $f : X \rightharpoondown Y, g : Y \rightharpoondown Z$ and $h : Z \rightharpoondown W$, $(h \circ g) \circ f = h \circ (g \circ f)$. A technical lemma is needed first.

Lemma 4. *Let* $f : X \rightharpoondown Y, g : Y \rightharpoondown Z$. *Then* $(g \circ f)^* = g^* f^*$.

Proof. It suffices to show that, for every $N \subset_{\mathrm{f}} \omega$, $\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, $(g \circ f)^*_N(\phi) = g^*_N(f^*_N(\phi))$. We use induction on the structure of X-formulas.

If $i, j \in N$, $(g \circ f)^*_N(\mathrm{v}_i \approx \mathrm{v}_j) = \mathrm{v}_i \approx \mathrm{v}_j = g^*_N(f^*_N(\mathrm{v}_i \approx \mathrm{v}_j))$.

If $x \in X_N$, $(g \circ f)^*_N(\overline{x}) = (g \circ f)_N(x) = (g^* f)_N(X) = g^*_N f_N(x) = g^*_N(f^*_N(\overline{x}))$.

If $\phi \in \mathrm{Fm}_{\mathcal{L}}(N)$,

$$\begin{aligned}(g \circ f)^*_N(\neg\phi) &= \neg(g \circ f)^*_N(\phi)\\ &= \neg g^*_N(f^*_N(\phi) \quad \text{(by the induction hypothesis)}\\ &= g^*_N(\neg f^*_N(\phi))\\ &= g^*_N(f^*_N(\neg\phi)).\end{aligned}$$

The case of $\wedge$ is handled similarly. For $\exists_k$, if $\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_{N\cup\{k\}}$, we have

$$\begin{aligned}(g \circ f)^*_N(\exists_k\phi) &= \exists_k(g \circ f)^*_{N\cup\{k\}}(\phi)\\ &= \exists_k g^*_{N\cup\{k\}}(f^*_{N\cup\{k\}}(\phi)) \quad \text{(by the induction hypothesis)}\\ &= g^*_N(\exists_k f^*_{N\cup\{k\}}(\phi))\\ &= g^*_N(f^*_N(\exists_k\phi)).\end{aligned}$$

□

If $f : X \rightharpoondown Y, g : Y \rightharpoondown Z$ and $h : Z \rightharpoondown W$ we have

$$\begin{aligned}h \circ (g \circ f) &= h^*(g \circ f)\\ &= h^*(g^* f)\\ &= (h^* g^*) f\\ &= (h \circ g)^* f \quad \text{(by Lemma 4)}\\ &= (h \circ g) \circ f,\end{aligned}$$

whence $\circ$ is associative as claimed.

Now define $j_X : X \rightharpoondown X$, given by $j_{X_N} : X_N \to \mathrm{Fm}_{\mathcal{L}}(X)_N$, with

$$j_{X_N}(x) = \overline{x}, \quad \text{for all } x \in X_N.$$

It is not hard to prove that, given $f : X \rightharpoondown Y$ and $g : Z \rightharpoondown X$ we have $f \circ j_X = f$ and $j_X \circ g = g$.

The discussion above shows that **Sign**, with collection of objects |**HSet**| and collection of morphisms

$$\mathbf{Sign}(X, Y) = \{f : X \rightharpoondown Y : f \in \mathbf{HSet}(X, \mathrm{Fm}_{\mathcal{L}}(Y))\},$$

for all $X, Y \in |\mathbf{HSet}|$, with composition $\circ$ and X-identity j_X, is a category.

The Syntax

In formalizing a logical system, its syntactic component has to be defined first. In classical deductive systems this consists of defining the well-formed formulas and the substitutions of formulas for individual variables, which are metalogical operations, i.e., take place in the metalanguage rather than the object language itself. In a multi-signature system, on the other hand, like the system of first-order logic that we are about to construct, a set of well-formed formulas for each relational signature has to be defined and the effect of the different possible interpretations of one signature in another on formulas has to be specified. The choice of the institution structure as the underlying formalism in this context makes it possible to unify the setting by considering a functor SEN : **Sign** $\rightarrow$ **Set**, whose object part gives the X-formulas, for each chosen signature X, and whose morphism part specifies the effect of signature interpretations on formulas.

At the object level, for every $X \in |\mathbf{Sign}|$, $\mathrm{SEN}(X) = \mathrm{Fm}_{\mathcal{L}}(X)_{\emptyset}$.

At the morphism level, given $f : X \rightharpoonup Y \in \mathrm{Mor}(\mathbf{Sign})$, we define $\mathrm{SEN}(f) : \mathrm{SEN}(X) \rightarrow \mathrm{SEN}(Y)$ by letting, for all $\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_{\emptyset}$,

$$\mathrm{SEN}(f)(\phi) = f^*_{\emptyset}(\phi).$$

It is not difficult to see that SEN is indeed a functor.

The Semantics

We start by describing first the model functor MOD : **Sign** $\rightarrow$ $\mathbf{CAT}^{\mathrm{op}}$ at the object level. Let A be a set. By $\mathrm{Rel}(A) \in |\mathbf{HSet}|$ we denote the h-set whose N-th level $\mathrm{Rel}_N(A)$ consists of all relations $r \subseteq A^{\omega}$ that depend only on the individual variables indexed by the elements of N. Let $X \in |\mathbf{HSet}|$. By an X-**structure** $\mathbf{A} = \langle A, X^{\mathbf{A}} \rangle$ we mean a pair consisting of a set A and an **HSet**-morphism $X^{\mathbf{A}} : X \rightarrow \mathrm{Rel}(A)$. As is customary, the notation $x^{\mathbf{A}}$ will be used in place of $X^{\mathbf{A}}(x)$, for $x \in X$. Given two X-structures $\mathbf{A} = \langle A, X^{\mathbf{A}} \rangle, \mathbf{B} = \langle B, X^{\mathbf{B}} \rangle$, by an X-**structure morphism** $h : \mathbf{A} \rightarrow \mathbf{B}$ we mean a bijective **Set**-map $h : A \rightarrow B$, such that, for all $N \subset_{\mathrm{f}} \omega, x \in X_N$,

$$\vec{a} \in x^{\mathbf{A}} \quad \text{if and only if} \quad h(\vec{a}) \in x^{\mathbf{B}}, \quad \text{for every } \vec{a} \in A^{\omega}.$$

X-structures with X-structure morphisms between then form a category, denoted by $\mathrm{MOD}(X)$.

Given an X-structure $\mathbf{A} = \langle A, X^{\mathbf{A}} \rangle$, the interpretation of the relation symbols in X in $\mathrm{Rel}(A)$ may be extended to an interpretation of all formulas in $\mathrm{Fm}_{\mathcal{L}}(X)$ in $\mathrm{Rel}(A)$. More precisely, define the **HSet**-map ${}^{\mathbf{A}} : \mathrm{Fm}_{\mathcal{L}}(X) \rightarrow \mathrm{Rel}(A)$ by induction on the structure of X-formulas as follows:

- $(\mathrm{v}_i \approx \mathrm{v}_j)^{\mathbf{A}} = \{\vec{a} \in A^{\omega} : a_i = a_j\}$, for all $i, j \in N$,
- $\overline{x}^{\mathbf{A}} = x^{\mathbf{A}}$, for every $x \in X_N$,
- $(\neg\phi)^{\mathbf{A}} = A^{\omega} - \phi^{\mathbf{A}}$, $(\phi_1 \wedge \phi_2)^{\mathbf{A}} = \phi_1^{\mathbf{A}} \cap \phi_2^{\mathbf{A}}$, for all $\phi, \phi_1, \phi_2 \in \mathrm{Fm}_{\mathcal{L}}(X)_N$,
- $(\exists_k\phi)^{\mathbf{A}} = \{\vec{b} \in A^{\omega} : b_i = a_i \ \forall\ i \neq k \ \text{ and } \ \vec{a} \in \phi^{\mathbf{A}}\}$, for all $\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_{N \cup \{k\}}$.

The following lemma makes explicit the fact that truth sets of formulas are preserved by model morphisms. It may be proved by an easy induction on the structure of X-formulas. Note that the presence of equality in the language immediately excludes the consideration of non injective model morphisms. These would not satisfy the lemma.

Lemma 5. *Let* $\mathbf{A} = \langle A, X^{\mathbf{A}}\rangle, \mathbf{B} = \langle B, X^{\mathbf{B}}\rangle \in |\mathrm{MOD}(X)|, h : \mathbf{A} \to \mathbf{B} \in \mathrm{Mor}(\mathrm{MOD}(X))$ *and* $\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_N$. *Then*

$$\vec{a} \in \phi^{\mathbf{A}} \quad \textit{if and only if} \quad h(\vec{a}) \in \phi^{\mathbf{B}}, \quad \textit{for all } \vec{a} \in A^{\omega}.$$

Next, we define MOD at the morphism level. If $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{Sign})$, $\mathrm{MOD}(f) : \mathrm{MOD}(Y) \to \mathrm{MOD}(X)$ is the functor defined as follows: Given $\mathbf{A} = \langle A, Y^{\mathbf{A}}\rangle \in |\mathrm{MOD}(Y)|$,

$$\mathrm{MOD}(f)(\mathbf{A}) = \langle A, X^{\mathrm{MOD}(f)(\mathbf{A})}\rangle, \quad \text{with}$$

$$x^{\mathrm{MOD}(f)(\mathbf{A})} = f(x)^{\mathbf{A}}, \quad \text{for all } x \in X.$$

Moreover, given a morphism $h : \langle A, Y^{\mathbf{A}}\rangle \to \langle B, Y^{\mathbf{B}}\rangle \in \mathrm{Mor}(\mathrm{MOD}(Y))$, $\mathrm{MOD}(f)(h) : \langle A, X^{\mathrm{MOD}(f)(\mathbf{A})}\rangle \to \langle B, X^{\mathrm{MOD}(f)(\mathbf{B})}\rangle$ is given by

$$\mathrm{MOD}(f)(h) = h.$$

Lemma 5 may be used to show that $\mathrm{MOD}(f)$ is well-defined at the morphism level. It is then immediate that $\mathrm{MOD} : \mathbf{Sign} \to \mathbf{CAT}^{\mathrm{op}}$, as defined above, is a functor.

Syntax, Semantics and Satisfaction

The syntax and the semantics of first-order logic without terms were defined in the previous two subsections. Now it remains to see how these two interact. This is the most important feature of the logic, since it allows the specification of a deductive apparatus. This interaction takes the form of a satisfaction relation between models and sentences. Following the work of Tarski, one has to specify when a sentence of the logic is satisfied by a given model. Since a multi-signature system is under consideration, a collection of such satisfaction relations will be defined. More precisely, for each signature X, one has to define what it means for an X-structure $\mathbf{A}$ to satisfy an X-sentence ϕ. Using the institution framework we proceed by completing the definition of the appropriate institution representing the system of first-order logic without terms.

Define $\mathcal{FOL} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ by letting **Sign** be the category defined in “The Signatures” subsection, $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be the functor defined in “The Syntax” subsection, $\mathrm{MOD} : \mathbf{Sign} \to \mathbf{CAT}^{\mathrm{op}}$ be the functor defined in “The Semantics” subsection and, for every $X \in |\mathbf{Sign}|$, $\models_X \subseteq |\mathrm{MOD}(X)| \times \mathrm{SEN}(X)$ be defined by

$$\langle A, X^{\mathbf{A}}\rangle \models_X \phi \quad \text{if and only if} \quad \phi^{\mathbf{A}} = A^{\omega},$$

for all $\mathbf{A} = \langle A, X^{\mathbf{A}}\rangle \in |\mathrm{MOD}(X)|$, $\phi \in \mathrm{SEN}(X)$.

The institution formalism requires showing that "truth is invariant under change of notation". This slogan means that, if one signature gets interpreted in another via a signature morphism, the induced transformations on the sentences and the models will be such that satisfaction will not be affected. Formally, if $f : X \rightharpoonup Y \in \text{Mor}(\mathbf{Sign})$, $\phi \in \text{SEN}(X)$ and $\mathbf{A} = \langle A, Y^{\mathbf{A}}\rangle \in |\text{MOD}(Y)|$, we must have

$$\langle A, X^{\text{MOD}(f)(\mathbf{A})}\rangle \models_X \phi \quad \text{if and only if} \quad \langle A, Y^{\mathbf{A}}\rangle \models_Y \text{SEN}(f)(\phi). \qquad (1)$$

To show that this equivalence holds in the present context a technical lemma, Lemma 6, is needed first. It expresses the fact that the interpretation of a given closed sentence in a given model under the image of an assignment equals the interpretation of the image of the closed sentence in the model under the same assignment. In other words, changing the assignment is tantamount to changing the formula using the same signature morphism. Lemma 6 may be proved by a simple induction on the structure of X-sentences, so its proof will be omitted.

Lemma 6. *Let* $X, Y \in |\mathbf{Sign}|$, $f : X \rightharpoonup Y \in \text{Mor}(\mathbf{Sign})$, $\phi \in \text{Fm}_{\mathcal{L}}(X)_{\emptyset}$ *and* $\mathbf{A} = \langle A, Y^{\mathbf{A}}\rangle \in |\text{MOD}(Y)|$. *Then*

$$\phi^{\text{MOD}(f)(\mathbf{A})} = f^*_{\emptyset}(\phi)^{\mathbf{A}}.$$

Lemma 6 may now be used to prove that the satisfaction relation (1) holds:

$$\begin{aligned}\langle A, X^{\text{MOD}(f)(\mathbf{A})}\rangle \models_X \phi \ &\text{iff } \phi^{\text{MOD}(f)(\mathbf{A})} = A^{\omega}\\ &\text{iff } f^*_{\emptyset}(\phi)^{\mathbf{A}} = A^{\omega}\\ &\text{iff } \langle A, Y^{\mathbf{A}}\rangle \models_Y f^*_{\emptyset}(\phi)\\ &\text{iff } \langle A, Y^{\mathbf{A}}\rangle \models_Y \text{SEN}(f)(\phi),\end{aligned}$$

as claimed. $\mathcal{FOL}$ is called the **institution of first-order logic (without terms)**.

3. First-Order Algebra Without Terms

In this section, it is shown how one may construct in a very natural way an algebraic theory whose algebras may be used to "simulate" the deductive apparatus of first-order logic without terms. In the institution of first-order logic without terms, to each chosen relational signature X, there are associated X-sentences and X-structures that are related to each other via the X-satisfaction relation. Each signature is related to the remaining signatures via signature interpretations, i.e., mappings in **Sign**, which also affect the sentences and the structures accordingly. These relations between the signatures impose a certain uniformity in the structure of the sentences and the models. This makes it possible to "unite" the different relational structures, by exploiting the common features in the construction. Surveying the signatures individually, it is easy to observe that all the sentences, regardless of signature, use common connectives and quantifiers and that substitution operations of formulas of one signature for basic relation symbols of another are performed uniformly. The most important difference between signatures is in the number and the arity of relation symbols used to construct the formulas of each signature. Grouping the common features together, an algebraic theory in **HSet**, representing "algebras of a single type" will be obtained, such that, the set of variables, used to construct this single type terms, will correspond to basic relations of the relational structures of the institution of first-order logic. Moreover, substitution of a term for a variable in this single type context will correspond to interpreting a signature into another signature in the institution of first-order logic, i.e., to a substitution of a formula of one signature for a basic relation symbol in another signature.

The Adjunction

The well-known correspondence between algebraic theories and adjunctions is the key to the modern theory of categorical abstract algebraic logic as developed in [15] (see also [16] and [17]). First, it provides the right framework for generalizing the algebraization process, previously restricted to the category of sets, to arbitrary categories. Second, it makes it possible, by incorporating the substitution operators in the object language and following the path from adjunctions to algebraic theories, to naturally construct algebraic counterparts of multi-signature logical systems without the need for the traditional ad-hoc preprocessing of the system.

In this subsection, it is shown how to naturally extract an adjunction out of the construction of the signature category **Sign** of the institution of first-order logic without terms.

First, define a functor $F : \mathbf{HSet} \to \mathbf{Sign}$ by

$$F(X) = X, \quad \text{for all } X \in |\mathbf{HSet}|,$$

and, given $f : X \to Y \in \mathrm{Mor}(\mathbf{HSet})$,

$$F(f) = j_Y f : X \rightharpoonup Y.$$

Next, define a functor $U : \mathbf{Sign} \to \mathbf{HSet}$ by

$$U(X) = \mathrm{Fm}_{\mathcal{L}}(X), \quad \text{for all } X \in |\mathbf{Sign}|,$$

and, given $f : X \rightharpoonup Y \in \mathrm{Mor}(\mathbf{Sign})$,

$$U(f) = f^* : \mathrm{Fm}_{\mathcal{L}}(X) \to \mathrm{Fm}_{\mathcal{L}}(Y).$$

Then, if $f : X \rightharpoonup Y, g : Y \rightharpoonup Z \in \mathrm{Mor}(\mathbf{Sign})$, we have

$$\begin{aligned} U(g \circ f) &= (g \circ f)^* \\ &= g^* f^* \quad \text{(by Lemma 4)} \\ &= U(g)U(f), \end{aligned}$$

i.e., U is indeed a functor, as claimed.

Finally, define natural transformations $\eta : I_{\mathbf{HSet}} \to UF$ by $\eta_X : X \to \mathrm{Fm}_{\mathcal{L}}(X)$, with

$$\eta_X = j_X, \quad \text{for all } X \in |\mathbf{HSet}|,$$

and $\epsilon : FU \to I_{\mathbf{Sign}}$ by $\epsilon_X : \mathrm{Fm}_{\mathcal{L}}(X) \rightharpoonup X$, with

$$\epsilon_X = i_{\mathrm{Fm}_{\mathcal{L}}(X)}, \quad \text{for all } X \in |\mathbf{Sign}|.$$

It is now shown that η and ϵ are in fact natural transformations. To this end, let $f : X \to Y \in \mathrm{Mor}(\mathbf{HSet})$. It suffices to show that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathrm{Fm}_{\mathcal{L}}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (j_Y f)^*} \\ Y & \xrightarrow[\eta_Y]{} & \mathrm{Fm}_{\mathcal{L}}(Y) \end{array}$$

We have

$$(j_Y f)^*_N(\eta_{X_N}(x)) = (j_Y f)^*_N(\overline{x}) = (j_Y f)_N(x) = \eta_{Y_N}(f_N(x)).$$

For ϵ, let $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{Sign})$. It suffices to show the commutativity of

$$\begin{array}{ccc} \mathrm{Fm}_{\mathcal{L}}(X) & \xrightarrow{\epsilon_X} & X \\ {\scriptstyle j_{\mathrm{Fm}_{\mathcal{L}}(Y)} f^*}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Fm}_{\mathcal{L}}(Y) & \xrightarrow[\epsilon_Y]{} & Y \end{array}$$

We have

$$f \circ \epsilon_X = f^* \epsilon_X = f^* = \epsilon_Y f^* = \epsilon^*_Y j_{\mathrm{Fm}_{\mathcal{L}}(Y)} f^* = \epsilon_Y \circ F(U(f)).$$

The next theorem may now be proved that shows that the functors $F : \mathbf{HSet} \to \mathbf{Sign}$ and $U : \mathbf{Sign} \to \mathbf{HSet}$ are adjoints with unit η and counit ϵ. This will conclude the first stage in the construction of the algebraic counterpart of the institution of first-order logic.

Theorem 7. $\langle F, U, \eta, \epsilon \rangle : \mathbf{HSet} \to \mathbf{Sign}$ *is an adjunction.*

Proof. It only remains to show that the following diagrams commute

$$\begin{array}{ccc} \mathrm{Fm}_{\mathcal{L}}(X) & \xrightarrow{\eta_{\mathrm{Fm}_{\mathcal{L}}(X)}} & \mathrm{Fm}_{\mathcal{L}}(\mathrm{Fm}_{\mathcal{L}}(X)) \\ & {\scriptstyle i_{\mathrm{Fm}_{\mathcal{L}}(X)}}\searrow & \downarrow{\scriptstyle i^*_{\mathrm{Fm}_{\mathcal{L}}(X)}} \\ & & \mathrm{Fm}_{\mathcal{L}}(X) \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\eta_{\mathrm{Fm}_{\mathcal{L}}(Y)} \eta_Y} & \mathrm{Fm}_{\mathcal{L}}(Y) \\ & {\scriptstyle \eta_Y}\searrow & \downarrow{\scriptstyle i_{\mathrm{Fm}_{\mathcal{L}}(Y)}} \\ & & Y \end{array}$$

Commutativity of the first triangle follows directly from the fact that the morphism $j_{\mathrm{Fm}_{\mathcal{L}}(X)}$ is the identity morphism of $\circ$. For the second diagram, we have

$$\begin{aligned} i_{\mathrm{Fm}_{\mathcal{L}}(Y)} \circ (j_{\mathrm{Fm}_{\mathcal{L}}(Y)} j_Y) &= i^*_{\mathrm{Fm}_{\mathcal{L}}(Y)}(j_{\mathrm{Fm}_{\mathcal{L}}(Y)} j_Y) \\ &= (i^*_{\mathrm{Fm}_{\mathcal{L}}(Y)} j_{\mathrm{Fm}_{\mathcal{L}}(Y)}) j_Y \\ &= i_{\mathrm{Fm}_{\mathcal{L}}(Y)} j_Y \\ &= j_Y. \end{aligned}$$

□

The Algebraic Theory

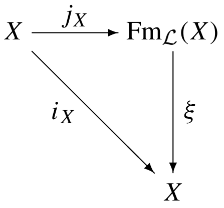
In this subsection, we briefly review how the adjunction $\langle F, U, \eta, \epsilon\rangle : \mathbf{HSet} \to \mathbf{Sign}$ gives rise to an algebraic theory $\mathbf{T} = \langle T, \eta, \mu\rangle$ in monoid form in **HSet**. As already mentioned, this process of extracting the algebraic theory **T** out of the adjunction relating the signature category of the multi-signature logic with its "underlying category" is the gist of the modern, categorical theory of algebraizability. It is the process of natural, automatic abstraction of the common features relating the different syntax components of the logical system that are, however, interrelated via the **Sign**-morphisms, which may be viewed as uniformly applicable substitution operators. Once this is done, to complete the algebraization, it only remains to investigate whether the semantical (or syntactical) deduction of the logical system can be simulated via the semantical deduction induced by some class of **T**-algebras.

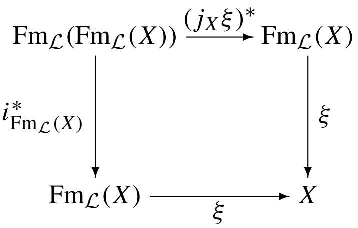
To create the algebraic theory, we set $T = UF$ and $\mu = U\epsilon_F$. Since the Kleisli category of an algebraic theory has, by definition, as objects the same objects with the underlying category of the theory and as morphisms from an object X to an object Y all the morphisms in the underlying category from X to $T(Y)$, with composition $\circ_K$ given by $g \circ_K f = \mu_Y T(g) f$, for all $f : X \to T(Y)$, $g : Y \to T(Z) \in \mathrm{Mor}(\mathbf{HSet})$, it is easy to see that in this case $\mathbf{HSet_T} = \mathbf{Sign}$ and that the Kleisli comparison functor $K = I_{\mathbf{Sign}}$. For instance, the following calculation shows that the Kleisli composition $\circ_K$ coincides with the composition $\circ$.

$$g \circ_K f = \mu_Z T(g) f = U(\epsilon_{F(Z)}) U(F(g)) f = U(\epsilon_{F(Z)} \circ F(g)) f = \\ U(i^*_{\mathrm{Fm}_{\mathcal{L}}(F(Z))} j_Z g) f = U(g) f = g^* f = g \circ f.$$

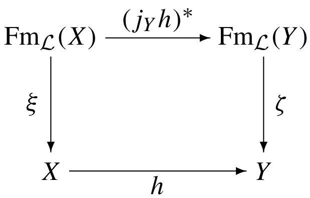
Therefore **Sign** is the category of all free algebras of the algebraic theory **T** in **HSet**.

A **T**-algebra $\mathbf{X} = \langle X, \xi\rangle$ in **HSet** is now a pair consisting of an h-set X together with an **HSet**-morphism $\xi : \mathrm{Fm}_{\mathcal{L}}(X) \to X$, such that the following diagrams commute





Given two **T**-algebras $\mathbf{X} = \langle X, \xi\rangle$ and $\mathbf{Y} = \langle Y, \zeta\rangle$, a **T**-algebra homomorphism $h : \mathbf{X} \to \mathbf{Y}$ is an **HSet**-morphism $h : X \to Y$, such that the following diagram commutes



The Eilenberg-Moore category of **T**-algebras in **HSet** is denoted by $\mathbf{HSet^T}$.

The Algebras of Relations

In this subsection, it is shown how, given a set A, a **T**-algebra $\mathbf{A}^*$, that corresponds to all relations on A, may be associated with it. This association is very important for several reasons. First, it gives a concrete example of what **T**-algebras look like. Intuitively speaking, $\mathbf{A}^*$ will have as universe the set of all finitary relations on A and its structure map will show how these relations behave under the operations of complementation, intersection and cylindrification. Second, it will be shown that, roughly speaking, the X-formulas that an X-structure $\mathbf{A}$, with universe A, satisfies in the institution of first-order logic are in one-to-one correspondence with the X-equations that are satisfied by $\mathbf{A}^*$ in an algebraic institution based on the algebraic theory **T**, once we "make fundamental operations in $\mathbf{A}^*$ agree with basic relations of $\mathbf{A}$". We call $\mathbf{A}^*$ the **algebra of relations on** A. This process will allow the construction of a class of algebras of relations whose semantical equational entailment will be used to algebraize first-order logic without terms.

Recall that, given a set A, $\mathrm{Rel}(A)$ denotes the h-set whose N-th level $\mathrm{Rel}_N(A)$ consists of all relations $r \subseteq A^\omega$ that depend only on the individual variables indexed by elements in N. Given such a set A, we define $\mathbf{A}^* = \langle \mathrm{Rel}(A), \xi_A \rangle$, where ξ_A is the map that extends the interpretation of equality and of all the remaining finitary relations to all formulas over relations. More formally, $\xi_A : \mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A)) \to \mathrm{Rel}(A) \in \mathrm{Mor}(\mathbf{HSet})$ is determined by $\xi_{A_N} : \mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_N \to \mathrm{Rel}_N(A)$, defined by recursion on the structure of $\mathrm{Rel}(A)$-formulas as follows:

- $\xi_{A_N}(\mathrm{v}_i \approx \mathrm{v}_j) = \{\vec{a} \in A^\omega : a_i = a_j\}$, for all $i, j \in N$,
- $\xi_{A_N}(\overline{x}) = x$, for every $x \in \mathrm{Rel}_N(A)$,
- $\xi_{A_N}(\neg\phi) = A^\omega - \xi_{A_N}(\phi)$, $\xi_{A_N}(\phi_1 \wedge \phi_2) = \xi_{A_N}(\phi_1) \cap \xi_{A_N}(\phi_2)$, for all $\phi, \phi_1, \phi_2 \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_N$,
- $\xi_{A_N}(\exists_k\phi) = \{\vec{b} \in A^\omega : a_i = b_i \ \forall i \neq k \quad \text{and} \quad \vec{a} \in \xi_{A_{N\cup\{k\}}}(\phi)\}$.

It is not very hard to verify that ξ_A is indeed an **HSet**-morphism. It will be shown that $\mathbf{A}^*$ is a **T**-algebra.

Theorem 8. $\mathbf{A}^* = \langle \mathrm{Rel}(A), \xi_A \rangle$ *is a* **T**-*algebra.*

Proof. By the definition of a **T**-algebra, it suffices to check commutativity of the following diagrams

$$
\begin{array}{ccc}
\mathrm{Rel}(A) & \xrightarrow{\eta_{\mathrm{Rel}(A)}} & \mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A)) \\
& {\scriptstyle i_{\mathrm{Rel}(A)}} \searrow & \downarrow {\scriptstyle \xi_A} \\
& & \mathrm{Rel}(A)
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Fm}_{\mathcal{L}}(\mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))) & \xrightarrow{(\eta_{\mathrm{Rel}(A)}\xi_A)^*} & \mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A)) \\
{\scriptstyle i^*_{\mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))}} \downarrow & & \downarrow {\scriptstyle \xi_A} \\
\mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A)) & \xrightarrow[\xi_A]{} & \mathrm{Rel}(A)
\end{array}
$$

For the triangle, let $N \subset_{\mathrm{f}} \omega$, $r \in \mathrm{Rel}_N(A)$. Then

$$
\begin{aligned}
\xi_{A_N}(\eta_{\mathrm{Rel}(A)_N}(r)) &= \xi_{A_N}(\overline{r}) \\
&= r.
\end{aligned}
$$

For the rectangle, induction on the structure of $\mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))$-formulas is used. If $i, j \in N$, then

$$\begin{aligned}\xi_{A_N}((\eta_{\mathrm{Rel}(A)}\xi_A)^*_N(\mathrm{v}_i \approx \mathrm{v}_j)) &= \xi_{A_N}(\mathrm{v}_i \approx \mathrm{v}_j)\\ &= \xi_{A_N}(i^*_{\mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_N}(\mathrm{v}_i \approx \mathrm{v}_j)).\end{aligned}$$

If $x \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_N$, then

$$\begin{aligned}\xi_{A_N}((\eta_{\mathrm{Rel}(A)}\xi_A)^*_N(\overline{x})) &= \xi_{A_N}((\eta_{\mathrm{Rel}(A)}\xi_A)_N(x))\\ &= \xi_{A_N}(\eta_{\mathrm{Rel}(A)_N}(\xi_{A_N}(x)))\\ &= \xi_{A_N}(x) \quad \text{(by commutativity of the triangle)}\\ &= \xi_{A_N}(i^*_{\mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_N}(\overline{x})).\end{aligned}$$

If $\phi \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A)))_N$, then

$$\begin{aligned}\xi_{A_N}((\eta_{\mathrm{Rel}(A)}\xi_A)^*_N(\neg\phi)) &= \xi_{A_N}(\neg(\eta_{\mathrm{Rel}(A)}\xi_A)^*_N(\phi))\\ &= A^\omega - \xi_{A_N}((\eta_{\mathrm{Rel}(A)}\xi_A)^*_N(\phi))\\ &= A^\omega - \xi_{A_N}(i^*_{\mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_N}(\phi))\\ &\quad \text{(by the induction hypothesis)}\\ &= \xi_{A_N}(\neg i^*_{\mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_N}(\phi))\\ &= \xi_{A_N}(i^*_{\mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_N}(\neg\phi)).\end{aligned}$$

The case of $\wedge$ is handled similarly. Finally, if $\phi \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A)))_{N\cup\{k\}}$,

$$\xi_{A_N}((\eta_{\mathrm{Rel}(A)}\xi_A)^*_N(\exists_k\phi)) =$$

$$\begin{aligned}&= \xi_{A_N}(\exists_k(\eta_{\mathrm{Rel}(A)}\xi_A)^*_{N\cup\{k\}}(\phi))\\ &= \{\vec{b} \in A^\omega : b_i = a_i\ \forall i \neq k\\ &\quad \text{and} \quad \vec{a} \in \xi_{A_{N\cup\{k\}}}((\eta_{\mathrm{Rel}(A)}\xi_A)^*_{N\cup\{k\}}(\phi))\}\\ &= \{\vec{b} \in A^\omega : b_i = a_i\ \forall i \neq k\\ &\quad \text{and} \quad \vec{a} \in \xi_{A_{N\cup\{k\}}}(i^*_{\mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_{N\cup\{k\}}}(\phi))\}\\ &\quad \text{(by the induction hypothesis)}\\ &= \xi_{A_N}(\exists_k(i^*_{\mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_{N\cup\{k\}}}(\phi)))\\ &= \xi_{A_N}(i^*_{\mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_N}(\exists_k\phi)).\end{aligned}$$ □

4. The Algebraization of First-Order Logic

Roughly speaking, algebraizing a logical system means associating with it an algebraic system in such a way that, first, each system may be syntactically interpreted in the other and, second, the entailment of each system may be simulated by the entailment of the other under the chosen syntactical interpretations. More specifically, the algebraization process of a multi-signature logic consists of two main components. A type of algebras has to be chosen that abstracts the syntactical features of the logic common to all its signature components. This choice makes possible the syntactical interpretation of the logic into the algebraic system and vice-versa. Once the type has been chosen, a class of algebras of that type has to be selected in such a way that the semantical consequence relation induced by it may simulate and be simulated by the consequence relation of the logical system under the previously chosen syntactical interpretations.

In the institution context, given an institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ that represents the multi-signature logical system to be algebraized, the following steps have to be carried out. An algebraic theory $\mathbf{T}$ in monoid form in a category $\mathbf{C}$ has to be chosen that corresponds to the choice of a single type of algebras. A full subcategory $\mathbf{L}$ of the Kleisli category $\mathbf{C_T}$ and a subcategory $\mathbf{Q}$ of the Eilenberg-Moore category $\mathbf{C^T}$ of $\mathbf{T}$-algebras have to be selected in such a way that an institution $\mathcal{I}^{\mathbf{L}}_{\mathbf{Q}} = \langle \mathbf{L}, \mathrm{EQ}, \mathrm{ALG}, \models \rangle$ may be constructed that is deductively equivalent to $\mathcal{I}$. This means that there exist functors $F : \mathbf{Sign} \to \mathbf{L}$ and $G : \mathbf{L} \to \mathbf{Sign}$, that are part of an adjoint equivalence $\langle F, G, \eta, \epsilon \rangle : \mathbf{Sign} \to \mathbf{L}$, modelling the syntactic interpretations, and natural transformations $\alpha : \mathrm{SEN} \to \mathcal{P}\mathrm{EQ}F$, $\beta : \mathrm{EQ} \to \mathcal{P}\mathrm{SEN}G$, such that, for every choice of $\Sigma \in |\mathbf{Sign}|$, $L \in |\mathbf{L}|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, $\Psi \cup \{\psi\} \subseteq \mathrm{EQ}(L)$, the following relations hold

$$\phi \in \Phi^c \quad \text{if and only if} \quad \alpha_\Sigma(\phi) \subseteq \alpha_\Sigma(\Phi)^c, \tag{2}$$

$$\psi \in \Psi^c \quad \text{if and only if} \quad \beta_L(\psi) \subseteq \beta_L(\Psi)^c, \tag{3}$$

$$\beta_{F(\Sigma)}(\alpha_\Sigma(\phi))^c = \mathrm{SEN}(\eta_\Sigma)(\phi)^c \quad \text{and} \quad \mathrm{EQ}(\epsilon_L)(\alpha_{G(L)}(\beta_L(\psi)))^c = \{\psi\}^c. \tag{4}$$

Thus, α and β simulate the deduction mechanism of $\mathcal{I}$ into that of $\mathcal{I}^{\mathbf{L}}_{\mathbf{Q}}$ and vice-versa and are inverses of each other.

By a result of [15], relation (2) and the second equality in (4) are sufficient for deductive equivalence.

For the special case of the institution of first-order logic $\mathcal{FOL} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ the algebraic theory is chosen to be the theory $\mathbf{T} = \langle T, \eta, \mu \rangle$ in $\mathbf{HSet}$ that was constructed in the previous section. Set $\mathbf{L} = \mathbf{Sign} = \mathbf{HSet_T}$ and let $\mathbf{Q}$ be the full subcategory of $\mathbf{HSet^T}$ with collection of objects

$$\{\mathbf{A}^* = \langle \mathrm{Rel}(A), \xi_A \rangle : A \in |\mathbf{Set}|\}.$$

Construct the institution $\mathcal{I}_{\mathbf{Q}} = \langle \mathbf{Sign}, \mathrm{EQ}, \mathrm{ALG}, \models \rangle$ as follows:

(i) $\mathrm{EQ} = \mathrm{SEN}^2$.

(ii) For every $X \in |\mathbf{Sign}|$, $\mathrm{ALG}(X)$ is the category with objects pairs $\langle \mathbf{A}^*, f \rangle$, $\mathbf{A}^* \in |\mathbf{Q}|$, $f : X \rightharpoondown \mathrm{Rel}(A) \in \mathrm{Mor}(\mathbf{Sign})$, and morphisms $h : \langle \mathbf{A}^*, f \rangle \to \langle \mathbf{B}^*, g \rangle$, $\mathbf{T}$-algebra homomorphism $h : \mathbf{A}^* \to \mathbf{B}^*$, such that $g = h \circ f$. Moreover, given $k : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{Sign})$, $\mathrm{ALG}(k) : \mathrm{ALG}(Y) \to \mathrm{ALG}(X)$ is the functor that maps an object $\langle \mathbf{A}^*, f \rangle \in |\mathrm{ALG}(Y)|$ to $\langle \mathbf{A}^*, f \circ k \rangle \in |\mathrm{ALG}(X)|$ and a morphism $h : \langle \mathbf{A}^*, f \rangle \to \langle \mathbf{B}^*, g \rangle$ in $\mathrm{MOD}(Y)$ to the morphism $\mathrm{MOD}(k)(h) : \langle \mathbf{A}^*, f \circ k \rangle \to \langle \mathbf{B}^*, g \circ k \rangle$ in $\mathrm{MOD}(X)$, with $\mathrm{MOD}(k)(h) = h$.

(iii) Finally, satisfaction in $\mathcal{I}_{\mathbf{Q}}$ is defined, for every $X \in |\mathbf{Sign}|$, by

$$\langle \mathbf{A}^*, f \rangle \models_X \phi \approx \psi \quad \text{if and only if} \quad \xi_A(f^*(\phi)) = \xi_A(f^*(\psi)),$$

for all $\langle \mathbf{A}^*, f \rangle \in |\mathrm{ALG}(X)|$, $\phi \approx \psi \in \mathrm{EQ}(X)$.

Before stating and proving the main result of the paper on the deductive equivalence of $\mathcal{FOL}$ and $\mathcal{I}_{\mathbf{Q}}$ the following lemma, that will be needed in the proof, is formulated. It formally expresses the fact that algebraic evaluation of a formula in a given algebra of relations over a set A under a specific algebraic interpretation yields the logical truth set of the formula in the set $\mathrm{Rel}(A)$ under a suitably chosen logical interpretation. The proof is by routine induction on the structure of formulas and will therefore be omitted.

Lemma 9. *Let* $X \in |\mathbf{Sign}|$, $N \subset_{\mathrm{f}} \omega$, $\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_N$ *and* $\langle\langle \mathrm{Rel}(A), \xi_A\rangle, f\rangle \in |\mathrm{ALG}(X)|$. *Then, if* $\mathbf{A} = \langle A, \xi_A f\rangle$,

$$\phi^{\mathbf{A}} = \xi_{A_N}(f_N^*(\phi)).$$

Given $X \in |\mathbf{Sign}|$, $N \subset_{\mathrm{f}} \omega$, $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, define

$$T(\phi) = \neg(\phi \wedge \neg\phi), \quad \phi \to \psi = \neg(\phi \wedge \neg\psi) \quad \text{and}$$

$$\phi \leftrightarrow \psi = (\phi \to \psi) \wedge (\psi \to \phi).$$

With the help of Lemma 9, the following theorem, the main result of the paper, may now be proved.

Theorem 10. $\mathcal{FOL} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ *and* $\mathcal{I}_{\mathbf{Q}} = \langle \mathbf{Sign}, \mathrm{EQ}, \mathrm{ALG}, \models\rangle$ *are deductively equivalent institutions.*

Proof. Take $F = G = I_{\mathbf{Sign}}$ as the signature functors and define the natural transformations $\alpha : \mathrm{SEN} \to \mathcal{P}\mathrm{EQ}$ and $\beta : \mathrm{EQ} \to \mathcal{P}\mathrm{SEN}$ by $\alpha_X : \mathrm{SEN}(X) \to \mathcal{P}(\mathrm{EQ}(X))$, with

$$\alpha_X(\phi) = \{\phi \approx T(\phi)\}, \quad \text{for all } \phi \in \mathrm{SEN}(X),$$

and $\beta_X : \mathrm{EQ}(X) \to \mathcal{P}(\mathrm{SEN}(X))$, with

$$\beta_X(\phi \approx \psi) = \{\phi \leftrightarrow \psi\}, \quad \text{for all } \phi \approx \psi \in \mathrm{EQ}(X).$$

It is straightforward to check that α and β are indeed natural transformations. It remains to show that (2) and the second equation in (4) hold.

For (2), let $X \in |\mathbf{Sign}|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(X)$. We need to show that

$$\phi \in \Phi^{C^{\mathcal{FOL}}} \quad \text{if and only if} \quad \phi \approx T(\phi) \in \{\psi \approx T(\psi) : \psi \in \Phi\}^{C^{\mathcal{I}_{\mathbf{Q}}}}.$$

If $\phi \in \Phi^{C^{\mathcal{FOL}}}$, then, for all $\mathbf{A} \in |\mathrm{MOD}(X)|$, we have

$$\mathbf{A} \models_X \Phi \quad \text{implies} \quad \mathbf{A} \models_X \phi. \tag{5}$$

Now, let $\langle\langle \mathrm{Rel}(A), \xi_A\rangle, f\rangle \in |\mathrm{ALG}(X)|$, such that $\langle\langle \mathrm{Rel}(A), \xi_A\rangle, f\rangle \models_X \{\psi \approx T(\psi) : \psi \in \Phi\}$. Thus, $\xi_A(f^*(\psi)) = \xi_A(f^*(T(\psi)))$, for all $\psi \in \Phi$. Hence, by Lemma 9, if $\mathbf{A} = \langle A, \xi_A f\rangle$, $\psi^{\mathbf{A}} = T(\psi)^{\mathbf{A}}$, for all $\psi \in \Phi$. Therefore $\mathbf{A} \models_X \psi$, for all $\psi \in \Phi$, and, hence, by (5), $\mathbf{A} \models_X \phi$. Reversing the steps in the deduction above gives $\langle\langle \mathrm{Rel}(A), \xi_A\rangle, f\rangle \models_X \phi \approx T(\phi)$, i.e., $\phi \approx T(\phi) \in \{\psi \approx T(\psi) : \psi \in \Phi\}^{C^{\mathcal{I}_{\mathbf{Q}}}}$, as was to be shown.

Conversely, if $\phi \approx T(\phi) \in \{\psi \approx T(\psi) : \psi \in \Phi\}^{C\mathcal{I}_{\mathbf{Q}}}$, then, for all $\langle\langle \mathrm{Rel}(A), \xi_A\rangle, f\rangle \in |\mathbf{ALG}(X)|$, we have

$$\langle\langle \mathrm{Rel}(A), \xi_A\rangle, f\rangle \models_X \psi \approx T(\psi), \text{ for all } \psi \in \Phi, \tag{6}$$

$$\text{implies } \langle\langle \mathrm{Rel}(A), \xi_A\rangle, f\rangle \models_X \phi \approx T(\phi).$$

Let $\mathbf{A} \in |\mathbf{MOD}(X)|$, such that $\mathbf{A} \models_X \Phi$. Then $\psi^{\mathbf{A}} = T(\psi)^{\mathbf{A}}$, for all $\psi \in \Phi$, i.e., $\xi_A(f^*(\psi)) = \xi_A(f^*(T(\psi)))$, for all $\psi \in \Phi$. Thus, $\langle\langle \mathrm{Rel}(A), \xi_A\rangle, f\rangle \models_X \psi \approx T(\psi)$, for all $\psi \in \Phi$, and, therefore, using (6), $\langle\langle \mathrm{Rel}(A), \xi_A\rangle, f\rangle \models_X \phi \approx T(\phi)$. Reversing the steps, we get $\mathbf{A} \models_X \phi$, i.e., $\phi \in \Phi^{C\mathcal{F}OL}$, as required.

For the second equality in (4), given $\phi, \psi \in \mathrm{SEN}(X)$, it suffices to show that

$$\{\phi \approx \psi\}^{C\mathcal{I}_{\mathbf{Q}}} = \{\phi \leftrightarrow \psi \approx T(\phi \leftrightarrow \psi)\}^{C\mathcal{I}_{\mathbf{Q}}}.$$

To this end, we show that, for all $\langle\langle \mathrm{Rel}(A), \xi_A\rangle, f\rangle \in |\mathbf{ALG}(X)|$,

$$\langle\langle \mathrm{Rel}(A), \xi_A\rangle, f\rangle \models_X \phi \approx \psi \quad \text{iff}$$

$$\langle\langle \mathrm{Rel}(A), \xi_A\rangle, f\rangle \models_X \phi \leftrightarrow \psi \approx T(\phi \leftrightarrow \psi).$$

We have

$$\begin{aligned}
&\langle\langle \mathrm{Rel}(A), \xi_A\rangle, f\rangle \models_X \phi \leftrightarrow \psi \approx T(\phi \leftrightarrow \psi) \quad \text{iff} \\
&\qquad \text{iff } \xi_A(f^*(\phi \leftrightarrow \psi)) = \xi_A(f^*(T(\phi \leftrightarrow \psi))) \\
&\qquad \text{iff } (\phi \leftrightarrow \psi)^{\mathbf{A}} = T(\phi \leftrightarrow \psi)^{\mathbf{A}} \quad \text{(by Lemma 9)} \\
&\qquad \text{iff } \phi^{\mathbf{A}} = \psi^{\mathbf{A}} \\
&\qquad \text{iff } \xi_A(f^*(\phi)) = \xi_A(f^*(\psi)) \quad \text{(by Lemma 9)} \\
&\qquad \text{iff } \langle\langle \mathrm{Rel}(A), \xi_A\rangle, f\rangle \models_X \phi \approx \psi.
\end{aligned}$$

□

Directions for Future Research

The investigation of the relationship between the variety of algebras represented by the algebraic theory over **HSet**, that was constructed in this paper, and the variety of cylindric algebras [11] was carried out in [20]. A similar relationship between a categorically constructed variety of clone algebras and the substitution algebras of Feldman [6] was studied in [19]. The work done in this direction has a double goal. The first is to enrich the abstract theory with specific examples that will, in turn, give new impetus to the development of the theory. The second is to use the examples as guides towards abstracting, unifying and incorporating new features into the general theory. Each of the following two directions represents one of these two goals.

First-order logic with terms (FOLT) is by far the most interesting example of a system that has not been treated yet. In [4], cylindric algebras were combined with the substitution algebras of Feldman [6] to provide an algebraization of FOLT. The formulation of a categorical algebraization of an institution representing a system of FOLT is a natural direction of investigation in the pursuit of more examples.

On the other hand, given the common features of the constructions in [18] and in the present work, it would be interesting from the theoretical viewpoint to investigate the existence of conditions, on a class of institutions containing both the institution of equational logic of [18] and $\mathcal{FOL}$, necessary and sufficient for members of these classes to be algebraizable. It would be nice if these conditions were given in a form of an intrinsic characterization of algebraizability similar to the intrinsic characterizations of algebraizable deductive systems given in [3].

References

1. Barr, M., Wells, C.: Category Theory for Computing Science, Third Edition, Les Publications CRM, Montréal, 1999
2. Blok, W.J., Pigozzi, D.: Protoalgebraic Logics. Studia Logica **45**, 337–369 (1986)
3. Blok, W.J., Pigozzi, D.: Algebraizable Logics. Memoirs Am. Math. Soc. **77**(396) (1989)
4. Cirulis, J.: An Algebraization of First-Order Logic With Terms. In: H. Andréka, J.D. Monk, I. Németi (eds.), Algebraic Logic (Proc. Conf. Budapest 1988), Colloq. Math. Soc. J. Bolyai, North-Holland, Amsterdam, 1991, pp. 125–146
5. Czelakowski, J.: Equivalential Logics I, II. Studia Logica **40**, 227–236, 355–372 (1981)
6. Feldman, N.: Axiomatization of Polynomial Substitution Algebras. J. Symbolic Logic **47**, 481–492 (1982)
7. Fiadeiro, J., Sernadas, A.: Structuring Theories on Consequence. In: D. Sannella, A. Tarlecki (eds.), Recent Trends in Data Type Specification, Lecture Notes in Computer Science, Vol. 332, Springer-Verlag, New York 1988, pp. 44–72
8. Goguen, J.A., Burstall, R.M.: Introducing Institutions. In: E. Clarke, D. Kozen (eds.), Proceedings of the Logic of Programming Workshop, Lecture Notes in Computer Science, Vol. 164, Springer-Verlag, New York, 1984, pp. 221–256
9. Goguen, J.A., Burstall, R.M.: Institutions: Abstract Model Theory for Specification and Programming. J. Asso. Comput. Mach. **39**(1) (1992) pp. 95–146
10. Halmos, P.R.: Algebraic Logic, Chelsea Publishing Company, New York, 1962
11. Henkin, L., Monk, J.D., Tarski, A.: Cylindric Algebras. Part I. Studies in Logic and the Foundations of Mathematics, 64, North-Holland, Amsterdam, 1985
12. Henkin, L., Monk, J.D., Tarski, A.: Cylindric Algebras. Part II. Studies in Logic and the Foundations of Mathematics, 115, North-Holland, Amsterdam, 1985
13. Mac Lane, S.: Categories for the Working Mathematician, Springer-Verlag, New York 1971
14. Manes, E.G.: Algebraic Theories, Springer-Verlag, New York 1976
15. Voutsadakis, G.: Categorical abstract algebraic logic. Doctoral Dissertation, Iowa State University, Ames, Iowa, 1998
16. Voutsadakis, G.: Categorical Abstract Algebraic Logic: Equivalent Institutions, Studia Logica **74**(1/2), 275–311 (2003)
17. Voutsadakis, G.: Categorical Abstract Algebraic Logic: Algebraizable Institutions, Applied Categorical Structures, **10**(6), 531–568 (2002)
18. Voutsadakis, G.: Categorical Abstract Algebraic Logic: Categorical Algebraization of Equational Logic. To appear in the Logic Journal of the IGPL
19. Voutsadakis, G.: A Categorical Construction of a Variety of Clone Algebras. Scientiae Mathematicae Japonicae **8**, 215–225 (2003)
20. Voutsadakis, G.: On the Categorical Algebras of First-Order Logic. Scientiae Mathematicae Japonicae **10**(16) (2004)